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Quantum Measurement Trees, II:
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Abstract:

Given a quantum state in the finite-dimensional Hilbert space $\mathbb{C}^n$, the range of possible values of a quantum observable is usually identified with the discrete spectrum of eigenvalues of a corresponding Hermitian matrix. Here any such observable is identified with: (i) an "ortho-measurable" function defined on the Boolean "ortho-algebra" generated by the eigenspaces that form an orthogonal decomposition of $\mathbb{C}^n$; (ii) a "numerically identified" orthogonal decomposition of $\mathbb{C}^n$. The latter means that each subspace of the orthogonal decomposition can be uniquely identified by its own attached real number, just as each eigenspace of a Hermitian matrix can be uniquely identified by the corresponding eigenvalue. Furthermore, any density matrix on $\mathbb{C}^n$ is identified with a Bayesian prior "ortho-probability" measure defined on the linear subspaces that make up the Boolean ortho-algebra induced by its eigenspaces. Then any pure quantum state is identified with a degenerate density matrix, and any mixed state with a probability measure on a set of orthogonal pure states. Finally, given any quantum observable, the relevant Bayesian posterior probabilities of measured outcomes can be found by the usual trace formula that extends Born's rule. `[193 words]`

Keywords: Quantum measurement trees, quantum contexts, numerically identified orthogonal decompositions, ortho-measurable functions, density matrices, ortho-probability measures.

1 Introduction and Outline

1.1 Quantum Measurement Trees

Following Hammond (2025), this is the second paper related to a research project on quantum measurement trees. These are similar to the decision trees considered by Raiffa (1968), which are one-player versions of the games in extensive form considered by von Neumann (1928). Following Hammond (1988, 2022), apart from decision nodes. as well as terminal nodes to which consequences are assigned, decision trees can include: (i) *chance nodes*, where what Anscombe and Aumann (1963) call a roulette lottery, with risk described by "objective" or hypothetical probabilities, is resolved; (ii) *event nodes*, where an uncertain event with "subjective" or personal probabilities occurs, depending on the outcome of what Anscombe and Aumann (1963) call a "horse lottery" of the kind considered by Savage (1954).

In fact, a quantum measurement tree is like a decision tree in which, in addition to chance and event nodes: (i) any decision node has become a preparation node at which relevant details of a quantum experiment are determined; (ii) at any terminal node, a possible real measurement of some quantum observable is determined. This project addresses the question of how far quantum measurement trees, with classical probabilities applying at each chance or event node, are able to describe processes that occur in an appropriate Hilbert space of solutions to Schrödinger's wave equation.[1]

The first paper presented two examples that can be described without using any Hilbert space. Here we consider a simple form of quantum measurement tree, where there is only one measurement node on each branch of the tree. The tree starts with a preparation node where an experimental configuration is selected. This configuration has usually been modelled as the combination of: (i) a quantum observable in the form of a Hermitian or self-adjoint matrix; (ii) a quantum state in the form of a density matrix. In combination with the quantum state, that Hermitian matrix determines at the immediately succeeding measurement chance node the specified hypothetical or objective probabilities in a roulette lottery that determines which eigenspace in its spectral decomposition occurs randomly.

[1] As is well known, these solutions in the complex Hilbert space $\mathbb{C}^n$ typically involve unitary matrices. The relevant mathematics of real eigenvalues and orthogonal eigenvectors for Hermitian matrices which represent observables and density matrices in $\mathbb{C}^n$ turns out to be very similar to that for symmetric matrices which, at least in principle if not in physical reality, could represent observables and density matrices whose domain is limited to the real Euclidean space $\mathbb{R}^n$.

1.2 Characterizing Observables and Probability Densities

The contribution of this paper is to describe, for the n-dimensional complex Hilbert space $\mathbb{C}^n$, some relevant natural bijections:

1. first, between each pair of the three spaces of:
   - Hermitian or self-adjoint $n \times n$ matrices $\mathbf{A}$, each with a spectrum $s^{\mathbf{A}} \subset \mathbb{R}$ of eigenvalues;
   - numerically identified orthogonal decompositions of $\mathbb{C}^n$, as described below in Definition 4.7 of Section 4.3;[2]
   - suitably defined functions $\mathbb{C}^n \ni \mathbf{x} \mapsto f^{\mathbf{A}}(\mathbf{x}) \in \mathbb{R} \cup \{*\}$ whose co-domain is a suitable one-point extension of the real line, and which are "ortho-measurable" in the sense of being measurable with respect to the Boolean "ortho-algebra" generated by the linear subspaces that form the spectral orthogonal decomposition $\bigoplus_{\lambda \in s^{\mathbf{A}}} L_\lambda$ of $\mathbb{C}^n$, with the property that $(f^{\mathbf{A}})^{-1}(\lambda) = L_\lambda \setminus \{\mathbf{0}\}$ for all $\lambda \in s^{\mathbf{A}}$;
2. second, between appropriate subspaces of the three spaces specified above that correspond to:
   - $n \times n$ density matrices $\boldsymbol{\rho}$;
   - orthogonal decompositions of $\mathbb{C}^n$ that are numerically identified by probability, as described below in Definition 6.7 of Section 6.2;
   - suitably defined "ortho-probability" measures which are defined on one of the "ortho-measurable" spaces defined above.

1.3 The Quantum Challenge

As discussed in Hammond (2025), it has been widely recognized that the essence of the "quantum challenge" can be seen as the impossibility of accommodating more than a small set of observable quantum phenomena within a single contextual probability space in the sense of Kolmogorov. The main significance of the results concerning bijections in this paper is that the relevant concepts of measurable set and probability measure can remain entirely classical. To make this possible, however, requires using an appropriate version of Vorob′ev's (1962) notion of a family of probability

[2] The term "numerically identified" seems preferable to "numbered" or "enumerated", since those suggest counting using members of the set $\mathbb{N}$ of natural numbers rather than being identified by a real number.

measures defined on a "multi-measurable space". Each of the Boolean or σ-algebras involved can be regarded as its own quantum context. Then staying within just one context is apparently an instance of what Griffiths (2002) calls the "single framework rule".[3] Which context or framework is ultimately relevant depends, of course, on what quantum observable (or set of quantum observables) is selected when an experimental configuration is being created.

1.4 Outline of Paper

Section 2 recapitulates relevant results concerning Hermitian or self-adjoint matrices. In particular, it introduces the notion of an "eigenpair" combining an eigenvalue with its associated eigenspace of eigenvectors. It also emphasizes the orthogonal decomposition of the space $\mathbb{C}^n$ into eigenspaces associated with different eigenvalues.

The following Section 3 considers orthogonal projection matrices, along with ortho-partitions, and ortho-algebras, especially those induced by the spectral decomposition of a Hermitian matrix. This leads to the concept of an ortho-measurable function.

After these essential preliminaries, Section 4 is devoted to the first bijections between the three sets of: (i) Hermitian matrices; (ii) numerically identified orthogonal decompositions; and (iii) ortho-measurable functions.

Next, Section 5 moves on to probabilities, starting with how they relate to wave functions — or rather wave vectors, when time is ignored. Specifically, a normalized wave vector which be regarded as a "pre-probability" latent variable in $\mathbb{C}^n$ that subsequently determines actual probabilities. The latter are given by Born's rule or, in the special case of an ortho-algebra based on the canonical orthonormal basis of $\mathbb{C}^n$, by the squared modulus rule.

Density matrices are the subject of Section 6, which also constructs the second family of bijections between: (i) density matrices; (ii) orthogonal decompositions identified numerically by probability; and (iii) ortho-probability measures restricted to a measurable space which constitutes an ortho-algebra. Then any pure quantum state is identified with a degenerate density matrix, and any mixed state that can be represented by a general density matrix with a probability mixture of orthogonal pure states.[4]

[3] See also Hohenberg (2010) as well as Friedberg and Hohenberg (2014).

[4] This departs from what seems to be some physicists' concept of a mixed state, which does not require orthogonality of the pure states that have positive probability.

The concluding Section 7 begins by summarizing the key features of the quantum measurement tree that this paper has shown how to construct. These include details of how to construct the measurable and probability metaspaces of the kind that were used to describe the randomness that occurs in the two examples presented in Hammond (2025). Finally, it also briefly touches on some key issues that deserve answers in future work.

2 Eigenpairs of Hermitian Matrices in $\mathbb{C}^n$

Most of the definitions and results in this section and the next are standard.

2.1 Hilbert Space and Adjoint Matrices

The n-dimensional linear space $\mathbb{C}^n$ over the algebraic field $\mathbb{C}$ has as its typical member the *column n-vector* $\mathbf{x} = (x_i)_{i=1}^n$ whose n components are complex numbers $x_i \in \mathbb{C}$. Given any complex number $c = a + \mathrm{i}\, b \in \mathbb{C}$, where $a, b \in \mathbb{R}$ and $\mathrm{i}^2 = -1$, its *complex conjugate* is $\bar{c} = a - \mathrm{i}\, b \in \mathbb{C}$.

Definition 2.1. *The* adjoint $\mathbf{A}^*$ *of any $m \times n$ matrix $\mathbf{A} = (a_{ij})_{m \times n}$ is the $n \times m$ transposed conjugate matrix $\mathbf{A}^* = (a^*_{ij})_{n \times m}$ whose ij element satisfies $a^*_{ij} = \bar{a}_{ji}$, implying that $\mathbf{A}^*$ is the transpose $\bar{\mathbf{A}}^\top$ of the matrix $\bar{\mathbf{A}}$ whose ij element is the complex conjugate $\bar{a}_{ij}$ of the element a_{ij} of matrix $\mathbf{A}$.*

An important property of adjoint matrices is that, for all pairs $\mathbf{A}, \mathbf{B}$ of $n \times n$ matrices, one has $(\mathbf{AB})^* = \mathbf{B}^* \mathbf{A}^*$.

The space $\mathbb{C}^n$ becomes a *Hilbert space* when equipped with the complex-valued *inner product* which is defined for all pairs $\mathbf{x} = (x_i)_{i=1}^n$ and $\mathbf{y} = (y_i)_{i=1}^n$ of column n-vectors by $< \mathbf{x}, \mathbf{y} > := \sum_{i=1}^n \bar{x}_i y_i$. This Hilbert space has a real-valued *norm* $\|\mathbf{x}\| \geq 0$ whose square is defined for all n-vectors $\mathbf{x} = (x_i)_{i=1}^n$ by

$$\|\mathbf{x}\|^2 := < \mathbf{x}, \mathbf{x} > = \sum\nolimits_{i=1}^n \bar{x}_i x_i = \sum\nolimits_{i=1}^n |x_i|^2$$

Remark 2.2. *Our notation allows the inner product $< \mathbf{x}, \mathbf{y} > = \sum_{i=1}^n \bar{x}_i y_i$ of any pair of column n-vectors $\mathbf{x} = (x_i)_{i=1}^n$ and $\mathbf{y} = (y_i)_{i=1}^n$ to be rewritten more concisely as the 1×1 "matrix" product $\mathbf{x}^* \mathbf{y}$ of the $1 \times n$ adjoint row matrix $\mathbf{x}^* = ((\bar{x}_i)_{i=1}^n)^\top = \bar{\mathbf{x}}^\top$ with the $n \times 1$ column matrix $\mathbf{y} = (y_i)_{i=1}^n$.*

The two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ are *orthogonal* just in case $\mathbf{x}^* \mathbf{y} = 0$.

The *unit sphere* of $\mathbb{C}^n$ is the set

$$\mathbb{S} := \{\mathbf{x} \in \mathbb{C}^n \mid \|\mathbf{x}\|^2 = 1\} = \{\mathbf{x} \in \mathbb{C}^n \mid \mathbf{x}^* \mathbf{x} = 1\}$$

2.2 Hermitian Matrices and Their Eigenpairs: A Review

The following two definitions extend to $\mathbb{C}^n$ the respective definitions of symmetric and orthogonal matrices in $\mathbb{R}^n$.

Definition 2.3.
- *The* $n \times n$ *matrix* $\mathbf{A}$ *is* Hermitian *or* self-adjoint *just in case its adjoint satisfies* $\mathbf{A}^* = \mathbf{A}$.
- *The* $n \times n$ *matrix* $\mathbf{U}$ *is* unitary *just in case its adjoint satisfies* $\mathbf{U}^* = \mathbf{U}^{-1}$.

The next two definitions are of eigenvalues, eigenvectors, eigenpairs, and then the spectrum of a Hermitian matrix.

Definition 2.4.
- *The pair* $(\lambda, \mathbf{x}) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{\mathbf{0}\})$ *is an* eigenpair *of* $\mathbf{A}$ *just in case* $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, *so* λ *is an* eigenvalue *and* $\mathbf{x} \neq \mathbf{0}$ *is a corresponding* eigenvector.
- *The* spectrum *of a matrix* $\mathbf{A}$ *is the finite set* $s^{\mathbf{A}}$ *of its eigenvalues.*

The following two results are well known properties of the eigenvalues and eigenvectors of a Hermitian matrix.

Proposition 2.5. *Suppose that* $\mathbf{A}$ *is any Hermitian matrix on* $\mathbb{C}^n$ *and that* $(\lambda, \mathbf{x})$ *with* $\mathbf{x} \neq \mathbf{0}$ *is any eigenpair. Then* $\lambda \in \mathbb{R}$.

Proof. Because $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ and so $\mathbf{x}^*\mathbf{A}^* = \bar{\lambda}\mathbf{x}^*$, it follows from $\mathbf{A} = \mathbf{A}^*$ that

$$(\lambda - \bar{\lambda})\mathbf{x}^*\mathbf{x} = \mathbf{x}^*(\lambda\mathbf{x}) - (\bar{\lambda}\mathbf{x}^*)\mathbf{x} = \mathbf{x}^*(\mathbf{A}\mathbf{x}) - (\mathbf{x}^*\mathbf{A}^*)\mathbf{x} = \mathbf{x}^*(\mathbf{A} - \mathbf{A}^*)\mathbf{x} = 0$$

But $\mathbf{x} \neq \mathbf{0}$ implies that $\mathbf{x}^*\mathbf{x} > 0$. It follows that $\lambda = \bar{\lambda}$, so the eigenvalue λ is real. □

Proposition 2.6. *If* $(\lambda, \mathbf{x})$ *are* $(\mu, \mathbf{y})$ *are any two eigenpairs with* $\lambda, \mu \in \mathbb{R}$ *and* $\lambda \neq \mu$, *then* $\mathbf{x}^*\mathbf{y} = 0$.

Proof. Because $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{A}\mathbf{y} = \mu\mathbf{y}$, and $\mathbf{A} = \mathbf{A}^*$, one has

$$(\lambda - \mu)\mathbf{x}^*\mathbf{y} = (\lambda\mathbf{x}^*)\mathbf{y} - \mathbf{x}^*(\mu\mathbf{y}) = (\mathbf{A}\mathbf{x})^*\mathbf{y} - \mathbf{x}^*(\mathbf{A}\mathbf{y}) = \mathbf{x}^*\mathbf{A}\mathbf{y} - \mathbf{x}^*\mathbf{A}\mathbf{y} = 0$$

But if $\lambda \neq \mu$ then $\lambda - \mu \neq 0$, so $\mathbf{x}^*\mathbf{y} = 0$. □

2.3 Orthogonal Decompositions in $\mathbb{C}^n$

Definition 2.7. • *A* linear subspace $L \subset \mathbb{C}^n$ *is a subset that is algebraically closed under linear combinations — i.e., if* $\mathbf{x}, \mathbf{y} \in L$ *and* $\alpha, \beta \in \mathbb{C}$*, then* $\alpha\mathbf{x} + \beta\mathbf{y} \in L$*.*

- *Given any set* $S \subset \mathbb{C}^n$ *of vectors, the set* span S *is the smallest linear subspace* $L \subset \mathbb{C}^n$ *such that* $S \subseteq L$*.*

- *Two linear subspaces* L *and* $\tilde{L}$ *of* $\mathbb{C}^n$ *are* orthogonal *just in case one has* $\mathbf{x}^*\mathbf{y} = 0$ *for all* $\mathbf{x} \in L$ *and* $\mathbf{y} \in \tilde{L}$*.*

- *Given any* $\mathbf{e} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$*, let*

$$[\mathbf{e}] := \mathrm{span}(\{\mathbf{e}\}) := \{\mathbf{x} \in \mathbb{C}^n \mid \exists c \in \mathbb{C} : \mathbf{x} = c\,\mathbf{e}\}$$

  denote the one-dimensional linear subspace of $\mathbb{C}^n$ *that is spanned by the non-zero vector* $\mathbf{e}$*.*

Definition 2.8. *The labelled finite family* $\mathcal{L}^D = \{L_d\}_{d \in D}$ *of linear subspaces* L_d *is:*

- mutually orthogonal *just in case, whenever* $d, d' \in D$ *with* $d \neq d'$*, the two spaces* L_d *and* $L_{d'}$ *are orthogonal;*

- *an* orthogonal decomposition *of* $\mathbb{C}^n$ *just in case the spaces are mutually orthogonal and the* direct *or* vector *sum*

$$\bigoplus\nolimits_{d \in D} L_d := \left\{ \mathbf{x}^+ \in \mathbb{C}^n \;\middle|\; \forall d \in D; \exists \mathbf{x}_d \in L_d : \mathbf{x}^+ = \sum\nolimits_{d \in D} \mathbf{x}_d \right\}$$

  of all the subspaces L_d *in* $\mathcal{L}^D$ *— or equivalently, the linear space* span $\mathcal{L}^D$ *spanned by* $\{L_d\}_{d \in D}$ *— is equal to the whole of* $\mathbb{C}^n$*.*

Definition 2.9. *For each natural number* $m \in \mathbb{N}$*, let* $\mathbb{N}_m \subset \mathbb{N}$ *denote the set* $\{1, 2, \ldots m\}$*.*

Example 2.10. *Let* $\{\mathbf{b}^k\}_{k \in \mathbb{N}_n}$ *be any orthonormal basis of* $\mathbb{C}^n$*, and let* $\{M_r\}_{r \in \mathbb{N}_m}$ *be any partition of* $\mathbb{N}_n$ *into* m *pairwise disjoint non-empty sets. For each* $r \in \mathbb{N}_m$*, define* L_r *as the linear space of dimension* $\#M_r$ *spanned by the set* $\{\mathbf{b}^k \mid k \in M_r\}$ *of basis vectors. Then* $\bigoplus_{r \in \mathbb{N}_m} L_r$ *is an orthogonal decomposition into* m *subspaces.*

Except in the special case when $\#M_r = 1$ *for all* $r \in \mathbb{N}_m$*, this is entirely different from the orthogonal decomposition* $\bigoplus_{k \in \mathbb{N}_n} L_{[\mathbf{b}^k]}$ *of* $\mathbb{C}^n$ *into the collection of* n *one-dimensional subspaces that are each spanned by one of the basis vectors.*

3 Orthogonal Projections in $\mathbb{C}^n$

3.1 Orthogonal Projection Matrices

Definition 3.1. *The* orthogonal projection $\mathbf{x}_L^\perp$ *of any* $\mathbf{x} \in \mathbb{C}^n$ *onto any linear subspace* L *of* $\mathbb{C}^n$ *is the unique* closest point *of* L *to* $\mathbf{x}$ *— i.e., it satisfies* $\{\mathbf{x}_L^\perp\} := \arg\min_{\mathbf{y}\in L} (\mathbf{x}-\mathbf{y})^*(\mathbf{x}-\mathbf{y})$.

The following is a standard result on orthogonal projection matrices, whose proof is omitted.

Proposition 3.2. *Let* L *be any linear subspace of* $\mathbb{C}^n$*.*

1. *For any* $\mathbf{x} \in \mathbb{C}^n$*, its* orthogonal projection $\mathbf{x}_L^\perp$ *onto* L *is the unique point of* L *that satisfies* $(\mathbf{x}-\mathbf{x}_L^\perp)^*(\mathbf{x}_L^\perp - \mathbf{y}) = 0$ *for all* $\mathbf{y} \in L$*.*
2. *The mapping* $\mathbb{C}^n \ni \mathbf{x} \mapsto \mathbf{x}_L^\perp \in L$ *is linear, so there exists a* projection matrix $\mathbf{\Pi}_L$ *such that* $\mathbf{\Pi}_L\mathbf{x} = \mathbf{x}_L^\perp$ *for all* $\mathbf{x} \in \mathbb{C}^n$*.*
3. *The projection matrix* $\mathbf{\Pi}_L$ *satisfies* $\mathbf{\Pi}_L^2 = \mathbf{\Pi}_L = \mathbf{\Pi}_L^*$*.*
4. *If* $\tilde{L}$ *is any linear subspace of* $\mathbb{C}^n$ *that is orthogonal to* L*, then* $\mathbf{\Pi}_L + \mathbf{\Pi}_{\tilde{L}} = \mathbf{\Pi}_{L\oplus\tilde{L}}$*.*
5. *If* $\{\mathbf{b}^k\}_{k=1}^m$ *is any orthonormal basis of* L*, then* $\mathbf{\Pi}_L = \sum_{k=1}^m \mathbf{\Pi}_{[\mathbf{b}^k]}$ *where each* $[\mathbf{b}^k]$ *denotes the one-dimensional subspace spanned by the basis vector* $\mathbf{b}^k$*.*
6. *For all* $\mathbf{x} \in \mathbb{C}^n$*, one has* $\mathbf{x}^*\mathbf{\Pi}_L\mathbf{x} = \mathbf{x}^*\mathbf{\Pi}_L^*\mathbf{\Pi}_L\mathbf{x} = \|\mathbf{\Pi}_L\mathbf{x}\|^2 \geq 0$*.*

3.2 One-Dimensional Orthogonal Projections

The following proposition characterizes orthogonal projections onto one-dimensional linear subspaces of $\mathbb{C}^n$.

Proposition 3.3. *Given any* n*-vector* $\mathbf{e}$ *in the unit sphere* $\mathbb{S}$ *of* $\mathbb{C}^n$*, the* $n\times n$ *matrix* $\mathbf{P} := \mathbf{e}\,\mathbf{e}^*$ *is Hermitian and represents the orthogonal projection* $\Pi_{[\mathbf{e}]}$ *of* $\mathbb{C}^n$ *onto the one-dimensional subspace* $[\mathbf{e}] = \mathrm{span}(\{\mathbf{e}\})$*.*

Proof. For each $\mathbf{e} \in \mathbb{S}$, the $n \times n$ matrix $\mathbf{P} := \mathbf{e}\,\mathbf{e}^*$ satisfies:

1. $\mathbf{P}^* = (\mathbf{e}\,\mathbf{e}^*)^* = (\mathbf{e}^*)^*\mathbf{e}^* = \mathbf{e}\,\mathbf{e}^* = \mathbf{P}$, so $\mathbf{P}$ is Hermitian;

2. Because matrix multiplication satisfies the associative law, and also $\mathbf{e}^*\mathbf{e} = 1$, one has

$$\mathbf{P}^2 = (\mathbf{e}\,\mathbf{e}^*)(\mathbf{e}\,\mathbf{e}^*) = \mathbf{e}(\mathbf{e}^*\mathbf{e})\mathbf{e}^* = \mathbf{e}\,\mathbf{e}^* = \mathbf{P}$$

   This implies that $\mathbf{P}$ is an orthogonal projection.

3. For all $\mathbf{x} \in \mathbb{C}^n$, because matrix multiplication satisfies the associative law, and $c := \mathbf{e}^*\mathbf{x}$ is a scalar in $\mathbb{C}$, one has

$$\mathbf{P}\,\mathbf{x} = (\mathbf{e}\,\mathbf{e}^*)\mathbf{x} = \mathbf{e}\,(\mathbf{e}^*\mathbf{x}) = \mathbf{e}\,c = c\,\mathbf{e} \in [\mathbf{e}]$$

   It follows that $\mathbf{P}$ is the orthogonal projection $\Pi_{[\mathbf{e}]}$ of $\mathbb{C}^n$ onto $[\mathbf{e}]$. □

3.3 Ortho-Partitions and Ortho-Measurability in $\mathbb{C}^n$

Proposition 3.4. *Given any orthogonal decomposition $\bigoplus_{d\in D} L_d$ of $\mathbb{C}^n$:*

1. *any two different subsets $L_d \setminus \{\mathbf{0}\}$ and $L_{d'} \setminus \{\mathbf{0}\}$ in the finite family $\cup_{d\in D} \{L_d \setminus \{\mathbf{0}\}\}$ are disjoint;*

2. *any family of vectors $\{\mathbf{x}_d\}_{d\in D}$ with $\mathbf{x}_d \in L_d \setminus \{\mathbf{0}\}$ for each $d \in D$ is linearly independent, and $\#D \le n$.*

Proof. First, suppose that $\mathbf{x} \in L_d \cap L_{d'}$ where $d \neq d'$. Because L_d and $L_{d'}$ are orthogonal, one has $\mathbf{x}^*\mathbf{x} = 0$, and so $\mathbf{x} = \mathbf{0}$. It follows that $L_d \setminus \{\mathbf{0}\}$ and $L_{d'} \setminus \{\mathbf{0}\}$ are disjoint.

Second, suppose that $\mathbf{0} = \sum_{d\in D} \alpha_d \mathbf{x}_d$ where $\alpha_d \in \mathbb{C}$ and $\mathbf{x}_d \in L_d \setminus \{\mathbf{0}\}$ for all $d \in D$. Then, because orthogonality implies that $\mathbf{x}_d^*\mathbf{x}_{d'} = 0$ whenever $d \neq d'$, one has

$$\begin{aligned} 0 = \left(\sum\nolimits_{d\in D} \alpha_d \mathbf{x}_d\right)^* \left(\sum\nolimits_{d\in D} \alpha_d \mathbf{x}_d\right) &= \sum\nolimits_{d\in D} (\bar{\alpha}_d \alpha_d) \mathbf{x}_d^* \mathbf{x}_d \\ &= \sum\nolimits_{d\in D} |\alpha_d|^2 \mathbf{x}_d^* \mathbf{x}_d \end{aligned}$$

But $\mathbf{x}_d^*\mathbf{x}_d > 0$ for all $d \in D$, so $\alpha_d = 0$ for all $d \in D$. According to the standard definition, therefore, the family $\{\mathbf{x}_d\}_{d\in D}$ of vectors is linearly independent. Then, because the dimension n is the maximum size of any linearly independent subset of $\mathbb{C}^n$, one must have $\#D \le n$. □

Definition 3.5. *Let $\bigoplus_{d\in D} L_d$ be any orthogonal decomposition of $\mathbb{C}^n$.*

1. *The* residual set *R^D of all vectors that are omitted from all the sets $\cup_{d\in D} \{L_d \setminus \{\mathbf{0}\}\}$ in the orthogonal decomposition is defined by $R^D := \mathbb{C}^n \setminus \cup_{d\in D}(L_d \setminus \{\mathbf{0}\})$.*

2. *The* ortho-partition *of* $\mathbb{C}^n$ *induced by* $\bigoplus_{d\in D} L_d$ *is the partition*

$$\mathfrak{P}^D := \{R^D\} \cup (\cup_{d\in D}\{L_d \setminus \{\mathbf{0}\}\}) \tag{1}$$

3. *The* ortho-algebra Σ^D *is the* σ*-algebra* $\sigma(\mathfrak{P}^D)$ *of subsets of* $\mathbb{C}^n$ *generated by the cells in the ortho-partition* $\mathfrak{P}^D$*.*

Because $\#D \leq n$ and so D is finite, evidently Σ^D equals the power set $\mathcal{P}(\mathfrak{P}^D) = 2^{\mathfrak{P}^D}$ of the set of all cells in the ortho-partition $\mathfrak{P}^D$ of $\mathbb{C}^n$.

3.4 The Spectral Decomposition of a Hermitian Matrix

Definition 3.6. *Given any* $n \times n$ *Hermitian matrix* $\mathbf{A}$ *and any of its eigenvalues* $\lambda \in s^{\mathbf{A}}$*, let:*

1. $E_\lambda := \{\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\} \mid \mathbf{A}\mathbf{x} = \lambda\mathbf{x}\}$ *be the corresponding non-empty set of its eigenvectors;*

2. $L_\lambda := E_\lambda \cup \{\mathbf{0}\}$ *be the corresponding (linear)* eigenspace*.*

Evidently Definition 3.6 implies that $E_\lambda = L_\lambda \setminus \{\mathbf{0}\}$ for all $\lambda \in s^{\mathbf{A}}$.

Our next result, which we give without proof, is a version of the standard spectral theorem that applies in the finite-dimensional space $\mathbb{C}^n$.

Proposition 3.7. *Any* $n \times n$ *Hermitian matrix* $\mathbf{A}$*:*

1. *induces a numerically identified orthogonal decomposition* $\bigoplus_{\lambda\in s^{\mathbf{A}}} L_\lambda$ *of* $\mathbb{C}^n$ *into linear subspaces* $L_\lambda = E_\lambda \cup \{\mathbf{0}\}$ *that correspond to the eigenspaces* E_λ *of* $\mathbf{A}$*, one for each* $\lambda \in s^{\mathbf{A}}$ *in the spectrum of* $\mathbf{A}$*;*

2. *has a* spectral decomposition *into the eigenvalue-weighted linear combination*

$$\mathbf{A} = \sum\nolimits_{\lambda\in s^{\mathbf{A}}} \lambda\, \mathbf{\Pi}_{L_\lambda} \tag{2}$$

of orthogonal projections $\mathbf{\Pi}_{L_\lambda}$ *onto the corresponding mutually orthogonal eigenspaces* L_λ*.*

4 Quantum Observables as Measurable Functions

4.1 From a Hermitian Matrix to Its Eigen-Pairing

Our construction of an ortho-measurable function that corresponds to a quantum observable in the form of a Hermitian matrix $\mathbf{A}$ will involve an extension of the real line $\mathbb{R}$ that adds an extra element $*$. Following terminology which is common in computer science, the extra element $*$ could be read as "not a number", often denoted by "NaN". The element $*$ is used in defining an eigenpairing as a mapping $\mathbb{C}^n \ni \mathbf{x} \mapsto f^{\mathbf{A}}(\mathbf{x})$ whose value is only a well-defined real number in case $\mathbf{x}$ is an eigenvector of $\mathbf{A}$.

Definition 4.1. *Given any $n \times n$ Hermitian matrix $\mathbf{A}$ with spectral decomposition $\mathbf{A} = \sum_{\lambda \in s^{\mathbf{A}}} \lambda \, \mathbf{\Pi}_{L_\lambda}$, as in* (2)*, define:*

1. *for each eigenvalue $\lambda \in s^{\mathbf{A}}$ and unique associated linear eigenspace L_λ, the set $E_\lambda := L_\lambda \setminus \{\mathbf{0}\}$ of corresponding eigenvectors, as well as the* indicator function $\mathbb{C}^n \ni \mathbf{x} \mapsto 1_{E_\lambda}(\mathbf{x}) \in \{0, 1\}$ *which is defined so that $1_{E_\lambda}(\mathbf{x}) = 1 \iff \mathbf{x} \in E_\lambda$;*

2. *the* residual set $R^{\mathbf{A}} := \mathbb{C}^n \setminus \cup_{\lambda \in s^{\mathbf{A}}} E_\lambda$ *of n-vectors (including $\mathbf{0}$) that are not eigenvectors of $\mathbf{A}$, for any of its eigenvalues;*

3. *the* one-point extension $\mathbb{R} \cup \{*\}$ *of the real line, where $* \notin \mathbb{R}$;*

4. *the* eigen-pairing *of $\mathbf{A}$ as the map $\mathbb{C}^n \ni \mathbf{x} \mapsto f^{\mathbf{A}}(\mathbf{x}) \in \mathbb{R} \cup \{*\}$ defined by*

$$f^{\mathbf{A}}(\mathbf{x}) := \begin{cases} \sum_{\lambda \in s^{\mathbf{A}}} \lambda \, 1_{E_\lambda}(\mathbf{x}) & \textit{if } \mathbf{x} \in \mathbb{C}^n \setminus R^{\mathbf{A}}; \\ * & \textit{if } \mathbf{x} \in R^{\mathbf{A}}. \end{cases} \tag{3}$$

By Proposition 2.6, the eigenspaces $E_\lambda = L_\lambda \setminus \{\mathbf{0}\}$ of any $n \times n$ Hermitian matrix $\mathbf{A}$ are orthogonal, and so disjoint, for different values of λ. Hence (3) implies that

$$f^{\mathbf{A}}(\mathbf{x}) = \lambda \iff \mathbf{x} \in E_\lambda \iff \mathbf{x} \neq \mathbf{0} \text{ and } \mathbf{A}\mathbf{x} = \lambda \mathbf{x} \tag{4}$$

It follows that the mapping $\mathbf{x} \mapsto f^{\mathbf{A}}(\mathbf{x})$ pairs each $\mathbf{x} \in \mathbb{C}^n$ with the extra point $*$ in case $\mathbf{x}$ belongs to the residual set $R^{\mathbf{A}}$, but with the eigenvalue λ in case $\mathbf{x} \in \mathbb{C}^n \setminus R^{\mathbf{A}}$ is an eigenvector in the eigenspace E_λ. In particular, the following result holds.

Lemma 4.2. *The range $f^{\mathbf{A}}(\mathbb{C}^n)$ of the map defined by* (3) *equals the finite set $s^{\mathbf{A}} \cup \{*\} \subset \mathbb{R} \cup \{*\}$.*

Remark 4.3. *An alternative definition of the function $\mathbf{x} \mapsto f^{\mathbf{A}}(\mathbf{x})$ would exclude from its domain the residual set $R^{\mathbf{A}}$ specified in part 2 of Definition 4.1. It seems more in the spirit of probability theory, however, to define $\mathbb{C}^n \ni \mathbf{x} \mapsto f^{\mathbf{A}}(\mathbf{x})$ on all of $\mathbb{C}^n$, but then to attach probability zero to the residual set.*

4.2 Making the Eigen-Pairing Measurable

We begin with a preliminary lemma.

Lemma 4.4. *Consider the smallest σ-algebra $\sigma(\mathcal{B} \cup \{*\})$ on the co-domain $\mathbb{R} \cup \{*\}$ that includes the singleton set $\{*\}$ in addition to all sets in the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$. Then $\sigma(\mathcal{B} \cup \{*\})$ is the union*

$$\Sigma := \mathcal{B} \cup \mathcal{B}^* \tag{5}$$

of the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$ with the family $\mathcal{B}^* := \{B \cup \{*\} \mid B \in \mathcal{B}\}$.

Proof. First, any σ-algebra that includes all the Borel sets $B \in \mathcal{B}$ as well as the singleton set $\{*\}$ must obviously include every set in the family Σ defined by (5). It remains only to prove that Σ is itself a σ-algebra.

Evidently the family Σ includes the whole co-domain $\mathbb{R} \cup \{*\}$ as well as, for each Borel set $B \in \mathcal{B}$, the respective complements of the two sets B and $B \cup \{*\}$ in $\mathbb{R} \cup \{*\}$, which are

$$\begin{array}{rrcl} & (\mathbb{R} \cup \{*\}) \setminus B & = & (\mathbb{R} \setminus B) \cup \{*\} \\ \text{and} & (\mathbb{R} \cup \{*\}) \setminus (B \cup \{*\}) & = & \mathbb{R} \setminus B \end{array} \tag{6}$$

Because $(\mathbb{R} \setminus B) \cup \{*\} \in \mathcal{B}^*$ and $(\mathbb{R} \cup \{*\}) \setminus (B \cup \{*\}) \in \mathcal{B}$, it follows that the complement of any set in Σ also belongs to Σ.

Finally, consider the union of any countable family $\mathcal{F}$ of sets in Σ. This union is: either (i), in case no set in $\mathcal{F}$ has $*$ as a member, a union $\cup_{k \in K} B_k$ of a countable family $\{B_k\}_{k \in K}$ of sets in $\mathcal{B}$; or (ii), in case at least one set in $\mathcal{F}$ has $*$ as a member, a union $(\cup_{k \in K} B_k) \cup \{*\}$. In either case, the countable union $\cup_{F \in \mathcal{F}} F$ belongs either to $\mathcal{B}$ or to $\mathcal{B}^*$, and so to Σ.

This completes the confirmation that Σ defined by (5) is a σ-algebra on $\mathbb{R} \cup \{*\}$. □

Theorem 4.5. *Let $\Sigma^{\mathbf{A}}$ denote the ortho-algebra on $\mathbb{C}^n$ which results from applying part 3 of Definition 3.5 to the orthogonal decomposition $\bigoplus_{\lambda \in s^{\mathbf{A}}} L_\lambda$. Then the eigen-pairing $\mathbb{C}^n \ni \mathbf{x} \mapsto f^{\mathbf{A}}(\mathbf{x}) \in \mathbb{R} \cup \{*\}$ defined by (3) yields a measurable function from the measurable space $(\mathbb{C}^n, \Sigma^{\mathbf{A}})$ to the measurable space $(\mathbb{R} \cup \{*\}, \sigma(\mathcal{B} \cup \{*\}) = (\mathbb{R} \cup \{*\}, \Sigma)$.*

Proof. Given any Borel set $B \subset \mathbb{R}$, it follows from (4) that $(f^{\mathbf{A}})^{-1}(B) = \cup_{\lambda \in B \cap s^{\mathbf{A}}} E_\lambda$ and also that $(f^{\mathbf{A}})^{-1}(B \cup \{*\})$ is the union $(f^{\mathbf{A}})^{-1}(B) \cup R^{\mathbf{A}}$ of $(f^{\mathbf{A}})^{-1}(B)$ with the residual set $R^{\mathbf{A}}$. Then both the sets $(f^{\mathbf{A}})^{-1}(B)$ and $(f^{\mathbf{A}})^{-1}(B) \cup R^{\mathbf{A}}$ are obviously $\Sigma^{\mathbf{A}}$-measurable, as unions of finitely many $\Sigma^{\mathbf{A}}$-measurable subsets of $\mathbb{C}^n$. □

4.3 Hermitian Matrices as Ortho-Measurable Functions

This section is devoted to the first main theorem of the paper. It involves functions meeting the following definition:

Definition 4.6. *A function $\mathbb{C}^n \ni \mathbf{x} \mapsto f(\mathbf{x}) \in \mathbb{R} \cup \{*\}$ is* ortho-measurable *just in case there exists an orthogonal decomposition $\bigoplus_{d\in D} L_d$ of $\mathbb{C}^n$ for which:*

1. *for all $\lambda \in \mathbb{R} \cap f(\mathbb{C}^n)$, there exists a unique $d \in D$ and so a unique space L_d of the decomposition such that $f^{-1}(\{\lambda\}) = L_d \setminus \{\mathbf{0}\}$;*
2. *$f^{-1}(\{*\})$ is the residual set $R = \mathbb{C}^n \setminus \cup_{d\in D}(L_d \setminus \{\mathbf{0}\})$.*

The proof of this first main theorem will make use of numerically identified orthogonal decompositions, which are defined as follows:

Definition 4.7. *The orthogonal decomposition $\bigoplus_{d\in D} L_d$ of $\mathbb{C}^n$ is* numerically identified *just in case there exist a finite subset $\Lambda \subset \mathbb{R}$ and a bijection $D \ni d \longleftrightarrow \lambda_d \in \Lambda$.*

With this additional definition, the theorem can be stated as follows:

Theorem 4.8. *There exist bijections between each pair of the three sets of:*

1. *quantum observables in the form of $n \times n$ Hermitian matrices $\mathbf{A}$;*
2. *numerically identified orthogonal decompositions $\bigoplus_{\lambda\in\Lambda} L_\lambda$ of $\mathbb{C}^n$;*
3. *ortho-measurable functions $\mathbb{C}^n \ni \mathbf{x} \mapsto f(\mathbf{x}) \in \mathbb{R} \cup \{*\}$ satisfying $f^{-1}(\{\lambda\}) = E_\lambda$ for all $\lambda \in \Lambda$.*

Proof. First, given any $n \times n$ Hermitian matrix $\mathbf{A}$, the spectral theorem 3.7 implies the existence of the unique numerically identified orthogonal decomposition $\bigoplus_{\lambda\in\Lambda} L_\lambda$ of $\mathbb{C}^n$, where Λ is the finite spectrum $s^{\mathbf{A}}$ of $\mathbf{A}$.

Second, let $\mathbb{C}^n \ni \mathbf{x} \mapsto f(\mathbf{x}) \in \mathbb{R} \cup \{*\}$ be any ortho-measurable function. Now relabel the sets L_d in the orthogonal decomposition $\bigoplus_{d\in D} L_d$ of $\mathbb{C}^n$ of Definition 4.6 according to the function values λ in the real part $\Lambda = \mathbb{R}\cap f(\mathbb{C}^n)$ of the range of f, which is a finite set. For different values of λ, the corresponding spaces L_λ are orthogonal. So the resulting labelled orthogonal decomposition $\bigoplus_{\lambda\in\Lambda} L_\lambda$ is numerically identified. Then the decomposition $\bigoplus_{\lambda\in\Lambda} L_\lambda$ serves as a parameter of the function f defined by

$$f\left(\bigoplus\nolimits_{\lambda\in\Lambda} L_\lambda; \mathbf{x}\right) := \begin{cases} \lambda & \text{if } \mathbf{x} \in E_\lambda; \\ * & \text{if } \mathbf{x} \in \mathbb{C}^n \setminus \cup_{\lambda\in\Lambda} E_\lambda. \end{cases} \tag{7}$$

Next, consider the intersection of the graph of this function, which is a subset of the Cartesian product $\mathbb{C}^n \times (\mathbb{R} \cup \{*\})$, with the product set $(\mathbb{C}^n \setminus \{\mathbf{0}\}) \times \mathbb{R}$. This intersection is the restricted graph made up of the finite union of pairwise disjoint sets given by

$$\Gamma\left(\bigoplus\nolimits_{\lambda\in\Lambda} L_\lambda\right) := \bigcup\nolimits_{\lambda\in\Lambda} (E_\lambda \times \{\lambda\}) \tag{8}$$

Finally, recalling the spectral decomposition (2) and putting $\Lambda = s^{\mathbf{A}}$, we have the following chain of bijections

$$\mathbf{A} = \sum\nolimits_{\lambda\in s^{\mathbf{A}}} \lambda\, \mathbf{\Pi}_{L_\lambda} \longleftrightarrow \bigoplus\nolimits_{\lambda\in\Lambda} L_\lambda \longleftrightarrow \Gamma\left(\bigoplus\nolimits_{\lambda\in\Lambda} L_\lambda\right) \tag{9}$$

This chain makes the result evident. □

4.4 A Contextual Multi-Measurable Space

Let $\mathcal{D}$ denote the family of all orthogonal decompositions of $\mathbb{C}^n$. Then, for each orthogonal decomposition $D \in \mathcal{D}$ and associated ortho-algebra Σ^D, the pair $(\mathbb{C}^n, \Sigma^D)$ is a measurable space that depends on the orthogonal decomposition D, regarded as a *context*. So, according to the definitions in Hammond (2025), the pair $(\mathbb{C}^n, (\Sigma^D)_{D\in\mathcal{D}})$ with the complete family of all ortho-algebras is a *multi-measurable space.*

5 Wave Vectors as Pre-Probabilities

5.1 Preliminary Definitions

In quantum theory with the quantum state space $\mathbb{C}^n$, a "normalized wave function", according to the usual terminology, is a mapping $T \ni t \mapsto \psi(t) \in \mathbb{S}$ from a time interval $T \subset \mathbb{R}$ to the unit sphere $\mathbb{S}$. Since the current focus is on events at one fixed time, we use the following definition.

Definition 5.1. *A* wave vector *is an element of the Hilbert space* $\mathbb{C}^n$*. The wave vector* $\boldsymbol{\psi} \in \mathbb{C}^n$ *is* normalized *just in case it is an element of the unit sphere* $\mathbb{S}$ *in* $\mathbb{C}^n$*.*

Definition 5.2. *Let* $\mathcal{B} = (\mathbf{b}^j)_{j=1}^n$ *be any orthonormal basis of* $\mathbb{C}^n$*.*

1. *Given any* $j \in \mathbb{N}_n$ *and any corresponding* $\mathbf{b}^j \in \mathcal{B}$*, let:*
    - $L_j := [\mathbf{b}^j]$ *denote the one-dimensional linear space spanned by the basis vector* $\mathbf{b}^j$*;*

- $\mathbf{\Pi}_{L_j}$ *denote the* $n \times n$ *Hermitian matrix* $\mathbf{b}^j(\mathbf{b}^j)^*$ *which, by Proposition 3.3, represents the orthogonal projection mapping onto* L_j*.*

2. *Let* $D^{\mathcal{B}} := \bigoplus_{j=1}^n L_j$ *denote the associated* basic *orthogonal decomposition of* $\mathbb{C}^n$*.*

3. *Let* $\mathfrak{P}^{\mathcal{B}}$ *denote the associated* basic *ortho-partition whose cells are, as in Definition 3.5, the* n *sets* $L_j \setminus \{\mathbf{0}\}$*, together with the residual set* $R^{\mathcal{B}} = \mathbb{C}^n \setminus \cup_{j=1}^n (L_j \setminus \{\mathbf{0}\})$*.*

4. *Let* $\Sigma^{\mathcal{B}} = \sigma(\mathfrak{P}^{\mathcal{B}})$ *denote the associated* basic *ortho-algebra generated by the cells which constitute the basic ortho-partition* $\mathfrak{P}^{\mathcal{B}}$*.*

5.2 Wave Vectors as Pre-Probabilities: Special Case

Consider the special case of a basic ortho-algebra $\Sigma^{\mathcal{B}}$ associated with an orthonormal basis $\mathcal{B} = (\mathbf{b}^j)_{j=1}^n$ of $\mathbb{C}^n$. Any normalized wave vector $\boldsymbol{\psi} \in \mathbb{S}$ can be used to construct a probability measure $\mathbb{P}_{\boldsymbol{\psi}}^{\mathcal{B}}$ on the measurable space $(\mathbb{C}^n, \Sigma^{\mathcal{B}})$, according to *Born's rule*, treating $\boldsymbol{\psi}$ as a parameter. This rule requires that, for each $j \in \mathbb{N}_n$ and $L_j = [\mathbf{b}^j]$, the probability $\mathbb{P}_{\boldsymbol{\psi}}^{\mathcal{B}}(L_j \setminus \{\mathbf{0}\})$ of the basic set $L_j \setminus \{\mathbf{0}\} \in \Sigma^{\mathcal{B}}$ is given by $\boldsymbol{\psi}^* \mathbf{\Pi}_{L_j} \boldsymbol{\psi}$.[5] But $\mathbf{\Pi}_{L_j} = \mathbf{b}^j(\mathbf{b}^j)^*$ in this special case, so

$$\mathbb{P}_{\boldsymbol{\psi}}^{\mathcal{B}}(L_j \setminus \{\mathbf{0}\}) = \boldsymbol{\psi}^* \mathbf{b}^j (\mathbf{b}^j)^* \boldsymbol{\psi} = |\boldsymbol{\psi}^* \mathbf{b}^j|^2 \tag{10}$$

In the even more special case of the *canonical orthonormal basis* $\mathcal{B}$, which consists of the n columns of the $n \times n$ identity matrix, one has $\boldsymbol{\psi}^* \mathbf{b}^j = \psi_j$. Then Born's rule formula (10) evidently reduces to $\mathbb{P}_{\boldsymbol{\psi}}^{\mathcal{B}}(L_j \setminus \{\mathbf{0}\}) = |\psi_j|^2$, which is the *squared modulus rule* for the probability of each basic set $L_j \setminus \{\mathbf{0}\}$ in the orthogonal decomposition $\bigoplus_{j=1}^n (L_j \setminus \{\mathbf{0}\})$. Because normalization implies that the components of the wave vector satisfy $\sum_{j=1}^n |\psi_j|^2 = 1$, these probabilities do sum to one.

5.3 Wave Vectors as Pre-Probabilities: General Case

Let $\bigoplus_{d \in D} L_d$ be any orthogonal decomposition of $\mathbb{C}^n$, with associated ortho-partition $\mathfrak{P}^D$ and ortho-algebra Σ^D. Born's rule requires the probability of each non-residual cell $L_d \setminus \{\mathbf{0}\}$ in the ortho-partition $\mathfrak{P}^D$ to satisfy

$$\mathbb{P}_{\boldsymbol{\psi}}^D(L_d \setminus \{\mathbf{0}\}) = \boldsymbol{\psi}^* \mathbf{\Pi}_{L_d} \boldsymbol{\psi} \tag{11}$$

[5] Note that when the vector $\boldsymbol{\psi}$ is not normalized, this formula is replaced by the *Rayleigh quotient* $\boldsymbol{\psi}^* \mathbf{\Pi}_{L_j} \boldsymbol{\psi} / |\boldsymbol{\psi}|^2$.

where, unlike in (10), the orthogonal projection represented by the matrix $\mathbf{\Pi}_{L_d}$ may be onto a space L_d whose dimension exceeds one.

Note that for an orthogonal decomposition, the sum $\sum_{d \in D} \mathbf{\Pi}_{L_d}$ of all the orthogonal projections onto the component subspaces L_d equals the identity matrix $\mathbf{I}$. It follows that

$$\begin{aligned} \sum_{d \in D} \mathbb{P}^D_{\boldsymbol{\psi}}(L_d \setminus \{\mathbf{0}\}) &= \sum_{d \in D} \boldsymbol{\psi}^* \mathbf{\Pi}_{L_d} \boldsymbol{\psi} \\ &= \boldsymbol{\psi}^* \left(\sum_{d \in D} \mathbf{\Pi}_{L_d} \right) \boldsymbol{\psi} = \boldsymbol{\psi}^* \mathbf{I} \boldsymbol{\psi} = \boldsymbol{\psi}^* \boldsymbol{\psi} = 1 \end{aligned} \tag{12}$$

Also $\mathbb{P}^D_{\boldsymbol{\psi}}(R^D) = 0$ for the residual set $R^D = \mathbb{C}^n \setminus \cup_{d \in D}(L_d \setminus \{\mathbf{0}\})$.

5.4 The CDF of an Observable for a Given Wave Vector

Consider any normalized wave vector $\boldsymbol{\psi} \in \mathbb{S}$, along with any quantum observable in the form of a Hermitian matrix $\mathbf{A}$ whose spectral decomposition is the eigenvalue-weighted sum $\mathbf{A} = \sum_{\lambda \in s^{\mathbf{A}}} \lambda \, \mathbf{\Pi}_{L^{\mathbf{A}}_\lambda}$ of the family $\left\{ \mathbf{\Pi}_{L^{\mathbf{A}}_\lambda} \mid \lambda \in s^{\mathbf{A}} \right\}$ of projection matrices. Then the pair $(\boldsymbol{\psi}, \mathbf{A})$ induces a random variable $\mathbb{C}^n \ni \mathbf{x} \mapsto f^{\mathbf{A}}(\mathbf{x}) \in \mathbb{R} \cup \{*\}$ on the ortho-measurable space $(\mathbb{C}^n, \Sigma^{\mathbf{A}})$ with a *cumulative distribution function* (or CDF) $\mathbb{R} \ni r \mapsto F^{\mathbf{A}}_{\boldsymbol{\psi}}(r) \in [0,1]$ which, as usual, specifies for each $r \in \mathbb{R}$ the probability that the random variable satisfies $f^{\mathbf{A}}(\mathbf{x}) \leq r$. This CDF takes the form

$$F^{\mathbf{A}}_{\boldsymbol{\psi}}(r) = \mathbb{P}^{\mathbf{A}}_{\boldsymbol{\psi}} \left((f^{\mathbf{A}})^{-1}(-\infty, r] \right) = \sum_{\lambda \in s^{\mathbf{A}}} 1_{\lambda \leq r}(\lambda) \, \boldsymbol{\psi}^* \mathbf{\Pi}_{L^{\mathbf{A}}_\lambda} \boldsymbol{\psi} \tag{13}$$

Because $\bigoplus_{\lambda \in s^{\mathbf{A}}} L^{\mathbf{A}}_\lambda = \mathbb{C}^n$, it follows from (13) and then (12) that

$$F^{\mathbf{A}}_{\boldsymbol{\psi}}(+\infty) = \mathbb{P}^{\mathbf{A}}_{\boldsymbol{\psi}} \left((f^{\mathbf{A}})^{-1}(\mathbb{R}) \right) = \sum_{\lambda \in s^{\mathbf{A}}} \boldsymbol{\psi}^* \mathbf{\Pi}_{L^{\mathbf{A}}_\lambda} \boldsymbol{\psi} = 1 \tag{14}$$

This, of course, implies that the CDF gives probability zero to the residual event that $f^{\mathbf{A}}(\mathbf{x}) = *$.

Because of the spectral decomposition $\mathbf{A} = \sum_{\lambda \in s^{\mathbf{A}}} \lambda \, \mathbf{\Pi}_{L^{\mathbf{A}}_\lambda}$ of $\mathbf{A}$ and linearity, the *expectation* of the induced random variable $f^{\mathbf{A}}$ is

$$\mathbb{E}_{\boldsymbol{\psi}} f^{\mathbf{A}} = \sum_{\lambda \in s^{\mathbf{A}}} \lambda \boldsymbol{\psi}^* \mathbf{\Pi}_{L^{\mathbf{A}}_\lambda} \boldsymbol{\psi} = \boldsymbol{\psi}^* \mathbf{A} \boldsymbol{\psi}$$

5.5 A Multi-Probability Space for a Given Wave Vector

Given any fixed normalized wave vector $\boldsymbol{\psi} \in \mathbb{S}$, we can now use the family $(\mathbb{P}^D_{\boldsymbol{\psi}})_{D \in \mathcal{D}}$ of probability measures we have just defined in order to extend:

- the previous *multi-measurable* space $(\mathbb{C}^n, (\Sigma^D)_{D \in \mathcal{D}})$ defined in Section 4.4, with a complete family of ortho-algebras Σ^D, one for each orthogonal decomposition $D \in \mathcal{D}$;

- into a *multi-probability* space $(\mathbb{C}^n, (\Sigma^D, \mathbb{P}^D_{\boldsymbol{\psi}})_{D \in \mathcal{D}})$, with a complete family of *contextual probability* spaces $(\mathbb{C}^n, \Sigma^D, \mathbb{P}^D_{\boldsymbol{\psi}})$, one for each orthogonal decomposition $D \in \mathcal{D}$.

6 Density Matrices and the Trace Formula

6.1 Key Properties of the Trace of a Matrix

Recall that the *trace* $\operatorname{tr} \mathbf{A}$ of any $n \times n$ matrix $\mathbf{A}$ is the sum $\sum_{j=1}^n a_{jj}$ of the elements on its principal diagonal. In case $\mathbf{A}$ is Hermitian, these diagonal elements are all real, and so therefore is the trace.

Lemma 6.1. *Let* $\mathbf{A} = (a_{ij})_{n \times n}$ *and* $\mathbf{B} = (b_{ji})_{n \times n}$ *be complex* $n \times n$ *matrices. Then: (i)* $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$*; (ii) if* $\mathbf{B}^{-1}$ *exists, then* $\operatorname{tr}(\mathbf{B}^{-1}\mathbf{AB}) = \operatorname{tr} \mathbf{A}$*.*

Proof. For part (i), one has

$$\operatorname{tr}(\mathbf{AB}) = \sum\nolimits_{i=1}^n \sum\nolimits_{j=1}^n a_{ij} b_{ji} = \sum\nolimits_{j=1}^n \sum\nolimits_{i=1}^n b_{ji} a_{ij} = \operatorname{tr}(\mathbf{BA})$$

For part (ii), put $\mathbf{U} = \mathbf{B}^{-1}$ and $\mathbf{V} = \mathbf{AB}$. Because part (i) implies that $\operatorname{tr}(\mathbf{UV}) = \operatorname{tr}(\mathbf{VU})$, one has $\operatorname{tr}(\mathbf{B}^{-1}\mathbf{AB}) = \operatorname{tr}(\mathbf{ABB}^{-1}) = \operatorname{tr} \mathbf{A}$. □

Proposition 6.2. *Suppose that* $\mathbf{A}$ *is an* $n \times n$ *Hermitian matrix whose spectral decomposition of* $\mathbf{A}$*, as in* (2)*, is* $\mathbf{A} = \sum_{\lambda \in s^{\mathbf{A}}} \lambda \, \boldsymbol{\Pi}_{L_\lambda}$*. For each* $\lambda \in s^{\mathbf{A}}$*, let* m_λ *denote the dimension of the linear space* L_λ*. Then*

$$\operatorname{tr} \mathbf{A} = \sum\nolimits_{\lambda \in s^{\mathbf{A}}} \lambda \, m_\lambda \tag{15}$$

Proof. A well-known property of any Hermitian matrix $\mathbf{A}$ is that it can be diagonalized, meaning that there exist an $n \times n$ unitary matrix $\mathbf{U}$ and an $n \times n$ diagonal matrix $\mathbf{D}$ such that $\mathbf{UAU}^{-1} = \mathbf{D}$, and so $\mathbf{U}^{-1}\mathbf{DU} = \mathbf{A}$. Moreover, the diagonal entries of $\mathbf{D}$ are the eigenvalues of $\mathbf{A}$. To allow for repeated eigenvalues, note that for each $\lambda \in s^{\mathbf{A}}$, the dimension m_λ of L_λ equals the number of times that λ appears on the diagonal of $\mathbf{D}$. So it follows from part (ii) of Lemma 6.1 that

$$\operatorname{tr} \mathbf{A} = \operatorname{tr}(\mathbf{U}^{-1}\mathbf{DU}) = \operatorname{tr} \mathbf{D} = \sum\nolimits_{\lambda \in s^{\mathbf{A}}} \lambda \, m_\lambda \qquad \square$$

The right-hand side of (15) can be described as the *dimensionally* or *multiplicity weighted sum* of the eigenvalues.

Proposition 6.3. *Let L be any linear subspace of $\mathbb{C}^n$ whose dimension is m_L. Then the trace* $\operatorname{tr} \mathbf{\Pi}_L$ *of the orthogonal projection* Π_L *onto L is m_L.*

Proof. Consider any diagonalization $\mathbf{D} = \mathbf{U}\mathbf{\Pi}_L\mathbf{U}^{-1}$ of $\mathbf{\Pi}_L$, where $\mathbf{U}$ is a unitary matrix. It is easy to see that if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{\Pi}_L\mathbf{x} = \lambda\mathbf{x}$, then: either (i) $\lambda = 1$ and $\mathbf{\Pi}_L\mathbf{x} = \mathbf{x}$, implying that $\mathbf{x} \in L$; or (ii) $\lambda = 0$ and $\mathbf{\Pi}_L\mathbf{x} = \mathbf{0}$, implying that $\mathbf{x} \in L^\perp$, the orthogonal complement of L that satisfies $L \bigoplus L^\perp = \mathbb{C}^n$. It follows that $\mathbf{D}$ has m_L diagonal elements equal to 1, with the remaining $n - m_L$ diagonal elements all equal to 0. But then Lemma 6.1 implies that $\operatorname{tr} \mathbf{\Pi}_L = \operatorname{tr}(\mathbf{U}^{-1}\mathbf{D}\mathbf{U}) = \operatorname{tr} \mathbf{D} = m_L$. □

6.2 Density Matrices and Their Spectrum

The following is yet another standard definition.

Definition 6.4. *An $n \times n$ Hermitian matrix $\boldsymbol{\rho}$ is:*

1. positive semi-definite *just in case $\boldsymbol{\psi}^* \boldsymbol{\rho} \boldsymbol{\psi} \geq 0$ for all $\boldsymbol{\psi} \in \mathbb{C}^n$;*[6]

2. *a* density matrix *just in case it is positive semi-definite and* $\operatorname{tr} \boldsymbol{\rho} = 1$.

Definition 6.5. *Given any $n \times n$ density matrix $\boldsymbol{\rho}$ and eigenvalue $\lambda \in s^{\boldsymbol{\rho}}$, let $m_\lambda \in \mathbb{N}$ denote the dimension of the corresponding eigenspace L_λ.*

The following result gives a well known characterization.

Proposition 6.6. *The $n \times n$ Hermitian matrix $\boldsymbol{\rho}$ is a density matrix if and only if: (i) all its eigenvalues $\lambda \in s^{\boldsymbol{\rho}}$ satisfy $\lambda \geq 0$; (ii) together the eigenvalues also satisfy the unit trace equation*

$$\sum\nolimits_{\lambda \in s^{\boldsymbol{\rho}}} m_\lambda \lambda = 1 \tag{16}$$

Proof. Given the $n \times n$ Hermitian matrix $\boldsymbol{\rho}$, let $\mathbf{D} = \mathbf{U}\boldsymbol{\rho}\mathbf{U}^{-1}$ be any diagonalization.

1. It is well-known that the quadratic form $\boldsymbol{\psi}^* \boldsymbol{\rho} \boldsymbol{\psi}$ is positive semi-definite if and only if all the diagonal elements of $\mathbf{D}$ are non-negative. Since the diagonal elements of $\mathbf{D}$ are the eigenvalues of $\boldsymbol{\rho}$, it follows that the matrix $\boldsymbol{\rho}$ is positive semi-definite if and only if $\lambda \geq 0$ for all $\lambda \in s^{\boldsymbol{\rho}}$.

[6] Most physicists and some mathematicians say that such a matrix $\boldsymbol{\rho}$ is *positive*.

2. By Proposition 6.2 applied to $\boldsymbol{\rho}$ rather than to $\mathbf{A}$, as well as part (ii) of Lemma 6.1, it follows that $\operatorname{tr} \boldsymbol{\rho} = 1$ and so $1 = \operatorname{tr} \boldsymbol{\rho} = \operatorname{tr} \mathbf{D} = \sum_{\lambda \in \Lambda} m_\lambda \lambda$ if and only if (16) is satisfied.

The proposition follows immediately. □

6.3 Quantum States as Ortho-Probability Measures

This section shows how any quantum state, in the form of a density matrix $\boldsymbol{\rho}$ on $\mathbb{C}^n$, can be identified with an "ortho-probability" measure over the components of an appropriate orthogonal decomposition of $\mathbb{C}^n$ into the different eigenspaces of $\boldsymbol{\rho}$. The following two definitions appear to be novel.

Definition 6.7. *Say that the orthogonal decomposition $\bigoplus_{\lambda \in \Lambda} L_\lambda$ of $\mathbb{C}^n$ is* numerically identified by probability *just in case the finite set Λ of numerical identifiers are all non-negative and, together with the dimensions m_λ of each subspace L_λ, satisfy the unit trace equation* (16).

Definition 6.8. *Consider any probability space $(\mathbb{C}^n, \Sigma^D, \mathbb{P}^D)$ with Σ^D as the ortho-algebra on $\mathbb{C}^n$ generated by the orthogonal decomposition $\bigoplus_{d \in D} L_d$, as specified in Definition 3.5, and with $\mathbb{P}^D$ as a probability measure on Σ^D. The triple $(\mathbb{C}^n, \Sigma^D, \mathbb{P}^D)$ is an* ortho-probability space *with* ortho-probability measure *$\mathbb{P}^D$ just in case, given the dimension $m_d := \dim L_d \in \mathbb{N}$ for each $d \in D$, there exists a bijection $D \ni d \longleftrightarrow \lambda_d \in \Lambda \subset [0, 1]$ such that:*

1. *for all $d \in D$, one has $\mathbb{P}^D(L_d \setminus \{\mathbf{0}\}) = m_d \, \lambda_d$;*
2. *$\sum_{d \in D} m_d \, \lambda_d = 1$.*

Note that the requirement that the mapping $d \longleftrightarrow \lambda_d$ is a bijection ensures that the orthogonal decomposition $\bigoplus_{d \in D} L_d$ is numerically identified by its probability, with each set $L_d \setminus \{\mathbf{0}\}$ as the unique eigenspace corresponding to the eigenvalue λ_d.

Here is the second main result of the paper.

Theorem 6.9. *There are natural bijections between the sets of:*

1. *$n \times n$ density matrices $\boldsymbol{\rho}$;*
2. *orthogonal decompositions $\bigoplus_{\lambda \in \Lambda} L_\lambda$ of $\mathbb{C}^n$ that are numerically identified by probability;*
3. *ortho-probability spaces $\left(\bigoplus_{d \in D} (L_d \setminus \{\mathbf{0}\}), \Sigma^D, \mathbb{P}^D\right)$ in $\mathbb{C}^n$.*

Proof. Given the spectrum $\Lambda = s^{\boldsymbol{\rho}}$ of any Hermitian matrix $\boldsymbol{\rho}$, we rewrite the bijections in (9) as

$$\boldsymbol{\rho} = \sum\nolimits_{\lambda \in s^{\boldsymbol{\rho}}} \lambda \, \boldsymbol{\Pi}_{L_\lambda} \longleftrightarrow \bigoplus\nolimits_{\lambda \in \Lambda} L_\lambda \longleftrightarrow \Gamma\left(\bigoplus\nolimits_{\lambda \in \Lambda} L_\lambda\right) \tag{17}$$

Here these are bijections between the three spaces of: (i) spectrally decomposed Hermitian matrices; (ii) numerically identified orthogonal probabilities; and (iii) ortho-measurable functions.

By Proposition 6.6, the Hermitian matrix $\boldsymbol{\rho}$ is a density matrix if and only if all its eigenvalues λ: (i) are non-negative; (ii) and together satisfy the unit trace equation (16). But this double condition exactly matches both: (i) the same double condition on the spectrum $\Lambda = s^{\boldsymbol{\rho}}$ as that used in Definition 6.7 of an orthogonal decomposition $\bigoplus_{\lambda \in \Lambda} L_\lambda$ which is numerically identified by probability; (ii) the double condition used in Definition 6.8 of an ortho-probability space. □

6.4 Mixed versus Pure Quantum States

We have just shown how a general quantum state, as represented by a density matrix $\boldsymbol{\rho}$, can be identified with an ortho-probability measure $\mathbb{P}^{\boldsymbol{\rho}}$ defined on the ortho-algebra $\Sigma^{\boldsymbol{\rho}}$ generated by the orthogonal decomposition $\bigoplus_{\lambda \in s^{\boldsymbol{\rho}}} E_\lambda$ of $\mathbb{C}^n$ into the eigenspaces of $\boldsymbol{\rho}$. In general, one may regard $\mathbb{P}^{\boldsymbol{\rho}}$ as describing a *mixed* quantum state. The special case of a *pure* quantum state occurs when the spectrum of $\boldsymbol{\rho}$ satisfies $s^{\boldsymbol{\rho}} = \{0, 1\}$ and there exists a unique normalized wave vector $\boldsymbol{\phi} \in \mathbb{S}$ such that the spectral decomposition $\boldsymbol{\rho} = \sum_{\lambda \in s^{\boldsymbol{\rho}}} \lambda \, \boldsymbol{\Pi}_{L_\lambda}$ of $\boldsymbol{\rho}$ reduces to the single non-zero term $\boldsymbol{\rho} = \boldsymbol{\Pi}_{[\boldsymbol{\phi}]} = \boldsymbol{\phi}\,\boldsymbol{\phi}^*$ involving the projection $\boldsymbol{\Pi}_{[\boldsymbol{\phi}]}$ onto the one-dimensional subspace $[\boldsymbol{\phi}]$ spanned by $\boldsymbol{\phi}$.

Consider a general measurable space $(\Omega, \mathcal{A})$ in which, for each $\omega \in \Omega$, one has $\{\omega\} \in \mathcal{A}$. Then there is evidently a convex set of possible probability measures $\mathbb{P}$ on $(\Omega, \mathcal{A})$ whose extreme points are the degenerate probability measures δ_ω on $(\Omega, \mathcal{A})$ which, for each $\omega \in \Omega$, satisfy $\delta_\omega(\{\omega\}) = 1$.

By contrast, consider the multi-measurable space $(\Omega, (\Sigma^D)_{D \in \mathcal{D}})$ where each σ-algebra Σ^D may be a based on a different orthogonal decomposition $\bigoplus_{d \in D} L_d$ of $\mathbb{C}^n$. Consider the mixture $\boldsymbol{\rho} = \sum_{k=1}^m \alpha_k \boldsymbol{\rho}_k$ of the set $\{\boldsymbol{\rho}_k\}_{k=1}^m$ of m different pure states $\boldsymbol{\rho}_k = \boldsymbol{\Pi}_{[\boldsymbol{\phi}_k]}$, where $\alpha_k > 0$ for $k = 1, 2, \ldots, m$ and $\sum_{k=1}^m \alpha_k = 1$. For $\boldsymbol{\rho}$ to be a mixed state in the sense of a density matrix, and so an ortho-probability measure, there must exist one single common orthogonal decomposition $\bigoplus_{d \in D} L_d$ of $\mathbb{C}^n$ which includes all the spaces $[\boldsymbol{\phi}_k]$. This implies mutual orthogonality of all the normalized wave vectors $\boldsymbol{\phi}_k$, as well as of the corresponding projections $\boldsymbol{\Pi}_{[\boldsymbol{\phi}_k]}$ onto the one-dimensional subspaces that they span.

6.5 Consistent Multi-Probability Spaces

The following definition applies to probability measures the consistency notion that Vorob′ev (1962) applied to measures more generally.

Definition 6.10. *Given the family $\mathcal{D}$ of orthogonal decompositions of $\mathbb{C}^n$, the multi-probability space $(\mathbb{C}^n, (\Sigma^D, \mathbb{P}^D)_{D\in\mathcal{D}})$ is* consistent *just in case, whenever D_1 and D_2 both belong to $\mathcal{D}$, and the linear space L has the property that $L\setminus\{\mathbf{0}\}$ belongs to both σ-algebras Σ^{D_1} and Σ^{D_2}, then the two contextual probability measures $\mathbb{P}^{D_1}$ and $\mathbb{P}^{D_2}$ satisfy $\mathbb{P}^{D_1}(L \setminus \{\mathbf{0}\}) = \mathbb{P}^{D_2}(L \setminus \{\mathbf{0}\})$.*

6.6 Quantum Probability Distributions over Projections

Let $\mathcal{P} := \{\Pi_L \mid L \in \mathcal{L}\}$ denote the domain of all orthogonal projections onto linear subspaces L of $\mathbb{C}^n$.

Definition 6.11. *A* quantum probability distribution *is a mapping $\mathcal{P} \ni \Pi \mapsto \mu(\Pi) \in [0,1]$ with the property that, if $\{\Pi_k\}_{k\in K}$ is a family of mutually orthogonal projection matrices, then $\mu\left(\sum_{k\in K} \Pi_k\right) = \sum_{k\in K} \mu(\Pi_k)$.*[7]

6.7 Characterizing Consistent Multi-Probability Spaces

Proposition 6.12. *The multi-probability space $(\mathbb{C}^n, (\Sigma^D, \mathbb{P}^D)_{D\in\mathcal{D}})$ is consistent if and only if there exists a quantum probability distribution $\mathcal{P} \ni \Pi \mapsto \mu(\Pi) \in [0,1]$ such that, whenever $L \in \Sigma^D$, then $\mathbb{P}^D(L) = \mu(\Pi_L)$, independent of the contextual orthogonal decomposition D.*

Proof. The result is an immediate implication of the two Definitions 6.10 and 6.11. □

Suppose the dimension n of $\mathbb{C}^n$ satisfies $n \geq 3$. Then a corollary of Gleason's (1957) theorem due to Parthasarathy (1992, Theorem 9.18) implies that there is a density matrix $\boldsymbol{\rho}$ satisfying $\mu(\Pi) = \text{tr}(\boldsymbol{\rho}\,\Pi)$ for all projections $\Pi \in \mathcal{P}$. This implies the *trace rule* stating that, for all $L \in \mathcal{L}$ and all $D \in \mathcal{D}$, one has $\mathbb{P}^D(L) = \mu(\Pi_L) = \text{tr}(\boldsymbol{\rho}\,\Pi_L)$.

7 Conclusion

7.1 A Quantum Measurement Tree

Shafer and Vovk (2001, pp. 189–191) present a convenient and concise mathematical description of a typical simple quantum experiment based on a

[7] This is a finite-dimensional version of the definition on p. 31 of Parthasarathy (1992).

finite-dimensional Hilbert space $\mathbb{C}^n$ where observables can be represented by Hermitian matrices rather than self-adjoint operators. Where time is not explicitly involved, this paper has laid out an alternative and possibly more informative description in the form of a very simple quantum measurement tree. This takes the form of a tree graph in which each path through the tree has three nodes, as follows.

1. First there is an initial *preparation node* n_0 where an experimental configuration or context c in a finite domain C is determined.[8] Each context $c \in C$ combines:
   - a *quantum observable* in the form of a Hermitian matrix $\mathbf{A}$ whose eigenvectors and eigenvalues determine a numerically identified orthogonal decomposition of $\mathbb{C}^n$, with an associated *ortho-algebra* of ortho-measurable events, and whose spectral decomposition corresponds to an *ortho-measurable function* $\mathbb{C}^n \ni \boldsymbol{\psi} \mapsto f(\boldsymbol{\psi}) \in \mathbb{R}$ that associates each eigenvalue $\lambda \in s^{\mathbf{A}}$ with its *eigenspace* $E_\lambda = f^{-1}(\lambda)$ of corresponding eigenvectors;
   - a *quantum density matrix* $\boldsymbol{\rho}$ in the form of an orthogonal decomposition of $\mathbb{C}^n$ identified by probability.

2. Second, the immediate successors of the initial node n_0 consist, for each contextual pair $(\mathbf{A}, \boldsymbol{\rho})$, a unique *measurement node* $n_{\mathbf{A},\boldsymbol{\rho}}$ in the form of a chance node where there is a roulette lottery, with a probability mass function $\mathbb{P}^{\mathbf{A},\boldsymbol{\rho}}$ defined on the finite set $\{E_\lambda \mid \lambda \in s^{\mathbf{A}}\}$ of eigenspaces of $\mathbf{A}$ by the *trace rule* requiring that $\mathbb{P}^{\mathbf{A},\boldsymbol{\rho}}(E_\lambda) = \mathrm{tr}(\boldsymbol{\rho}\,\Pi_{E_\lambda})$ for each eigenvalue $\lambda \in s^{\mathbf{A}}$.

3. Third, the immediate successors of each measurement node $n_{\mathbf{A},\boldsymbol{\rho}}$ consist, for each potential value $\lambda \in s^{\mathbf{A}}$ of the observable $\mathbf{A}$, of a unique terminal *observable node* $n_{\mathbf{A},\boldsymbol{\rho},\lambda}$ of the tree that is identified with the potential observation $\lambda \in s^{\mathbf{A}}$.

7.2 Reduction to a Probability Metaspace

In Hammond (2025) two particular multi-probability spaces were each reduced to a single probability metaspace whose sample space included a vari-

[8] We insist on the domain C of contexts being finite only to ensure that the measurement tree is finite, in the sense of having a finite set of nodes. Extensions allowing C to be infinite only present difficulties if, for instance, we want to introduce a general measurable random process which determines the experimental configuration $c \in C$.

able σ-algebra. Each reduction involved a finite domain C of possible contexts, together with a probability mass function $C \ni c \mapsto q_c \in [0,1]$ in the set $\Delta(C)$ of those mappings that satisfy $\sum_{c \in C} q_c = 1$. Applying a similar reduction to the quantum measurement tree described in Section 7.1 results in a probability metaspace $(\Omega^M, \Sigma^M, \mathbb{P}_q^M)$ where:

1. The sample space Ω^M is the Cartesian product set $\mathbb{C}^n \times \{\Sigma^{\mathbf{A}_c} \mid c \in C\}$ of pairs $(\boldsymbol{\psi}, \Sigma)$ that combine a wave vector $\boldsymbol{\psi}$ with a contextual orthoalgebra Σ on $\mathbb{C}^n$ selected from the finite set $\{\Sigma^{\mathbf{A}_c} \mid c \in C\}$.

2. Let $\mathfrak{P}^M$ be the partition of Ω^M into the finite collection

$$\cup_{c \in C} \left(\cup_{E \in \Sigma^{\mathbf{A}_c}} \{E \times \{\Sigma^{\mathbf{A}_c}\} \right) \tag{18}$$

of cells which, for each context $c \in C$ and then for every orthomeasurable set $E \in \Sigma^{\mathbf{A}_c}$, take the form of the Cartesian product set $E \times \{\Sigma^{\mathbf{A}_c}\}$. Then the σ-algebra Σ^M is the power set $\sigma(\mathfrak{P}^M) = 2^{\mathfrak{P}^M}$ of the set of cells in the partition $\mathfrak{P}^M$.

3. For each probability mass function $q \in \Delta(C)$, the probability measure $\mathbb{P}_q^M$ on the finite σ-algebra Σ^M is the unique probability mass function which, for each context $c \in C$ and measurable set $E \times \{\Sigma^{\mathbf{A}_c}\} \in \Sigma^M$, and given each residual set $R^{\mathbf{A}_c}$ in $\mathbb{C}^n$, satisfies

$$\mathbb{P}_q^M(E \times \{\Sigma^{\mathbf{A}_c}\}) = \begin{cases} 0 & \text{if } E = R^{\mathbf{A}_c} \\ q_c \operatorname{tr}(\boldsymbol{\rho}\, \Pi_E) & \text{if } E \in \Sigma^{\mathbf{A}_c} \setminus \{R^{\mathbf{A}_c}\} \end{cases} \tag{19}$$

7.3 Concluding Remarks and Future Research

Consider any quantum observable represented by the $n \times n$ Hermitian matrix $\mathbf{A}$. Note that each eigenvalue λ in the spectrum $s^{\mathbf{A}}$ of $\mathbf{A}$ is only a *potential* observation. Even if the eigenspace E_λ is the realized result of the roulette lottery at the measurement node $n_{\mathbf{A},\boldsymbol{\rho}}$ of the quantum measurement tree described in Section 7.1, whether the eigenvalue λ is *actually* observed depends on whether the experimental configuration includes some device for making that observation.

Note too that in this paper the roulette lotteries with probabilities specified by the trace rule have been limited to those with a finite set of outcomes in the real line, whose corresponding measurement operators are Hermitian matrices. In future work it is planned to consider both: (i) more general real-valued observables with an infinite range of possible values, possibly unbounded; (ii) vector-valued observables with a range of multidimensional simultaneously observed measurements, whose different components are described by commuting Hermitian matrices.

Finally, one reason to consider quantum measurement trees is that, in principle, they should be able to describe the effects of a sequence of quantum measurements. Then, of course, it is important to model how any quantum measurement affects the unitary evolution of a quantum state. This is especially true when, on the basis of Theorem 6.9, we regard any quantum state as an ortho-probability measure on the components of the appropriate orthogonal decomposition of $\mathbb{C}^n$ into the eigenspaces of the usual Hermitian density matrix.

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