

# A Theory of Protest Voting

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Originally: July 2012. Last modified: September 2012.<sup>1</sup>

**Abstract.** A supporter of a candidate for office may wish to restrict her power or to send her a message by casting a protest vote against her. A sufficiently large protest may convince the candidate to accept the protesters' demands after winning the election; however, if the protest is too large then it risks causing the candidate to lose to a disliked opponent. I study a model of protest voting in which there is uncertainty about the true electorate-wide enthusiasm for the protest relative to the conflicting desire to ensure that the candidate is safely elected. I find that protest votes are strategic substitutes, and that protest voting reacts negatively to voters' expectations about the true enthusiasm for the protest. An increase in the candidate's popularity (and so a reduction in the desire for a successful protest relative to the wish to see her elected) is offset by increased protest voting. Greater popularity can increase protest voting by enough to harm a candidate's performance at the ballot box.

## 1. THE LOGIC OF PROTEST VOTING

In a two-horse-race election, a voter's incentives seem straightforward: if he wishes his favorite to win then he should vote for her. Nevertheless, sometimes voters support the leading opponent, they vote for an out-of-the-running third candidate, or they spoil their ballot papers. In this paper I use a theoretical model to study how these acts of protest voting respond to the electoral environment, to beliefs about the candidate's popularity, and to voters' anticipation of the candidate's reaction to the election result. Amongst other results, I find that any gain in a candidate's popularity relative to the popular enthusiasm for a successful protest is offset by an endogenous increase in protest voting. Furthermore, if the candidate responds endogenously to the election (by this I mean that she infers her popularity from the outcome, and then decides whether to give in to the protest's demands) then, perhaps surprisingly, the increase in protest voting is large enough to result in a net fall in her ballot-box performance. Empirically, there is evidence that protest voting is quantitatively significant. Protest voting can be considered a strategic vote in the broad sense that a voter switches away from

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<sup>1</sup>I thank Jean-Pierre Benoît and Chris Wallace for their comments.

his first-choice candidate. It is well known that a sizeable fraction of supporters of trailing candidates in multi-candidate plurality-rule elections do this, even when debates about the measurement methodology are recognized (Niemi, Whitten, and Franklin, 1992, 1993; Evans and Heath, 1993; Fisher, 2004). However, those who prefer the leading candidates do this too. In the context of the 1987 UK general election, Franklin, Niemi, and Whitten (1994) found that voters who switched away from their first-choice candidates were roughly evenly split between classical “instrumental” types who abandoned a trailing contender and what they called “expressive” strategic voters who turn away from one of the two leaders.

The UK context described above involves genuine multi-party competition; however, votes for minority candidates also occur in the classic bipartite environment of the United States. For example, the presidential elections of 1968, 1980, and 1992 featured the presence of the independent candidates Wallace, Anderson, and Perot (Abramson, Aldrich, Paolino, and Rohde, 1995). In 1992, Ross Perot received more votes than the margin between the leading candidates (Lacy and Burden, 1999); potentially the Perot voters could have switched the outcome, given that some (Alvarez and Nagler, 1995) have suggested that he drew more votes from Bush than from Clinton. Furthermore, it has been argued (Gold, 1995) that relative neutrality between major party candidates can open up opportunities for the disaffected to switch to a candidate such as Perot.<sup>2</sup> Even if a third-party candidate is absent, voters may actively engage in dissent by spoiling their ballot papers; notably, Rosenthal and Sen (1973) documented instances of such blank ballots in the context of the French Fifth Republic.

Here, I join Kselman and Niou (2011) in thinking of a protest vote as “a targeted signal of dissatisfaction to one’s most-preferred political party.” Such a vote can make sense when a voter cares about aspects of the election result beyond the identity of the winner. Concretely, I study a stylized situation in which a voter (“he”) would like his favorite candidate (“she”) to win, but he prefers to avoid a critically large winning margin; he wishes to prevent a landslide win so that he may (Franklin, Niemi, and Whitten, 1994, p. 552) “humble a party that is poised to win by an overwhelming margin.” A desire to constrain the winner may arise

<sup>2</sup>Such third-party opportunities arise elsewhere; for example Bowler and Lanoue (1992) studied the implications for Canadian voting behaviour. Shifts in votes away from established leading parties are also a feature beyond national boundaries. For example, elections to the European Parliament do not determine the identity of a ruling government, and this can enable voters “to express their opposition to a particular government” or “to signal their preferences on a particular policy issue they care about which the main parties are ignoring, such as the environment, or immigration” (Hix and Marsh, 2011, p. 5).

when a voter wishes to send a message to her. For example, a re-elected incumbent politician may choose to modify an unpopular policy if her perceived popularity is sufficiently low.

In this situation, a vote can be pivotal in two different ways: it may tip the balance to enable the candidate to win; however, a protest vote may prevent the landslide win. A voter contemplates the relative likelihood of these events. This computation is closely related to that used in a strategic-voting scenario. In a classic three-candidate plurality-rule strategic-voting situation, a voter compares the probability that a sincere vote will enable his favorite to win to the probability that a strategic vote for a less-preferred candidate may defeat a disliked third opponent. The fundamental force is one of strategic complements: if others vote strategically, then this (heuristically, at least) enhances a voter's incentive to join them. In contrast, the force in a protest-voting scenario is one of strategic substitutes: if others engage in protest voting, then a voter becomes more concerned with ensuring that his preferred candidate wins the election. In the context of my model, I confirm this; however, when I extend the model to allow voters to receive private signals of the candidate's electorate-wide popularity, I find that the situation is more nuanced: a greater response by others to their private signals (a stronger tendency to cast a protest vote when a signal reveals the candidate's greater popularity) can induce a voter to respond more strongly to his own private signal.

The most interesting findings concern the effect of a candidate's popularity. A voter's preference for the candidate is determined by his payoff from seeing the candidate win rather than lose relative to his payoff gain from a successful protest. I define the candidate's popularity as the average of this payoff ratio across the electorate. The direct effect of an increase in a candidate's popularity is to increase her success at the ballot box. However, greater perceived popularity shifts voters toward greater protest voting, and this naturally offsets, at least partially, the candidate's increased popularity; the offset effect becomes complete as voters' beliefs about the candidate's popularity become very precise.

As noted above, a rationale for a protest vote is that a voter may wish to send a message to a winning candidate. As Franklin, Niemi, and Whitten (1994, p. 552) observed, "a voter might expressively vote for a small party in order to show support for the policies espoused by that party in the hopes that the voter's preferred party might be induced to adopt them." The situation I have in mind is one in which the supporters of a politician wish her to drop an

unpopular policy. She will do so if she believes that the strength of feeling against the policy is sufficiently great; equivalently, she does this if her perceived popularity falls below a critical threshold. The candidate's perception of her true popularity is determined by the election result. Thus, ballot-box support is an informative signal of popularity, and a protest vote is an act of signal jamming. Naturally, a candidate for office understands this, and accounts for the endogenous presence of signal-jamming protest votes when she makes inferences about her own popularity. There is a feedback effect here: anticipated protest voting makes a candidate less willing to react to the protest.

The feedback effect from the signal-jamming role of protest votes can be strong enough to ensure that increased popularity can actively harm the prospects of a candidate for office. Imagine a situation in which a politician drops a disliked policy if and only if the amount of protest voting crosses a line in the sand. The direct effect of an increase in her popularity is to reduce the number of protest votes; however, the strategic-substitutes logic described above feeds back into the behavior of voters leading to greater protest voting. Voters are now more willing to cast a protest vote, and so a significantly sized protest does not necessarily indicate greater true underlying disquiet; the candidate does not interpret a large loss in support as a reflection of fundamental unpopularity. This endogenously moves the line in the sand; a larger protest vote is needed to persuade the candidate to abandon the disliked policy. Of course, this movement of the line further encourages greater protest voting. The overall effect can be enough to generate a net loss in ballot-box support.

The paper follows a conventional structure. Following a brief discussion of recent related research (Section 2), I describe the model specification (Section 3). I then characterize voting equilibria when voters have either common beliefs about the candidate's popularity (Section 4) and when they update their beliefs based on introspective observation of their own preferences (Section 5). After characterizing equilibria, I present a full set of comparative-static results (Section 6), before extending the model in two directions. Firstly, I consider a signal-jamming world in which the candidate responds endogenously to her inferred popularity (Section 7). Secondly, I study a voting game in which voters receive separate private signals of the candidate's popularity (Section 8). Proofs are relegated to Appendix B.

## 2. RELATED LITERATURE

Over the last dozen years, several formal theoretical studies of protest voting (and related phenomena) have appeared; my paper contributes to this literature. I discuss here a small selection of related papers. Briefly, my paper is closely related to work which considers the signal-jamming role of election results; the key contributions include those by Piketty (2000), Castanheira (2003), Razin (2003), and Meirowitz and Shotts (2009).<sup>3</sup> The modeling technology uses techniques from recent analyses of strategic voting in the presence of aggregate uncertainty; the relevant papers are those by Myatt (2007) and Dewan and Myatt (2007).

Recent contributions to a broader theory of voting have identified different routes via which a vote may be instrumental. For example, Castanheira (2003) observed that a vote may be “outcome pivotal” (so that it changes the outcome of an election) and also “communication pivotal” (it changes others’ future behavior by influencing how they learn about the world). Piketty (2000) identified three channels for communicative voting: firstly, voters may wish to induce policy shifts by mainstream parties; secondly, they may wish to learn about candidates in order to assist the coordination of votes in future elections; and, thirdly, voters may wish to use their votes to influence others’ opinions and so others’ future votes. His work concentrated on the third of these channels; here, however, my model is focused on the first channel, and other recent contributions to the literature share that focus.

Shotts (2006) studied a two-election model in which office-motivated left-wing and right-wing candidates infer the preferred policy of the median voter from the outcome of a first election, and move to that policy ready for the second election; some voters face an incentive to engage in signal-jamming in the first election. He described an equilibrium in which moderate voters abstain in the first-election. Such abstention was ruled out by Meirowitz and Shotts (2009); furthermore, in the context of the Shotts (2006) model, Hummel (2011) demonstrated that abstention vanishes in a large election. Meirowitz and Shotts (2009) found that the long-run signal-jamming incentive (to influence candidates’ future policies) dominates the short-run instrumental incentive (to elect the favored candidate in the first election) when the electorate is

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<sup>3</sup>Beyond the contributions discussed here, there are other less closely related recent contributions. For example, Kselman and Niu (2011) extended their earlier work on strategic voting (Kselman and Niu, 2009) and described the situations in which a protest vote might make sense, but they did not conduct a game-theoretic analysis; Kang (2004) discussed the possible application of the “exit and voice” ideas of Hirschman (1970) to protest voting; and Smirnov and Fowler (2007) considered the influence of margins of victory on candidates’ future positions.

large. The authors of all of these papers specify a model in which there is no aggregate uncertainty; voters types are independent draws from a known distribution. My paper shares with these the feature that voters may engage in signal jamming to influence a politician's future behavior; however, unlike these papers I use a model in which there is aggregate uncertainty.<sup>4</sup>

Aggregate uncertainty features in the model of Razin (2003). In his common-value world, centrist voters may be hit with a shock that moves their (common) position to the left or right. Left-wing and right-wing candidates respond (when they win) to the inferred shock but also incorporate their own biases in their policy choices. Voters receive private binary signals of the shock. Razin (2003) identified the important tension between the signaling (moving the policy) and pivotal (choosing the right winner) motivations for vote choices. A distinction between his work and mine is that Razin (2003) considered a common-value environment whereas I consider a private-value world; there are extensive modeling differences too.<sup>5</sup> Aggregate uncertainty is also present in the model of Castanheira (2003). Actors learn about the preference of the median voter by observing the outcome of an election; this observation determines subsequent policy positions. However, the nature of the model specification means that (Castanheira, 2003, p. 1208) "observing the vote results of only two parties is not sufficient to learn where the median voter stands" and so "the vote share of losers thus reveals additional information." This generates votes for extremists (via voters who pursuing a communicative objective) and the anticipation of this can influence the positions of mainstream candidates.

The modeling technology of this paper exploits the relationship between protest voting and strategic voting. The model specification has the following features: if all voters support the candidate, then she wins but the protest fails; if they all cast protest votes then the protest succeeds but their favored candidate loses; and if their votes are split then the protest can succeed without causing the candidate to lose. In essence, they play an anti-coordination game; if a large fraction of them are expected to protest then an individual voter would prefer

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<sup>4</sup>Relatedly, Meirowitz and Tucker (2007) proposed a three-voter model in which a poor showing for a candidate induces her to increase her effort (that is, she accumulates valence) rather than change her policy position. Other more distantly related contributions to this strand of the literature include the analysis of voters' strategic responses to polls (Meirowitz, 2005b) and the analysis of voting and candidate behavior in primaries when those primaries reveal information relevant to a subsequent general election (Meirowitz, 2005a).

<sup>5</sup>The common vs. private-value distinction arises in a comparison with work by McMurray (2012). He considered voters who receive private signals of a common ideal policy, and candidates learn from the election outcome; his work is related to the "swing voter's curse" jury-voting analyses of Feddersen and Pesendorfer (1996, 1997, 1998).

to refrain from doing so. This can be compared to the classic strategic-voting setting in which voters choose between two challengers when they wish to defeat a disliked opponent. Coordination behind either challenger produces a good outcome, but a split allows the disliked opponent to sneak through. Older analyses of the strategic-voting game (Palfrey, 1989; Myerson and Weber, 1993; Cox, 1994) used specifications without aggregate uncertainty, and predicted (if knife-edge unstable equilibria are put aside) the full coordination of voters. More recently, however, Myatt (2007) presented a model with aggregate uncertainty about the popularity of the candidates and he predicted multi-candidate support; with a common-value specification, Dewan and Myatt (2007) used a closely related modeling framework to study the coordinating effects of party leadership. This paper uses many elements of these antecedents, but where payoffs are structured to reward anti-coordination rather than coordination.

### 3. A MODEL OF PROTEST VOTING

**Players.** I consider a game played by  $n$  voters (for whom I use the pronoun “he”) and a single candidate for office (the pronoun “she”). For now I fix exogenously the behavior of the candidate, and so I focus on a simultaneous-move game played by the voters. In Section 7 of the paper I extend the model to allow the candidate to become an active player of the game.

**Moves and Outcomes.** Each voter either votes for the candidate, or he casts a protest vote. I write  $b$  for the number of ballots cast for the candidate. There are three possible outcomes: (i) if  $b$  is small then the candidate loses; (ii) if  $b$  is moderate then she wins with a relatively small winning margin; and (iii) if  $b$  is large then she enjoys a landslide victory. Formally, there are two thresholds  $p_L$  and  $p_H$  satisfying  $0 < p_L < p_H < 1$  such that

$$\text{outcome for the candidate} = \begin{cases} \text{lose} & \text{if } \frac{b}{n} < p_L, \\ \text{win} & \text{if } p_L < \frac{b}{n} < p_H, \text{ and} \\ \text{landslide} & \text{if } p_H < \frac{b}{n}. \end{cases} \quad (1)$$

Before specifying payoffs, I interpret this specification. The voters are natural supporters of the candidate, but are willing to withhold their votes in order to limit the candidate’s overall support. The lower threshold  $p_L$  is the ballot-box support needed to win; it will depend upon

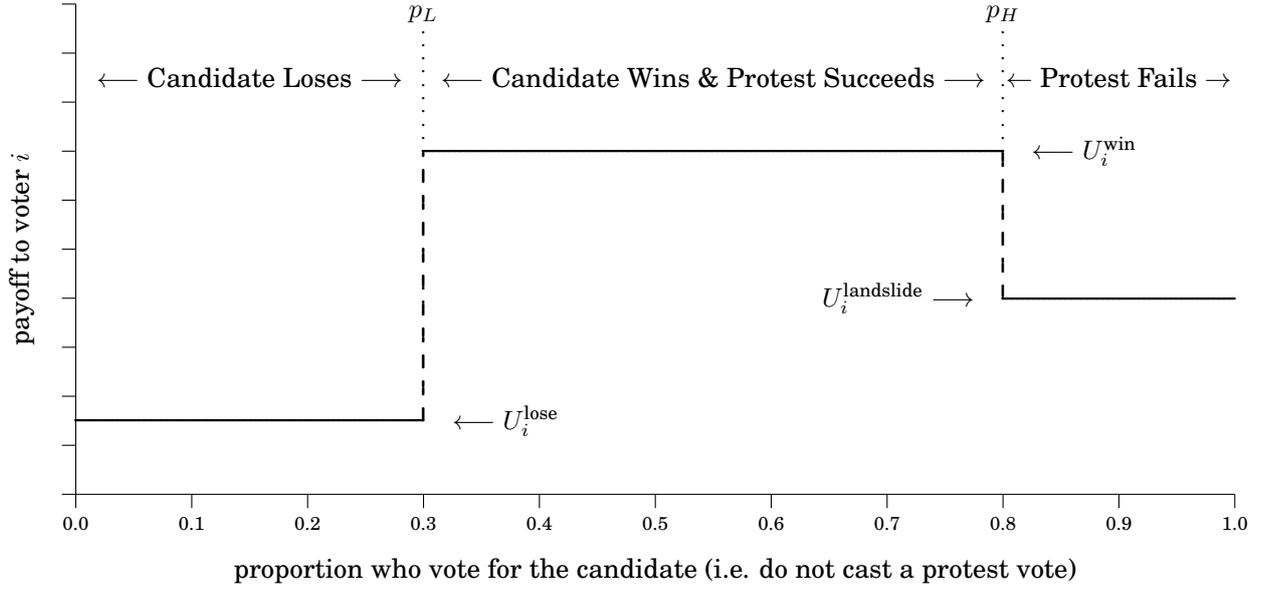


FIGURE 1. Outcomes and Voters' Payoffs

the support of other candidates in the election. The higher threshold  $p_H$  has various interpretations, and I describe three here. Firstly, it could be the support needed for the candidate to achieve greater formal power; think of decisions in which a super-majority is required. Secondly, it could be the support needed for a candidate or party to avoid sharing power in a coalition. Thirdly, it could be a “line in the sand” below which the candidate’s behavior changes; for example a re-elected incumbent may abandon an unpopular policy if her vote share is sufficiently low. Under this third interpretation (which I study further in Section 7) the threshold  $p_H$  can be endogenous; the candidate may use the election outcome to infer the underlying satisfaction with her policies, and this inference is determined in equilibrium.

**Voters' Payoffs.** Voters would like the candidate to win, but dislike a landslide win. Formally, the payoff of voter  $i$  is determined solely by the election outcome, and I assume that

$$U_i^{\text{win}} > \max \left\{ U_i^{\text{lose}}, U_i^{\text{landslide}} \right\}, \quad (2)$$

where the notation should be obvious; this is illustrated graphically in Figure 1. The desire to avoid a landslide provides the incentive for protest voting. This must be balanced against the desire to avoid an outright loss. Bringing these two elements together,

$$u_i \equiv \log \left[ \frac{U_i^{\text{win}} - U_i^{\text{lose}}}{U_i^{\text{win}} - U_i^{\text{landslide}}} \right] \quad (3)$$

is the preference type of voter  $i$ . When  $u_i$  is lower a voter cares more about a successful protest relative to ensuring a win for the candidate; hence, voters with lower preference types are more willing to cast a protest vote. Voters' preferences are heterogeneous. Specifically, there is an average preference type  $\theta$ , and individual preference types are normally distributed around this:

$$u_i | \theta \sim N(\theta, \sigma^2), \quad (4)$$

where types are conditionally independent, and so  $\text{cov}[u_i, u_{i'} | \theta] = 0$  for  $i \neq i'$ . The average  $\theta$  is the underlying popularity of the candidate; equivalently,  $-\theta$  represents the underlying electorate-wide desire to see a successful protest. If  $\theta$  were known then types would be IID draws from a known distribution. However,  $\theta$  is unknown, and so there is aggregate uncertainty about electorate-wide preferences; unconditionally voters' types are correlated.

**Information.** I consider three different model variants, according to voters' information about the candidate's popularity (equivalently, about the enthusiasm for a successful protest).

*Public Information.* In the public information case, all voters believe that

$$\theta \sim N\left(\mu, \frac{\sigma^2}{\psi}\right). \quad (5)$$

A voter derives his beliefs about other voters' types from this. Hence, voter  $i$ 's beliefs about the preference types of voters  $j$  and  $j'$  are joint normal, with moments

$$E[u_j] = E[u_{j'}] = \mu, \quad \text{var}[u_j] = \text{var}[u_{j'}] = \frac{(1 + \psi)\sigma^2}{\psi}, \quad \text{and} \quad \text{cov}[u_j, u_{j'}] = \frac{\sigma^2}{\psi}. \quad (6)$$

If an observer begins with a flat prior over  $\theta$  and observes a random sample of  $\psi$  preference realisations, then his posterior beliefs will have precision  $\psi/\sigma^2$ , and so  $\psi$  (rather than  $\psi/\sigma^2$ ) is the appropriate measure of the accuracy of information about underlying preferences.<sup>6</sup>

*Introspective Voters.* Under the public-information specification, voters' beliefs about  $\theta$  do not differ; this is by assumption. This differs from a situation in which voters share a common prior (before the realisation of their preference types) which each voter updates introspectively using his own type realisation. A voter's preference type  $u_i$  is a signal of  $\theta$  with precision

<sup>6</sup>The results of Section 6 apply to the public-information specification if  $\psi > 1$ . This says that beliefs are at least as precise as those obtained following the observation of a single preference realization.

$1/\sigma^2$ . The prior described in equation (5) has precision  $\psi/\sigma^2$ . Updating beliefs:

$$\theta | u_i \sim N\left(\frac{\psi\mu + u_i}{\psi + 1}, \frac{\sigma^2}{\psi + 1}\right). \quad (7)$$

A voter's expectation of others' preferences is now related to his own preference. Furthermore,

$$E[u_j | u_i] = \frac{\psi\mu + u_i}{\psi + 1}, \quad \text{var}[u_j | u_i] = \frac{(\psi + 2)\sigma^2}{\psi + 1}, \quad \text{and} \quad \text{cov}[u_j, u_{j'} | u_i] = \frac{\sigma^2}{\psi + 1}. \quad (8)$$

*Private Information.* Under a private information specification, I expand a voter's type to a pair  $(u_i, s_i)$  where  $s_i$  is a private signal of  $\theta$ . Conditional on  $\theta$ , these are joint normally distributed, and are (conditionally) independent of others' types. I have already observed that  $u_i$  is itself an informative signal of  $\theta$ . I incorporate this into the signal  $s_i$  so that  $s_i$  is a sufficient statistic for  $(u_i, s_i)$  when conducting inference about  $\theta$ . Formally:

$$s_i | \theta \sim N\left(\theta, \frac{\sigma^2}{\lambda}\right) \quad \text{and} \quad u_i | (\theta, s_i) \sim N\left(s_i, \frac{(\lambda - 1)\sigma^2}{\lambda}\right). \quad (9)$$

This implies that  $u_i | \theta \sim N(\theta, \sigma^2)$ , as before. Note that  $\lambda$  is the precision of a voter's private signal. Even if other information sources are weak, the availability of introspection results in the regularity assumption that  $\lambda > 1$ . Updating the prior belief from equation (5),

$$\theta | (u_i, s_i) \sim N\left(\frac{\psi\mu + \lambda s_i}{\psi + \lambda}, \frac{\sigma^2}{\psi + \lambda}\right). \quad (10)$$

**Solution Concept.** A strategy  $v_i(u_i, s_i) : \mathcal{R}^2 \mapsto [0, 1]$  for voter  $i$  is the probability that he votes for the candidate (otherwise he protests) conditional on his type. The notation here is for the private information case; in the other cases the dependence on  $s_i$  is dropped. I restrict to type-symmetric strategy profiles (there is very little loss in generality from doing this) which can be stated as a single strategy  $v(u_i, s_i)$ . I write  $\text{BR}[v(\cdot, \cdot) | (u_i, s_i)]$  for the set of best replies; usually I will restrict to strict best replies, so that  $\text{BR}[v(\cdot, \cdot) | (u_i, s_i)]$  is a singleton.

The usual solution concept would be a type-symmetric (Bayesian) Nash equilibrium. Here, however, the primary focus is on behavior in large electorates. One reason for this is to ensure that uncertainties over idiosyncratic type realizations do not drive things. A second (pragmatic) reason is that the solutions simplify appreciably when  $n$  is large. One approach in a large-electorate context would be to find an equilibrium (assuming that one exists) for

each  $n$ , and to examine the limiting properties of a sequence of equilibria as  $n \rightarrow \infty$ . Here, however, I follow earlier work (Myatt, 2007; Dewan and Myatt, 2007) by defining a solution concept over the sequence of voting games indexed by the electorate's size. The idea is to seek a single voting strategy that specifies a best reply whenever the electorate is large enough.

**Definition 1** (Voting Equilibria). *A voting equilibrium is a voting strategy that, if used by everyone, specifies for almost every type (that is, all types except possibly a zero-probability subset) a best reply in all electorates that are sufficiently large. Formally,  $v(\cdot, \cdot)$  satisfies*

$$\Pr \left[ v(u_i, s_i) \in \lim_{n \rightarrow \infty} \text{BR}[v(\cdot, \cdot) | (u_i, s_i)] \right] = 1. \quad (11)$$

For any finite electorate size, a voting equilibrium is less stringent than the Nash concept because there may be some types who do not play a best reply. In this sense, a voting equilibrium is a kind of  $\varepsilon$ -equilibrium. However, the set of types who can profitably deviate shrinks as  $n$  increases, and the payoff gain from a deviation falls in an appropriate sense. Myatt (2007) and Dewan and Myatt (2007) discussed and justified this solution concept more fully. However, for the public-information specification I report (in Appendix A) the orthodox approach: a sequence of Bayesian Nash equilibria converges to the voting equilibrium as  $n \rightarrow \infty$ .

Definition 1 allows for weak best replies, and so allows for fully coordinated equilibria in which all voters ignore their type realizations and unconditionally choose the same action. For example, if everyone unanimously votes in favor of the candidate then no individual voter can change the outcome, and so anything is a best reply.

For most of the paper, however, my attention focuses on strict best replies. Such equilibria involve voting strategies in which voters' actions respond to their type realizations. Amongst such strategies are those that induce a positive relationship between a candidate's popularity and the probability of a vote for her. I refer to these as monotonic strategies.

**Definition 2** (Strict, Monotonic, and Cutpoint Equilibria). *An equilibrium is strict if the best replies are strict. A voting strategy is monotonic if the probability  $P(\theta) \equiv \mathbb{E}[v(u_i, s_i) | \theta]$  of a vote for the candidate is increasing in her popularity  $\theta$  and if  $\lim_{\theta \rightarrow \infty} P(\theta) = 1$  and  $\lim_{\theta \rightarrow -\infty} P(\theta) = 0$ . A voter uses a cutpoint strategy if there is a cutpoint  $u^*$  such that he votes for the candidate if  $u_i > u^*$  and casts a protests vote if  $u_i < u^*$ .*

#### 4. EQUILIBRIA WITH PUBLIC INFORMATION

Here I consider voting equilibria under the public-information specification: voters share the same beliefs  $\theta \sim N(\mu, \sigma^2/\psi)$  about the popularity of the candidate. Any dependence on  $s_i$  can be dropped, and so a type-symmetric voting strategy is a function  $v(u_i) : \mathcal{R} \mapsto [0, 1]$ .

**Optimal Voting.** A voter's decision matters only when his vote is pivotal. There are two ways this can happen: a vote for the candidate may push her support up above the lower threshold  $p_L$ , turning a loss into a win; or, it may push support above the higher threshold  $p_H$ , enabling a landslide win. The first effect yields a gain of  $U_i^{\text{win}} - U_i^{\text{lose}}$  whereas the second effect generates a loss of  $U_i^{\text{win}} - U_i^{\text{landslide}}$ . It is strictly optimal to cast a protest vote if and only if

$$\Pr[\text{Pivotal at } L \mid v(\cdot)] \left( U_i^{\text{win}} - U_i^{\text{lose}} \right) < \Pr[\text{Pivotal at } H \mid v(\cdot)] \left( U_i^{\text{win}} - U_i^{\text{landslide}} \right), \quad (12)$$

where the probabilities are evaluated conditional on all other voters using a particular voting strategy  $v(\cdot)$ . Under public information, voters share the same beliefs about pivotal events, and so the probabilities in this inequality apply to everyone. A strict equilibrium involves positive pivotal probabilities, and so for such equilibria this inequality is equivalent to

$$u_i < \log \left[ \frac{\Pr[\text{Pivotal at } H \mid v(\cdot)]}{\Pr[\text{Pivotal at } L \mid v(\cdot)]} \right]. \quad (13)$$

A voter's beliefs are independent of his type, and so the right-hand side of this inequality is a constant. Hence a voter uses a cutpoint strategy; Lemma 1 confirms this. For this result and others that follow, the claims apply to almost all voters; that is, the set of types to whom the claims do not apply has zero probability.

**Lemma 1 (Thresholds).** *In a strict voting equilibrium, voters use a cutpoint strategy, where*

$$u^* = \lim_{n \rightarrow \infty} \log \left[ \frac{\Pr[\text{Pivotal at } H \mid u^*]}{\Pr[\text{Pivotal at } L \mid u^*]} \right], \quad (14)$$

*and where the pivotal probabilities are evaluated given that all voters use the cutpoint  $u^*$ .*

This does not say what happens when  $u_i = u^*$ . To ease exposition, I assume that in such cases a vote is cast for the candidate. That is, a voter casts a protest vote if and only if  $u_i < u^*$ .

**Properties of Pivotal Probabilities.** Given that all voters use a cutpoint strategy, the probability  $p$  that a voter casts her ballot for the candidate is

$$p = P(\theta) \quad \text{where} \quad P(\theta) \equiv \Phi\left(\frac{\theta - u^*}{\sigma}\right), \quad (15)$$

and where  $\Phi(\cdot)$  is the cumulative distribution function of the standard normal distribution. Of course,  $\theta$  is uncertain, and therefore so is  $p$ . The electorate's (common) beliefs about  $\theta$  transform into beliefs about  $p$ , which I represent by the density  $f(p)$ .

A voter does not care directly about  $p$  but instead cares about the likelihood of pivotal events. Conditional on  $p$ , the votes of others are a draw from a binomial distribution with parameters  $p$  and  $n - 1$ . Taking expectations over  $p$ , the voter can evaluate the probability that his vote is required for the candidate to win rather than lose:

$$\Pr[\text{Pivotal at } L] = \binom{n-1}{\lfloor pLn \rfloor} \int_0^1 p^{\lfloor pLn \rfloor} (1-p)^{(n-1)-\lfloor pLn \rfloor} f(p) dp, \quad (16)$$

where  $\lfloor pLn \rfloor$  indicates the greatest integer that is weakly smaller than  $pLn$ . A similar expression holds for the other pivotal probability. As  $n$  grows this pivotal probability shrinks. More subtly, perhaps, the polynomial term in the integrand is sharply peaked around  $p_L$ , and so as  $n$  increases only value of the density around  $p_L$  matters. In fact, the application of a slight variant of Proposition 1 from Chamberlain and Rothschild (1981), which in turn is a variant of a similar result of Good and Mayer (1975), generates a useful lemma.

**Lemma 2** (Pivotal Probabilities). *If votes for the candidate are cast with conditionally independent probability  $p$ , where  $p \sim f(\cdot)$ , and if the density  $f(\cdot)$  is positive and continuous around  $p_L$  and  $p_H$ , then  $\lim_{n \rightarrow \infty} n \Pr[\text{Pivotal at } L] = f(p_L)$  and  $\lim_{n \rightarrow \infty} n \Pr[\text{Pivotal at } H] = f(p_H)$ .*

If voters use a cutpoint strategy then the density  $f(\cdot)$  has full support on the unit interval, and so the conditions of Lemma 2 are satisfied. A corollary is readily obtained.

**Corollary** (to Lemma 2). *If a voter believes that others vote for the candidate with conditionally independent probability  $p$ , where  $p \sim f(\cdot)$  with full support on the unit interval, then*

$$\lim_{n \rightarrow \infty} \log \left[ \frac{\Pr[\text{Pivotal at } H]}{\Pr[\text{Pivotal at } L]} \right] = \log \left[ \frac{f(p_H)}{f(p_L)} \right]. \quad (17)$$

Writing  $f(p | u^*)$  for a voter's beliefs about  $p$  given the use of a cutpoint  $u^*$  by others (this is derived from her beliefs about  $\theta$ ), Lemma 1 and 2 combine to yield a corollary.

**Corollary** (to Lemmas 1 and 2). *In a strict equilibrium, voters use a cutpoint strategy where*

$$u^* = \log \left[ \frac{f(p_H | u^*)}{f(p_L | u^*)} \right], \quad (18)$$

where  $f(p | u^*)$  is the density of the shared beliefs about  $p$  given the use of the cutpoint  $u^*$ . This density is derived from the common belief  $\theta \sim N(\mu, \sigma^2/\psi)$ , where  $p = \Phi((\theta - u^*)/\sigma)$ .

**Equilibrium.** The probability of a vote for the candidate is  $p = P(\theta) \equiv \Phi((\theta - u^*)/\sigma)$ ; equivalently, the average voter type that induces the probability  $p$  is  $\theta = u^* + \sigma\Phi^{-1}(p)$ . If beliefs about  $\theta$  are described by a density  $g(\theta)$ , changing variables straightforwardly yields

$$f(p | u^*) = \frac{g(u^* + \sigma\Phi^{-1}(p))}{\sigma\phi(\Phi^{-1}(p))}, \quad (19)$$

where the notation  $\phi(\cdot)$  indicates the density of the standard normal distribution. Substituting in the usual formula for this density,

$$\log \left[ \frac{f(p_H | u^*)}{f(p_L | u^*)} \right] = \frac{z_H^2 - z_L^2}{2} + \log \left[ \frac{g(u^* + \sigma z_H)}{g(u^* + \sigma z_L)} \right] \text{ where } z_H \equiv \Phi^{-1}(p_H) \text{ and } z_L \equiv \Phi^{-1}(p_L). \quad (20)$$

The log odds term  $\log[f(p_H | u^*)/f(p_L | u^*)]$  reflects the relative likelihood a vote's influence on preventing a landslide versus enabling a regular win for the candidate, and so it reflects the incentive to cast a protest vote. It is decreasing in  $u^*$  if  $g(\cdot)$  is log concave; this is a mild regularity condition. Of course, an increase in the cutpoint  $u^*$  corresponds to an increase in protest voting by others. In summary, this means that protest votes are strategic substitutes: more protest voting by others (so less support for the candidate) weakens the incentive for a best-responding voter to cast a protest vote.

**Lemma 3** (Strategic Substitutes). *If the density  $g(\theta)$  of beliefs about the candidate's popularity  $\theta$  is log concave, then more protest voting by others reduces the incentive for protest voting.*

If the log odds term is decreasing in  $u^*$  then there is a unique solution to equation (18) and hence a unique equilibrium. (Note that the various normality assumptions are not needed

here.) Given the normal density for  $g(\theta)$ , following from the specification  $\theta \sim N(\mu, \sigma^2/\psi)$ , it is straightforward to obtain a closed-form solution for the unique equilibrium. In fact,

$$\log \left[ \frac{g(u^* + \sigma z_H)}{g(u^* + \sigma z_L)} \right] = \frac{\psi(z_L^2 - z_H^2)}{2} - \frac{\psi(u^* - \mu)(z_H - z_L)}{\sigma}, \quad (21)$$

where  $z_L$  and  $z_H$  are as defined previously, and so the log odds term is linear in  $u^*$ .

**Proposition 1** (Equilibrium with Public Information). *If voters believe that  $\theta \sim N(\mu, \sigma^2/\psi)$  then there is a unique strict voting equilibrium. Voters use a cutpoint strategy where*

$$u^* = \frac{\psi(z_H - z_L)}{\psi(z_H - z_L) + \sigma} \left[ \mu - \frac{\psi - 1}{\psi} \frac{\sigma(z_H + z_L)}{2} \right], \quad (22)$$

and where  $z_L \equiv \Phi^{-1}(p_L)$ ,  $z_H \equiv \Phi^{-1}(p_H)$ , and  $\Phi(\cdot)$  is the standard normal distribution function.

Before conducting comparative-static exercises, I consider the impact of voter introspection.

## 5. EQUILIBRIA WITH INTROSPECTIVE VOTERS

In the public-information case considered above, voters' beliefs do not differ. Here I move on to consider the introspective case, where voters share a common prior which they then update following the realizations of their idiosyncratic preference types.

**Optimal Voting with Introspection.** Much of the logic used in the public-information case continues to apply when voters are introspective. A strict voting equilibrium involves strictly positive pivotal probabilities, and so the criterion for voter  $i$  to cast a protest vote is

$$u_i < \log \left[ \frac{\Pr[\text{Pivotal at } H \mid v(\cdot), u_i]}{\Pr[\text{Pivotal at } L \mid v(\cdot), u_i]} \right]. \quad (23)$$

This differs from the inequality (13) because the right-hand side depends upon the realization of a voter's preference type. A voter's preference is a signal of the candidate's true underlying popularity, and so usefully informs a voter about the relative likelihood of different pivotal events. In the public-information world, Lemma 1 confirms that a voter optimally casts a protest vote if and only if  $u_i$  falls below a cutpoint. Here, this conclusion cannot be reached, simply because the right-hand side of the inequality may be increasing in (23).

Nevertheless, any strict and monotonic voting equilibrium involves the use of a cutpoint, and so here I consider the optimal behavior of a voter given that others cast protests vote if and only if their realized preference types fall below a cutpoint  $u^*$ . If this is the case, then the probability that a voter casts her ballot for the candidate is  $p = P(\theta)$  where  $P(\theta) = \Phi((\theta - u^*)/\sigma)$ . Such a strategy is naturally monotonic, and so Lemma 2 and its corollaries apply to a voter  $i$  so long as the density  $f(p)$  is replaced with the conditional density  $f(p | u_i)$ .

**Lemma 4** (Characterization of Equilibrium Threshold). *If voters are introspective, then in a monotonic equilibrium voters use a cutpoint strategy which satisfies*

$$u^* = \log \left[ \frac{f(p_H | u^*, u_i = u^*)}{f(p_L | u^*, u_i = u^*)} \right] \quad (24)$$

where the posterior density  $f(p | u^*, u_i)$  is derived from the prior  $\theta \sim N(\mu, \sigma^2/\psi)$  updated via the voter's preference realization  $u_i$ , and the equation  $p = \Phi((\theta - u^*)/\sigma)$ .

The key here is the density  $f(p | u^*, u_i)$ . Equation (20) continues to hold here, so that

$$\log \left[ \frac{f(p_H | u^*, u_i)}{f(p_L | u^*, u_i)} \right] = \frac{z_H^2 - z_L^2}{2} + \log \left[ \frac{g(u^* + \sigma z_H | u_i)}{g(u^* + \sigma z_L | u_i)} \right] \quad (25)$$

where  $z_L$  and  $z_H$  are as before; the difference is that a voter's beliefs about the candidate's popularity depend upon  $u_i$ . Given the various normality assumptions, this expression—which determines the incentive to cast a protest vote—is (linearly) increasing in  $u_i$ . In fact,

$$\log \left[ \frac{g(u^* + \sigma z_H | u_i)}{g(u^* + \sigma z_L | u_i)} \right] = \frac{\sigma(E[\theta | u_i] - u^*)(z_H - z_L)}{\text{var}[\theta | u_i]} + \frac{\sigma^2(z_L^2 - z_H^2)}{2 \text{var}[\theta | u_i]} \quad (26)$$

$$= \frac{(1 + \psi)(z_H - z_L)}{\sigma} \left( \frac{\psi\mu + u_i}{\psi + 1} - u^* \right) - \frac{(1 + \psi)(z_H^2 - z_L^2)}{2}, \quad (27)$$

where the second inequality stems from  $E[\theta | u_i] = (\psi\mu + u_i)/(1 + \psi)$  and  $\text{var}[\theta | u_i] = \sigma^2/(1 + \psi)$ . Hence, if other voters use a threshold  $u^*$  then voter  $i$  optimally responds, in electorates that are sufficiently large, by casting a protest vote if and only if

$$u_i < \frac{z_H - z_L}{\sigma} \left[ u_i - u^* + \psi(\mu - u^*) - \frac{\psi\sigma(z_H + z_L)}{2} \right]. \quad (28)$$

Two conflicting forces determine a voter's optimal response to the use of cutpoint strategy by others. The left-hand side is increasing in  $u_i$ : a voter who cares more about a win for the

candidate relative to a successful protest faces a weakened incentive to protest. However, the right-hand side is also increasing in  $u_i$ : an enthusiastic fan of the candidate expects others to like her too, and so anticipates reduced protest voting amongst voters; this (via the logic of strategic substitutes highlighted in Lemma 3) increases his incentive to cast a protest vote. A voter's best reply is to use a cutpoint rule (that is, cast a protest vote if and only if  $u_i$  falls below a critical value) if and only if the left-hand of the inequality (28) is more responsive to  $u_i$  than the right-hand side. By inspection, this is true if and only if  $(z_H - z_L) < \sigma$ .

**Equilibrium.** The discussion above concerned the reply (in a large electorate) of a voter to the use of a cutpoint  $u^*$  by others. This reply is itself a (monotonic) cutpoint strategy if and only if  $(z_H - z_L) < \sigma$ . If voters are too similar, so that  $\sigma < (z_H - z_L)$  then the best reply is turned upside down: voters who are most enthusiastic about the candidate cast a protest vote, whereas those who care most about the protest come out for the candidate. Naturally, this all unravels; there is no cutpoint equilibrium (and there is no monotonic equilibrium) if  $\sigma < (z_H - z_L)$ . If, however,  $\sigma > (z_H - z_L)$  then an equilibrium can be obtained by evaluating inequality (28) as an equality for  $u_i = u^*$ . That is, an equilibrium cutpoint satisfies

$$u^* = \frac{z_H - z_L}{\sigma} \left[ \psi(\mu - u^*) - \frac{\psi\sigma(z_H + z_L)}{2} \right]. \quad (29)$$

This (linear) equation solves easily to generate the next result.

**Proposition 2** (Equilibrium with Introspection). *Suppose that voters are introspective. If  $\sigma < z_H - z_L$  then a monotonic voting equilibrium does not exist. However, if  $\sigma > z_H - z_L$  then there is a unique monotonic voting equilibrium in which voters use the cutpoint*

$$u^* = \frac{(z_H - z_L)\psi}{(z_H - z_L)\psi + \sigma} \left[ \mu - \frac{\sigma(z_H + z_L)}{2} \right]. \quad (30)$$

*Relative to the public-information case, there are fewer protest votes, if and only if  $p_L > 1 - p_H$ .*

The final statement compares the equilibrium cutpoint here with that obtained in the public-information case from Proposition 1. In fact, the difference between the equilibrium thresholds reported in equations (30) and (22) is

$$\Delta u^* = -\frac{\psi(z_H - z_L)}{\psi(z_H - z_L) + \sigma} \frac{\sigma(z_H + z_L)}{2\psi}, \quad (31)$$

and so the use of introspection by voters results in less protest voting if and only if  $z_H + z_L > 0$ , which is equivalent to  $p_L > 1 - p_H$ . Notice that  $p_L$  is the fraction of voters that need to coordinate behind the candidate if she is to avoid defeat, whereas  $1 - p_H$  is the fraction of voters that need to coordinate behind a protest in order to avoid an unwanted landslide. Hence, the inequality  $p_L > 1 - p_H$  says that coordination to avoid the candidate's defeat is harder than the coordination needed to avoid a landslide.

## 6. COMPARATIVE-STATIC EXERCISES

A cutpoint  $u^*$  determines the willingness of voters to engage in protest voting. I will say that a change in a parameter “increases protest voting” if and only if it increases the equilibrium cutpoint. For simplicity of exposition, here I focus on the introspective specification, for which

$$u^* = \frac{\psi(z_H - z_L)}{\psi(z_H - z_L) + \sigma} \left[ \mu - \frac{\sigma(z_H + z_L)}{2} \right], \quad (32)$$

where  $z_L \equiv \Phi^{-1}(p_L)$  and  $z_H \equiv \Phi^{-1}(p_H)$  are transformations of the proportions of the electorate needed for the candidate to enjoy a win and a landslide win, respectively. This solution readily generates a suite of comparative-static claims. Slight variations of all of these claims also apply to the public-information specification. I assume that  $\sigma > z_H - z_L$  so that a strictly monotonic equilibrium exists (Proposition 2); the results refer to that equilibrium.

**The Need for Coordination.** The first claims concern the effect of the thresholds  $p_H$  and  $p_L$ . Recall that  $p_L$  is the coordination required to prevent the candidate from losing, whereas  $1 - p_H$  is the coordination (in the opposite direction) required to prevent a landslide win.

One perspective is obtained by noting that  $u^*$  depends upon the two terms  $z_H - z_L$  and  $z_H + z_L$ . The former term measures the gap between the thresholds  $p_H$  and  $p_L$ ; effectively, it is the size of the region within which voters' collectively achieve their preferred outcome. As this gap narrows (fixing  $z_H + z_L$ ) the absolute size of  $u^*$  (whether positive or negative) falls; that is, the equilibrium cutpoint moves toward zero. When  $p_H$  and  $p_L$  become close the pivotal events with which a voter is concerned become equally likely, and so a voter makes his decision based upon whether he thinks that a successful protest is more or less important than a win for the candidate; such a decision criterion corresponds to  $u^* = 0$ . Fixing the gap  $z_H - z_L$ , an increase

in  $z_H + z_L$  corresponds to a simultaneous increase in the coordination needed for a candidate win ( $p_L$  is larger) and a reduction in the coordination needed for a successful protest ( $1 - p_H$  is smaller). This naturally leads to greater protest voting. Hence, comparative-static predictions using  $z_H + z_L$  and  $z_H - z_L$  as parameters are very straightforward.

Changes in the individual parameters  $p_L$  and  $p_H$  are a little more involved. For example, an increase in  $p_L$  increases  $z_H + z_L$  which weakens protest voting; however, it also narrows the gap  $z_H - z_L$  which shrinks the absolute size of the equilibrium cutpoint. If  $u^* > 0$  then the two effects play out in the same direction; however, if  $u^* < 0$  then they conflict. The next proposition confirms the overall effects of changes in  $p_L$  and  $p_H$ .

**Proposition 3** (Need for Coordination). *(i) If the candidate's expected popularity is sufficiently high, so that  $\mu > \sigma z_H$ , then protest voting falls, and votes shift toward the candidate, following a rise in the coordination  $p_L$  needed for her to win. However, if her popularity is low, so that  $\mu < \sigma z_H$ , then protest voting is increasing in  $p_L$  if  $p_L$  is sufficiently close to  $p_H$ .*

*(ii) Similarly, if  $\mu < \sigma z_L$ , so that her expected popularity is low, then protest voting is increasing (she loses support) in the coordination  $1 - p_H$  needed for a successful protest. However, if  $\mu > \sigma z_L$ , then protest voting is decreasing in  $1 - p_H$  if  $p_L$  and  $p_H$  are sufficiently close.*

If  $p_H$  and  $p_L$  are not too close (so that the region allowing the voters' preferred outcome is large) then the influence of either threshold is straightforward. Increasing  $p_L$  makes the candidate's position more precarious; protest voting is more costly. Similarly, a reduction in  $p_H$  makes the prospect of a (disliked) landslide more likely, and so increases the pressure for a successful protest. Of course, what is important for a voter is the relative likelihood of the pivotal events. The odds move in the natural direction as the need-for-coordination parameters change. However, if  $p_H$  and  $p_L$  are close the effect of a parameter on the gap  $z_H - z_L$  can dominate.

**Voters' Heterogeneity.** Noting that voters' preference types are distributed  $u_i \sim N(\theta, \sigma^2)$ , the two moments which describe the pattern of their preferences are the mean  $\theta$  and variance  $\sigma^2$ . Of course, an increase the underlying average popularity of the candidate (fixing voters' prior beliefs about  $\theta$ ; that is, fixing  $\mu$ ) leads to more support for her. Here, then, I consider

the impact of changes in  $\sigma^2$ . This variance parameter measures the degree of heterogeneity amongst the electorate. The effect of increased heterogeneity can go either way.

**Proposition 4** (Heterogeneity of Preferences). *Protest voting is increasing in the heterogeneity  $\sigma^2$  of voters' preferences if and only if  $\mu < \psi(z_L^2 - z_H^2)$ , which holds if and only if  $u^* > \mu$ .*

Notice that the condition  $u^* > \mu$  holds if and only if protest votes are expected to be more numerous than votes for the candidate.

The effect of heterogeneity can be understood by writing the equilibrium cutpoint as

$$u^* = \frac{\psi(z_H - z_L)\mu + \sigma\psi(z_L^2 - z_H^2)}{\psi(z_H - z_L) + \sigma}, \quad (33)$$

which is a weighted average of candidate's expected popularity and the term  $\psi(z_L^2 - z_H^2)$ ; this latter term measures the relative difference in the barriers to the coordination. As  $\sigma^2$  grows, and voters become more heterogeneous, the true average preference matters less.

**Voters' Beliefs.** Finally, I consider the effect of voters' beliefs about the candidate's popularity. These beliefs are determined by the mean  $\mu$  and the precision parameter  $\psi$ .

**Proposition 5** (Beliefs about the Candidate's Popularity). *Protest voting is strictly increasing in the expected popularity  $\mu$  of the candidate. Protest voting is strictly increasing in the precision  $\psi$  of voters' prior beliefs about the candidate's popularity if and only if*

$$\mu > \frac{\sigma(z_L + z_H)}{2}. \quad (34)$$

*Furthermore, the equilibrium cutpoint  $u^*$  is supermodular in the mean and precision of voters' beliefs:  $\partial^2 u^* / \partial \psi \partial \mu > 0$ , and so an increase in a candidate's perceived popularity has a greater effect on protest voting whenever voters' beliefs are more precise.*

The first claim is straightforward: an increase in the candidate's expected popularity makes an undesired loss less likely and an unwanted landslide more likely; this raises the incentive to cast a protest vote. The effect of the precision of beliefs is more intricate. It is easiest to understand when  $p_H = 1 - p_L$ , so that coordination needed for the candidate to win is the same as coordination needed for a successful protest. In this case  $z_L + z_H = 0$ , and so the inequality

(34) reduces to  $\mu > 0$ , and the solution for  $u^*$  satisfies  $0 < u^* < \mu$ . Hence, an indifferent voter thinks that other voters are more likely than not to be more in favor of the candidate than he is; he concludes that they are more likely to vote for the candidate, and so the pivotal event  $H$  is relatively more likely than the pivotal event  $L$ . As the precision of beliefs increases (so  $\psi$  rises) the pivotal event  $H$  becomes relatively more likely, and so protest voting increases.

Perhaps the most interesting finding which emerges from Proposition 5 is the competing effects of a candidate's popularity. Greater popularity is a double-edged sword for a candidate: an increase in her true popularity (that is, an increase in the true value  $\theta$  of the average relative preference amongst the electorate) is helpful; however, an increase in her perceived popularity harms her chances via an increase in protest voting. The competing effects are seen most sharply when voters' beliefs are very precise.

**Proposition 6** (Protest Voting with Precise Beliefs). *Allowing beliefs to become precise,*

$$\lim_{\psi \rightarrow \infty} u^* = \mu - \frac{\sigma(z_H + z_L)}{2}. \quad (35)$$

*In the unique strict voting equilibrium, the probability of a protest vote satisfies*

$$\lim_{\psi \rightarrow \infty} \Pr[u_i < u^*] = 1 - \Phi\left(\frac{z_H + z_L}{2}\right) \quad \text{and so} \quad p_L < \lim_{\psi \rightarrow \infty} \Pr[u_i > u^*] < p_H, \quad (36)$$

*where  $z_H \equiv \Phi^{-1}(p_H)$  and  $z_L \equiv \Phi^{-1}(p_L)$  as before. Hence, in the limit as beliefs become precise, the amount of protest voting and the election outcome are independent of voters' preferences.*

This proposition offers a sharp take-home message from the paper so far. When voters' beliefs about the candidate's popularity are very precise (so that, in essence, there is relatively mild aggregate uncertainty) then the amount of protest voting (and so the election outcome) is independent of how voters feel about the candidate and about the value of a successful protest.

The logic behind Proposition 6 is clearest when  $p_L = 1 - p_H$ , so that  $z_H + z_L = 0$ ; for this case, the coordination required to elect the candidate is the same as the coordination required for a successful protest. As beliefs become very precise (so that  $\psi \rightarrow \infty$ ) the relative likelihood of one pivotal event versus the other diverges unless the probability of a vote for the candidate satisfies  $p \rightarrow \frac{1}{2}$ . More generally, the limiting split between votes for the candidate and protest votes is tied down by the need to prevent the divergence of the ratio of pivotal probabilities.

## 7. ENDOGENOUS CANDIDATE RESPONSE

So far, the candidate has played no active role in the game. In this section I allow her to react endogenously to the protest votes cast by the electorate. I continue (as in the last section) to assume that voters introspectively use their own preferences to update their beliefs; once again, the results I present here also hold in the public-information setting.

**The Candidate's Policy Choice.** I consider a situation in which the candidate chooses whether to keep or to drop an unpopular policy from her manifesto. She wishes to drop the policy (and so she caves in to the protesters' demand) if and only if she perceives her popularity to be sufficiently low; equivalently, she does this if she sees sufficient popular disquiet.

As a concrete example, consider an incumbent politician contemplating an environmentally unfriendly policy. The electorate comprises voters with environmental concerns, and so a protest vote might be cast for a specialist green party candidate. The number of such voters is large enough to cause (potentially) the incumbent to lose to her leading challenger, but is insufficient for the green candidate to win. Hence, a green vote is an attempt to jam the signal of the candidate as she infers the intensity of environmentalism amongst the electorate.

Formally, the candidate wishes to keep the unpopular policy if and only if her popularity lies above a critical value  $\theta^\dagger$ , and so I assume that she receives a zero payoff if she drops the policy and a payoff of  $\theta - \theta^\dagger$  from keeping it. I consider strategies for which she drops the policy if and only if her observed support is sufficiently low; that is, she chooses the threshold  $p_H \in [p_L, 1]$ . For voters I consider the use of a cutpoint strategy.

**Definition 3** (Equilibrium with an Endogenous Candidate Response). *In the voting-and-policy game, a strategy profile is a real-valued pair  $(u^*, p_H) \in \mathcal{R} \times [p_L, 1)$  where a voter protests if and only if  $u_i < u^*$  and the candidate drops the disliked policy if and only if her support falls below  $p_H$ . This is an equilibrium if, for almost all voter types and almost all election outcomes, voters and the candidate play a best reply so long as the electorate is large enough.*

This definition insists upon a finite value for  $u^*$ , and so I am ruling out equilibria in which voters ignore their type realizations and all take the same action.<sup>7</sup>

<sup>7</sup>Such fully coordinated equilibria always exist. Note, however, that when  $u^*$  is large but finite then the candidate responds by setting  $p_H = p_L$  (see the discussion below) which in turn leads back to  $u^* = 0$ . Hence a fully

**Equilibrium.** The equilibrium behavior of voters can be built upon the earlier results. If  $1 > p_H > p_L$  then the equilibrium from Proposition 2 applies here. If  $p_H = p_L$ , so that the candidate always keeps the unpopular policy, then a voter optimally votes for her if and only if  $u_i > 0$ ; this corresponds to  $u^* = 0$ , and equation (30) continues to hold.

The candidate understands that the probability of a vote for her is  $p = \Phi((\theta - u^*)/\sigma)$ ; or, inverting this, if the probability of a vote for her is  $p$  then her popularity is  $\theta = u^* + \sigma\Phi^{-1}(p)$ . There is a feedback effect here: if protest voting rises (an increase in  $u^*$ ) then (given  $p$ ) the candidate holds a more optimistic view of her popularity. In a large electorate, the observed proportion of those voting for the candidate will converge to  $p$ . Hence (so long as the electorate size  $n$  is sufficiently large) the candidate abandons the unpopular policy if and only if  $u^* + \sigma\Phi^{-1}(p) < \theta^\dagger$  or, equivalently, if and only if  $p < p_H$  where

$$p_H = \Phi\left(\frac{\theta^\dagger - u^*}{\sigma}\right), \quad (37)$$

so long as  $p_H > p_L$ ; otherwise, the candidate never drops the policy which is equivalent to choosing  $p_H = p_L$ . The ballot-box support above which the candidate keeps her disliked policy falls as protest voting rises; but of course the increased coordination needed for a successful protest can (and certainly will if  $\mu < \sigma z_L$ ; see Proposition 3) further increase protest voting. Before exploring this logic, I report the conditions for an equilibrium in the next lemma.

**Lemma 5** (Equilibrium Conditions). *In an equilibrium with endogenous candidate response in which voters do not unanimously take a single type-independent action,  $u^*$  and  $p_H$  satisfy*

$$u^* = \frac{(z_H - z_L)\psi}{(z_H - z_L)\psi + \sigma} \left[ \mu - \frac{\sigma(z_H + z_L)}{2} \right] \text{ and } p_H = \Phi(z_H) \text{ where } z_H = \max \left\{ \frac{\theta^\dagger - u^*}{\sigma}, z_L \right\}. \quad (38)$$

I have noted that greater enthusiasm for protest voting dissuades the candidate from reaction, and this may feed back into further protest voting. This suggests that there may be an equilibrium in which the candidate never drops the policy. Such an equilibrium exists whenever  $\theta^\dagger$  (the popularity that convinces the candidate to press ahead) is not too large.

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coordinated equilibrium in which every voter protests (that is,  $u^* = \infty$ ) is not robust to a slight shift away to a situation with very high (but nevertheless incomplete) levels of protest voting. There are also problems with fully coordinated equilibria in which no voter protests.

**Proposition 7** (Equilibrium with an Ignored Protest). *If  $\theta^\dagger < \sigma z_L$  then there is an equilibrium in which the candidate always ignores the outcome of the election and keeps the disliked policy, so that  $p_H = p_L$ , and each voter  $i$  casts a protest vote if and only if  $u_i < 0$ .*

The other case to consider is when  $p_H > p_L$  so that  $z_H = (\theta^\dagger - u^*)/\sigma$ . The key result is simplest to state when voters' beliefs are precise. Allowing  $\psi \rightarrow \infty$  the equilibrium conditions become

$$u^* = \mu - \frac{\sigma(z_H + z_L)}{2} \quad \text{and} \quad z_H = \frac{\theta^\dagger - u^*}{\sigma}. \quad (39)$$

Notice again the feedback here: anything which pushes up protest voting (an increase in  $u^*$ ) induces a fall in  $z_H$  (equivalently, a fall in  $p_H$ ) and so a further increase in  $u^*$ . Nevertheless, solving these equations simultaneously yields an equilibrium which is characterized here.

**Proposition 8** (Equilibria with a Responsive Candidate). *If  $\theta^\dagger > \sigma z_L$  and  $\sigma^2 > 2(\theta^\dagger - \mu)$  then there is a unique equilibrium in which  $p_H > p_L$ . If  $\theta^\dagger > \mu$  then this equilibrium satisfies*

$$\lim_{\psi \rightarrow \infty} u^* = 2\mu - \theta^\dagger - \sigma z_L \quad \text{and} \quad \lim_{\psi \rightarrow \infty} (z_H - z_L) = \frac{2(\theta^\dagger - \mu)}{\sigma}, \quad (40)$$

*but if  $\theta^\dagger < \mu$  then  $\lim_{\psi \rightarrow \infty} u^* = \lim_{\psi \rightarrow \infty} (z_H - z_L) = 0$ . If  $\theta^\dagger < \sigma z_L$  then there can exist equilibria in which  $p_H > p_L$ . If  $\theta^\dagger < \mu$  then such equilibria also satisfy  $\lim_{\psi \rightarrow \infty} u^* = \lim_{\psi \rightarrow \infty} (z_H - z_L) = 0$ .*

The condition  $\sigma^2 > 2(\theta^\dagger - \mu)$  corresponds to the condition  $\sigma > z_H - z_L$  from Proposition 2; it is needed for the existence of a monotonic voting equilibrium. It can be dropped if the public-information specification of Section 4 is used.

Proposition 8 offers the message that protest voting falls as the candidate becomes amenable to the protest, but range of election outcomes for which the protest succeeds widens.

**The Effect of Popularity.** The theme of this section is the interaction between voters' equilibrium engagement in a protest-voting movement and a candidate's willingness to respond to the protest. Perhaps the most interesting implication of this is the fact that greater popularity for a candidate can, overall, have a negative effect on her ballot-box performance.

Recall that when the precision of voters' beliefs is high, and the candidate's behavior is fixed, an increase in a candidate's expected popularity translates one-for-one into increased protest

voting. Here, however, things are more extreme: the reduced willingness of the candidate to react to the protest increases  $u^*$  still further. In fact, for the case where  $\theta^\dagger > \max\{\sigma z_L, \mu\}$ ,

$$\lim_{\psi \rightarrow \infty} \frac{\partial u^*}{\partial \mu} = 2, \quad (41)$$

and so the rise in expected popularity has a double effect on the willingness of voters to engage in protest voting. When beliefs are precise the candidate's expected popularity goes hand-in-hand with her actual popularity. Equation (41) implies that the effect of increased protest voting following an increase in popularity is greater than the direct effect.

**Proposition 9** (The Effect of Beliefs with an Endogenous Candidate Response). *Assume that  $\theta^\dagger > \sigma z_L$  and  $\sigma^2 > 2(\theta^\dagger - \mu)$ . In the unique equilibrium, if  $\theta^\dagger > \mu$  then*

$$\lim_{\psi \rightarrow \infty} \Pr[u_i < u^*] = 1 - \Phi\left(z_L + \frac{\theta^\dagger - \mu}{\sigma}\right) \quad \text{and} \quad p_L < \lim_{\psi \rightarrow \infty} \Pr[u_i > u^*] < \lim_{\psi \rightarrow \infty} p_H. \quad (42)$$

*Hence the size of the protest is increasing in the candidate's popularity; equivalently, it is decreasing in the enthusiasm of the electorate for the protest. Protest voting is insufficient to cause the candidate to lose; however, she drops the policy when she wins. If  $\theta^\dagger < \mu$  then*

$$\lim_{\psi \rightarrow \infty} \Pr[u_i < u^*] = 1 - \Phi\left(\frac{\mu}{\sigma}\right) \quad \text{and} \quad p_L = \lim_{\psi \rightarrow \infty} p_H < \lim_{\psi \rightarrow \infty} \Pr[u_i > u^*]. \quad (43)$$

*Hence the proportion who protest is decreasing in the candidate's popularity. Protest voting is insufficient to cause her to lose, and she keeps the disliked policy when she wins.*

The key message here is that the twin endogenous responses to an increase in popularity are enough to harm a candidate's performance at the ballot box; however, those responses are not enough for her to lose. If her popularity becomes sufficiently strong (when  $\mu = \theta^*$ ) then she always drops the policy; beyond this an increase her popularity reduces protest voting.

**Corollary** (to Proposition 9). *If  $\theta^\dagger > \sigma z_L$  then protest voting is maximized when  $\mu = \theta^\dagger$ .*

Notice that the condition  $\mu = \theta^\dagger$  says that prior to the election the candidate is indifferent between keeping and dropping the policy. A moment's reflection suggests that this is a likely case of interest. Suppose, for instance, that  $\mu$  represents the existing status quo; think of it as the average voter preference before a shock to preferences. A candidate's policy is likely

to be in tune with the status quo. Thus, offered any opportunity to shift her policy up or down by a small amount she will be indifferent; this corresponds to the condition  $\theta^\dagger = \mu$ . This (admittedly hand-waving) argument suggests that protest voting is likely to be large.

## 8. EQUILIBRIA WITH PRIVATE INFORMATION

Here I consider the expanded model in which voters receive their own information about the candidate's popularity, and so a type-symmetric voting strategy  $v(u_i, s_i) : \mathcal{R}^2 \mapsto [0, 1]$  depends not only on a voter's preference type but also on his signal realization.

**Optimal Voting.** Just as in the public information case, if voters' decisions vary then pivotal probabilities are positive and so a voter finds it strictly optimal to cast a protest vote if and only if  $u_i$  falls below the log odds of the two pivotal events. That is,

$$u_i < \log \left[ \frac{\Pr[\text{Pivotal at } H \mid v(\cdot, \cdot), s_i]}{\Pr[\text{Pivotal at } L \mid v(\cdot, \cdot), s_i]} \right]. \quad (44)$$

This differs from inequality (13) because the probabilities are now conditioned on the signal  $s_i$ . (The probabilities also depend on the strategy  $v(\cdot, \cdot)$  used by others.) From the perspective of voter  $i$  this inequality takes the form  $u_i < U(s_i)$  where  $U(s_i)$  is a signal-based cutpoint, and so an equilibrium strategy  $v(u_i, s_i)$  is derived from  $U(s_i)$ .

**Lemma 6** (Signal-Based Cutpoints). *In a strict equilibrium, a voter  $i$  casts a protest vote if and only if  $u_i < U(s_i)$ , where  $U(s_i)$  depends upon his private signal realization. This satisfies*

$$U(s_i) = \lim_{n \rightarrow \infty} \log \left[ \frac{\Pr[\text{Pivotal at } H \mid U(\cdot), s_i]}{\Pr[\text{Pivotal at } L \mid U(\cdot), s_i]} \right], \quad (45)$$

where the probabilities are evaluated given that others use the cutpoint function  $U(\cdot)$ .

**Monotonicity and Linearity.** Given that each voter protests if and only if  $u_i < U(s_i)$ , the probability  $p$  of a vote for the candidate, conditional on  $\theta$ , is

$$p = P(\theta) \quad \text{where} \quad P(\theta) \equiv \Pr[u_i \geq U(s_i) \mid \theta]. \quad (46)$$

$U(s_i)$  can be increasing in  $s_i$  and so  $P(\theta)$  may increase or decrease in response to changes in  $\theta$ . However, if the response of  $U(s_i)$  is not too strong (for example, if  $U'(s_i) < 1$ ) then  $P(\theta)$  is

monotonic. Given monotonicity, a voter's beliefs about  $p$  are given by the density

$$f(p | s_i, U(\cdot)) = \frac{g(\theta | s_i)}{P'(\theta)} \quad \text{where } p = P(\theta), \quad (47)$$

where  $g(\theta | s_i)$  is the density of a voter's posterior beliefs about the candidate's popularity.

If  $P(\theta)$  is increasing then there are two critical values of the candidate's popularity,  $\theta_H$  and  $\theta_L$ , which generate support reaching  $p_L$  and  $p_H$  respectively. In fact,

$$\log \left[ \frac{f(p_H | s_i, U(\cdot))}{f(p_L | s_i, U(\cdot))} \right] = \log \left[ \frac{P'(\theta_L)}{P'(\theta_H)} \right] + \log \left[ \frac{g(\theta_H | s_i)}{g(\theta_L | s_i)} \right] \\ \text{where } \theta_H = P^{-1}(p_H) \quad \text{and} \quad \theta_L = P^{-1}(p_L). \quad (48)$$

Given the normality assumptions, the log likelihood ratio  $\log[g(\theta_H | s_i)/g(\theta_L | s_i)]$  is linear in  $s_i$ .

**Lemma 7 (Linearity).** *In a monotonic voting equilibrium, voter  $i$  casts a protest vote if and only if  $u_i < U(s_i)$ , where the cutpoint function is linear:  $U(s_i) = u^\dagger + rs_i$ , where  $0 < r < 1$ .*

Henceforth I restrict attention to monotonic linear strategies. That is, strategies where each voter casts a protest vote if and only if  $u_i < u^\dagger + rs_i$  where the coefficient  $r$  satisfies  $0 < r < 1$ .

**Feedback Effects.** A voter's posterior beliefs about  $\theta$  are normally distributed with mean  $(\psi\mu + \lambda s_i)/(\psi + \lambda)$  (this is the precision-weighted average of the prior mean and the signal) and variance  $\sigma^2/(\psi + \lambda)$ . Hence, straightforwardly, and as the proof of Lemma 7 confirms,

$$\log \left[ \frac{g(\theta_H | s_i)}{g(\theta_L | s_i)} \right] = \frac{\theta_H - \theta_L}{\sigma^2} \left[ \lambda(s_i - \mu) + (\psi + \lambda) \left( \mu - \frac{(\theta_H + \theta_L)}{2} \right) \right]. \quad (49)$$

The coefficient on the voter's signal is  $\lambda(\theta_H - \theta_L)/\sigma^2$ , and so his response to that signal is stronger when his signal is more precise (so that  $\lambda$  is higher) and when the gap between the critical popularities  $\theta_H$  and  $\theta_L$  is large. Those critical popularities depend upon the behavior of others, and they depend on the response of others to their own signals.

I now calculate those critical popularities. A vote is cast for the candidate if and only if  $u_i \geq U(s_i)$ , or equivalently if  $u_i - U(s_i) \geq 0$ . Given that  $U(s_i) = u^\dagger + rs_i$ , an increase in the underlying value of  $\theta$  shifts  $u_i - U(s_i)$  with the coefficient  $(1 - r)$ . Hence, if others respond relatively strongly to their own signals (that is, if  $r$  is higher) then in aggregate their behavior

responds relatively sluggishly to changes in the candidate's popularity. Of course, a change in  $r$  also changes the conditional variance of  $u_i - U(s_i)$ , but nevertheless the overall effect of an increase in  $r$  is maintained. In fact

$$u_i - U(s_i) | \theta \sim N \left( (1-r)\theta - u^\dagger, \frac{\sigma^2(\lambda - 1 + (1-r)^2)}{\lambda} \right), \quad (50)$$

and so the probability of a vote for the candidate, conditional on  $\theta$ , is

$$P(\theta) = \Phi \left( \frac{\theta - [u^\dagger/(1-r)]}{(\sigma/\lambda)S(r)} \right) \quad \text{where} \quad S(r) \equiv \sqrt{\lambda + \frac{\lambda(\lambda-1)}{(1-r)^2}}. \quad (51)$$

The response of  $p$  (the probability of a vote for the candidate) to  $\theta$  (her popularity) is inversely related to the term  $S(r)$ . This term is increasing in  $r$ , and so an increase in  $r$  (a greater responsiveness to private signals) implies that the ballot-box support for a candidate responds more weakly to her popularity. Inverting  $P(\theta)$  and evaluating it at  $p_L$  and  $p_H$  yields, for  $r < 1$ ,

$$\theta_L = \frac{u^\dagger}{1-r} + \frac{\sigma z_L S(r)}{\lambda} \quad \text{and} \quad \theta_H = \frac{u^\dagger}{1-r} + \frac{\sigma z_H S(r)}{\lambda}, \quad (52)$$

where  $z_L$  and  $z_H$  are defined as before. An increase in  $r$  (and so in  $S(r)$ ) pushes apart the critical popularities  $\theta_H$  and  $\theta_L$ . A best-responding voter contemplates the relative likelihood of values for  $\theta$  which are further apart, and he places more weight on the informative signals at his disposal. Indeed, with these solutions in hand the coefficient on the private signal  $s_i$  in the log likelihood ratio term is

$$\frac{\lambda(\theta_H - \theta_L)}{\sigma^2} = \frac{(z_H - z_L)S(r)}{\sigma}. \quad (53)$$

As already noted,  $S(r)$  is increasing in  $r$  and so a heightened response of others to their signals further encourages a voter to react to his own signal. An increase in  $r$  is an increased willingness to protest when the candidate is seen as popular, and so one aspect of protest voting can involve strategic complements; this contrasts with the finding of Lemma 3.

**Lemma 8 (Best Replies with Private Signals).** *If voters use a linear cutpoint  $U_i(s) = u^\dagger + r s_i$  where  $0 < r < 1$  then the unique best reply is to use the cutpoint  $\hat{U}_i(s_i) = \hat{u}^\dagger + \hat{r} s_i$  where*

$$\hat{r} = \frac{(z_H - z_L)S(r)}{\sigma} \quad \text{and} \quad \hat{u}^\dagger = \frac{\hat{r}}{\lambda} \left[ \psi \mu - \frac{(\psi + \lambda)u^\dagger}{1-r} \right] - \frac{z_H^2 - z_L^2}{2\lambda} \left[ \psi + \frac{(\psi + \lambda)(\lambda - 1)}{(1-r)^2} \right], \quad (54)$$

where  $S(r)$  is an increasing function of  $r$ . An increase in the response of others to their signals raises a voter's response to his own signal; however, an unconditional increase in protest voting by others (an increase in  $u^\dagger$ ) lowers a voter's own tendency to protest ( $\hat{u}^\dagger$  falls).

**Corollary** (to Lemma 8). *Raw protest voting (determined by  $u^\dagger$ ) is a strategic substitute; however, the response of voters to their private signals (determined by  $r$ ) is a strategic complement.*

**Equilibrium.** Given the nature of best replies reported in Lemma 8, the characterization of a monotonic voting equilibrium is straightforward: from equation (54) such an equilibrium corresponds to a pair  $(r, u^\dagger)$  satisfying  $0 < r < 1$  and

$$r = \frac{(z_H - z_L)S(r)}{\sigma} \quad \text{and} \quad u^\dagger = \frac{r}{\lambda} \left[ \psi\mu - \frac{(\psi + \lambda)u^\dagger}{1 - r} \right] - \frac{z_H^2 - z_L^2}{2\lambda} \left[ \psi + \frac{(\psi + \lambda)(\lambda - 1)}{(1 - r)^2} \right]. \quad (55)$$

The second of these equations is linear in  $u^\dagger$ , and is solved straightforwardly. The first equation, however, does not necessarily have a solution. For  $r \in (0, 1)$  the function  $S(r)$  is positive, increasing, convex, and satisfies  $S(r) \rightarrow \infty$  as  $r \rightarrow 1$ . This means that  $(z_H - z_L)S(r) > \sigma r$  for all  $r \in (0, 1)$  if the gap  $z_H - z_L$  is sufficiently large. If not, then there is a solution to this equation; however, the properties mentioned here ensure that if there is a solution (so that a monotonic voting equilibrium exists) then there is also a second solution.

**Proposition 10** (Equilibrium with Private Information). *For  $\lambda > 1$ , define*

$$\bar{\sigma} \equiv (z_H - z_L) \min_{r \in [0,1]} \left[ \frac{S(r)}{r} \right] \quad \text{where} \quad S(r) \equiv \sqrt{\lambda + \frac{\lambda(\lambda - 1)}{(1 - r)^2}}, \quad (56)$$

where  $z_L \equiv \Phi^{-1}(p_L)$  and  $z_H \equiv \Phi^{-1}(p_H)$ . *If heterogeneity is low, so that  $\sigma < \bar{\sigma}$  then there a monotonic equilibrium does not exist. If  $\sigma > \bar{\sigma}$ , then there are two monotonic equilibria. In each equilibrium, a voter protests if and only if  $u_i < u^\dagger + rs_i$  where  $\sigma r = (z_H - z_L)S(r)$  and*

$$u^\dagger = \frac{1 - r}{\lambda + r\psi} \left( r\psi\mu - \frac{z_H^2 - z_L^2}{2} \left[ \psi + \frac{(\psi + \lambda)(\lambda - 1)}{(1 - r)^2} \right] \right). \quad (57)$$

Note that  $\bar{\sigma}$  is increasing in  $p_H$ , decreasing in  $p_L$ , and increasing in the signal precision  $\lambda$ .

Proposition 10 deals with case where each voter's private information goes beyond introspection, so that  $\lambda > 1$ . A special case of interest is when a voter has no additional private information, so that  $\lambda = 1$ . To reach this case, I take the limit as  $\lambda \downarrow 1$ . Notice that  $S(r) \rightarrow 1$

for  $r < 1$  and so, taking the lowest equilibrium solution for  $r$ ,

$$r \rightarrow \frac{z_H - z_L}{\sigma} \quad \text{and} \quad \frac{u^\dagger}{1-r} \rightarrow \frac{\psi(z_H - z_L)}{\psi(z_H - z_L)\psi + \sigma} \left[ \mu - \frac{\sigma(z_H + z_L)}{2} \right]. \quad (58)$$

A voter protests if and only if  $u_i < u^\dagger + r s_i$ . In the limiting case  $\lambda = 1$ , the signal  $s_i$  is equal to  $u_i$  and so this inequality becomes  $u_i < u^\dagger + r u_i$ , or equivalently  $u_i < u^\dagger / (1 - r)$ . From equation (58), observe that  $u^\dagger / (1 - r) \rightarrow u^*$  where  $u^*$  is the cutpoint from Proposition 2.

**The Effect of the Precision of Beliefs.** I conclude this section by investigating the effect of precise beliefs. I begin by considering the properties of equilibria as  $\psi \rightarrow \infty$ , so that the prior becomes sharp. An equilibrium response  $r$  to the private signal does not depend on  $\psi$ . However, the constant  $u^\dagger$  does, and (from the proof of Proposition 11) satisfies

$$\lim_{\psi \rightarrow \infty} \frac{u^\dagger}{1-r} = \mu - \frac{(z_H + z_L) \sigma S(r)}{2 \lambda}. \quad (59)$$

Examining the probability of a protest vote as  $\psi \rightarrow \infty$  generates the next proposition. This result verifies that Proposition 6 continues to hold in a world with general private signals.

**Proposition 11** (Protest Voting with a Precise Prior). *In either monotonic equilibrium,*

$$\lim_{\psi \rightarrow \infty} \Pr[u_i < u^\dagger + r s_i] = 1 - \Phi\left(\frac{z_H + z_L}{2}\right) \quad \text{and so} \quad p_L < \lim_{\psi \rightarrow \infty} \Pr[u_i > u^\dagger + r s_i] < p_H, \quad (60)$$

*and so in the limit as prior aggregate uncertainty is eliminated, the fraction of voters who protest and the election outcome are independent of voters' preferences.*

A take-home message from Section 6 is that the amount of protest voting (and so the election outcome) is independent of how voters feel about the candidate and about the value of a protest whenever aggregate uncertainty is mild. This message holds more generally, when voters have better private signals than simple introspection.

Nevertheless, increasing the private information of voters is problematic. Notice that  $S(r) \rightarrow \infty$  as  $\lambda \rightarrow \infty$ , and this implies that  $\bar{\sigma} \rightarrow \infty$ . Recall that  $\bar{\sigma}$  is the critical value of preference heterogeneity from Proposition 10; if  $\sigma < \bar{\sigma}$  then there is no monotonic equilibrium.

**Corollary** (to Proposition 10). *If the precision of voters' private signals about the popularity of the candidate is sufficiently high, then a monotonic voting equilibrium does not exist.*

When private signals are very precise then voters react strongly to them; this means that they refrain from protest voting (anticipating that others will do so) when they see the candidate as very popular. Of course, this leads to a back-to-front situation in which a protest succeeds only when the candidate's popularity is high, and so a monotonic equilibrium falls apart.

## 9. CONCLUDING REMARKS

I have considered a model of protest voting in which voters face an anti-coordination problem: ideally, they would like the protest to be large enough to succeed, but not large enough to prevent their preferred candidate from winning the election. Heuristically, at least, protest votes are strategic substitutes: if others are likely to engage in widespread protest voting then, other things equal, an individual voter prefers to back the favored candidate.

If voters base their decisions solely on their preferences then protest voting does exhibit the properties of a strategic substitute. Perhaps surprisingly, however, the reaction to a private signal of the candidate's popularity exhibits the properties of a strategic complement. If others react strongly to their information, then they refrain from protesting when the candidate is seen as more popular, simply because they believe that others are more likely to protest. However, this slows the reaction of protest voting to the candidate's true standing amongst the electorate; technically, it pushes apart the critical pivotal events which enter into a voter's calculations, and he reacts by placing greater emphasis on his private information. The self-reinforcing nature of reactions to private signals can (when such signals are precise) prevent the existence of equilibria in which there is a monotonic relationship between a candidate's popularity and the votes which she receives.

The strategic substitutability and complementarity of the two components of a protest voter's strategy contrast with the classic strategic-voting problem. Suppose that voters must coordinate behind one of two challengers in order to defeat a disliked incumbent candidate. Here, strategic votes are strategic complements: if other voters shift toward one of the challengers, then others should do so too. This suggests a strong force toward Duvergerian equilibria with full coordination behind one of the challengers. Despite this, Myatt (2007) showed that voters' responses to their private signals of the challengers' popularities exhibit the properties of a strategic substitute: if other voters respond strongly to their private signals (so that they

back whomever their signals indicate is the stronger challenger) then the incentive of a voter to react to his own signal is weakened rather than strengthened. What is happening here is that the pivotal events which enter into his decision making become closer and so more evenly matched, which drives him to follow his personal preference rather than act strategically.

A conclusion emerging here is that the protest voting and strategic voting are very closely related; the inverted nature of the findings arises because (heuristically, at least) strategic voting is a coordination game whereas protest voting is an anti-coordination game.

Putting aside the nature of strategic interaction in the protest-voting game, I now turn attention to the key properties of equilibrium behavior. Many results are very natural. For example, the comparative-static analysis shows that voters usually respond to the need for coordination: protest voting increases as the need to coordinate for a successful protest rises, but falls as more coordination is required for the candidate to win rather than lose. Perhaps the most interesting results, however, concern the candidate's perceived popularity: an increase in her perceived popularity generates a greater enthusiasm for protest voting.

The double-edged nature of enhanced popularity is sharpest when voters' prior beliefs are precise. In this case, any increase in a candidate's popularity is offset by increased strategic voting. This offset effect is even more dramatic when the candidate's reaction to a protest is determined endogenously. If she anticipates a greater enthusiasm for protest voting, then a poor showing in the election is less indicative of a genuine loss in her popularity, and so she is less willing to react; this generates further protest voting. The net effect is that an increase in a candidate's popularity results in a net fall in her ballot-box performance. Protest voting is at its highest when, prior to the election, a candidate is indifferent between accepting and rejecting the demands of the protesters. The take-home message here is that popular candidates do not necessarily do better; indeed, whereas a candidate likes to be truly popular, she prefers to be seen as unpopular (and so vulnerable to defeat by a disliked competing candidate) in order to scare off any widespread protest vote against her.

#### APPENDIX A. EQUILIBRIA IN FINITE ELECTORATES

This paper uses the concept of a voting equilibrium from Definition 1. This is a single voting strategy, defined over an entire sequence of voting games, which specifies a best reply for

almost every player type so long as the electorate is sufficiently large. Here I consider orthodox Bayesian Nash equilibria in finite electorates under the public-information specification; what I say here can also be extended to apply to the introspective specification.

Fixing  $n$ , consider a symmetric Bayesian Nash equilibrium in which voters respond to their types. If decisions are type-responsive then the probabilities of the pivotal outcomes are positive. Given symmetry, voters share the same beliefs, and so a voter protests if and only if equation (13) holds. This is a cutpoint strategy, and so a symmetric type-responsive equilibrium in a finite electorate corresponds to a cutpoint  $u_n^*$  which satisfies  $u_n^* = L_n(u_n^*)$  where

$$L_n(\hat{u}) \equiv \log \left[ \frac{\Pr[\text{Pivotal at } H \mid \hat{u}]}{\Pr[\text{Pivotal at } L \mid \hat{u}]} \right] \quad (61)$$

is the log odds of the pivotal events given that others use a cutpoint  $\hat{u}$ . Finding a Bayesian Nash equilibrium corresponds to finding a fixed point of  $L_n(\hat{u})$ . Now,  $L_n(\hat{u})$  is continuous in  $\hat{u}$ , and based on the analysis conducted within the main paper,

$$\lim_{n \rightarrow \infty} L_n(\hat{u}) = \frac{(1 - \psi)(z_H^2 - z_L^2)}{2} - \frac{\psi(\hat{u} - \mu)(z_H - z_L)}{\sigma}. \quad (62)$$

This limit is decreasing in  $\hat{u}$ . It lies strictly above  $\hat{u}$  if  $\hat{u} < u^*$  (where  $u^*$  is the equilibrium cutpoint described in the paper) and it lies strictly below  $\hat{u}$  if  $\hat{u} > u^*$ . Hence, for any  $\varepsilon > 0$ , if  $n$  is sufficiently large then  $L_n(\hat{u})$  has a fixed point satisfying  $u_n^* \in (u^* - \varepsilon, u^* + \varepsilon)$ . From this, a sequence of equilibrium cutpoints can be constructed which satisfy  $u_n^* \rightarrow u^*$  as  $n \rightarrow \infty$ . A similar argument establishes that any other sequence of cutpoints cannot converge to a finite limit; any other equilibrium sequence (if such a sequence exists) must diverge to  $\pm\infty$ .

## APPENDIX B. OMITTED PROOFS

*Proof of Lemma 1.* Requiring strict replies ensures that the probability of either action is positive, and so pivotal probabilities are positive. Hence, for each electorate size  $n$  I may define

$$L_n \equiv \log \left[ \frac{\Pr[\text{Pivotal at } H \mid v(\cdot)]}{\Pr[\text{Pivotal at } L \mid v(\cdot)]} \right]. \quad (63)$$

If  $\lim_{n \rightarrow \infty} L_n = \pm\infty$  then all types share the same best reply as  $n \rightarrow \infty$ , leading back to a type-independent pure voting strategy; this is not a strict equilibrium. If  $L_n$  does not converge, then

there is a positive-probability interval of types for whom the best reply repeatedly switches as  $n \rightarrow \infty$ ; this cannot be an equilibrium. So, if  $v(\cdot)$  is a strict equilibrium strategy then the limit  $u^* = \lim_{n \rightarrow \infty} L_n$  exists. If  $u_i > u^*$  (respectively,  $u_i < u^*$ ) then voting for the candidate (respectively, casting a protest vote) is the unique best reply for all  $n$  sufficiently large.  $\square$

*Proof of Lemma 2.* Good and Mayer (1975) and Chamberlain and Rothschild (1981) proved this for  $p_L = \frac{1}{2}$ . The technique of Chamberlain and Rothschild (1981) extends to other values of  $p_L$  and  $p_H$ . The result here is a special case of Lemma 1 of Myatt (2012); he generalized the Good-Mayer-Chamberlain-Rothschild finding to more candidates and more pivotal events.  $\square$

*Proof of Lemma 3.* This result follows from the discussion in the text and equation (20).  $\square$

*Proof of Proposition 1.* Applying the corollary to Lemmas 1 and 2, together with equation (20),

$$u^* = \frac{z_H^2 - z_L^2}{2} + \log \left[ \frac{g(u^* + \sigma z_H)}{g(u^* + \sigma z_L)} \right]. \quad (64)$$

Given that  $\theta \sim N(\mu, \sigma^2/\psi)$ , the density  $g(\cdot)$  is of course

$$g(\theta) = \frac{1}{\sqrt{2\pi\sigma^2/\psi}} \exp \left( -\frac{\psi(\theta - \mu)^2}{2\sigma^2} \right). \quad (65)$$

This readily yields equation (21) in the text; combining this with equation (64),

$$u^* = \frac{z_H^2 - z_L^2}{2} + \frac{\psi(z_L^2 - z_H^2)}{2} - \frac{\psi(u^* - \mu)(z_H - z_L)}{\sigma}. \quad (66)$$

This is linear in  $u^*$ , and solves to give the solution reported in the proposition.  $\square$

*Proof of Lemma 4.* This is a special case of results from the private-information model.  $\square$

*Proof of Proposition 2.* This early claims follow from the discussion in the text, the solution for  $u^*$  from re-arranging equation (29), and the final claim from an inspection of equation (31).  $\square$

*Proof of Proposition 3.* Recall that the solution for the equilibrium cutpoint is

$$u^* = \frac{\psi(z_H - z_L)}{\psi(z_H - z_L) + \sigma} \left[ \mu - \frac{\sigma(z_H + z_L)}{2} \right]. \quad (67)$$

Differentiation with respect to  $z_L$  yields

$$\frac{\partial u^*}{\partial z_L} = -\frac{\sigma}{\psi(z_H - z_L)^2 + \sigma(z_H - z_L)} \left[ u^* + \frac{\psi(z_H - z_L)^2}{2} \right] \quad (68)$$

$$\text{and } \frac{\partial^2 u^*}{\partial z_L^2} \Big|_{\partial u^*/\partial z_L=0} = \frac{\sigma\psi(z_H - z_L)}{\psi(z_H - z_L)^2 + \sigma(z_H - z_L)} > 0, \quad (69)$$

and so  $u^*$  is a quasi-convex function of  $z_L$ . Hence, it is either everywhere decreasing in  $z_L$ , everywhere increasing in  $z_L$ , or first decreasing and then increasing. To investigate further, I evaluate the first derivative at upper bound of the range for  $z_L$ :

$$\frac{\partial u^*}{\partial z_L} \Big|_{z_L \rightarrow z_H} = -\frac{\psi}{\sigma} [\mu - \sigma z_H] < 0 \quad \Leftrightarrow \quad \mu > \sigma z_H. \quad (70)$$

So, if  $\mu > \sigma z_H$  then  $u^*$  must be decreasing in  $z_L$  across its entire range. However, if  $\mu < \sigma z_H$  then  $u^*$  is increasing in  $z_L$  if  $z_L$  is close to  $z_H$ , or equivalently if  $p_H - p_L$  is sufficiently small.

Similarly, differentiation with respect to  $z_H$  yields:

$$\frac{\partial u^*}{\partial z_H} = \frac{\sigma}{\psi(z_H - z_L)^2 + \sigma(z_H - z_L)} \left[ u^* - \frac{\psi(z_H - z_L)^2}{2} \right] \quad (71)$$

$$\text{and } \frac{\partial^2 u^*}{\partial z_H^2} \Big|_{\partial u^*/\partial z_H=0} = -\frac{\sigma\psi(z_H - z_L)}{\psi(z_H - z_L)^2 + \sigma(z_H - z_L)} < 0, \quad (72)$$

and so  $u^*$  is a quasi-concave function of  $z_H$ . Hence, it is either everywhere decreasing in  $z_L$ , everywhere increasing in  $z_H$ , or first increasing and then decreasing. Now:

$$\frac{\partial u^*}{\partial z_H} \Big|_{z_H \downarrow z_L} = \frac{\psi}{\sigma} [\mu - \sigma z_L] < 0 \quad \Leftrightarrow \quad \mu < \sigma z_L. \quad (73)$$

So, if  $\mu < \sigma z_L$  then  $u^*$  must be decreasing in  $z_H$  across its entire range; equivalently, this means that  $u^*$  is increasing in  $1 - p_H$  across its range. However, if not then  $u^*$  is increasing in  $z_H$  (and so decreasing in the need for coordination  $1 - p_H$ ) if  $p_H - p_L$  is sufficiently small.  $\square$

*Proof of Proposition 4.* The proof follows from the discussion in the text.  $\square$

*Proof of Propositions 5–6.* The claims follow straightforwardly from an inspection of  $u^*$ .  $\square$

*Proof of Lemma 5.* The first equilibrium condition is equation (30) of Proposition 2. The second condition follows from the discussion of the candidate's optimal behavior.  $\square$

*Proof of Proposition 7.* Ignoring the election outcome is equivalent to choosing  $p_H = p_L$ . For voters, the two pivotal events  $L$  and  $H$  coincide. So, a pivotal vote simultaneously enables the win and the protest's failure. Hence, a voter optimally protests if and only if  $u_i < 0$ . Given that  $u^* = 0$ , and the inequality  $\theta^\dagger < \sigma z_L$  holds, then the second condition of Lemma 5 confirms that the choice  $p_H = p_L$  is optimal for the candidate.  $\square$

*Proof of Proposition 8.* The conditions from Lemma 5 may be written as

$$u^* = \frac{(z_H - z_L)\psi}{(z_H - z_L)\psi + \sigma} \left[ \mu - \sigma z_L - \frac{\sigma(z_H - z_L)}{2} \right] \quad \text{and} \quad z_H - z_L = \frac{\theta^\dagger - \sigma z_L - u^*}{\sigma}. \quad (74)$$

These conditions combine to eliminate  $u^*$  and give a quadratic in  $z_H - z_L$ :

$$\frac{\psi\sigma(z_H - z_L)^2}{2} + [\sigma^2 + \psi(\mu - \theta^\dagger)](z_H - z_L) + \sigma(\sigma z_L - \theta^\dagger) = 0. \quad (75)$$

A sufficient condition for this to have real roots is if the constant term is negative, which is so if and only if  $\sigma z_L < \theta^\dagger$ . Note that in this case there cannot be an equilibrium in which  $p_H = p_L$ . Moreover, in this case, there is a unique positive root. This is

$$z_H - z_L = \frac{\psi(\theta^\dagger - \mu) - \sigma^2 + \sqrt{[\psi(\theta^\dagger - \mu) - \sigma^2]^2 + 2\psi\sigma^2(\theta^\dagger - \sigma z_L)}}{\psi\sigma}. \quad (76)$$

Allowing  $\psi \rightarrow \infty$  generates the claims of the proposition. If  $\theta^\dagger < \sigma z_L$  then the quadratic may still have solutions. In fact, for  $\theta^\dagger \neq \mu$  there are real solutions if  $\psi$  is sufficiently large. Given that solutions exist, the limits of the roots as  $\psi \rightarrow \infty$  are

$$\lim_{\psi \rightarrow \infty} (z_H - z_L) = \theta^\dagger - \mu \pm |\theta^\dagger - \mu|. \quad (77)$$

By inspection, if  $\theta^\dagger < \mu$  then the positive root converges to zero.  $\square$

*Proof of Proposition 8.* If  $\psi \rightarrow \infty$  then  $\Pr[u_i > u^*] \rightarrow \Pr[u_i > u^* | \theta = \mu]$  and

$$\Pr[u_i > u^* | \theta = \mu] = P(\mu) = \Phi\left(\frac{\mu - u^*}{\sigma}\right) \rightarrow \Phi\left(\frac{\theta^\dagger - \mu + \sigma z_L}{\sigma}\right), \quad (78)$$

where the final step applies Proposition 8. This yields the claims of equation (42), and the claims of equation (43) are obtained a similar way.  $\square$

*Proof of Lemma 6.* This follows from the approach taken in the proof of Lemma 1.  $\square$

*Proof of Lemma 7.* Given the normality assumptions, the density  $g(\theta | s_i)$  takes the form

$$g(\theta | s_i) = \frac{1}{\sqrt{2\pi \text{var}[\theta | s_i]}} \exp\left(-\frac{(\theta - \mathbb{E}[\theta | s_i])^2}{2 \text{var}[\theta | s_i]}\right). \quad (79)$$

A few straightforward algebraic steps yield:

$$\log \left[ \frac{g(\theta_H | s_i)}{g(\theta_L | s_i)} \right] = \frac{(\theta_L - \mathbb{E}[\theta | s_i])^2 - (\theta_H - \mathbb{E}[\theta | s_i])^2}{2 \text{var}[\theta | s_i]} \quad (80)$$

$$= \frac{\theta_L^2 - \theta_H^2}{2 \text{var}[\theta | s_i]} + \frac{(\theta_H - \theta_L) \mathbb{E}[\theta | s_i]}{\text{var}[\theta | s_i]} \quad (81)$$

$$= \frac{\theta_H - \theta_L}{\text{var}[\theta | s_i]} \left( \mathbb{E}[\theta | s_i] - \frac{\theta_H + \theta_L}{2} \right). \quad (82)$$

The posterior precision is the sum of the prior precision and signal precision, and the posterior mean is the precision-weighted average of the prior mean and the signal. That is,

$$\mathbb{E}[\theta | s_i] = \frac{\psi\mu + \lambda s_i}{\psi + \lambda} \quad \text{and} \quad \text{var}[\theta | s_i] = \frac{\sigma^2}{\psi + \lambda}. \quad (83)$$

Substituting in these expressions yields equation (49) given in the main text. Combining equation (45) from Lemma 6 with equations (48) and (49) yields

$$U(s_i) = \lim_{n \rightarrow \infty} \log \left[ \frac{\Pr[\text{Pivotal at } H | U(\cdot), s_i]}{\Pr[\text{Pivotal at } L | U(\cdot), s_i]} \right] \quad (84)$$

$$= \log \left[ \frac{f(p_H | s_i, U(\cdot))}{f(p_L | s_i, U(\cdot))} \right] \quad (85)$$

$$= \log \left[ \frac{P'(\theta_L)}{P'(\theta_H)} \right] + \log \left[ \frac{g(\theta_H | s_i)}{g(\theta_L | s_i)} \right] \quad (86)$$

$$= \log \left[ \frac{P'(\theta_L)}{P'(\theta_H)} \right] + \frac{\theta_H - \theta_L}{\sigma^2} \left[ \lambda(s_i - \mu) + (\psi + \lambda) \left( \mu - \frac{(\theta_H + \theta_L)}{2} \right) \right], \quad (87)$$

where the second equality follows from an application of Lemma 2. Defining

$$u^\dagger = \log \left[ \frac{P'(\theta_L)}{P'(\theta_H)} \right] + \frac{\theta_H - \theta_L}{\sigma^2} \left[ \psi\mu - \frac{(\psi + \lambda)(\theta_H + \theta_L)}{2} \right] \quad \text{and} \quad r = \frac{\lambda(\theta_H - \theta_L)}{\sigma^2}. \quad (88)$$

it is clear that this takes the linear form  $U(s_i) = u^\dagger + r s_i$ , as required.  $\square$

*Proof of Lemma 8.* I begin by verifying the various calculations in the text leading up to the statement of the lemma. Notice that  $u_i - U(s_i) = (1 - r)s_i - u^\dagger + (u_i - s_i)$ . Recall that

$$s_i | \theta \sim N\left(\theta, \frac{\sigma^2}{\lambda}\right) \quad \text{and} \quad (u_i - s_i) | (\theta, s_i) \sim N\left(0, \frac{(\lambda - 1)\sigma^2}{\lambda}\right). \quad (89)$$

Conditional on  $\theta$ , the mean and variance are

$$\mathbb{E}[u_i - U(s_i) | \theta] = (1 - r)\theta - u^\dagger \quad \text{and} \quad \text{var}[u_i - U(s_i) | \theta] = \frac{(1 - r)^2\sigma^2 + (\lambda - 1)\sigma^2}{\lambda}, \quad (90)$$

which can be used to give the statement in equation (50). The probability of a vote for the candidate, conditional on her true underlying popularity  $\theta$ , is

$$P(\theta) = \Phi\left(\frac{(1 - r)\theta - u^\dagger}{\sigma\sqrt{(\lambda - 1 + (1 - r)^2)/\lambda}}\right), \quad (91)$$

and from this the expression in equation (51) is obtained. Inverting this,

$$\theta = \frac{u^\dagger}{1 - r} + \frac{\sigma S(r)\Phi^{-1}(p)}{\lambda}, \quad (92)$$

which yields the solutions for  $\theta_L$  and  $\theta_H$  given in equation (52). Furthermore,

$$\theta_H - \theta_L = \frac{\sigma S(r)(z_H - z_L)}{\lambda} \quad \text{and} \quad \frac{\theta_H + \theta_L}{2} = \frac{u^\dagger}{1 - r} + \frac{\sigma S(r)(z_H + z_L)}{2\lambda}. \quad (93)$$

Also, differentiating  $P(\theta)$  and applying equation (92),

$$P'(\theta) = \frac{\lambda}{\sigma S(r)}\phi\left(\frac{\theta - [u^\dagger/(1 - r)]}{(\sigma/\lambda)S(r)}\right) = \frac{\lambda}{\sigma S(r)}\phi(\Phi^{-1}(p)), \quad (94)$$

where  $\psi(\cdot)$  is the density of the standard normal distribution. Using this,

$$\log\left[\frac{P'(\theta_L)}{P'(\theta_H)}\right] = \frac{z_H^2 - z_L^2}{2}. \quad (95)$$

Now recall from equation (88) in the proof of Lemma 7 that

$$u^\dagger = \log\left[\frac{P'(\theta_L)}{P'(\theta_H)}\right] + \frac{\theta_H - \theta_L}{\sigma^2}\left[\psi\mu - \frac{(\psi + \lambda)(\theta_H + \theta_L)}{2}\right] \quad \text{and} \quad r = \frac{\lambda(\theta_H - \theta_L)}{\sigma^2}. \quad (96)$$

If all other voters use a threshold  $U(s_i) = u^\dagger + rs_i$ , then clearly (as claimed) a best reply is to use a threshold  $\hat{U}(s_i) = \hat{u}^\dagger + \hat{r}s_i$ . Using the expression above and substituting in for  $\theta_H - \theta_L$ ,

$$\hat{r} = \frac{\lambda(\theta_H - \theta_L)}{\sigma^2} = \frac{S(r)(z_H - z_L)}{\sigma}. \quad (97)$$

Similarly, for the intercept of the threshold function,

$$\hat{u}^\dagger = \log \left[ \frac{P'(\theta_L)}{P'(\theta_H)} \right] + \frac{\theta_H - \theta_L}{\sigma^2} \left[ \psi\mu - \frac{(\psi + \lambda)(\theta_H + \theta_L)}{2} \right] \quad (98)$$

$$= \frac{z_H^2 - z_L^2}{2} + \frac{S(r)(z_H - z_L)}{\lambda\sigma} \left[ \psi\mu - (\psi + \lambda) \left( \frac{u^\dagger}{1-r} + \frac{\sigma S(r)(z_H + z_L)}{2\lambda} \right) \right] \quad (99)$$

$$= (z_H - z_L) \left( \frac{z_H + z_L}{2} + \frac{S(r)}{\lambda\sigma} \left[ \psi\mu - (\psi + \lambda) \left( \frac{u^\dagger}{1-r} + \frac{\sigma S(r)(z_H + z_L)}{2\lambda} \right) \right] \right) \quad (100)$$

$$= (z_H - z_L) \left( \frac{z_H + z_L}{2} \left[ 1 - \frac{[S(r)]^2(\psi + \lambda)}{\lambda^2} \right] + \frac{S(r)}{\lambda\sigma} \left[ \psi\mu - \frac{(\psi + \lambda)u^\dagger}{1-r} \right] \right) \quad (101)$$

$$= (z_H - z_L) \left( \frac{S(r)}{\lambda\sigma} \left[ \psi\mu - \frac{(\psi + \lambda)u^\dagger}{1-r} \right] - \frac{z_H + z_L}{2\lambda} \left[ \psi + \frac{(\psi + \lambda)(\lambda - 1)}{(1-r)^2} \right] \right), \quad (102)$$

where the extra step of spotting  $\hat{r}$  yields the expression in the statement of the lemma.  $\square$

*Proof of Proposition 10.* From Lemma 7, a monotonic equilibrium involves the use of a linear cutpoint function  $U(s_i) = u^\dagger + rs_i$ , and using Lemma 8 the coefficients  $u^\dagger$  and  $r$  satisfy equation (55). Fixing  $r$ , the second equation is linear in  $u^\dagger$  and easily yields equation (57) in the statement of the proposition. However,  $r$  needs to satisfy  $0 < r < 1$  and

$$\sigma = \frac{(z_H - z_L)S(r)}{r}. \quad (103)$$

Clearly, there can be such a solution if and only if  $(z_H - z_L)S(r)/r$  falls below  $\sigma$  within the range  $(0, 1)$ ; this is so if and only if  $\sigma > \bar{\sigma}$ . A solution for  $r$  is a fixed point of the function  $(z_H - z_L)S(r)/\sigma$ . By inspection,  $S(r)$  is increasing in  $r$  and  $S(0) > 0$ , and  $S(r) \rightarrow \infty$  as  $r \rightarrow 1$ . So, if there is a solution then  $(z_H - z_L)S(r)/\sigma$  must cross  $r$  from above to below;  $S(r) \rightarrow \infty$  as  $r \rightarrow 1$  and so  $(z_H - z_L)S(r)/\sigma$  must subsequently cross  $r$  from below to above. I conclude that if there is a solution, then there must be a second solution. (The exception is when  $(z_H - z_L)S(r)/\sigma$  is tangent to  $r$  at the unique fixed point, which happens when  $\sigma = \bar{\sigma}$ .) There can, however, be no more than two fixed points, owing to the convexity of  $S(r)$ .  $\square$

*Proof of Proposition 11.* Substituting in for  $S(r)$ ,

$$\frac{u^\dagger}{1-r} = \frac{1}{\lambda + r\psi} \left( r\psi\mu - \frac{z_H^2 - z_L^2}{2} \left[ \frac{(\psi + \lambda)[S(r)]^2}{\lambda} - \lambda \right] \right) \quad (104)$$

$$= \frac{1}{\lambda + r\psi} \left( r\psi\mu - \frac{(z_H + z_L)(z_H - z_L)}{2} \left[ \frac{\sigma r(\psi + \lambda)S(r)}{\lambda(z_H - z_L)} - \lambda \right] \right), \quad (105)$$

where the second equality follows from substitution of  $S(r) = \sigma r / (z_H - z_L)$  from the equilibrium condition for  $r$ . Taking the limit as  $\psi \rightarrow \infty$  yields equation (59) from the main text. Evaluated at  $\theta = \mu$ , the probability of a vote for the candidate is

$$\Pr[u_i > u^\dagger + r s_i | \theta = \mu] = P(\mu) = \Phi\left(\frac{\mu - [u^\dagger / (1 - r)]}{(\sigma/\lambda)S(r)}\right) \rightarrow \Phi\left(\frac{z_H + z_L}{2}\right). \quad (106)$$

If  $\psi \rightarrow \infty$  then  $E[P(\theta) | \mu] \rightarrow P(\mu)$ , yield the proposition's claims.  $\square$

#### REFERENCES

- ABRAMSON, P. R., J. H. ALDRICH, P. PAOLINO, AND D. W. ROHDE (1995): "Third-Party and Independent Candidates in American Politics: Wallace, Anderson, and Perot," *Political Research Quarterly*, 110(3), 349–367.
- ALVAREZ, R. M., AND J. NAGLER (1995): "Economics, Issues and the Perot Candidacy: Voter Choice in the 1992 Presidential Election," *American Journal of Political Science*, 39(3), 714–744.
- BOWLER, S., AND D. J. LANOUE (1992): "Strategic and Protest Voting for Third Parties: the Case of the Canadian Ndp," *Political Research Quarterly*, 45(2), 485–499.
- CASTANHEIRA, M. (2003): "Why Vote for Losers?," *Journal of the European Economic Association*, 1(5), 1207–1238.
- CHAMBERLAIN, G., AND M. ROTHSCHILD (1981): "A Note on the Probability of Casting a Decisive Vote," *Journal of Economic Theory*, 25(1), 152–162.
- COX, G. W. (1994): "Strategic Voting Equilibria under the Single Nontransferable Vote," *American Political Science Review*, 88(3), 608–621.
- DEWAN, T., AND D. P. MYATT (2007): "Leading the Party: Coordination, Direction, and Communication," *American Political Science Review*, 101(4), 827–845.
- EVANS, G., AND A. HEATH (1993): "A Tactical Error in the Analysis of Tactical Voting: A Response to Niemi, Whitten and Franklin," *British Journal of Political Science*, 23(1), 131–137.
- FEDDERSEN, T. J., AND W. PESENDORFER (1996): "The Swing Voter's Curse," *American Economic Review*, 86(3), 408–424.

- (1997): “Voting Behavior and Information Aggregation in Elections with Private Information,” *Econometrica*, 65(5), 1029–1058.
- (1998): “Convicting the Innocent: The Inferiority of Unanimous Jury Verdicts under Strategic Voting,” *American Political Science Review*, 92(1), 23–35.
- FISHER, S. D. (2004): “Definition and Measurement of Tactical Voting: The Role of Rational Choice,” *British Journal of Political Science*, 34(1), 152–166.
- FRANKLIN, M. N., R. G. NIEMI, AND G. WHITTEN (1994): “The Two Faces of Tactical Voting,” *British Journal of Political Science*, 24(4), 549–557.
- GOLD, H. J. (1995): “Third Party Voting in Presidential Elections: A Study of Perot, Anderson, and Wallace,” *Political Research Quarterly*, 48(4), 751–773.
- GOOD, I. J., AND L. S. MAYER (1975): “Estimating the Efficacy of a Vote,” *Behavioral Science*, 20(1), 25–33.
- HIRSCHMAN, A. (1970): *Exit, Voice, and Loyalty*. Harvard University Press, Cambridge, MA.
- HIX, S., AND M. MARSH (2011): “Second-Order Effects plus Pan-European Political Swings: An Analysis of European Parliament Elections Across Time,” *Electoral Studies*, 30(1), 4–15.
- HUMMEL, P. (2011): “Abstention and Signaling in Large Repeated Elections,” *Games and Economic Behavior*, 72(2), 586–593.
- KANG, W.-T. (2004): “Protest Voting and Abstention under Plurality Rule Elections,” *Journal of Theoretical Politics*, 16(1), 79–102.
- KSELMAN, D., AND E. NIOU (2009): “Strategic Voting in Plurality Elections,” *Political Analysis*, 18(2), 227–244.
- (2011): “Protest Voting in Plurality Elections: A Theory of Voter Signaling,” *Public Choice*, 148(3–4), 395–418.
- LACY, D., AND B. C. BURDEN (1999): “The Vote-Stealing and Turnout Effects of Ross Perot in the 1992 Election,” *American Journal of Political Science*, 43(1), 233–255.
- MCMURRAY, J. C. (2012): “Ideology as Opinion: A Spatial Model of Common-Value Elections,” working paper.
- MEIROWITZ, A. (2005a): “Informational Party Primaries and Strategic Ambiguity,” *Journal of Theoretical Politics*, 17(1), 107–136.

- (2005b): “Polling Games and Information Revelation in the Downsian Framework,” *Games and Economic Behavior*, 51(2), 464–489.
- MEIROWITZ, A., AND K. W. SHOTTS (2009): “Pivots Versus Signals in Elections,” *Journal of Economic Theory*, 144(2), 744–771.
- MEIROWITZ, A., AND J. A. TUCKER (2007): “Run Boris Run: Strategic Voting in Sequential Elections,” *Journal of Politics*, 69(1), 88–99.
- MYATT, D. P. (2007): “On the Theory of Strategic Voting,” *Review of Economic Studies*, 74(1), 255–281.
- (2012): “A Rational Choice Theory of Voter Turnout,” working paper.
- MYERSON, R. B., AND R. J. WEBER (1993): “A Theory of Voting Equilibria,” *American Political Science Review*, 87(1), 102–114.
- NIEMI, R. G., G. WHITTEN, AND M. N. FRANKLIN (1992): “Constituency Characteristics, Individual Characteristics and Tactical Voting in the 1987 British General Election,” *British Journal of Political Science*, 22(2), 229–240.
- (1993): “People Who Live in Glass Houses: A Response to Evans and Heath’s Critique of Our Note on Tactical Voting,” *British Journal of Political Science*, 23(4), 549–553.
- PALFREY, T. R. (1989): “A Mathematical Proof of Duverger’s Law,” in *Models of Strategic Choice in Politics*, ed. by P. C. Ordeshook, pp. 69–92. University of Michigan Press, Ann Arbor.
- PIKETTY, T. (2000): “Voting as Communicating,” *Review of Economic Studies*, 67(1), 169–191.
- RAZIN, R. (2003): “Signaling and Election Motivations in a Voting Model,” *Econometrica*, 71(4), 1083–1119.
- ROSENTHAL, H., AND S. SEN (1973): “Electoral Participation in the French Fifth Republic,” *American Political Science Review*, 67(1), 29–54.
- SHOTTS, K. W. (2006): “A Signaling Model of Repeated Elections,” *Social Choice and Welfare*, 27(2), 251–261.
- SMIRNOV, O., AND J. H. FOWLER (2007): “Policy-Motivated Parties in Dynamic Political Competition,” *Journal of Theoretical Politics*, 19(1), 9–31.