A Traffic Jam Theory of Recessions*

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`Comments and suggestions are extremely welcome`

Abstract

I construct a dynamic economy in which agents are interconnected: the output produced by one agent is the consumption good of another. I show that this economy can generate recessions which resemble traffic jams. At the micro level, each individual agent waits for his own income to increase before he increases his spending. However, his spending behavior affects the income of another agent. Thus, the spending behavior of agents during recessions resembles the stop-and-go behavior of vehicles during traffic jams. Furthermore, these traffic jam recessions are not caused by large aggregate shocks. Instead, in certain parts of the parameter space, a small pertubation or individual shock is amplified as its impact cascades from one agent to another. These dynamics eventually result in a stable recessionary equilibrium in which aggregate output, consumption, and employment remain low for many periods. Thus, much like in traffic james, agents cannot identify any large exogenous shock that caused the recession. Finally, I provide conditions under which these traffic jam recessions are most likely to occur.

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1 Introduction

Are recessions similar to traffic jams? Consider the following two anecdotal observations.

First, driver behavior seems similar to that of economic agents. In traffic jams, one often gets the feeling that if all cars just drove forward at a slow but steady pace, we would all get out of the traffic jam. However, this takes coordination and it is in fact not a good description of how drivers actually behave. Instead, in traffic jams, we observe what is known as "stop-and-go" behavior. An individual driver waits for the car in front to move forward before he moves forward. This opens up space for the car behind him, in which case that car moves forward. Hence, in traffic jams, all cars are simply waiting for the space to open up ahead of them before they move. One sees clearly that the actions of these drivers are not based on the entire state of the highway[1], but instead are based on their own very local conditions. Similarly, in recessions we observe another form of "stop-and-go" behavior. Households wait for their income to increase before they increase their consumption spending. Firms wait for sales to pick up before they increase production or employ more workers. It seems as though the actions of economic agents, too, are not based on the entire state of the aggregate economy, but instead are based on their own individual situations or constraints. And again, one gets the feeling that if all households simply spent more and if all firms simply employed more workers, the recession would come to an end. Yet, this takes coordination; instead, for each individual economic agent and for each individual driver, local interactions matter first and foremost.

Second, traffic james, like recessions, do not seem to always be driven by large exogenous shocks. Sometimes traffic jams are caused by something fundamental–an obstruction on the road or a car crash. However, more often than not traffic jams seem to occur spontaneously, or at least without any underlying cause–perhaps due to some slight, unobserved perturbation.[2] The traffic engineers call these "phantom jams" as drivers in the jam cannot seem to identify any particular cause of the jam. Furthermore, these phantom jams seem more likely to occur when traffic dense.

Similiarly, the underlying causes of business cyles seem to be equally elusive. While the standard approach to modelling business cycles is to build dynamic models of rational agents and then to analyze the model's equilibrium response to exogenous aggregate shocks, this approach is in some ways unsatisfactory. As John Cochrane (1994) writes, "it is difficult to find large, identifiable, exogenous shocks" in the data. Modigliani (1977) and Hall (1980)

[1] This could be due either to the fact that drivers don't know what's going on in the entire highway, or the simple physical constraint that they can't hit the car ahead of them.

[2] In fact, this has also been shown in some experiments.

contend that standard equilibrium models may leave too much unexplained. Furthermore, during and after actual recessions, it is not as if firm executives, consumers, central bankers, or even economists are easily able to identify the large aggregate shocks driving each episode.[3] Thus, although standard general equilibrium models rely on aggregate shocks as the main drivers of fluctuations, it is difficult both through introspection and by observation of the data to be fully satisfied with this modelling approach. Much like traffic jams, macroeconomic recessions are often "phantom".

In this paper I construct a model in which recessions resemble traffic jams in these two respects. Agents are arranged in a network such that the output produced by one agent is the consumption good of another. During normal times agents receive steady streams of income and as a result their consumption is a steady flow. However, during recessions, agents exhibit stop-and-go behavior: each agent i waits for his own income to increase before increasing his spending. But, this implies that agent $i-1$, who produces the consumption good for agent i, is experiencing a drop in income, and hence also not spending. If agent $i-1$ isn't spending, this affects the income of agent $i-2$, and so on. Thus, agents are all locally waiting for their prospects to improve, while their non-spending behavior is affecting the income of others. Thus, the model in some way shares the same spirit of the earlier literature on Keynesian coordination failures, but through a very different mechanism and modeling technique.

Second, in this model recessions are driven not by large aggregate shocks, but instead by small perturbations, or local shocks. These individual-specific or local shocks may have reverberating effects so that the economy eventually finds itself in a recession. However, these perturbations could be so small that they would not be identified as aggregate shocks in the data, nor would all the agents in the model be aware of them. Furthermore, in this model small perturbations do not always lead to recessions. Under certain conditions, these perturbations die out and the equilibrium converges back to the "normal times" equilibrium. Under certain other conditions, however, these perturbations are amplified, leading to prolonged traffic jam recessions. Thus, in sharp contrast to standard equilibrium models, this model could potentially identify conditions under which recessions are more likely to occur, rather than simply attributing them to unpredictable exogenous shocks. Furthermore, this model may allow for new policy insights designed to end the traffic-jam recession and bring the economy back to the normal times regime.

[3] Sure, for certain recessions, such as the oil price recessions of the 70s or the Volcker recession in the early 80s, we have some idea of the large aggregate shocks behind these aggregate declines. However, I would argue that for most business cyles fluctuations this is not the case. As Hall (1977) points out, only rarely do we find obvious candidates such as the oil shocks in the 70s. Even if we consider the latest recession, the fall in the value of the housing market was only a negligible fraction of total U.S. GDP.

Framework. First, I draw on the literature on traffic flow in engineering. In this literature one of the most successful and widely accepted models of simulating traffic is called the Optimal Velocity Model introduced by Bando et al (1995). This is a car-following model in which N cars follow each other on a circular road of length L; car i follows car $i+1$. The bumper-to-bumper distance between car i and car $i+1$ is called car i's "headway". In car-following models, cars are given a behavioral equation which dictates their acceleration or speed as a function of nearest-neighbor stimuli (see survey of the literature by Orosz et al 2006). The innovation in Bando et al (1995) is the introduction of a particular form for this behavioral equation–it imposes that each car's acceleration is an increasing function of its headway. If a car's headway is very large, the car speeds up, if it is too small, the car slows down and potentially comes to a stop.

The results of this simple model are quite striking. This model can produce both uniform traffic flow as well as a stop-and-go waves which resemble traffic jams. In the uniform-flow equilibrium, all cars follow each other around the circle at equal velocity and at equal speed. This equilibrium is unique and globally stable in a particular region of the parameter space, implying that the effects of any small perturbation eventually die out and the system converges back to uniform flow. The uniform flow equilibrium, however, loses stability when a certain parameter is varied; at this point a Hopf bifurcation of the dynamical system occurs meaning that an individual vehicle limit cycle becomes stable.[4] Here, what emerges instead are travelling waves which resemble the stop-and-go behavior in traffic jams. Individual cars converge to a limit cycle: cars oscillate between facing low headway and slowing down to a stop (entering a traffic jam), and facing large headway and speeding up (exiting the traffic jam). In this equilibrium, there are many cars sitting in the traffic jam, waiting for their headway to increase before moving forward, implying that aggregate velocity has decreased relative to that in the uniform-flow. Furthermore, due to the instability of the uniform-flow equilibrium and the stability of the stop-and-go solution, the transition path seems compelling: small perturbations develop into large traffic jams as their effects cascade down the line of cars.

With this model in mind, I then build a similar model within an economic environment. I construct a dynamic economy in which agents are inter-connected: the output produced by one agent is the consumption good of another. I then show how this environment is similar to that in the traffic model. In this analogy, the expenditure of each agent is similar to their velocity. Given this interpretation, I show that headway in the model is equal to cash-on-hand at the beginning of the period. Thus, the resources an agent spends on consumption

[4] However, note that the aggregate behavior is not in a limit cycle. Only that of individual cars.

in a given period becomes the income for the next agent (the producer of that good) the following period. This increases the latter agent's cash-on-hand in the following period, which he may then choose to spend on consumption, therefore moving those resources to the next agent. And so on. This is analagous to the idea that whenever a car moves forward, this increases the headway for the car behind him, in which case that car may move forward.

Now, in the traffic model there is a behavioral equation which dictates the behavior of cars–cars are supposed to accelerate when headway is large, and decelerate when headway is low. The next step in the economic model then is to see whether the behavior of the agents in the model can match the behavior of cars in the traffic model. Here, I take two approaches. First, in the economic model I start by allowing for arbitrary consumption functions and then derive under what conditions these functions can lead to traffic jam recessions. To understand this, note that in the traffic model, depending on the parameters of the behavioral equation, either the uniform flow equilibrium or the stop-and-go solution is stable. In particular, what matters is the slope of the acceleration of the car with respect to the headway. When this slope is sufficiently low, uniform flow is stable; when this slope is sufficiently high, uniform flow loses stability and the traffic jam occurs. This slope is analogous in the economic model to the marginal propensity to consume out of current cash-in-hand. I formalize this condition, and show that when the marginal propensity to consume out of cash-in-hand is very high, the economy can fall into a traffic jam recession. I then simulate the economy and analyze the transitional paths. I find this preliminar exercise useful–once one understands the general properties consumption functions must have in order to generate traffic jams recessions, I can then provide guidance as to what conditions in terms of microfoundations: preferences, information, constraints, etc. would allow for policy functions of this shape as an optimal response to the household's problem.

Second, I then attempt to construct from micro-foundations optimal household policy functions such that the consumption function satisfies these properties. The starting point is a model without any credit or borrowing constraints. I show that with permanent income consumers, one can acheive a policy function which is similar to the behavior equation in the traffic model. This is because whenever an agent observes an income shock, if he believes income is a random walk, his consumption will also increase as an optimal response to the increase in his permanent income. As in Hall (1977), under certain preferences, this implies that his own consumption follows a random walk, which therefore implies that the income of the following agent is a random walk. In this model, however, the slope of this consumpion policy function is not high enough to generate traffic jams. In order to generate traffic jams, a higher marginal propensity to consume is needed. I thus explore the case of quasi-hyperbolic agents. In this case, I show that depending upon parameters, one can obtain a high enough marginal propensity to consume such that a traffic jam recession occurs.

Finally, I consider a variant with borrowing constraints. In my opinion, this is the most natural microfoundation, as we well know that this leads to high marginal propensities to consume when agents are close to their borrowing constraints. The model here is similar to a consumption savings model with idiosyncratic income (labor) risk, as in Aiyagari, Huggett, Bewley. However, in contrast to these papers, the income risk here is endogenous–the income of one agent depends on the consumption behavior of another. In this version of the model the state space unfortunately blows up as agents are trying to forecast the shocks of all other agents and must keep track of entire distributions. Hence, in order to simplify the problem, I assume that agents have a constrained information capacity as in Sims (2003), Gabaix (2011), Woodford (2012). Households thus cannot keep track of entire state of the world, and instead can only keep track and form expectations over a finite number of moments. I thus define an approximate equilibrium as in Krussell-Smith () and then simulate the economy with borrowing constraints. I show that this environment can easily lead to traffic jam recessions.

Related literature. This paper is firstly related to the engineering literature on traffic flow. Finally, in terms of the traffic literature, I borrow the models of Traffic Bando et. al. (1995). This model has been used extensively throught that literature. See, e.g. Gasser et. al. (2004), Orosz Stepan (2006), Orosz et. al. (2009) In car-following models, discrete entities move in continuous time and continuous space[5]

In economics, my paper is most closely related to Jovanovic (1987 and working paper 1983) and the "sandpile" models Scheinkman and Woodford (1994) and Bak, Chen, Scheinkman, Woodford (1993). In fact, in his 1983 working paper version, Jovanovic explores an environoment very similar to this one: agents are arranged in a circle and each agent consumes the good produced by the agent to his left. Jovanovic shows that with independent agent-specific preference shocks and without any aid of aggregate shocks, in this economy he can produce aggregate fluctuations!

This paper is also related to the self-organized criticality literature. The "Sandpile Model" of Scheinkman and Woodford (1994) and Bak, Chen, Scheinkman, Woodford (1993). In these models there is some low frequency movement that takes you into the Bifurcation range. Stresses the importance of supply chain linkages.

Furthermore, the results of this model have the flavor of Keynesian Coordination Failures; it thus complements the literature on multiple equilibria and sunspot fluctuations. See, e.g. Shell (1977), Azariadis (1981), Azariadis and Guesnerie (1986), Benhabib and Farmer (1994,

[5]There is another literature called continuum or macroscopic models. These models characterize traffic in terms of density and velocity fields use partial differential equations.

1999), Cass and Shell (1983), Cooper and John (1988), Farmer (1993), Farmer and Woodford (1997), and Woodford (1991). The results of the traffic model can be interpreted as a coordination failure: the network structure and decentralized trading prevents households from coordinating on spending more and generating more income. However, unlike this previous literature, the coordination failure does not originate from any of the familiar sources (externalities and non-convexities), nor is there ever more than one stable equilibria. Also, Roberts (), and Jones and Manuelli ().

Furthermore, the methodology used in this paper is that of dynamical systems, limit cycles and Hopf Bifurcations; it is thus partly related to an older literature in dynamic general equilibrium theory, studying whether rational behavior can give rise to endogenous aggregate fluctuations. See, for example, Magill (1979), Boldrin and Montrucchio (1986), Scheinkman (1984) Boldrin and Deneckere (1987). Turnpike theorem. This work is surveyed in Boldrin and Woodford (1990). These papers look at representative agent growth models with a unique perfect-foresight equilbrium. They find that deterministic dynamical systems can generate both periodic limit cycles as well as chaotic dynamics that can look very irregular. In this model, rather, on the aggregate there are no endogenous fluctuations–there are limit cycles only at the individual level.

Finally, in this paper fluctuations are driven by small shocks to individual agents, rather than aggregate shocks. In this sense, this paper shares the spirit of the early literature on real business cycles and the role of intersectoral linkages and sectoral shocks. Beginning with Long and Plosser's (1983) multi-sectoral model of real business cycles, a debate then ensued between Horvath (1998, 2000) and Dupor (1999) over whether sectoral shocks could lead to strong observable aggregate TFP shocks. More recently, this work has been extended and generalized by Acemoglu et al. (2011), for arbitrary production networks. Finally, the results of the Acemoglu et. al. paper are related to that of Gabaix (2011), who shows that firm level shocks may translate into aggregate fluctuations when the firm size distribution is power law distributed, i.e. sufficiently heavy-tailed. La'O and Bigio (2013) build on the production network literature and show how financial frictions within firms affect other firms within the network. Finally, there is the Credit Chains model of kiyotaki moore.

Layout. This paper is organized as follows. Section 2 first introduces the canonical traffic model from the traffic literature. Section 3 then sets up the environment of the economic model and further shows how the economic model is in fact related to the traffic model. Section 4 characterizes the equilibrium and presents the uniform flow equilibrium. Section () then looks at the stability of this model and explores whether "traffic-jam" like recessions can occur. Section 6 then considers a variant with borrowing constraints and explains how one will get the policy functions for individual households in this environment. Section 8 then concludes. All proofs are in the Appendix.

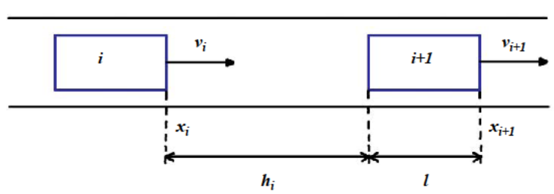
2 The Traffic Model

In this section I present the simple traffic model that can produce both uniform flow and stop-and-go traffic. There are two general approaches to modeling traffic. One is continous models in which traffic is described via a continuous density distribution and a continuous velocity distribution over location and time.[6] The other method of modelling traffic is to consider a car-following model. In car-following models, discrete entities move in continuous time and continuous space. I follow the latter approach. The rest of this section mirrors the exposition on car-following models found in Orosz et al (2006, 2009).

Consider a model of N cars indexed by $i \in \{1, 2, \ldots, N\}$. Here, car i follows car $i+1$. Let $x_{i,t}$ denote the position of car i at time t, let $v_{i,t}$ denote the velocity of car i at time t and let $\dot{v}_{i,t}$ denote the acceleration of car i at time t. Finally, let $h_{i,t}$ be bumper-to-bumper distance between car i and car $i+1$, also calle dthe headway:

$$h_{i,t} = x_{i+1,t} - x_{i,t} - l$$

where l is length of car. For simplicity and without loss of generality, we take $l \to 0$. See Figure 2.



[6]See, e.g. Lighthill & Whitham (1955).

One must also specify boundary conditions. For simplicity, we place these N cars on a circular road of length L. This yields the following equation $\sum_{i=1}^{N} h_{i,t} = L$.

Finally, to complete the model we need a car-following rule, that is, the velocity or the acceleration of each car has to be given as the function of stimuli–these are usually headway, the velocity difference, or the vehicle's own velocity. As economists, we can think of this as simply a behavioral equation for each car. Here, I will follow a class of models that has been extensively studied and widely accepted in the traffic literature called the "Optimal Velocity Model" (Bando et. al, 1995). See (Bando et al. 1998, Gasser et al 2004, Orosz et al. 2004) In this class of models, the acceleration of vehicle i is given by

$$\dot{v}_{i,t} = \alpha \left(V \left(h_{i,t} \right) - v_{i,t} \right) \tag{1}$$

where $\alpha > 0$ is a constant, and V is a continuous, monotonically increasing function of vehicle i's headway $h_{i,t}$.[7] This equation was proposed by Bando et al (1995) and has proved quite successful. Despite its simplicity, this model can produce qualitatively almost all kinds of traffic behaviour, including uniform traffic flow as well as stop-and-go waves.

Equation (1) deserves some comment. First, the assumption here is that the acceleration of vehicle i is a function only of nearby stimuli–the vehicle's own velocity and its headway (its distance to the nearest car). These are called nearest-neighbor interactions. That is, each car's individual state is strictly smaller than the aggregate state.[8]

Next, this model is entitled the optimal velocitiy model (OVM) and $V(\cdot)$ is called the optimal velocity function. However, note that in the usual economic sense, there is nothing necessarily "optimal" about it. That is, equation (1) is *not* the result of any optimization problem on the part of the agents nor a planner; instead, this behavior is simply imposed. The reason one might call it optimal is that $V(h_{i,t})$ can be thought of as the "optimal velocity" a driver would like to have given its current headway $h_{i,t}$. If this optimal velocity $V(h_{i,t})$ is greater than the car's current velocity $v_{i,t}$, the car speeds up. Conversely, if $V(h_{i,t})$ is less than the car's current velocity $v_{i,t}$, the car slows down. Finally, $\alpha > 0$ is called the relaxation parameter; it dictates how sensitive the driver's acceleration is to this difference in optimal and current velocity.

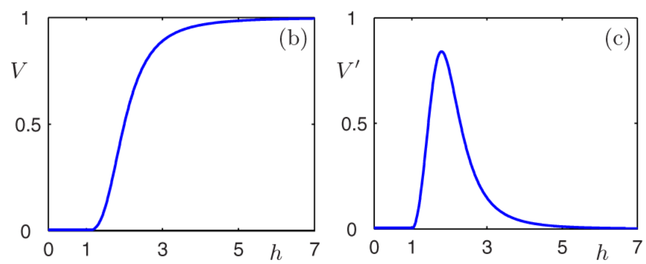
[7] A more general version often studied is given by $\dot{v}_{i,t} = \alpha \left(V \left(h_{i,t} \right) - v_{i,t} \right) + W \left(\dot{h}_{i,t} \right)$. Here, I follow Bando et. al. 1995 and Gasser et al. 2004 and set $W = 0$.

[8] There exist extensions in which stimuli also include next-nearest neighbour interactions (Wilson et al 2004). In muti-look-ahead models, drivers respond to the motion of more than one vehicle ahead. These can increase the linear stability of the uniform flow.

Finally, the optimal velocity function V satisfies the following properties: (i) it is contiuous, non-negative, and monotonically increasing, (ii) it approaches a maximum velocity for large headway $\lim_{h\to\infty} V(h) = v^0$ where v^0 acts as a desired speed limit, and (iii) it is zero for small headway. A simple example of the optimal velocity function is given by the following specification, used in Orosz et. al (2009)

$$V(h) = \begin{cases} 0 & \text{if} \quad h \in [0,1) \\ \frac{(h-1)^3}{1+(h-1)^3} & \text{if} \quad h \in [1,\infty) \end{cases}$$

This is rescaled by v^0. Figure 2 plots this function and its first derivative. Note that the rescaled speed limit is 1.



The optimal velocity function and its first derivative. Source: Orosz et al ()

Therefore, equilibrium of this traffic model is given by the following set of ODEs

$$h_{i,t} = x_{i-1,t} - x_{i,t}, \quad \forall i \in \{1,\ldots N\} \tag{2}$$

$$\sum_{i=1}^{N} h_{i,t} = L \tag{3}$$

$$\dot{v}_{i,t} = \alpha \left[V(h_{i,t}) - v_{i,t}\right], \quad \forall i \in \{1,\ldots N\} \tag{4}$$

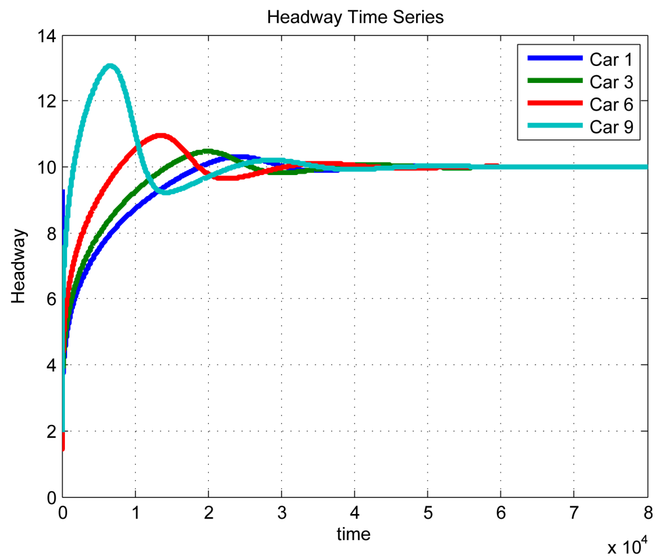
The first equation simply describes the relation between positions and headway, the second condition gives us periodic boundary conditions, and the third equation are the behavioral equations for the cars. Finally, as mentioned before This behavioral equation is useful as it can produce both uniform flow and stop-and-go traffic, which I will describe next.

Uniform Flow Equilibrium. This system admits a uniform flow equilibrium. The definition of the uniform flow equilibrium is an equilibrium which satisfies (2)-(4) in which the velocities and the headways of all cars are constant (time-independent): $h_{i,t} = h^*$ and $v_{it} = v^*, \forall i \in \{1,\ldots N\}$. In this equilibrium, all cars travel at same velocity, equally spaced. Characterizing the uniform flow is quite simple. If all cars are equally spaced, then $h^* = L/N$. Furthermore, in order for all cars to be travelling at constant velocity, in order for equation

(4) to hold, we must have that $0 = V(h^*) - v^*$. Thus, the uniform flow equilibrium is characterized by

$$h_{i,t} = h^* = L/N, \qquad v_{it} = v^* = V(L/N), \qquad \forall i \in \{1, \ldots N\}$$

As will be discussed next, the uniform flow equilibrium is unique and globally stable in part of parameter space. This implies that one may start cars in any position and at any velocity, and as long as they behave according to the optimal velocity equation, over time these cars will converge to the uniform flow equilibrium. This is demonstrated in the following figure.



Convergence to Uniform-Flow

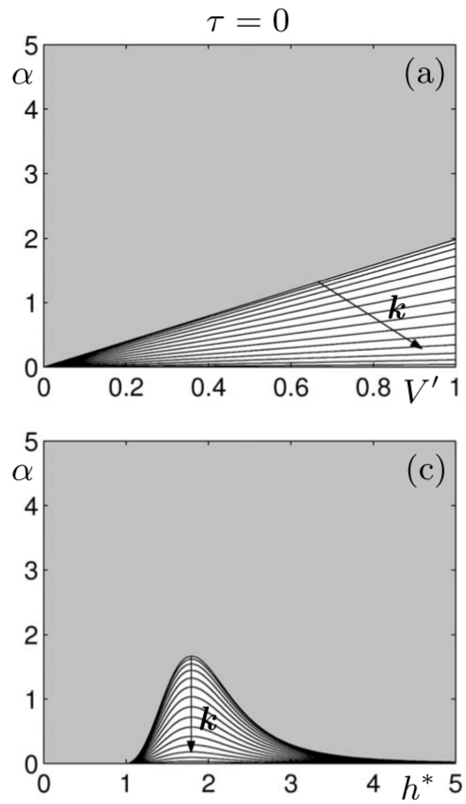
Bifurcations of the Uniform Flow. We now consider the stability of the uniform flow. We find that the uniform flow equilibrium is stable in part of parameter space, however the uniform equilibrium may lose stability when the parameter h^* is varied. In order to see this, one needs to linearize the system around uniform-flow equilibrium and consider the eigenvalues $\lambda \in \mathbb{C}$. To conserve on space, the linear stability analysis is restricted to the appendix; here, I will simply present the result.

Proposition 1. *The uniform flow equilibrium is stable if and only if*

$$V'(h^*) < \frac{1}{2}\alpha$$

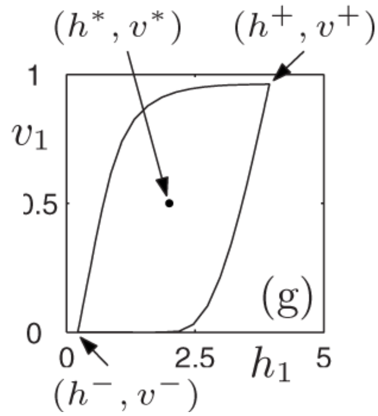
In the terminology of dynamical systems, when crossing the stability curve at $V'(h^*) = \frac{1}{2}\alpha$, a (subcritical) Hopf bifurcation takes place. At this point a pair of complex conjugate eigenvalues cross the imaginary axis, $\lambda = i\omega$. Once this occurs, the uniform flow becomes unstable and instead, travelling waves with frequency ω appear. That is, the stable equilibrium is a the limit cycle for each vehicle.

Figure 2 summarizes this information by ploting the linear stability diagrams. The top panel of Figure 2 plots the stability diagram in terms of the $(V'(h^*), \alpha)$ space. The domain in which the uniform flow is linearly stable is shaded. When $V'(h^*) < \frac{1}{2}\alpha$ the uniform flow equilibrium loses stability and a Hopf bifurcation occurs; the arrows represent the increase in wave number k. Using the derivative of the OV function, one may transform the stability diagrams from the $(V'(h^*), \alpha)$ plane to the (h^*, α) plane, thus the bottom panel of Figure 2 plots the linear stability in terms of the (h^*, α) space. From this, we see that when traffic is sufficiently dense, i.e. when h^* is low enough (approaching from above), the uniform flow equilibrium loses stability.

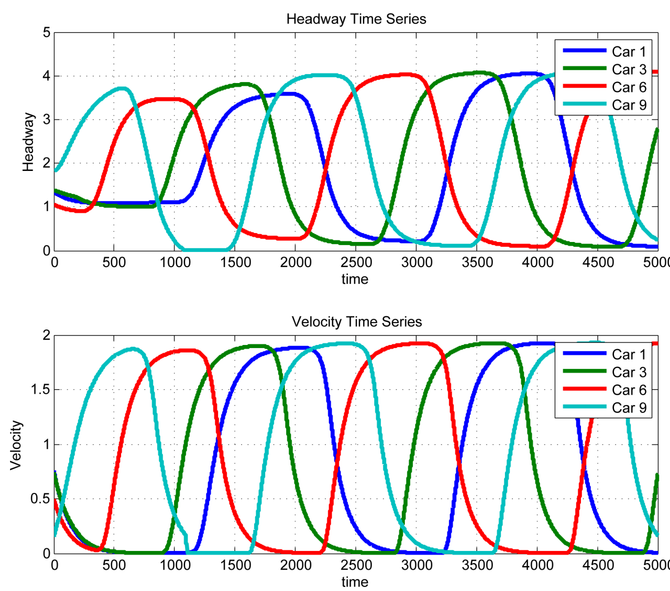


Source: Orosz et al. ()

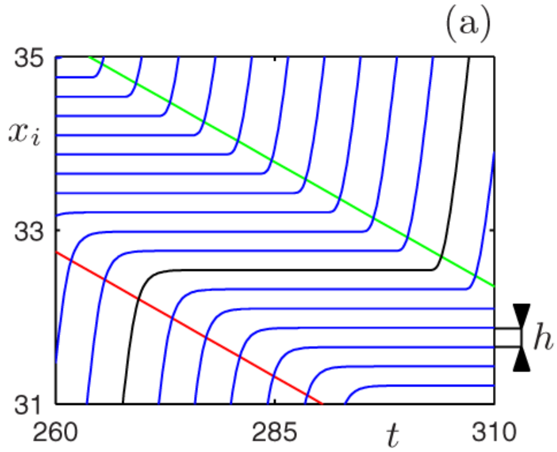
Stop-and-Go Waves. Thus, when $V'(h^*)$ is sufficiently high relative to α, the uniform flow equilibrium loses its stability. When this occurs, what emerges instead are travelling waves which resemble the stop-and-go behavior in traffic jams. Individual cars converge to a limit cycle, an oscillatory solution. See Figure (). Cars oscillate between facing low headway and slowing down to a very low speed or to a stop, sitting in a traffic jam waiting for their headway to increase, and then facing large headway and speeding up until they hit the traffic jam again.



Furthermore, this limit cycle is stable in this region, hence any small perturbation thus takes cars into the oscillatory solution. The following figure illustrates this convergence to an to an oscillatory solution.



Finally Figure 2 plots the trajectories of multiple vehicles. The y-axis is the position of each vehicle, plotted as a function of time t. Each blue line is the trajectory of an individual vehicle. The vehicle enters the traffic jam, is stuck there for a while, and then when its headway opens up, the car speeds up. The red line indicates the stop-front of the jam and the green line indicates the go-front of the jam.



To summarize, when $V'(h^*)$ is sufficiently high, or when traffic is sufficiently dense, a traffic jam can emerge. At the micro level, individual cars enter a traffic jam in which they wait for their headway to increase before moving. At the macro level, aggregate velocity and headway have fallen relative to the uniform flow equilibrium. Furthermore, small perturbations develop into large traffic jams; "tiny fluctuations may develop into stop-and-go waves as they cascade back along the highway, i.e. 'tiny actions have large effects'" (Orosz et al, 2009). The traffic engineering literature describes these as "phantom jams" in the sense that drivers cannot see any cause of the jam even after they've left the congested region.

3 The Economic Model

The purpose of this paper is to build a model in which recessions resemble traffic jams. As above,... Agents may care only about local interactions. Could optimizing agents follow similar behavioral rule? If so, perhaps the economy could generate behavior at the micro level that resembles stop-and-go traffic. Analogy. decrease in velocity is similar to a decrease in spending. wait for headway to increase ⇔ wait for income to increase.

With this model in mind, I then build a similar model within an economic environment. I construct a dynamic economy in which agents are inter-connected: the output produced by one agent is the consumption good of another. I then show how this environment is similar to that in the traffic model. In this analogy, the expenditure of each agent is similar to their velocity. Given this interpretation, I show that headway in the model is equal to cash-on-hand at the beginning of the period. Thus, the resources an agent spends on consumption in a given period becomes the income for the next agent (the producer of that good) the following period. This increases the latter agent's cash-on-hand in the following period, which he may then choose to spend on consumption, therefore moving those resources to the next agent. And so on. This is analagous to the idea that whenever a car moves forward, this increases the headway for the car behind him, in which case that car may move forward.

if so, could small perturbations lead to recessions? Perhaps we could then generate "phantom recessions" in which agents cannot identify large shock. Furthermore, from the the traffic model presented in Section 2, we see that the theory predicts that traffic jams are more likely to occur under certain conditions–conditions which take us to the other part of the parameter space. Thus, building an economic model may have implications for when recessions are more likely to occur.

In this section I construct a model to address these questions.

3.1 The Model

Time is discrete and indexed by t. There are N households composed of a producer and a consumer. Households are arranged in a circular network such that household i consumes the output produced by household $i-1$.[9]

Geography. There are N islands. The consumer of household i lives and consumes on island i while at the same time the producer of household i lives and produces on island $i+1$. Hence, for any island i, consumer i and producer $i-1$ co-habitate this island: producer $i-1$ produces on that island a consumption good for consumer i.

Commodity Space. There are $N+2$ goods. First, there are the N different commodities which the N households consume. As already mentioned before, household i consumes the commodity produced by household $i-1$. Furthermore, these commodities are perishable, and hence cannot be stored. Second, there are shells. Shells can be used for trade or to buy intermediate goods. Unlike the commodities, shells may be stored. There is a storage technology for shells with return $1+r$. Finally, there is uniform intermediate good which has a constant price in terms of shells. Thus, shells are used for trade however they do have an intrinsic value–they are used to buy intermediate goods. One can think of this as some outside worker sector, in which these laborers actually like consuming shells. This is shown in the appendix.

Timing. At the beginning of each period, each household receives profits from its producer from the previous period. Once each household receives last period's profits (in shells), the goods market on each island takes place. That is, the household makes consumption and savings decisions, the producer on that island produces the consumption good, and markets clear within each island. The household pays the producer for the consumption good in shells. Once these markets clear, the producer on each island sends his profits back to his

[9] And household 1 consumes the output of household N.

own consumer to be received at the beginning of next period.

Preferences. The utility of household i is given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_{it}\right)$$

where β is the discount factor and $u\left(c\right)$ is a strictly increasing, concave, one-period utility function satisfying the Inada conditions.

The household's budget constraint in shells is given by

$$p_{i-1,t} c_{it} + a_{i,t} = \left(1+r\right)\left[\pi_{i,t-1} + a_{it-1}\right] \tag{5}$$

The left hand side is expenditure on the consumption good and savings: $p_{i-1,t}$ is the price of its consumption good at time t (the price of the good produced by producer $i-1$) and a_{it} is the number of shells the household decides to save and carry into the next period. Here I have normalized the price of shells to 1. The right hand side is shells on hand at the beginning of the period: $\pi_{i,t-1}$ are the profits the household receives from its firm from last period, and a_{it-1} are the shells the household gets from saving last period. Both sources of shells from last period invested in some fund, in which they get some return $(1+r)$ next period.

Producers. Producer i's production function is given by

$$y_{it} = A_{it} n_{it}^{\alpha}$$

where n_{it} is an intermediate input whose marginal cost is χ shells. Each producer maximizes per-period profits given by

$$\pi_{it} = p_{it} y_{it} - \chi n_{it}$$

where p_{it} is the price of the good produced by firm i.

Market clearing. The consumption of any commodity must be equal to the amount produced since there is no storage

$$c_{it} = y_{i-1,t}, \forall i, t$$

Trade is facilitated across these households by exchanging shells for these consumption goods.

Idiosyncratic Shocks. I assume that there are island-specific productivity shocks

$$A_{i,t} = \rho A_{i,t-1} + \varepsilon_{i,t}$$

but no aggregate shocks. This leads to stochastic prices $p_{i,t}$ and expenditure $z_{i+1,t}$.

In terms of timing and information in the presense of these shocks, it works as follows. At the beginning of each period, producer i on island $i+1$ observes his own productivity A_{it}. Consumer $i+1$ also observes the productivity A_{it} of the producer on his island, but may not necessarily see the productivity of his own producer on the next island. Markets clear under this information set.

Remarks. First, note that I need consumption of household i to be equal to production of household $i-1$. This implies that the consumption and investment goods are different in order for islands to produce different amounts. To understand this, suppose the opposite: that production can be used either as consumption or investment. Now consider the following. Island i produces y_i. Island $i+1$ buys this production y_i and uses it either for consumption or investment. If island $i+1$ has its own income and its own saved goods, all of that is spent on consumption and investment. That is, suppose whatever cash-in-hand $i+1$ is $h_{i+1} = (1+r)(y_{i+1} + a_{i+1})$. Then household $i+1$ can spend this cash-on-hand on either either on c or a

$$\begin{aligned} c + a &= (1+r)(y+a) \\ c + (a - (1+r)a) &= y \\ c + x &= y \end{aligned}$$

Household $i+1$ purchases $c+x$ from household i. But this implies that household i must have produced y_i too. So in the end, all households produce the same ammount y. This is why I disconnect the consumption and investment goods from one another and therefore introduce another good used for storage (shells).

Furthermore, note that I allow for an intermediate good that is used in production rather than own household labor. The reason for this is that the state space blows up otherwise.

Equilibrium definition. I define an equilibrium as follows.

Definition 1. *A competitive approximate equilibrium is a collection of allocation and price functions such that*

(i) given current prices and expectations of future prices and income, allocations are optimal for households and firms given their expectations.

(ii) prices clear all markets

(iii) household and firm expectations are rational

This is a standard definition of equilibrium for this economy. Within each island is a walrasian equilibrium.

3.2 Relation to Traffic Model

In this section I show how the economic model outlined above is similar to the traffic model presented in Section 2. First, note that given the producers problem,

$$\max p_{it} y_{it} - \chi n_{it}$$

Producer optimality is given by $p_{it}\alpha \frac{y_{it}}{n_{it}} = \chi$. This implies that profits are proportional to expenditure on that good $\pi_{it} = (1-\alpha) p_{it} y_{it}$.

Now, let $z_{i,t} \equiv p_{i-1,t} c_{i,t}$ denote expenditure of household i on consumption in period t. Since consumption of household i is equal to output of household $i-1$, we have that

$$z_{i,t} \equiv p_{i-1,t} c_{i,t} = p_{i-1,t} y_{i-1,t}$$

The firm optimization and Cobb-Douglas production functions imply that equilibrium profits are equal to

$$\pi_{i,t} = (1-\alpha) p_{i,t} y_{i,t} = (1-\alpha) z_{i+1,t}$$

We can therefore rewrite the sequence of budget constraints in (5) as

$$z_{i,t} + a_{i,t} = (1+r) \left((1-\alpha) z_{i+1,t-1} + a_{it-1} \right)$$

Position, Velocity, and Acceleration.. Let x_{it} denote the value of all expenditure through period t

$$x_{it} \equiv \sum_{j=0}^{t} (1+r)^j z_{i,t-j} + \sum_{k=1}^{i-1} (1+r)^t a_{k,0}$$

Thus, I say that x_{it} denotes the "position" of household i at end of period t. One can think of this as the amount of numeraire the consumer has used. We can think of this position as if agents hold pieces of numeraire. Each unit of numeraire has a number on it, and as they trade, they increase in their number.

I define a discounted time-derivative operator as follows

$$\Delta \equiv 1 - (1+r) L$$

where L is the lag operator. It is straight-forward to show that the velocity of agent i at time t, or the first (discounted) time-derivative of x_{it}, is equal to expenditure this period.

$$v_{it} \equiv \Delta x_{i,t} = z_{i,t}$$

Finally, the acceleration of household i is simply just the household's change in expenditure

$$\Delta v_{it} = z_{it} - (1 + r)\, z_{it-1}$$

Headway. I now consider headway–that is, the bumper-to-bumper distance between these households. I define headway of household i at time t as a particular difference in position (distance) between that household and the household in front of it.

$$h_{it} \equiv (1 - \alpha)\, x_{i+1,t} - x_{i,t}$$

Given this definition, I obtain the following characterization of headway

Lemma 1. *Headway at the beginning of period t is equal to the household's resources before consuming or investing*

$$h_{i,t-1} = (1 - \alpha)\, z_{i+1,t-1} + a_{i,t-1}$$

See proof in the Appendix. We can thus rewrite the budget constraint as

$$z_{it} + a_{i,t} = (1 + r)\, h_{i,t-1}$$

Headway is thereby the household's shells-on-hand (or cash-on-hand) at the beginning of the period, before consuming and investing. This is similar to the state variables that we often see in many economic problems.

Discretized traffic model In summary, in the economic model we have that velocity is equivalent to expenditure, $v_{i,t} = z_{i,t}$, and that headway is equal to shells-on-hand at the beginning of the period, $h_{i,t-1} = (1 - \alpha)\, z_{i+1,t-1} + a_{i,t-1}$. The equations closely correspond to those in the discretized traffic model. See e.g. the discretized traffic version of the optimal velocity model in Tadaki et al. (1997); here, the optimal velocity equation in discrete time is given by

$$v_{i,t} - v_{i,t-1} = \alpha\left(V\left(h_{i,t-1}\right) - v_{i,t-1}\right)$$

What I want in my economic model is the following

$$z_{i,t} - (1 + r)\, z_{i,t-1} = G\left(h_{i,t-1}, z_{i,t-1}\right)$$

with $G_h > 0$, $G_z < 0$.[10] Thus, I want expenditure to be increasing in cash-on-hand. This is related to the household's marginal propensity to consume.

[10] This is where I apply the so-called want operator.

4 Equilibrium Characterization

In what follows, I show how one can obtain these types of functions. First, I have now decided to work on a model of permanent income consumers. I prove that this model is always stable. Thus, one needs a model with a higher marginal propensity to consume among consumers in order to generate traffic jam recessions. I then show that with quasi-hyperbolic consumers, one obtains more hand-to-mouth behavior, and under certain parameters, one can obtain traffic jam recessions.

In this section I characterize the equilibrium. I first solve the consumer's problem. I then show that under certain parameter values, the equilibrium is unstable.

4.1 Perfect Information and Foresight

Suppose there are no shocks at all. Then all agents are identical. Hence, the equilibrium is characterized by the following.

4.2 Permanent Income Consumers

I now characterize the optimal behavior for Permanent Income consumers. The consumer's problem is given by

$$\max \mathbb{E} \sum_{t=0}^{\infty} \delta^t u\left(c_{i,t}\right)$$

subject to the household's budget constraint.

$$p_{i-1,t} c_{it} + a_{i,t} = \pi_{i,t-1} + (1+r)\, a_{it-1}$$

Let $\beta^t \lambda_{i,t}$ be the Lagrange multiplier on the budget constraint. The FOCs are given by

$$\begin{aligned} \delta^t u'\left(c_{it}\right) - \lambda_{it} p_{i-1,t} &= 0 \\ -\lambda_{i,t} + \mathbb{E}_t \left(1+r\right) \lambda_{i,t+1} &= 0 \end{aligned}$$

Thus, the real marginal utility of consumption is given by $\lambda_{it} = \frac{\delta^t u'(c_{it})}{p_{i-1,t}}$. Combining these conditions gives us the following Euler Equation

$$\frac{u'\left(c_{it}\right)}{p_{i-1,t}} = \mathbb{E}_t \delta \left(1+r\right) \frac{u'\left(c_{i,t+1}\right)}{p_{i-1,t+1}}$$

I will first use quadratic utility as in Hall (1978)

$$u(c) = -\frac{1}{2}(\tilde{c} - c)^2$$

so that marginal utility is linear $u'(c) = \bar{c} - c$. The Euler Equation is thus given by

$$\frac{\bar{c} - c_{it}}{p_{i-1,t}} = \mathbb{E}_t \delta (1 + r) \frac{(\bar{c} - c_{it+1})}{p_{i-1,t+1}}$$

Using the fact that $\delta(1 + r) = 1$ and that $p_{i-1,t}/p_{i-1,t+1} = 1$, Furthermore, let us suppose that A_{it} follows a random walk. and $\mathbb{E}_t \left(\frac{p_{i-1,t}}{p_{i-1,t+1}} \right) = 1$. We thus obtain the obtain the random walk result

$$c_{it} = \mathbb{E}_t c_{it+1}$$

Consumption today is the optimal forecast of consumption tomorrow.

Finally, to pin down today's consumption, we merely look at the budget constraint

$$p_{i-1,t} c_{it} + a_{i,t} = (1 - \alpha) p_{i,t} y_{i,t} + (1 + r) a_{it-1}$$

where I have replaced profit with (). Iterating this forward, we have that the budget constraint must satisfy

$$\sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j p_{i-1,t+j} c_{it+j} = \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j (1 - \alpha) p_{i,t+j} y_{i,t+j} + (1 + r) a_{it-1}$$

taking expectations of both sides, we have that

$$p_{i-1,t} c_{it} + \sum_{j=1}^{\infty} \left(\frac{1}{1+r} \right)^j \mathbb{E}_t p_{i-1,t} c_{it} = (1 - \alpha) p_{i,t} y_{i,t} + \sum_{j=1}^{\infty} \left(\frac{1}{1+r} \right)^j (1 - \alpha) \mathbb{E}_t p_{i,t+j} y_{i,t+j} + (1 + r) a_{it-1}$$

But, given that income and consumption both follow a random walk, we have that $\mathbb{E}_t p_{i-1,t} c_{it} = p_{i-1,t} c_{it}$ and that $\mathbb{E}_t p_{i,t+j} y_{i,t+j} = p_{i,t} y_{i,t}$. This implies that the budget constraint reduces to

$$\sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j p_{i-1,t} c_{it} = \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j (1 - \alpha) p_{i,t} y_{i,t} + (1 + r) a_{it-1}$$

This implies that

$$p_{i-1,t} c_{it} = (1 - \alpha) p_{i,t} y_{i,t} + r a_{it-1}$$

or, in terms of expenditure,

$$z_{it} = (1 - \alpha)\, z_{i+1,t-1} + r a_{it-1} \tag{6}$$

As in the standard model, consumption is equal to the innovation in my permanent income... the annuity value of my accumulated assets. I doesn't matter what p is, expenditure will adjust so that this is true.

I now relate this to headway, expenditure, so that we can close the model. Note that headway at the beginning of the period is

$$h_{i,t-1} = (1 - \alpha)\, z_{i+1,t-1} + (1 + r)\, a_{i,t-1}$$

Combining this with the permanent income behavior of (), we have that the behavior of the household must satisfy

$$z_{it} = h_{i,t-1} - a_{i,t-1}$$

I consume whatever I made today: the interest off my asset holding and the increase in my headway coming from my income today. Or, I consume everything I have except the amount I want to keep bringing with me into the future (which is equal to what i came in with).

I want to get rid of the term $a_{i,t-1}$ so that i may just write this in terms of the different velocities of the agents. Note that my consumption yesterday and whatever i saved yesterday is equal to my headway yesterday. That is, I can lag () one period in order to obtain.

$$z_{it-1} + a_{i,t-1} = h_{i,t-2}$$

Plugging this in for $a_{i,t-1}$ in (), i get a new behavioral equation solely in terms of velocities and headways.

$$z_{it} = h_{i,t-1} - h_{i,t-2} + z_{it-1}$$

Now, allow me take the discounted time derivative of expenditure

$$z_{it} - (1 + r)\, z_{it-1} = h_{i,t-1} - h_{i,t-2} - r z_{it-1}$$

One can rewrite the right-hand side of this equation in the following way

$$h_{i,t-1} - h_{i,t-2} - r z_{it-1} = \frac{1}{1+r} \Delta h_{i,t-1} + r \left(\frac{1}{1+r} h_{i,t-1} - z_{it-1} \right)$$

Therefore

$$\begin{aligned}\Delta z_{it} &= \frac{1}{1+r}\Delta h_{i,t-1} + r\left(\frac{1}{1+r}h_{i,t-1} - z_{it-1}\right) \\ z_{it} - (1+r)\, z_{it-1} &= \frac{1}{1+r}\left(h_{i,t-1} - (1-r)\, h_{i,t-2}\right) + r\left(\frac{1}{1+r}h_{i,t-1} - z_{it-1}\right)\end{aligned}$$

I can rewrite this in continuous time as follows. Need to check what I do with the r and α. Let us now take the continuous time analog of these policy functions.

$$\dot{z}_{it} = \frac{1}{1+r}\dot{h}_{it} + r\left(\frac{1}{1+r}h_{i,t} - z_{it}\right)$$

This looks the same as the generalized traffic equations.

Proposition 2. *The expenditure policy function when agents are permanent income consumers is given by.*

$$\dot{z}_{it} = f\left(h, \dot{h}, z\right) = r\left(\frac{1}{1+r}h_{i,t} - z_{it}\right) + \frac{1}{1+r}\dot{h}_{it}$$

We now study the stability of this system. We can rewrite this as,

$$\dot{z}_{it} = f\left(h, \dot{h}, z\right) = r\left(\frac{1}{1+r}h_{i,t} - z_{it}\right) + \frac{1}{1+r}\dot{h}_{it}$$

This policy function implies that

$$\begin{aligned}F &= \partial_h f = r\left(\frac{1}{1+r}\right) \\ G &= \partial_{\dot{h}} f = \frac{1}{1+r} \\ H &= -\partial_z f = r\end{aligned}$$

Stability condition is given by

$$F < \frac{1}{2-\alpha}\left((2-\alpha)\, G + H\right) H$$

Proposition 3. *The stability condition when agents are permanent income consumers is*

$$0 < \frac{1}{2-\alpha} r\, (1+r)$$

Therefore, this system is always stable. Recall that $1 + r = 1/\delta$ and $\alpha < 1$. Thus, the right hand side is always positive. Is this intuitive or not? It would seem a bit strange if it were not stable. This is similar to Hall (1978) permanent income model. The only difference here is that there is not a representative consumer.

Equilibrium consumption In the permanent income model, demand from consumer i for good $i - 1$ is given by

$$p_{i-1,t}c_{it} = (1 - \alpha) z_{i+1,t-1} + ra_{it-1}$$

Supply is given by

$$p_{i-1,t} = \frac{\chi}{\alpha A_{i-1,t}^{1/\alpha}} y_{i-1,t}^{1/\alpha - 1}$$

Combining the two and using market clearing $c_{it} = y_{i-1,t}$, we have that

$$\frac{\chi}{\alpha A_{i-1,t}^{1/\alpha}} c_{it}^{1/\alpha} = (1 - \alpha) z_{i+1,t-1} + ra_{it-1}$$

therefore, equilibrium consumption is given by

$$c_{it} = \left(\frac{\alpha}{\chi}\right)^{\alpha} A_{i-1,t} \left[(1 - \alpha) z_{i+1,t-1} + ra_{it-1}\right]^{\alpha}$$

that is, equilibrium consumption is increasing in the productivity shock and in permanent income.

4.3 Quasi-Hyperbolic Consumers

Here I follow Laibson () and introduce Quasi-Hyperbolic Discounting into the preferences of the agents.Quasi-hyperbolic utility (also called Beta-Delta Utility) I do this in order to increase the hand-to-mouth behavior of these consumers.

$$\max u(c_{i,t}) + E_t \beta \sum_{j=1}^{\infty} \delta^t u(c_{i,t+j})$$

Indeed, the discount factor between periods t and $t + 1$ is $\beta\delta$, and between any two adjacent periods later it is δ.

subject to the household's budget constraint. The FOCs of this problem are given by

$$\begin{aligned} u'\left(c_{i,t}\right) - p_{i-1,t}\lambda_{i,t} &= 0 \\ \beta\delta^{t+j}u'\left(c_{i,t+j}\right) - p_{i-1,t+j}\lambda_{i,t+j} &= 0 \\ -\lambda_{i,t} + \left(1+r\right)\lambda_{i,t+1} &= 0 \end{aligned}$$

Thus, between future periods we get the usual Euler Equation

$$\frac{u'\left(c_{i,t+j}\right)}{p_{i-1,t+j}} = E_{i,t+j}\left(1+r\right)\delta\frac{u'\left(c_{i,t+j+1}\right)}{p_{i-1,t+j+1}}, \qquad \forall j = \{1, 2, \ldots\}$$

but between today and tomorrow the euler equation is given by

$$\frac{u'\left(c_{i,t}\right)}{p_{i-1,t}} = \beta E_{i,t}\left(1+r\right)\delta\frac{u'\left(c_{i,t+1}\right)}{p_{i-1,t+1}}$$

This implies that from the perspective today, with linear utility and as long as $(1+r)\delta = 1$, and prices are not expected to grow).

Suppose we have log utility

$$\frac{1}{p_{i-1,t}c_{i,t}} = \beta E_{i,t}\left(1+r\right)\delta\frac{1}{p_{i-1,t+1}c_{i,t+1}}$$

then we have that between future periods: consumers expect a random walk.

$$c_{i,t+j} = E_{i,t+j}c_{i,t+j+1}, \qquad \forall\ j = \{1, 2, \ldots\}$$

whereas between today and tomorrow

$$\frac{1}{p_{i-1,t}c_{i,t}} = \beta E_{i,t}\frac{1}{p_{i-1,t+1}c_{i,t+1}}$$

We have approximately that.

$$\beta p_{i-1,t}c_{i,t} = E_{i,t}p_{i-1,t+1}c_{i,t+1}$$

need to deal with log approximation, but it's cool for now.

Finally, to pin down today's consumption, we merely look at the budget constraint

$$p_{i-1,t}c_{it} + a_{i,t} = \left(1-\alpha\right)p_{i,t}y_{i,t} + \left(1+r\right)a_{it-1}$$

where I have replaced profit with (). Iterating this forward, we have that the budget constraint must satisfy

$$\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^{j} p_{i-1,t+j}c_{it+j} = \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^{j}(1-\alpha)\,p_{i,t+j}y_{i,t+j} + (1+r)\,a_{it-1}$$

taking expectations of both sides, we have that

$$p_{i-1,t}c_{it}+\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^{j}\mathbb{E}_t p_{i-1,t}c_{it} = (1-\alpha)\,p_{i,t}y_{i,t}+\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^{j}(1-\alpha)\,\mathbb{E}_t p_{i,t+j}y_{i,t+j}+(1+r)\,a_{it-1}$$

But, given that consumption after today and income follows a random walk, we have that $\mathbb{E}_t p_{i,t+j}y_{i,t+j} = p_{i,t}y_{i,t}$. This implies that the budget constraint reduces to

$$p_{i-1,t}c_{it}+\left(\frac{1}{1+r}\right)\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^{j}\mathbb{E}_t p_{i-1,t+1}c_{i,t+1} = \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^{j}(1-\alpha)\,p_{i,t}y_{i,t}+(1+r)\,a_{it-1}$$

This reduces to

$$p_{i-1,t}c_{it}+\frac{1}{r}\mathbb{E}_t p_{i-1,t+1}c_{i,t+1} = \frac{1+r}{r}(1-\alpha)\,p_{i,t}y_{i,t}+(1+r)\,a_{it-1}$$

From () we have that $\beta p_{i-1,t}c_{i,t} = E_{i,t}p_{i-1,t+1}c_{i,t+1}$. Plugging this in, we have that

$$p_{i-1,t}c_{it}+\frac{1}{r}\beta p_{i-1,t}c_{i,t} = \frac{1+r}{r}(1-\alpha)\,p_{i,t}y_{i,t}+(1+r)\,a_{it-1}$$

and solving for $p_{i-1,t}c_{i,t}$ we get that

$$p_{i-1,t}c_{i,t} = \frac{1+r}{\beta+r}(1-\alpha)\,p_{i,t}y_{i,t}+\frac{(1+r)}{\beta+r}ra_{it-1}$$

In terms of expenditure, this is given by

$$z_{it} = \frac{1+r}{\beta+r}(1-\alpha)\,z_{i+1,t-1}+\frac{(1+r)}{\beta+r}ra_{it-1}$$

Doing the same manipulations as before for Proposition (), we have that

$$\Delta z_{it} = \frac{1}{\beta+r}\Delta h_{i,t-1}+\frac{r}{\beta+r}h_{i,t-1}+(1+r)\left(\frac{1-(\beta+r)}{\beta+r}\right)z_{it-1}$$

Proposition 4. *The expenditure policy function when agents are quasi-hyperbolic*

$$\dot{z}_{it} = f\left(h, \dot{h}, z\right) = \frac{1}{\beta + r}\dot{h}_i(t) + \frac{r}{\beta + r}h_i(t) + (1 + r)\left(\frac{1 - (\beta + r)}{\beta + r}\right)z_i(t)$$

We now study the stability of this system. This policy function implies that

$$\begin{aligned} F &= \partial_h f = \frac{r}{\beta + r} \\ G &= \partial_{\dot{h}} f = \frac{1}{\beta + r} \\ H &= -\partial_z f = -(1 + r)\left(\frac{1 - (\beta + r)}{\beta + r}\right) \end{aligned}$$

The stability condition is given by

$$F < \frac{1}{2 - \alpha}\left((2 - \alpha)G + H\right)H$$

This leads us to the following proposition.

Proposition 5. *The stability condition when agents are quasi-hyperbolic is given by*

$$r < \frac{1}{2 - \alpha}\left(\frac{(2 - \alpha) + (1 + r)((\beta + r) - 1)}{\beta + r}\right)(1 + r)((\beta + r) - 1)$$

Now, check that there exists a β that violates this condition.

Let's look first at the extremes. First, suppose $\beta = 1$. Then this reduces to

$$0 < \frac{1}{2 - \alpha}r(1 + r)$$

Therefore, we nest the permanent income case. totally stable.

Now suppose $\beta = 0$. That is, agents do not care at all about future consumption. This is the other extreme of the quasi-hyperbolic discounting case. In this case, we get that

$$r < \frac{1}{2 - \alpha}\left(\frac{(2 - \alpha) + (1 + r)(r - 1)}{r}\right)(1 + r)(r - 1)$$

5 Equilibrium Traffic Dynamics

Diagrams of simulations. Transitional Dynamics in both regions.

6 Borrowing Constraints Model

Finally, I consider a variant with borrowing constraints. In my opinion, this is the most natural microfoundation, as we well know that this leads to increasing and concave policy functions as well as high marginal propensities to consume when agents are close to their borrowing constraints. The model here is similar to a consumption savings model with idiosyncratic income (labor) risk, as in Aiyagari, Huggett, Bewley. However, in contrast to these papers, the income risk here is endogenous–the income of one agent depends on the consumption behavior of another. The disadvantage: very ugly problem. In this version of the model the state space unfortunately blows up as agents are trying to forecast the shocks of all other agents and must keep track of entire distributions. Hence, in order to simplify the problem, I assume that agents have a constrained information capacity as in Sims (2003), Gabaix (2011), Woodford (2012). Households thus cannot keep track of entire state of the world, and instead can only keep track and form expectations over a finite number of moments. I thus define an approximate equilibrium as in Krussell-Smith () and then simulate the economy with borrowing constraints. I show that this environment can easily lead to traffic jam recessions.

I thus assume that the household faces borrowing constraint

$$a_{it} \geq -\phi$$

One can also think of this as a cash-in-hand constraint if $\phi = 0$. The equilibrium is defined as before, but with the added borrowing constraint (). Given this borrowing constraint, the household's problem is given by

$$V_{it}\left(p_{i-1,t}; \mathbf{z}_{i+1}; a_{i,t}\right) = \max_{c_i, a'_i} u\left(c_{i,t}\right) + \beta \mathbb{E}_{i,t} V_{i,t+1}\left(p'_{i-1}; z_{i+1}; a'_i\right)$$

subject to

$$\begin{aligned} p_{i-1,t} c_{i,t} + a_{i,t} &= (1-\alpha)\, z_{i+1,t-1} + (1+r)\, a_{it-1} \\ a_{i,t} &\geq -\phi \end{aligned}$$

and

$$\begin{aligned} \mathbf{z}_{i+1} &= \mathbf{z}_{i+1} \cup z_{i,t} \\ z_{i+1,t} &= \psi\left(p_{i-1,t}; \mathbf{z}_{i+2}; a_{i,t}\right) \\ p'_{i-1} &= p\left(A'_{i-1}, z_{i+1}, \bar{a}'_i\right) \end{aligned}$$

This is a very general formulation of the household's problem. It is easy to see how this problem becomes intractable. The main problem here is that agents must keep track of entire state.

To preserve tractability, I assume that each consumer perceives income $z_{i+1,t}$ as Markov

$$z_{i+1,t+1} = \psi\left(z_{i+1,t}\right)$$

This corresponds with a constrained information capacity of agents as in Sims (2003), Gabaix (2011), Woodford (2012). Households cannot keep track of entire state of the world. I thus revise

Definition 2. *A competitive approximate equilibrium is a collection of allocation and price functions such that*

(i) given current prices and expectations of future prices and income, allocations are optimal for households and firms

(ii) prices clear all markets

(iii) household expectations are based on perceived Markov process

$$z'_{i+1} = \psi\left(z_{i+1}\right)$$

where ψ is the best approximation of the true process

Part (iii) is similar to Krusell-Smith. have not yet defined "best approximation"

Household's Problem. The household's consumption-savings problem similar to Bewley economy

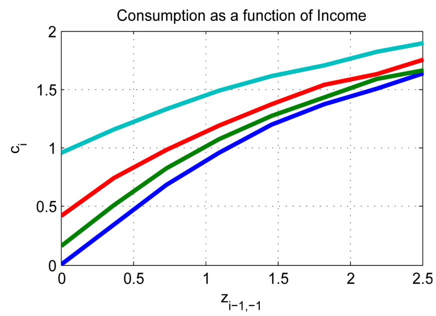
$$V\left(p_{i-1}; z_{i+1,-1}; a_i\right) = \max_{c_i, a'_i} u\left(c_i\right) + \beta \mathbb{E}_i V\left(p'_{i-1}; z_{i+1}; a'_i\right)$$

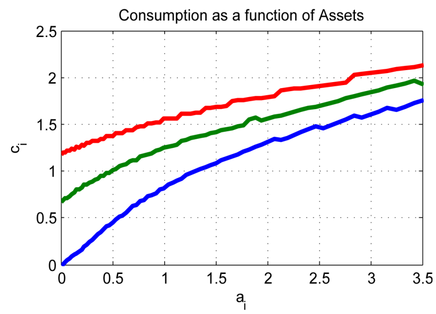
subject to

$$\begin{aligned} p_{i-1}c_i + a'_i &= (1-\alpha)\, z_{i+1,-1} + (1+r)\, a_i \\ a'_i &\geq -\phi \end{aligned}$$

and

$$\begin{aligned} z_{i+1} &= \psi\left(z_{i+1,-1}\right) \\ p'_{i-1} &= p\left(A'_{i-1}, z_{i+1}, \bar{a}'_i\right) \end{aligned}$$





PROBLEM: Right now in the code, the part dealing with tomorrow's value function in terms of the price is incorrecct. Need to fix this.

I thus obtain Policy Functions

$$\begin{aligned} a_i' &= d\left(p_{i-1}; z_{i+1,-1}; a_i\right) \\ c_i &= g\left(p_{i-1}; z_{i+1,-1}; a_i\right) \end{aligned}$$

c_i increasing in $z_{i+1,-1}, a_i$, decreasing in p_{i-1}

Numerical Simulation Parameter Values

$$u\left(c\right) = \frac{c^{1-\gamma}}{1-\gamma}$$

$\gamma = 1$, $\beta = .9$, $\phi = 0$

for now, exogenous interest rate $r = .02$

first, exogenous beliefs about income

$$z_{i+1}' = \rho z_{i+1} + \varepsilon, \quad \varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$$

numerically approximated with 8 states, $\rho = .2$, $\sigma_\varepsilon = .5$

Consumption is increasing in Assets and Income

Firm's ProblemFOC for firm

$$p_i \alpha \frac{y_i}{n_i} = \chi$$

thus, supply is increasing in p and A

$$y_{i-1}\left(p_{i-1}, A_{i-1}\right)$$

Equilibrium in the Goods Markets
demand as a function of price

$$c_i\left(p_{i-1}; z_{i+1,-1}; a_i\right)$$

supply as a function of price

$$y_{i-1}\left(p_{i-1}, A_{i-1}\right)$$

market clearing: equilibrium price p_{i-1}^* such that

$$y_{i-1}\left(p_{i-1}^*, A_{i-1}\right) = c_i\left(p_{i-1}^*; z_{i+1,-1}; a_i\right)$$

$$p_{i-1}^* = p\left(A_{i-1}; z_{i+1,-1}; a_i\right)$$

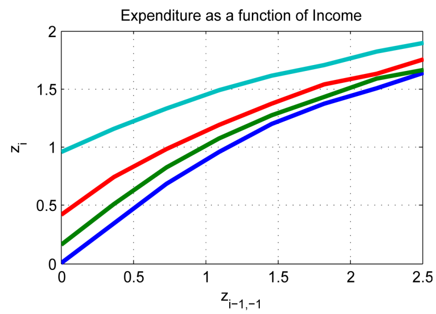
Therefore, the equilibrium price must satisfy

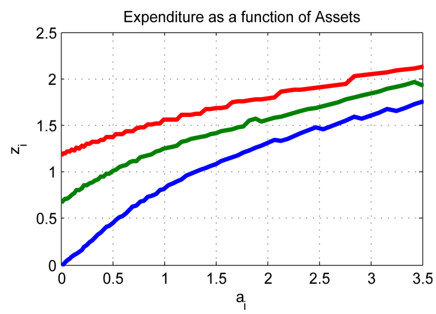
$$p^* \alpha \frac{y_i}{n_i} = \chi$$

$$\begin{aligned} p_{it} \alpha A \left(\frac{y}{A}\right)^{1-1/\alpha} &= \chi \\ p_{it} \alpha A_{it}^{1/\alpha} \left(y_{it}\right)^{1-1/\alpha} &= \chi \end{aligned}$$

thus

$$p = \frac{\chi}{\alpha A_{it}^{1/\alpha} \left(y_{it}\right)^{1-1/\alpha}}$$





Expenditure Policy Function

equilibrium expenditure of household i

$$z_i = p^*_{i-1} c_i \left(p^*_{i-1}; z_{i+1,-1}; a_i\right)$$

new equilibrium policy functions

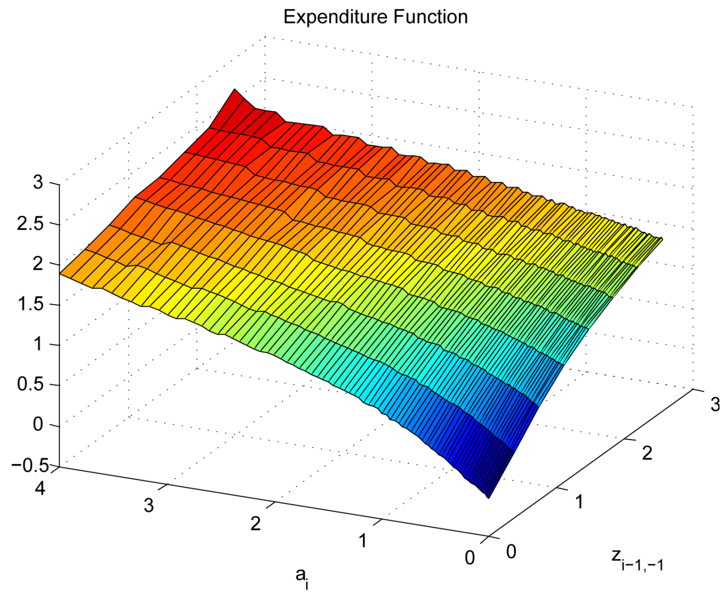
$$\begin{aligned} a_i' &= D\left(z_{i+1,-1}; a_i; A_{i-1}\right) \\ z_i &= G\left(z_{i+1,-1}; a_i; A_{i-1}\right) \end{aligned}$$

Expenditure is increasing in Assets and Income

Expenditure Policy Function

Next Step: Solve General Equilibrium Fixed Point

- start at non-stochastic equilibrium
- compute equilibrium with shocks to A_{it}
- approximate true process for z_{it} with some Markov process ψ
- use ψ for beliefs in next iteration



- iterate until ψ is "close to" true z process
  - will this converge? hopefully..

7 Empirical Implications

What are some of the empirical implications of this model? First, More Hand-to-Mouth behavior imply that Recessions more likely. Furthermore, when Agents close to borrowing constraint → Recessions more likely. This would potentially be a nice thing to test.

Which recessions could this model potentially apply to? The subprime, The 1907 recession was presumably caused by...

Furthermore, there is evidence... Reinhart Rogoff. Alan Taylor...

8 Conclusion

I construct a model in which recessions resemble traffic jams. The next steps in this are clearly two fold. First, I would like to find data on local interactions and see how to get a flux-like diagram like that in the traffic literature. This would give some empirical evidence for this mechanism. Second, it would be important to think about efficiency and policy.

Appendix

Proof of Stability in Traffic Model

$$\dot{v} = f\left(h, \dot{h}, v\right) = \frac{1}{T}\left(V\left(h\right) - v\right) + W\left(\dot{h}\right)$$

Linearize this about the uniform flow equilibrium. we then have

$$\dot{v}\left(t\right) = F\tilde{h}\left(t\right) + G\tilde{\dot{h}}\left(t\right) - H\tilde{v}\left(t\right)$$

where

$$\begin{aligned} F &= d_h f = \alpha V'\left(h\right) \\ G &= d_{\dot{h}} f = W'\left(h\right) \\ H &= -d_v f = \alpha \end{aligned}$$

...

$$\begin{aligned} \alpha V'\left(h\right) &< \frac{1}{2}\left(2W'\left(h\right) + \alpha\right)\alpha \\ V'\left(h\right) &< \frac{1}{2}\left(2W'\left(h\right) + \alpha\right) \end{aligned}$$

if $W'\left(h\right) = 0$, then the stability condition becomes

$$V'\left(h^*\right) < \frac{1}{2}\alpha$$

Proof of Worker Sector Claim The marginal cost can be attributed to an outside worker sector. There is a worker sector that consumes shells and works for the circle firms. The preferences of this worker sector are given by

$$U\left(\hat{c}_t, \hat{n}_t\right) = \hat{c}_t - \chi\hat{n}_t$$

where $\hat{c}_t$ is the consumption in shells and $\hat{n}_t$ is it's labor. The household maximizes his utility subject to its budget constraint.

$$\hat{c}_t = w_t\hat{n}_t$$

Thus the equilibrium wage in every period is a constant is given by

$$
\begin{aligned}
1-\lambda &= 0 \\
-\chi + w_t\lambda_t &= 0
\end{aligned}
$$

$$w_t = \chi$$

suppose utility is

$$\sum \hat{\beta}^t \left(\hat{c}_t - \chi\hat{n}_t\right)$$

and budget constraint is

$$\hat{c}_t + a_{t+1} = w_t\hat{n}_t + (1+r)\,a_t$$

then

$$
\begin{aligned}
1-\lambda_t &= 0 \\
-\chi + w_t\lambda_t &= 0 \\
-\hat{\beta}^t\lambda_t + (1+r)\,\hat{\beta}^{t+1}\lambda_{t+1} &= 0
\end{aligned}
$$

thus

$$1+r = \frac{1}{\hat{\beta}}$$

this pins down r. Thus, the workers pin down the interest rate. households and workers can trade in a mainland market for shells.

storage technology for shells. $s_{t+1} = f\left(s_t\right)$. this firm maximizes profits given by

$$f\left(s_t\right) - (1+r)\,s_t$$

foc

$$f'\left(s_t\right) = (1+r) = \frac{1}{\hat{\beta}}$$

this pins down the number of shells in the economy. Thus, the total number of shells in the economy remain constant.

adding up

$$
\begin{aligned}
\sum z_{i,t} + a_{i,t} &= (1-\alpha)\,z_{i+1,t-1} + (1+r)\,a_{it-1} \\
\hat{c}_t + \hat{a}_{t+1} &= \alpha\bar{z}_t + (1+r)\,\hat{a}_t \\
&=
\end{aligned}
$$

Proof of firm's problem First, let's characterize the firm's problem, which is very simple. Firm i's static problem in period t is given by

$$\max p_{it} y_{it} - \chi n_{it}$$

where $y_{it} = A_{i,t} n_{i,t}^{\alpha}$. Producer optimality is given by

$$p_{it} \alpha \frac{y_{it}}{n_{it}} = \chi$$

optimality for the firm

$$p_{it} = \frac{\chi}{\alpha \frac{y_{it}}{n_{it}}}$$

This implies that firm profits are given by

$$\begin{aligned} \pi_{i,t} &= p_{it} y_{it} - p_{it} \alpha \frac{y_{it}}{n_{it}} n_{it} \\ &= (1-\alpha) p_{it} y_{it} \end{aligned}$$

We can easily write the supply function of firms in terms of p_{it}

$$p_{it} \alpha A_{it} n_{it}^{\alpha-1} = \chi$$

where

$$n_{it} = \left(\frac{y_{it}}{A_{it}} \right)^{1/\alpha}$$

Plugging this into (), we get that

$$p_{it} \alpha A_{it}^{1/\alpha} (y_{it})^{1-1/\alpha} = \chi$$

Thus supply is given by

$$y_{it}^{1/\alpha - 1} = \left(\frac{\alpha}{\chi} \right) A^{1/\alpha} p_{it}$$

thus

$$y_{it}(p_{it}) = \left[\left(\frac{\alpha}{\chi} \right) A^{1/\alpha} p_{it} \right]^{\frac{1}{1/\alpha - 1}}$$

Proof of Lemma 1

$$\begin{aligned} v_{it} &\equiv \Delta x_{i,t} = x_{i,t} - (1+r)\, x_{i,t-1} \\ &= \left(z_{i,t} + (1+r) \sum_{j=0}^{\infty} (1+r)^j\, z_{i+1,t-1-j} \right) - (1+r) \sum_{j=0}^{\infty} (1+r)^j\, z_{i,t-1-j} \\ &= z_{i,t} \end{aligned}$$

The position of agent $i+1$ at time t is given by

$$x_{i+1,t} = \sum_{j=0}^{t} (1+r)^j\, z_{i+1,t-j} = z_{i+1,t} + (1+r) \sum_{j=0}^{\infty} (1+r)^j \left[z_{i+1,t-1-j} \right]$$

Multiplying this by $(1-\alpha)$ we have that

$$(1-\alpha)\, x_{i+1,t} = (1-\alpha)\, z_{i+1,t} + \sum_{j=0}^{\infty} (1+r)^j \left[(1+r)\,(1-\alpha)\, z_{i+1,t-1-j} \right]$$

Next, rearranging the budget constraint,

$$z_{it} + a_{i,t} = (1+r)\,(1-\alpha)\, z_{i+1,t-1} + (1+r)\, a_{it-1}$$

we obtain the following

$$(1+r)\,(1-\alpha)\, z_{i+1,t-1-j} = z_{i,t-j} + a_{i,t-j} - (1+r)\, a_{i,t-1-j}$$

Plugging this into () we get that

$$(1-\alpha)\, x_{i+1,t} = (1-\alpha)\, z_{i+1,t} + \sum_{j=0}^{\infty} (1+r)^j \left(z_{i,t-j} + a_{i,t-j} - (1+r)\, a_{i,t-1-j} \right)$$

Now, if we write out the position of agent i at time t, this is given by

$$(1+r)\, x_{i,t} = (1+r) \sum_{j=0}^{t} (1+r)^j\, z_{i,t-j}$$

Substituting () and () into our definition of headway,

$$h_{it} \equiv (1-\alpha)\, x_{i+1,t} - x_{i,t}$$

we have that

$$h_{it} = (1-\alpha)\, z_{i+1,t} + \sum_{j=0}^{t} (1+r)^j \left(z_{i,t-j} + a_{i,t-j} - (1+r)\, a_{i,t-1-j}\right) - \sum_{j=0}^{\infty} (1+r)^j\, z_{i,t-j}$$

Thus,

$$h_{it} = (1-\alpha)\, z_{i+1,t} + \sum_{j=0}^{\infty} (1+r)^j \left(a_{i,t-j} - (1+r)\, a_{i,t-1-j}\right)$$

Expanding the terms in the summation, we have that headway satisfies

$$\begin{aligned} h_{it} &= (1-\alpha)\, z_{i+1,t} + \left(a_{i,t} - (1+r)\, a_{i,t-1}\right) \\ &\quad + (1+r)\left(a_{i,t-1} - (1+r)\, a_{i,t-2}\right) \\ &\quad + (1+r)^2 \left(a_{i,t-2} - (1+r)\, a_{i,t-3}\right) + \cdots \end{aligned}$$

All of the $a_{i,t-j}$ cancel out except for $j = 0$. Thus, we have that

$$h_{it} = (1-\alpha)\, z_{i+1,t} + a_{i,t}$$

Rewriting this for $h_{i,t-1}$ we have that

$$h_{i,t-1} = (1-\alpha)\, z_{i+1,t-1} + a_{i,t-1}$$

QED.

change in headway Furthermore note that the change in headway may be written as

$$\begin{aligned} \Delta h_{i,t-1} &= h_{i,t-1} - (1+r)\, h_{i,t-2} \\ &= \left((1-\alpha)\, x_{i+1,t-1} - (1+r)\, x_{i,t-1}\right) - (1+r)\left((1-\alpha)\, x_{i+1,t-2} - (1+r)\, x_{i,t-2}\right) \\ &= \left[(1-\alpha)\, x_{i+1,t-1} - (1+r)(1-\alpha)\, x_{i+1,t-2}\right] - (1+r)\left[x_{i,t-1} - (1+r)\, x_{i,t-2}\right] \\ &= (1-\alpha)\, \Delta x_{i,t-1} - (1+r)\, \Delta x_{i,t-1} \\ &= (1-\alpha)\, z_{i,t-1} - (1+r)\, z_{i,t-1} \end{aligned}$$

Proof of Proposition Policy Function Permanent Income

$$\frac{1}{1-\frac{1}{1+r}} p_{i-1,t} c_{it} = \frac{1}{1-\frac{1}{1+r}} (1-\alpha)\, p_{i,t} y_{i,t} + (1+r)\, a_{it-1}$$

$$\begin{aligned}
\frac{1+r}{r} p_{i-1,t} c_{it} &= \frac{1+r}{r} (1-\alpha) p_{i,t} y_{i,t} + (1+r) a_{it-1} \\
p_{i-1,t} c_{it} &= (1-\alpha) p_{i,t} y_{i,t} + r a_{it-1}
\end{aligned}$$

We can rewrite the right-hand side of this equation in the following way

$$\begin{aligned}
h_{i,t-1} - h_{i,t-2} - r z_{it-1} &= h_{i,t-1} - h_{i,t-2} - r z_{it-1} + \frac{1}{1+r} h_{i,t-1} - \frac{1}{1+r} h_{i,t-1} \\
&= \left(\frac{1}{1+r} h_{i,t-1} - h_{i,t-2} \right) + h_{i,t-1} - \frac{1}{1+r} h_{i,t-1} - r z_{it-1} \\
&= \frac{1}{1+r} \left[\left(h_{i,t-1} - (1+r) h_{i,t-2} \right) + (1+r) h_{i,t-1} - h_{i,t-1} - (1+r) r z_{it-1} \right] \\
&= \frac{1}{1+r} \left[\Delta h_{i,t-1} + r h_{i,t-1} - (1+r) r z_{it-1} \right] \\
&= \frac{1}{1+r} \Delta h_{i,t-1} + \frac{1}{1+r} r h_{i,t-1} - r z_{it-1} \\
&= \frac{1}{1+r} \Delta h_{i,t-1} + r \left(\frac{1}{1+r} h_{i,t-1} - z_{it-1} \right)
\end{aligned}$$

Proof of Proposition Stability Permanent Income Follows from the main text.

In my model, this corresponds to

$$\begin{aligned}
r \left(\frac{1}{1+r} \right) &< \frac{1}{2-\alpha} \left((2-\alpha) \frac{1}{1+r} + r \right) r \\
\frac{1}{1+r} &< \frac{1}{2-\alpha} \left((2-\alpha) \frac{1}{1+r} + r \right) \\
\frac{1}{1+r} &< \frac{1}{1+r} + \frac{1}{2-\alpha} r \\
1 &< 1 + \frac{1}{2-\alpha} r (1+r)
\end{aligned}$$

Proof of Proposition Policy Function Quasi-Hyperbolic

$$\begin{aligned}
p_{i-1,t} c_{it} + \frac{1}{r} \beta p_{i-1,t} c_{i,t} &= \frac{1+r}{r} (1-\alpha) p_{i,t} y_{i,t} + (1+r) a_{it-1} \\
\left(1 + \frac{\beta}{r} \right) p_{i-1,t} c_{i,t} &= \frac{1+r}{r} (1-\alpha) p_{i,t} y_{i,t} + (1+r) a_{it-1} \\
\left(\frac{r+\beta}{r} \right) p_{i-1,t} c_{i,t} &= \frac{1+r}{r} (1-\alpha) p_{i,t} y_{i,t} + (1+r) a_{it-1} \\
p_{i-1,t} c_{i,t} &= \frac{1+r}{\beta+r} (1-\alpha) p_{i,t} y_{i,t} + \frac{(1+r)}{\beta+r} r a_{it-1}
\end{aligned}$$

$$z_{it} = \frac{1+r}{\beta+r}(1-\alpha) z_{it+1} + \frac{(1+r)}{\beta+r} r a_{it-1}$$
$$h_{i,t-1} = (1-\alpha) z_{i+1,t-1} + (1+r) a_{i,t-1}$$

thus

$$\begin{aligned} z_{it} &= \frac{1+r}{\beta+r}\left(h_{i,t-1} - (1+r) a_{i,t-1}\right) + \frac{(1+r)}{\beta+r} r a_{it-1} \\ z_{it} &= \frac{1+r}{\beta+r}\left(h_{i,t-1} - a_{i,t-1}\right) \end{aligned}$$

but note that my consumption yesterday and whatever i saved yesterday is equal to my headway yesterday.

$$z_{it-1} + a_{i,t-1} = h_{i,t-2}$$

thus, i get a new behavioral equation

$$z_{it} = \frac{1+r}{\beta+r}\left(h_{i,t-1} - h_{i,t-2} + z_{it-1}\right)$$

$$\begin{aligned} z_{it} - (1+r) z_{it-1} &= \frac{1+r}{\beta+r}\left(h_{i,t-1} - h_{i,t-2} + z_{it-1}\right) - (1+r) z_{it-1} \\ z_{it} - (1+r) z_{it-1} &= \frac{1+r}{\beta+r}\left(h_{i,t-1} - h_{i,t-2}\right) + (1+r)\left(\frac{1}{\beta+r} - 1\right) z_{it-1} \end{aligned}$$

we can rewrite the right-hand side in the following way

$$\begin{aligned} & \frac{1+r}{\beta+r}\left(h_{i,t-1} - h_{i,t-2} + \frac{1}{1+r} h_{i,t-1} - \frac{1}{1+r} h_{i,t-1}\right) + (1+r)\left(\frac{1}{\beta+r} - 1\right) z_{it-1} \\ &= \frac{1+r}{\beta+r}\left(\frac{1}{1+r} h_{i,t-1} - h_{i,t-2}\right) + \frac{1+r}{\beta+r} h_{i,t-1} - \frac{1+r}{\beta+r}\frac{1}{1+r} h_{i,t-1} + (1+r)\left(\frac{1}{\beta+r} - 1\right) z_{it-1} \\ &= \frac{1}{\beta+r}\left(h_{i,t-1} - (1+r) h_{i,t-2}\right) + \frac{1+r}{\beta+r} h_{i,t-1} - \frac{1}{\beta+r} h_{i,t-1} + (1+r)\left(\frac{1}{\beta+r} - 1\right) z_{it-1} \\ &= \frac{1}{\beta+r}\Delta h_{i,t-1} + \frac{r}{\beta+r} h_{i,t-1} + (1+r)\left(\frac{1}{\beta+r} - 1\right) z_{it-1} \\ \Delta z_{it} &= \frac{1}{\beta+r}\Delta h_{i,t-1} + \frac{r}{\beta+r} h_{i,t-1} + (1+r)\left(\frac{1-(\beta+r)}{\beta+r}\right) z_{it-1} \end{aligned}$$

before we had

$$= \frac{1}{1+r}\Delta h_{i,t-1} + r\left(\frac{1}{1+r} h_{i,t-1} - z_{it-1}\right)$$

Proof of Proposition Stability Quasi-Hyperbolic plugging these in

$$\frac{r}{\beta+r} < \frac{1}{2-\alpha}\left(\left(2-\alpha\right)\frac{1}{\beta+r} + \left(1+r\right)\left(\frac{\left(\beta+r\right)-1}{\beta+r}\right)\right)\left(1+r\right)\left(\frac{\left(\beta+r\right)-1}{\beta+r}\right)$$

$$\begin{aligned}
r &< \frac{1}{2-\alpha}\left(\left(2-\alpha\right)\frac{1}{\beta+r} + \left(1+r\right)\left(\frac{\left(\beta+r\right)-1}{\beta+r}\right)\right)\left(1+r\right)\left(\left(\beta+r\right)-1\right) \\
r &< \frac{1}{2-\alpha}\left(\frac{\left(2-\alpha\right) + \left(1+r\right)\left(\left(\beta+r\right)-1\right)}{\beta+r}\right)\left(1+r\right)\left(\left(\beta+r\right)-1\right)
\end{aligned}$$

beta equals zero case

$$\begin{aligned}
r &< \frac{1}{2-\alpha}\left(\frac{\left(2-\alpha\right) + \left(1+r\right)\left(r-1\right)}{r}\right)\left(1+r\right)\left(r-1\right) \\
\left(2-\alpha\right)r^2 &< \left(\left(2-\alpha\right) + \left(1+r\right)\left(r-1\right)\right)\left(1+r\right)\left(r-1\right) \\
\left(2-\alpha\right)r^2 &< -\left(\left(2-\alpha\right) - \left(1+r\right)\left(1-r\right)\right)\left(1+r\right)\left(1-r\right) \\
\left(2-\alpha\right)r^2 &< -\left(\left(2-\alpha\right) - \left(1-r^2\right)\right)\left(1-r^2\right) \\
\left(2-\alpha\right)r^2 &< -\left(2-\alpha\right)\left(1-r^2\right) + \left(1-r^2\right)\left(1-r^2\right) \\
\left(2-\alpha\right)r^2 &< -\left(2-\alpha\right) + \left(2-\alpha\right)r^2 + \left(1-r^2\right)\left(1-r^2\right) \\
0 &< -\left(2-\alpha\right) + \left(1-r^2\right)\left(1-r^2\right)
\end{aligned}$$

stable if

$$\left(1-r^2\right)\left(1-r^2\right) > \left(2-\alpha\right)$$

stable if

$$\begin{aligned}
\left(1-r^2\right)^2 &> \left(2-\alpha\right) \\
1-r^2 &> \pm\sqrt{\left(2-\alpha\right)}
\end{aligned}$$

$$r^2 > 1-\sqrt{2}$$

in general this is not true

$$\sqrt{\left(1-\sqrt{2}\right)}$$

exponential utility With exponential utility

$$u\left(c\right) = -\exp\left\{-\gamma c\right\}$$

then

$$u'(c) = \gamma \exp\{-\gamma c\}$$

the Euler Equation is thus given by

$$\frac{\gamma \exp\{-\gamma c_{it}\}}{p_{i-1,t}} = E_t \beta (1+r) \frac{\gamma \exp\{-\gamma c_{it+1}\}}{p_{i-1,t+1}}$$

$$\frac{\exp\{c_{it}\}}{\exp\{c_{it+1}\}} = \beta (1+r) \frac{p_{i-1,t}}{p_{i-1,t+1}}$$

$$c_{it} - c_{it+1} = \log[\beta (1+r)] + \log\left(\frac{p_{i-1,t}}{p_{i-1,t+1}}\right)$$