Incentive Efficient Price Systems in Economies with Adverse Selection*

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Abstract

We decentralize incentive efficient allocations in large adverse selection economies by introducing a Walrasian market for mechanisms, that is, for menus of contracts. Facing a budget constraint, informed individuals choose lotteries over mechanisms, while firms supply (slots at) mechanisms at given prices. An equilibrium requires that firms cannot favorably change, or cut, prices. We show that an equilibrium exists and is incentive efficient. We provide a way to compute an equilibrium as the solution to a recursive programming problem that selects the incentive efficient outcome preferred by the highest type within an appropriately defined set. For Rothschild and Stiglitz economies, this is the only equilibrium outcome.

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1 Introduction

In a large adverse selection economy, can informationally constrained efficient allocations be decentralized through a universal, detail-free system preserving anonymity?[1]

A first answer to these questions may be attempted by using a market solution. Indeed, Prescott and Townsend (1984a, hereafter PT; and 1984b) first laid out the question of whether constrained optimal allocations could be decentralized through prices, linking explicitly the problem of mechanism design to the description of a Walrasian market for contracts.

In this paper we continue this line of investigation, and introduce a Walrasian market for mechanisms, i.e., for platforms where a large number of agents meet and trade. A mechanism is a menu of contracts, one for each type of individuals in the population, and crucially may allow cross-subsidization across types participating the mechanism. As the baseline story, we will use a Rothschild and Stiglitz (1976; hereafter RS) insurance economy, where contracts are policies specifying insurance premium and coverage levels, and mechanisms are insurance policy menus.

Individuals choose lotteries[2] over mechanisms, while firms supply (slots at) mechanisms. Lotteries are priced linearly: the price of a lottery is the average price of its mechanisms. The price of a mechanism can be thought of as the fee for a slot at the mechanism, i.e., for a net trade (or contract) among the ones in the menu. For given prices, individuals buy budget feasible lotteries, while firms realize the profits from the sale of their offered mechanisms. Once lotteries are chosen, their mechanism outcome realizes, and agents choose which contract to commit to, here summarized by final state-contingent consumption bundles. Only one mechanism can be entered ex post, so that contracts are exclusive in the sense of PT. However, all sorts of mechanisms can be traded, or entered, at the ex-ante stage.

For this market structure, we introduce a notion of equilibrium which, beyond the standard requirements of optimization, rational expectations, and market clearing, captures the following essence of competition: prices are competitive if firms cannot favorably change them. A change is favorable when a firm cutting a price can make profits, expecting that other firms may

[1] These requirements are known as the Wilson's doctrine (R. Wilson (1987)) in mechanism design.

[2] For an interpretation of lotteries as sunspots or rationing, see PT (1984a,b) and Kehoe, Levine and Prescott (2002), or Gale (1996, 2001), respectively.

exit only if it is in their interest to do so, and stay otherwise. When prices are immune from price cuts, the price system achieves constrained efficiency: at equilibrium, prices are such that all agents participate the constrained efficient mechanism, so that a first welfare theorem holds. Moreover, an equilibrium always exists. With two types the equilibrium is also outcome-unique and coincides with the separating mechanism most preferred by the high-quality types among the constrained optimal allocations giving at least the RS contract utility to the low-quality types. Constrained optimality generally allows for cross-subsidies and, with type-dependent utilities, for randomizations, so our equilibrium outcome is not always the RS outcome.

Related literature

There is no agreed–upon notion of competition or decentralization in economies with adverse selection.[3] RS introduced a notion of competition among contract designers and obtained the disconcerting result that equilibrium may not exist and may not be constrained efficient[4]. Later, C. Wilson (1977), Riley (1979) and Hellwig (1987) did propose versions of the notion of competition that regained existence. Most closely related to this paper, Miyazaki (1977) (see also Crocker and Snow (1985)) studied a labor market economy with adverse selection (with only two types) where firms offer mechanisms. The C. Wilson equilibrium then coincides in allocation with ours. From this point of view, our contribution can be seen as a general competitive equilibrium translation of Miyazaki's strategic setting, and our notion of equilibrium a competitive version of Wilson's. Crucially, in those papers individuals can choose only among the few mechanisms offered by the firms, and the profit-maximizing firms only offer resource feasible mechanisms. A price system is absent. Instead, in our market system individuals can a priori choose among all possible mechanisms, resource feasible or not. The price system bears the burden of making the markets function, that is, clear, and guarantees feasibility.

Gale (1992, 1996, 2001) also explored a notion of competitive market

[3]In a similar vein, there is no natural definition of the core in this context. To wit, R. Wilson (1978) (and later Vohra, 1999, and related literature, with incentive constraints), Myerson (1983) and Boyd and Prescott (1986) all use different notions of core that can be applied to large adverse selection economies. Our equilibrium notion resonates with the idea of deviation but differs from both Myerson's and Boyd and Prescott's core ideas.

[4]Competition à la RS necessarily destroys cross-subsidies even when they would Pareto improve upon a separating contract. This takes the form of nonexistence when the pooling allocation, allowing cross-subsidies, dominates the separating contract.

for contracts and proposed some stability-based refinement of beliefs to pin down the equilibrium, delivering coherence of incentives and competition, i.e., existence —see also Dubey and Geanakoplos (2002) and Zame (2007). However, none of these market systems and associated refinements guarantees constrained optimality. Instead, our market system achieves constrained efficiency at competitive prices for two reasons: first, it prices mechanisms, not just contracts; second, it puts conditions on prices not via small action set perturbations, but by requiring that prices at inactive mechanisms cannot be favorably changed —a selection criterion over the set of (Walrasian) equilibria.[5]

Finally, following the PT work Bisin and Gottardi (2006) used a notion of Lindahl equilibrium to decentralize constrained efficient allocations. Nevertheless, their price system is coupled with restrictions on market participation, a fundamental ingredient for their optimality result (see Rustichini and Siconolfi, 2008). It also requires a distribution of consumption rights tailored to the actual distribution of types in the population. Both conditions clash with the goal of decentralization à la Wilson, that asks for the system to be universal –i.e., type distribution independent. In our construction, individuals only face prices for slots at various mechanisms, and no other institutional constraint limiting their choices. The price system is specified and works without requiring prior knowledge of the type distribution in the population. Of course, the cost we pay for a less informationally demanding mechanism is its computational complexity.

The paper is organized as follows. In Section 2 we introduce our insurance economies, the allocation problem as an assignment of individuals to mechanisms, and we define incentive efficiency. In Section 3 we describe our market for mechanisms. Section 4 defines competitive prices, i.e., the notion of equilibrium we use. Section 5 describes preliminary properties derived from optimization, market clearing and rational expectations –summarized in the notion of weak equilibrium. Section 6 discusses incentive efficiency of equilibrium, and Section 7 instead discusses existence, under some mild extra assumptions. Section 8 extends the results beyond insurance economies, and Section 9 discusses alternative market structures. Proofs are found in the

[5] A market that prices only contracts, and not mechanisms, may not support incentive efficient outcomes even if prices cannot be favorably changed (see the Discussion section). In a related note (Citanna and Siconolfi (2011)), we show that pricing mechanisms, but using stability-based refinements à la Gale and Dubey and Geanakoplos, may not result in incentive efficient equilibrium outcomes either.

Appendix.

2 The economies

We look at a simple, large economy with asymmetric information: there is only one physical consumption good, and a continuum of individuals with unobservable type $s \in S$, a finite set. We denote by π_s the fraction of the type s agents in the population, with $\pi_s > 0$ and $\sum_{s \in S} \pi_s = 1$. Individuals have type-invariant, uncertain endowments, subject to finitely many idiosyncratic shocks $\omega \in \Omega$. The individual endowment is e_ω with $e_\omega > e_{\omega'} > 0$ for any $\omega, \omega' \in \Omega$ with $\omega > \omega'$. There is an exogenously given and commonly known type-dependent probability distribution over idiosyncratic shocks $\pi(\omega|s)$. Without essential loss of generality the map $\mathbb{E}_s(e)$ is strictly increasing in s, where $\mathbb{E}_s(\xi) = \sum_\omega \pi(\omega|s)\xi_\omega$ for any random variable ξ on Ω. In line with many insurance models we interpret type $s = 1$ as the 'bad' type. Notice that first order stochastic dominance and therefore monotone likelihood ratio imply this condition on $\mathbb{E}_s(e)$, allowing us to also interpret the bad type as the high-risk type.

Individual preferences are represented by a von Neumann-Morgenstern utility function with cardinality index $v_s : \mathbb{R}_+ \to \mathbb{R}$, a continuous, strictly increasing and strictly concave map. Thus, the utility to a type-s individual generated by a net trade $z \equiv (z_\omega)_{\omega \in \Omega} \geq -e \equiv -(e_\omega)_{\omega \in \Omega}$ is

$$u(z, s) \equiv \mathbb{E}_s[v_s(z + e)].$$

Therefore, as in many insurance applications, individuals have private information about their risk profile or their risk aversion.

A special case of our setup is represented by what we call *Wilson economies* (Wilson, 1977). These are economies with type invariant cardinality indexes, $v_s = v$ for all s; moreover, $\Omega = 2$ (states are 'High', $s = 2$, or 'Low', $s = 1$), and there are $S \geq 2$ types. Note how the probability of the High state for type s is strictly increasing in s. Wilson economies are themselves a generalization of *RS economies* (where $S = 2$).

Mechanisms

A (direct) mechanism is a menu of possibly random[6] insurance contracts,

[6] As noted among others by PT (1984b) and Kehoe, Levine, Prescott (2002), randomizations can ease the incentive compatibility constraints when the cardinality index is type dependent.

giving rise to a probability distribution over state-contingent net trades or consumption. Individual trades are assumed to be fully verifiable and enforceable –i.e., contracts are exclusive.

More precisely, a *mechanism* is a collection of S probability measures, or contracts, $\zeta_s \equiv (\zeta_{\omega,s})_{\omega\in\Omega}$, over state-contingent net trades, i.e., ζ_s is a Borel probability distribution over a compact subset $K \subset \mathbb{R}^\Omega$ such that $z \geq -e$, for all $z \in K$, where $\zeta_{\omega,s}$ can be a random net trade assignment. Further conditions on K will be specified momentarily.

We denote the set of mechanisms by Z, with generic element $\zeta = (\zeta_s)_{s\in S}$. Define with some abuse of notation

$$u(\zeta_{s'}, s) = \int u(z, s) d\zeta_{s'} = \sum_{\omega} [\int v_s(z_\omega + e_\omega)] d\zeta_{\omega,s'}] \pi(\omega|s),$$

and

$$\mathbb{E}_s(\zeta) = \sum_{\omega} \pi(\omega|s) \int z_\omega d\zeta_{\omega,s}.$$

A type-s individual preferences for a mechanism ζ are

$$U_s(\zeta) \equiv \max_{s'} u(\zeta_{s'}, s),$$

reflecting the fact that individuals can hide their type and choose among probability measures. Without loss of generality we can restrict attention to (or define the set of mechanisms as) the set X of S-tuple of *incentive compatible* probability measures over net trades, that is,

$$X = \{\zeta \in Z : u(\zeta_s, s) \geq u(\zeta_{s'}, s), \text{ all } s, s' \in S\}.$$

A mechanism ζ where $\zeta_s = \zeta_{s'}$ for all s, s' is called a *pooling mechanism.* A mechanism ζ such that $\zeta = \delta_z$ where δ is the Dirac function and $z = (z_s)_{s\in S} \in \times_s K \subset \mathbb{R}^{S\Omega}$ is said to be *elementary.*

Allocations, lotteries, feasibility and incentive efficiency

The classical point of view sees an allocation problem as an assignment of net trades (i.e., insurance levels) to individuals. Because of incentive compatibility due to adverse selection, it is instead convenient to think of the allocation problem here as consisting in assigning agents to mechanisms. As in the PT tradition, the assignment problem can be recast in terms of lotteries, as follows. The consumption set of all individuals is $\Delta(X)$, the set of lotteries ν_s over X.[7] The utility from a lottery $\nu_s \in \Delta(X)$ is

$$U_s(\nu_s) = \int_X U_s(\zeta) d\nu_s,$$

where, since $\zeta \in X$ is an incentive compatible collection of probability measures over net trades, $U_s(\zeta) = u(\zeta_s, s)$.

Given the linearity of the map $U_s(\nu_s)$, hereafter we write $U_s\nu_s$. Since $U_s\delta_\zeta = U_s(\zeta)$, we feel free to switch from one to the other notation whenever more convenient. We also denote by $\Sigma(\nu_s)$ the support of the lottery ν_s. Notice that even if $\nu_s = \delta_\zeta$ is degenerate, it may allow randomizations within mechanism ζ.

Consider an allocation $\nu \in \Delta(X)^S$, that is, a collection of individual lotteries $\nu = (\nu_s)_{s \in S}$. An allocation ν is *feasible* if

$$\sum_s \pi_s \int_X \mathbb{E}_s(\zeta) d\nu_s \leq 0.$$

We let

$$Y = \{\nu \in \Delta(X)^S : \sum_s \pi_s \int_X \mathbb{E}_s(\zeta) d\nu_s \leq 0\}$$

be the set of feasible allocations. To allow for a nontrivial analysis, we choose K so that $\times_s K$ contains all feasible and incentive compatible elementary mechanisms as well as $\bar{z}$, a pooling (and therefore incentive compatible) elementary mechanism satisfying $\bar{z} \gg 0$ and

$$U_s(\bar{z}) > U_s\nu_s \text{ for all } s, \text{ all } \nu \in Y.$$

The last inequality implies that individual preferences are nonsatiated in the interior of the feasible set. Because of the structure of K, the set of mechanisms X is very rich and contains also mechanisms which are not budget balanced. Feasibility of allocations itself implies budget balance in the aggregate only: given a feasible allocation ν, it is still possible for a mechanism $\zeta \in \Sigma(\nu)$ to consume more resources than those belonging to its participants, i.e., $\sum_s \pi_s \mathbb{E}_s(\zeta) > 0$. With some abuse of notation we write $\zeta \in Y$ to denote a mechanism $\zeta \in X$ such that $\sum_s \pi_s \mathbb{E}_s(\zeta) \leq 0$, or $z \in Y$ for $\zeta = \delta_z$ elementary with $\sum_s \pi_s \mathbb{E}_s(z) \leq 0$.

An allocation ν is:

[7] These are Borel probability measures over X. If X is finite, $\mu \in \Delta(X)$ iff $\mu(z_i) \geq 0$ all $z_i \in X$ and $\sum_i \mu(z_i) = 1$.

individually rational if

$$U_s\nu_s \geq U_s(0), \text{ all } s;$$

(ex-ante) incentive compatible if

$$U_s\nu_s \geq U_s\nu_{s'}, \text{ all } s, s' \in S; \tag{1}$$

feasible allocations are ex-post incentive compatible because their support is restricted to X;

incentive efficient (or CPO, for *Constrained Pareto Optimal*) if it is feasible, individually rational, incentive compatible and there is no other feasible, incentive compatible allocation $\tilde{\nu}$ such that

$$U_s\tilde{\nu}_s \geq U_s\nu_s, \text{ all } s \in S,$$

with one strict inequality.

a strict CPO if it is feasible, individually rational, incentive compatible and there is no other feasible, incentive compatible allocation $\tilde{\nu}$ such that,

$$U_s\tilde{\nu}_s > U_s\nu_s \text{ for all } s \in S.$$

3 A price system

We now describe a market system for the assignment of agents to mechanisms. We allow agents to buy lotteries $\mu \in \Delta(X)$ over slots at mechanisms in X, given prices $p(\zeta)$ for an individual slot at mechanism ζ. Such prices p can be positive or negative –in principle because an agent may be required to provide more resources than he consumes when participating a mechanism. One can think of $p(\zeta)$ as the fee paid by an individual to participate mechanism ζ; if this number is negative, the agent receives money to enter the mechanism. Thus, given a price p, each agent faces the corresponding budget constraint

$$p\mu = \int_X p(\zeta) d\mu \leq 0.$$

Mechanisms, and slots at mechanisms, are offered by a large, but finite number of firms/intermediaries at such prices. Offering one unit of mechanism $\zeta \in X$ yields $p(\zeta)$ revenues to the firm. Firms guarantee that there will be resources to consume at each mechanism, and verify and enforce the agents' resulting net trades at a mechanism. The question is how much the per unit expected cost of this is. The answer depends on the proportions of the types in the mechanism. In order to gauge the net consumption of resources at mechanism ζ, firms must form beliefs $\beta(\zeta)$ about the type distribution that will be attracted to such a mechanism. As customary we assume that beliefs are common across firms.

The expected cost of having type s in mechanism ζ is $\mathbb{E}_s(\zeta)$. Thus, denoting by $\beta(s;\zeta)$ the believed fraction of type s in mechanism ζ, the expected cost of running such a mechanism is

$$\mathbb{E}_\beta(\zeta) = \sum_s \mathbb{E}_s(\zeta)\beta(s;\zeta)$$

Firms supply positive finite Borel measures over X, $\nu^* \in \mathcal{M}_+(X)$, i.e., offer (a nonnegative number ν^* of) mechanisms to the agents. Then, the total cost of offering a measure ν^* of mechanisms is $\int_X \mathbb{E}_\beta(\zeta)d\nu^*(\zeta)$. Notice that expected and effective unit costs coincide in any situation where the law of large numbers applies.

For later interpretation it is worth stressing that each offered mechanism is large, as it contains a large number of agents if active, and yet small as it may be offered by many firms. Also, these many firms may offer each a different mechanism, and mechanisms therefore compete with each other for agents/customers in this market. Crucially, observe that no agent has knowledge of the true population distribution of types, π; instead, mechanisms work as a function of the potential or attracted distribution β, and of prices p.

Finally, one may wonder if pricing only elementary mechanisms or, as in most of the literature, just individual contracts would not be enough for our purposes. We will address this modeling issue in the Conclusions after that all necessary tools have been presented.

To settle technical issues, we endow $\Delta(X)^S$ with the weak* topology and let prices be elements of $\mathbb{C}(X,\mathbb{R})$, the weak* continuous linear functionals over X.[8] Prices are linear in mechanisms, though generally not linear in ζ

[8] More precisely, we consider $\mathbb{C}(K,\mathbb{R})$ endowed with the sup-norm topology. Since $K \subset$

–or, a fortiori, in z. Beliefs are $\beta : X \to \Delta(S)$, a Borel-measurable function such that $\beta(s;\zeta) \geq 0$ for all $s \in S$ and $\sum_s \beta(s;\zeta) = 1$, for all $\zeta \in X$. Hereafter, $\mathcal{B}(X)$ denotes the Borel $\sigma-$algebra over X.

4 Equilibrium

We introduce a notion of competitive equilibrium to study the market allocation outcomes.

Weak equilibrium

The natural definition of competitive equilibrium must include a consistent collection of prices, allocations, and beliefs via three standard requirements: optimization, market clearing and rational expectations. These requirements are common to existing definitions in the literature (as in Gale (1992), Dubey and Geanakoplos (2002), and Zame (2007)), though our price system differs in view of what is tradeable.

A *weak equilibrium* is an array (ν, β, ν^*, p) of individual lotteries $(\nu_s)_{s\in S} \in \Delta(X)^S$, beliefs β, an aggregate supply of the firms $\nu^* \in \mathcal{M}_+(X)$, and a price function $p \in \mathbb{C}(X, \mathbb{R})$ such that:

(O) (Optimization) given p, ν_s solves

$$\max_{\mu\in\Delta(X)} U_s\mu \text{ subject to } p\mu \leq 0,$$

all $s \in S$; given p and β, ν^* solves

$$\max_{\mu\in\mathcal{M}_+(X)} \int_X [p(\zeta) - \mathbb{E}_\beta(\zeta)]d\mu;$$

(M) (Market clearing) $\sum_s \pi_s\nu_s(B) = \nu^*(B)$ for all $B \in \mathcal{B}(X)$;

(C) (Belief consistency) (ν, β, ν^*, p) are compatible, or

$\mathbb{R}^\Omega$ is compact, the norm dual of $\mathbb{C}(K, \mathbb{R})$ endowed with the sup-norm topology is $\mathcal{M}(K)$, the space of finite Borel measures over K (see, e.g., Aliprantis and Border (1999), Theorem 13.12). Then, we consider the (normed) dual pair $< \mathbb{C}(K, \mathbb{R}), \mathcal{M}(K) >$ and endow $\mathcal{M}(K)$ with the weak* topology. Also, $\Delta(K) \subset \mathcal{M}(K)$ is weak* compact (see, e.g., Aliprantis and Border (1999), Theorem 14.11), and so is the closed set $X \subset \Delta(K)^S$. When considering $\mathbb{C}(X, \mathbb{R})$, X is endowed with the weak* topology induced by $< \mathbb{C}(K, \mathbb{R}), \mathcal{M}(K) >$, while $\mathbb{C}(X, \mathbb{R})$ is endowed with the sup-norm topology. The norm dual of $\mathbb{C}(X, \mathbb{R})$ is therefore $\mathcal{M}(X)$, and $\Delta(X) \subset \mathcal{M}(X)$ is weak* compact (where the weak* is induced this time by the dual pair $< \mathbb{C}(X, \mathbb{R}), \mathcal{M}(X) >$).

$$\pi_s\nu_s(B) = \int_B \beta(s;\zeta)d(\sum_{s'} \pi_{s'}\nu_{s'}), \text{ all } s \in S,\ B \in \mathcal{B}(X).$$

The first equilibrium requirement, (O), is the standard behavioral restriction on individuals and firms. Its consequences are explored in the next subsection. Notwithstanding the large number of firms, the constant returns to scale feature of their production set $\mathcal{M}_+(X)$ and their common beliefs allow for collapsing the firms' choice problems into the "aggregate" profit maximization problem.

The second requirement, (M), or 'demand equals supply', highlights the allocation problem as an assignment problem, as mentioned above.

The third requirement, (C), is a rational expectations condition. It says that the precise proportions of types in any subset of mechanisms are ultimately determined at equilibrium by the optimal individual choices. That is, whenever $\nu^*(B) > 0$, and therefore via (M) $\sum_s \pi_s\nu_s(B) > 0$, beliefs are uniquely[9] determined by ν^* and they are consistent with Bayes' rule:

$$\beta(s;B) \equiv \frac{\int_B \beta(s;\zeta)d\nu^*(\zeta)}{\nu^*(B)} = \frac{\pi_s\nu_s(B)}{\sum_{s'} \pi_{s'}\nu_{s'}(B)}.$$

Thus, using (M) and (C), the *unit profit* of mechanism ζ can be written without any reference to β and ν^* as

$$\Pi(\zeta;(\nu,p)) = \begin{cases} p(\zeta) - \sum_s \mathbb{E}_s(\zeta)\frac{d\pi_s\nu_s}{d(\sum_{s'}\pi_{s'}\nu_{s'})}(\zeta), & \text{if } \zeta \in \Sigma(\sum_{s'}\pi_{s'}\nu_{s'}) \\ 0 & \text{otherwise,} \end{cases}$$

where $d\pi_s\nu_s/d(\sum_{s'}\pi_{s'}\nu_{s'})$ is the Radon-Nikodym derivative.

However, (C) does not pin down beliefs for inactive mechanisms, that is, for ζ outside sets of ν^* zero measure. Importantly, ν may not be (and typically will not be) absolutely continuous with respect to the Lebesgue measure. Thus, beliefs may be disconnected from allocations in large sets of the mechanisms space.

A standard competitive firm takes prices as given at equilibrium because it has no incentive to change prices. Why? By increasing prices the firm loses

[9] Obviously each probability measure ν_s is absolutely continuous with respect to the measure $\nu^* = \sum_s \pi_s\nu_s$. Thus, technically condition (C) says that β is restricted to be the Radon-Nykodim derivative of $\pi_s\nu_s$ with respect to ν^*. Given ν, if two systems of beliefs β and β' satisfy condition (C), β and β' coincide ν^*-a.e..

all of its customers, as they can get the same product from the competitors at a lower price. By decreasing prices, the firm is pricing below marginal cost thereby decreasing its profits. In our weak equilibrium the price of a mechanism also is equal to its marginal cost. However, in our economies the cost is defined endogenously, that is, the marginal cost of a mechanism ζ is equal to $\mathbb{E}_\beta(\zeta)$. No firm has an incentive to increase the price of any mechanism: all customers would disappear as a reaction to such a price increase. Instead, a firm operating an inactive mechanism may want to challenge (or not agree with) the market price and decrease the price of its mechanism. If by doing so the firm realizes a positive profit, that market price is in this very precise sense not competitive. The underlying belief and the corresponding price cannot be taken as given. This suggests a notion of competitive equilibrium where no firm running an inactive mechanism has an incentive to cut its price. This equilibrium notion requires a definition of what happens after the price cut. This is where we introduce the instrumental notion of a quasi equilibrium.

Quasi equilibrium

We define a state of the economy in which the price system will be after being challenged by a price cut. A quasi equilibrium is meant to describe such a state of the economy by requiring individual optimization and market clearing, and that *all tradeable* mechanisms realize nonnegative profits. Prices alone will in general not guarantee this condition. If we insisted that a price cut must be consistent with nonnegative profits for *all* mechanisms, we would substantially reduce the firms' ability to compete with each other. While this restriction on price cuts would work,[10] it would tamper on the firms' freedom to run their business, going against the idea of competitive markets. Instead, using a free exit rationale, we allow for mechanisms that could potentially lose money to be shut down. The extent to which mechanisms can be shut down will be limited only by the firms' interest to do so (a free entry rationale) via a notion of maximal shut-down.

A *market indicator* is a measurable map $\eta : X \to \{0, 1\}$: $\eta(\zeta) = 0$ means that a mechanism is shut down, while $\eta(\zeta) = 1$ means the opposite. We call mechanism ζ *tradeable (at* η*)* if $\eta(\zeta) = 1$, and denote with X_η the set of tradeable mechanisms generated by η. Given a price map p and a market

[10] In fact, as the proofs in the Appendix show, we can easily prove that weak equilibria which are immune only from these restricted price cuts always exist and are constrained efficient.

indicator η, individuals choose lotteries over tradeable mechanisms, that is they solve:

$$\max_{\mu \in \Delta(X_\eta)} U_s \mu \text{ subject to } p\mu \leq 0.$$

Let $M_s(p,\eta) \subset \Delta(X_\eta)$ be the set of optimal solutions and $M(p,\eta) = \times_s M_s(p,\eta)$. We let $\Pi(\zeta;(\nu,p,\eta))$ be the unit profit for any mechanism $\zeta \in X_\eta$ and $\nu \in M(p,\eta)$.

The market indicator η is *viable* at p if there exists $\nu \in M(p,\eta)$ such that $\Pi(\zeta;(\nu,p,\eta)) \geq 0$, for all $\zeta \in X_\eta$. The market indicator η is *maximal* at p if it is viable, and for all $\eta' \geq \eta$, $\eta'(\zeta) - \eta(\zeta) = 1$ implies $\Pi(\zeta;(\nu',p,\eta')) < 0$ for some $\nu' \in M(p,\eta')$.

An array (ν,p,η), $p \in \mathbb{C}(X,\mathbb{R})$ is a *quasi equilibrium* if:

(O') $\nu \in M(p,\eta)$;

(Π) $\Pi(\zeta;(\nu,p,\eta)) \geq 0$ for all $\zeta \in X_\eta$;

(T) η is maximal at p.

Two remarks are in order. First, beliefs do not appear in the definition of a quasi equilibrium, because at a quasi equilibrium condition (Π) directly states a no loss condition for all tradeable mechanisms, bypassing beliefs (compatible with allocations). Second, maximality implies that a mechanism ζ cannot shut down if, were it open, it would make money.

Price cuts

We allow firms to challenge the weak equilibrium beliefs by performing price cuts. The price cuts define a new price map and new market choices (optimal lotteries). At such new prices and choices, some mechanisms may experience losses and they shut down, exiting the market. If as a result of the price cuts, of the firms' exits and of the new individual choices the economy moves to a quasi equilibrium where some mechanisms may experience positive profits, we say the price cut is effective.

Given p, a *price cut* is a pair $(\bar{X},\phi)$ with $\bar{X} \in \mathcal{B}(X)$ and ϕ a measurable function on $\bar{X}$ with $\phi(\zeta) < p(\zeta)$ for all $\zeta \in \bar{X}$. Let p_ϕ be the price defined as $p_\phi(\zeta) = p(\zeta)$ for all $\zeta \notin \bar{X}$, and $p_\phi(\zeta) = \phi(\zeta)$ for all $\zeta \in \bar{X}$.

A price cut $(\bar{X},\phi)$ is *effective* against the weak equilibrium (ν,β,ν^*,p) if there exists a market indicator η and an allocation $\nu \in \Delta(X)^S$ such that (ν,p_ϕ,η) is a quasi equilibrium with $\eta(\zeta) = 1$ and $\Pi(\zeta;(\nu,p_\phi,\eta)) > 0$ for all $\zeta \in \bar{X}$.

The maximality of the market indicator η implies that the price cut is not effective in destroying a given weak equilibrium just because some mechanisms have been arbitrarily shut down.

Equilibrium

If a weak equilibrium is vulnerable to an effective price cut, prices can be favorably changed. At an equilibrium there cannot be favorable price changes, i.e., successful price cuts: an equilibrium is a weak equilibrium which is immune from price cuts. Formally:

An *equilibrium* is a (ν, p) such that: (i) (ν, β, ν^*, p) is a weak equilibrium for some β, ν^*; (ii) there does not exist an effective price cut against (ν, β, ν^*, p).

Thus, the notion of equilibrium we adopt echoes two ideas adapted from the early strategic literature: anticipatory expectations –via the beliefs sustaining the price cut as a quasi equilibrium; and free entry and exit –via the possibility of price cuts and the maximality of the map η, respectively.

5 Preliminary properties

Zero profits, feasibility, and a simpler definition of weak equilibrium

Since the firm(s) is profit maximizer and the technology displays constant returns to scale, at a weak equilibrium profits must be zero. Thus, prices must satisfy

$$p(\zeta) \leq \mathbb{E}_\beta(\zeta) \text{ for all } \zeta, \text{ and } p(\zeta) = \mathbb{E}_\beta(\zeta)\ \nu^*\text{-a.e.} \tag{ZP}$$

This break-even or zero-profit condition is common to related competition models (again, see Gale (1992), Dubey and Geanakoplos (2002), and Zame (2007)). Without loss of generality, condition (ZP) restricts weak equilibrium prices to the set

$$\mathbb{P} = \{p \in \mathbb{C}(X, \mathbb{R}) : p(\zeta) = \mathbb{E}_\beta(\zeta) \text{ all } \zeta, \text{ for some belief system } \beta\}.$$

Equivalently, $p \in \mathbb{P}$ if and only if $p(\zeta) \in [p_m(\zeta), p_M(\zeta)]$ for $p_m(\zeta) = \min_s \mathbb{E}_s(\zeta)$ and $p_M(\zeta) = \max_s \mathbb{E}_s(\zeta)$. This has two consequences.

First, weak equilibrium allocations are feasible. Let (ν, β, ν^*, p) be a weak equilibrium, then ν_s satisfies the budget constraints, that is, $\pi_s p \nu_s \leq 0$ for all s, and therefore

$$\int p(\sum_s \pi_s d\nu_s) = 0.$$

Then, using (ZP), (M) and (C),

$$0 \geq \sum_s \pi_s \int p(\zeta) d\nu_s = \int_X \mathbb{E}_\beta(\zeta) d\nu^* = \sum_s \pi_s \int_X \mathbb{E}_s(\zeta) d\nu_s,$$

and this is feasibility. If individual optimal choices satisfy the budget constraints with equality, then the weak equilibrium allocation ν satisfies feasibility with equality. That the premise holds true is stated in the next lemma, whose proof is obvious and therefore omitted. Let $M_s(p)$ be the set of optimal solutions for type-s individuals at prices $p \in \mathbb{C}(X, \mathbb{R})$.

Lemma 1 *Let $p \in \mathbb{P}$ be given. Then $M_s(p) \neq \varnothing$, and $p\mu = 0$ for all $\mu \in M_s(p)$.*

Hence, (weak) equilibrium allocations belong to $\bar{Y}$, the set of allocations satisfying feasibility with equality,

$$\bar{Y} = \{\mu \in Y : \int_X [\sum_s \pi_s \mathbb{E}_s(\zeta) d\mu_s] = 0\}.$$

With some abuse of notation we write $\zeta \in \bar{Y}$ meaning that $\sum_s \pi_s \mathbb{E}_s(\zeta) = 0$.

Second, when prices are in $\mathbb{P}$, β and ν^* can be eliminated from the definition of weak equilibrium by reading conditions (M) and (C) as the definitions of ν^* and β, respectively. More precisely, we can equivalently define a weak equilibrium as a pair (ν, p) such that $\nu \in \times_s M_s(p)$, $p \in \mathbb{P}$, and $\Pi(\zeta; (\nu, p)) = 0$ for all $\zeta \in X$. Beliefs generated by ν through equation (C) satisfy the zero profit condition (ZP) and since $p \in \mathbb{P}$, we can set them so that $p(\zeta) = \mathbb{E}_\beta(\zeta)$ for all $\zeta \in X$. Notice though that ν determines β only ν-a.e.; β and β' are compatible with the weak equilibrium pair (ν, p) not only if they are equal ν-a.e., but also if both satisfy $p(\zeta) = \mathbb{E}_\beta(\zeta) = \mathbb{E}_{\beta'}(\zeta)$, for all $\zeta \in X$. Hereafter, we will refer to a weak equilibrium simply as a pair (ν, p).

Existence and indeterminacy of weak equilibrium allocations

As in related work by Gale (1992), weak equilibria exist, but they may be severely indeterminate. Here, we give a precise characterization of such phenomenon.

We say that $p \in \mathbb{P}$ *supports mechanism* ζ if $\zeta \in M_s(p)$ for all s. Let $\mathbb{P}(\zeta) \subset \mathbb{P}$ the set of prices that support ζ. We claim that if ζ is a feasible mechanism in $\bar{Y}$ and $\mathbb{P}(\zeta) \neq \varnothing$, then (ζ, p) is a weak equilibrium for every $p \in \mathbb{P}(\zeta)$. In other words, the set of induced beliefs β satisfying condition (C) and the set of those satisfying $p(\zeta) = \mathbb{E}_\beta(\zeta)$, for all $\zeta \in X$, (which is nonempty because $\mathbb{P}(\zeta) \neq \varnothing$) have nonempty intersection when the candidate allocation ν is degenerate, that is, $\nu_s = \delta_\zeta$, all s. This straightforward preliminary observation is needed to show the indeterminacy of the weak equilibrium set. Its proof is left to the reader.

Lemma 2 *Let $\zeta \in \bar{Y}$ with $\mathbb{P}(\zeta) \neq \varnothing$. Then, for each $p \in \mathbb{P}(\zeta)$, there exists β_p solving $p(\zeta') = \sum_s \beta_p(s;\zeta')\mathbb{E}_s(\zeta')$ for all $\zeta' \in X\backslash\{\zeta\}$; $\beta_p(s;\zeta) = \pi_s$ for all s, and $(\zeta, \beta_p, \delta_\zeta, p)$ is a weak equilibrium.*

Starting with the next proposition we use the notion of *compound*, or *mean lottery*. For any lottery $\nu \in \Delta(X)$, the *mean of* ν is the mechanism $\mathbb{E}\nu \in X$ satisfying

$$\int f d\mathbb{E}\nu = \int \int f d\zeta d\nu, \text{ for all } f \in \mathbb{C}(K, \mathbb{R}).$$

which is well defined and weakly continuous in $\mathbb{C}(K, \mathbb{R})$, by the Riesz Representation Theorem for linear functionals over $\mathbb{C}(K, \mathbb{R})$ (see Aliprantis and Border (1999), Theorem 13.12). Since $\mathbb{E}\nu \in X$, it is $\mathbb{E}\nu = (\mathbb{E}\nu_s)_{s\in S}$, with $\mathbb{E}\nu_s \in \Delta(K)$. Intuitively, mechanism $\mathbb{E}\nu$ compounds lottery ν with mechanisms ζ in $\Sigma(\nu)$. By linearity of payoffs and resources, we have

$$U_s(\nu) = \int u(\zeta, s)d\nu = \int (\int u(z, s)d\zeta_s)d\nu = \int u(z, s)d\mathbb{E}\nu = U_s(\mathbb{E}\nu),$$

as well as

$$\int [\sum_s \pi_s \mathbb{E}_s(\zeta)]d\nu_s = \int [\int \sum_s \pi_s \mathbb{E}_s(z)d\zeta_s]d\nu_s = \int \sum_s \pi_s \mathbb{E}_s(z)d\mathbb{E}\nu_s,$$

so compounding preserves resources and payoffs. In the sequel, whenever convenient and without loss of generality we identify allocations with mechanisms.

In the next step we find the largest possible set of mechanisms in $\bar{Y}$ for which belief based supporting prices are nonempty, that is, $\mathbb{P}(\zeta) \neq \varnothing$. It turns out that such set is a strict subset of mechanisms in $\bar{Y}$, because in our economy individuals have access to an outside option providing a lower bound to the weak equilibrium utility. To this end, consider $\zeta^{RS} = (\zeta_s^{RS})_{s\in S} \in X$, the *generalized Rothschild and Stiglitz* (or RS) separating contracts, defined as follows. With

$$X^0 = \{\zeta \in X : \mathbb{E}_s(\zeta) \leq 0, \text{ all } s\},$$

for each s let $\zeta^*(s)$ be a solution to

$$\max_{\zeta \in X^0} U_s(\zeta). \tag{RS}$$

Then, let ζ_s^{RS} be the s-th entry of the vector $\zeta^*(s)$, that is, $\zeta_s^{RS} = \zeta_s^*(s)$, all s. Denote with X^{RS} the set of mechanisms ζ^{RS} so defined. Note that in general the set X^{RS} may contain many mechanisms, though all of them are payoff equivalent by construction.

Lemma 3 (generalized RS contract) *X^{RS} is a nonempty subset of X^0. Moreover, $\mathbb{E}_1(\zeta^{RS}) = 0$ for any $\zeta^{RS} \in X^{RS}$.*

Mechanisms ζ^{RS} are the "best" contracts in X^0 and serve as an outside option for each type. Clearly, any ζ^{RS} provides full insurance to the lowest types (i.e., $\zeta_1^{RS} = \delta_{z_1^{RS}}$ with $z_{\omega,1}^{RS} + e_\omega = x_1^{RS}$, a scalar for all ω) as for them ζ_1^{RS} is the optimum at their fair prices. Since $\zeta^{RS} \in X^0$, any belief-based price system gives non positive value to these contracts. Hence any type will always be able to buy them in the market. Therefore, the welfare of the weak equilibrium allocations is bounded below by $U_s(\zeta^{RS})$, all $s \in S$. Then, we limit attention to allocations contained in the set

$$\Delta' = \{\zeta \in X | U_s(\zeta) \geq U_s(\zeta^{RS}), \text{ some } \zeta^{RS} \in X^{RS}, \text{ all } s\}.$$

The next proposition shows that the set of weak equilibrium allocations coincides with the set $\Delta' \cap \bar{Y}$. It also shows that the payoff indeterminacy of weak equilibrium is entirely exhausted by such mechanisms.

Proposition 4 *Any mechanism $\zeta^* \in \Delta' \cap \bar{Y}$ is a weak equilibrium allocation. In particular, there exist a closed set Z_*, $X \supset Z_* \supset \{\zeta^*\}$ and a collection of scalars $(\lambda_s)_{s\in S} \gg 0$ such that the price function $\bar{p}$ defined as*

$$\bar{p}(\zeta) = \begin{cases} \max_{s\in S} \frac{U_s(\zeta) - U_s(\zeta^*)}{\lambda_s} & \text{if } z \in Z_*, \\ p_m(\zeta) & \text{otherwise} \end{cases}$$

is an element of $\mathbb{P}(\zeta^*)$. *Moreover, if* (ν, p) *is a weak equilibrium, then its mean lottery* $\mathbb{E}\nu$ *is a weak equilibrium supported by* $p' \in \mathbb{P}(E\nu)$.

Hence, pricing every mechanism essentially at utility value, but relative to a reference point $\zeta^* \in \Delta'$ and for the type that values a mechanism the most, delivers a weak equilibrium. Because of Proposition 4, hereafter, without loss of generality, we identify weak equilibrium allocations with elements of X, as opposed to with lotteries in $\Delta(X)$.

By Proposition 4, the notion of weak equilibrium pins down equilibrium payoffs only if the allocation ζ^{RS} is a CPO, otherwise equilibrium payoffs are indeterminate.

Price cuts: a simpler definition

Effective price cuts have two important, simple properties. First, there is no loss generality in restricting attention to price cuts where $\bar{X}$ is a singleton and the associated degenerate allocation is a mechanism in X. The trading maps are identical and the quasi equilibrium associated to the new price cut is payoff equivalent to the old one both for the firm operating the price cut as well as for all individuals. Second, the quasi equilibrium allocation associated to an effective price cut does not exhaust the total resources of the economy.

Lemma 5 *Let* $(\bar{X}, \phi)$ *be an effective price cut against the weak equilibrium* (ζ^*, p) *and* (ν, p_ϕ, η) *the associated quasi equilibrium. Then there is an effective price cut* $(\bar{X}', \phi')$ *with* $\bar{X}' = \{\zeta'\}$ *and* $\phi'(\zeta') = 0 \leq p(\zeta')$, *such that* $(\zeta', p_{\phi'}, \eta)$ *is an associated quasi equilibrium with* $U_s\nu_s = U_s(\zeta')$ *for all* s, $\int_X \Pi(\zeta; (\nu, p, \eta))(\sum_s \pi_s d\nu_s) = \Pi(\zeta'; (\zeta', p, \eta)) > 0$ *and* $\sum_s \pi_s \int \mathbb{E}_s(\zeta) d\nu_s = \sum_s \pi_s \mathbb{E}_s(\zeta') < 0$.

We leave the straightforward proof to the reader. As a consequence, hereafter without loss of generality we will use the following much simpler definition of effective price cut.

The pair (ζ', p_ϕ) is an *effective* price cut against the weak equilibrium (ζ, p) if: $p_\phi(\zeta) = p(\zeta)$ for all $\zeta \neq \zeta'$, while $p_\phi(\zeta') = 0 < p(\zeta')$; the array (ζ', p_ϕ, η) is a quasi equilibrium for some trading map η with $\eta(\zeta') = 1$; and $\Pi(\zeta'; (\zeta', p_\phi, \eta)) > 0$.

6 Incentive efficiency of equilibrium

With the preliminary properties stated and our definition of price cut immunity in place, we can now show that an equilibrium must be a strict CPO.

Theorem 6 *If ζ^* is an equilibrium allocation, then it is a strict CPO.*

Why? A strictly inefficient mechanism can only be sustained as a weak equilibrium by prices for strictly Pareto improving mechanisms —including a strictly feasible one such mechanism— that are too high, that is, positive. Competition across mechanisms (through price cuts) insures that such a situation is not stable, as firms can favorably change prices –from positive to zero– to increase their profits on the strictly feasible mechanism while every type will flock to this mechanism at zero price.

However, competition is not strong enough to destroy weak equilibrium allocations that are strict CPO, but not CPO. Indeed, because of the nature of the incentive compatibility constraints, strict monotonicity of preferences does not suffice to make strict CPO and CPO allocations equivalent. The argument in the absence of incentive constraints goes as follows: if ζ is a strict, but not a weak Pareto optimum, there is a feasible ζ' such that $U_s(\zeta') \geq U_s(\zeta)$ for all s, with strict inequality for at least one $\bar{s}$; then, by continuity, $U_{\bar{s}}(\zeta''_{\bar{s}}) > U_{\bar{s}}(\zeta)$ for some $\zeta''_{\bar{s}}$ with $\mathbb{E}_{\bar{s}}(\zeta'') < \mathbb{E}_{\bar{s}}(\zeta')$ arbitrarily close to $\zeta'_{\bar{s}}$; the gap $\mathbb{E}_{\bar{s}}(\zeta' - \zeta'') > 0$ can then be redistributed among the other individuals $s \neq \bar{s}$ so to find a feasible allocation $\bar{\zeta}$ that makes everybody better off, a contradiction to ζ being a strict Pareto optimum. However, in adverse selection economies this argument breaks down because $\bar{\zeta}$ may fail to be incentive compatible. It is known that if the cardinality indexes are type invariant, $v_s = v$ for all s, then strict CPO and CPO allocations coincide. Interestingly, the argument exploits the s invariance of v_s reducing lotteries over ζ and ζ' to payoff-equivalent elementary mechanisms that use less resources. At that point, and only then, it exploits strict monotonicity of preferences. It is an open question to find minimal conditions under which strict CPO and CPO allocations coincide.

7 Existence of equilibrium

When $\zeta^{RS} \in CPO$, equilibrium price systems always exist and the equilibrium allocation is payoff unique, as it is shown in the following lemma.

Lemma 7 *If X^{RS} is contained in the strict CPO set, then X^{RS} is contained in the set of equilibrium allocations. If $X^{RS} \subset CPO$, then the equilibrium set coincides with X^{RS}.*

This is because such an RS contract has $\mathbb{E}_s(\zeta^{RS}) = 0$ for all s. Therefore, it cannot be shut down after a price cut, due to the maximality requirement on η; and it cannot be left open because someone would prefer it to the equilibrium allocation, requiring that its price be above zero –clearly impossible.

To study the case when ζ^{RS} is not a strict CPO, we adopt the following two assumptions:

Monotonicity Let $\zeta \in \Delta(K)$ be such that $U_s(\zeta) \geq U_s(\zeta^{RS})$ and $\mathbb{E}_s(\zeta) \leq 0$ then $\mathbb{E}_\sigma(\zeta) \leq \mathbb{E}_s(\zeta) \leq 0$ for all $\sigma > s$.

Sorting Let $\zeta, \zeta' \in \Delta(K)$ be such that $U_s(\zeta) \geq U_s(\zeta')$, while $U_{s+1}(\zeta) \leq U_{s+1}(\zeta')$; then

$$\begin{aligned} U_\sigma(\zeta) &\leq U_\sigma(\zeta') \text{ all } \sigma > s, \text{ while} \\ U_{\sigma'}(\zeta) &\geq U_{\sigma'}(\zeta') \text{ all } \sigma' \leq s. \end{aligned}$$

Monotonicity and Sorting are widely used in the contract literature, even in recent equilibrium analysis of competition with contracts (see, e.g., Guerrieri, Shimer and Wright, 2010). Wilson economies satisfy Monotonicity and Sorting as proved for completeness at the end of the Appendix. We emphasize that these assumptions do not have bite when $S = 2$: Monotonicity holds immediately as $U_1(\zeta) \geq U_1(\zeta_1^{RS})$ and $\mathbb{E}_1(\zeta) \leq 0$ imply that $\zeta = \zeta_1^{RS}$; since ζ_1^{RS} provides full insurance and $\mathbb{E}_s(e)$ is already assumed to be increasing with s, the claim follows; Sorting, instead, is vacuous. Hence, when $S = 2$ and utilities are type-dependent, the usual 'single-crossing' condition may be violated, but our analysis goes through. More generally, for $S > 2$ an economy may fail 'single-crossing', but still satisfies Sorting.

S-CPO

The existence of equilibrium is proved by constructing a specific strict constrained efficient allocation that will be shown to be contained in the equilibrium allocation set. Consider the following recursive family of programming problems indexed by $s = 2, ..., S$. Let $V_1(1) \equiv U_1(\zeta^{RS})$. For $s \geq 2$, the s-th problem is appended to $s-1$ values $V_\sigma(\sigma)$, $\sigma \leq s-1$, and is defined as

$$
\begin{aligned}
V_s(s) \equiv \max\nolimits_{\zeta \in X} U_s(\zeta) \\
U_\sigma(\zeta) \geq V_\sigma(\sigma), \qquad \text{all } \sigma < s, \\
\textstyle\sum_{\sigma \leq s} \pi_\sigma \mathbb{E}_\sigma(\zeta) \leq 0.
\end{aligned}
$$

Let $\zeta(s)$ be (one of) the optimal solution(s) and let $V(s) = (V_\sigma(s))_{\sigma=1}^{s} \equiv (U_\sigma(\zeta(s)))_{\sigma=1}^{s}$ be the utilities assigned to the $\sigma \leq s$ types in the s-th programming problem. Notice that for each type $\sigma < s$ the value of the σ-th programming problem for such a type provides the reservation utility for s-th programming problem. In Appendix we show several key properties of the s-th programming problems, summarized in the following statement.

Lemma 8 *For each s:*

(i) there exists a solution $\zeta(s)$ to the s-th problem;

(ii) any solution $\zeta(s)$ satisfies $\sum_{s' \leq \sigma} \pi_{s'} \mathbb{E}_{s'}(\zeta(s)) \geq 0$ for all $\sigma \leq s$, with equality for $\sigma = s$.

(iii) $\zeta(S) \in \Delta' \cap \bar{Y}$;

(iv) any $\zeta(S)$ is a strict CPO, and there is a $\zeta(S)$ in the CPO set;

(v) if ζ^{RS} is a strict CPO, then ζ^{RS} is an optimal solution to the S-th problem;

(vi) when $S = 2$, all $\zeta(S)$ are in the CPO set;

(vii) if $v_s = v$ all s, then $\zeta(S)$ is unique and $\zeta(S) = z(S)$.

Property (ii) is key, and it states that any s-th problem is characterized by a specific cross-subsidization pattern. We call it the '*top-down*' property of cross-subsidies. It says that after partitioning types into two groups, those lower than and higher than an arbitrarily given type, respectively, the lower-type group never subsidizes the higher-type group. Let $X(S) \subset X$ be the set of $\zeta(S)$ which are CPOs. Property (iv) justifies calling $\zeta(S) \in X(S)$ an *S-CPO*, the CPO preferred by the highest type in the set of mechanisms satisfying $U_s(\zeta) \geq V_s(s)$, all $s < S$. Properties (vi) and (vii) identify two situations in which Pareto optimality and uniqueness, respectively, of $\zeta(S)$ are satisfied. Under Property (vii), $\zeta(S)$ is also elementary, i.e., only ω-dependent.

Existence and uniqueness

Our main result is stated as follows.

Theorem 9 *Under Monotonicity and Sorting, every $\zeta(S)$ is an equilibrium allocation for every economy.*

Theorem 9 not only establishes that price cut immunity is nonvacuous, but also provides a way to compute an equilibrium via the sequence of s-th programming problems. Notice that, by Lemma 8.v, ζ^{RS} is a $\zeta(S)$ allocation whenever it is a strict CPO, thereby reconciling Theorem 9 with Lemma 7. At equilibrium, transfers across types, if any, occur within the active mechanism. Prices (of mechanisms) not only signal to agents which mechanism they should select, guaranteeing feasibility (budget balancedness) in the process, but also reflect such transfers in that the price paid by a contributing (receiving) type is zero, when it should have been negative (positive) at his odds.

So under Monotonicity and Sorting $\zeta(S)$ is an equilibrium. Why? To provide the intuition, here we avoid unnecessary details (dealt with in the Appendix). In particular, we suppose that utility inequalities presented in the argument are strict, when needed, and that we are dealing with mechanisms ζ satisfying $p_m(\zeta) \leq 0 < p_M(\zeta)$. In the Appendix, we show that there exists $p \in \mathbb{P}(\zeta(S))$ such that $p(\zeta) = 0$ for all such ζ that moreover have $U_s(\zeta) \leq U_s(\zeta(S))$ for all s. Let $(\zeta(S), p)$ be a weak equilibrium with this property.

Now proceed by contradiction, and assume that there exists an effective price cut (ζ', p_ϕ). We construct a mechanism $\bar{\zeta}$ such that: (i) $U_s(\bar{\zeta}) \leq U_s(\zeta(S))$ for all s; (ii) for each s, either $U_s(\bar{\zeta}) > U_s(\zeta')$ or $U_s(\bar{\zeta}) < U_s(\zeta')$; and (iii) $\sum_{s: U_s(\bar{\zeta}) > U_s(\zeta')} \pi_s \mathbb{E}_s(\bar{\zeta}) < 0$. Under (i)-(iii), it is $p(\bar{\zeta}) = 0$. Therefore, $\bar{\zeta}$ cannot be shut down at the price cut, since if it were open it would realize positive profits, by (iii). Equivalently, an effective price cut cannot exist.

Mechanism $\bar{\zeta}$ is identified by a pair of threshold types, $\hat{\sigma}$ and s_*, defined as follows:

- If $U_S(\zeta(S)) > U_S(\zeta')$, let $\hat{\sigma} = S$; if $U_S(\zeta(S)) < U_S(\zeta')$, let $\hat{\sigma}$ be such that $U_{\hat{\sigma}}(\zeta(S)) > U_{\hat{\sigma}}(\zeta')$ and $\zeta(S) = \zeta(\hat{\sigma})$ is an optimal solution to the $\hat{\sigma}$-th problem (that such a type exists is again shown in the Appendix).

- Let s_* be the highest type $s \leq \hat{\sigma}$ such that $U_s(\zeta(S)) < U_s(\zeta')$.

Now $\bar{\zeta}$ is defined as follows:

$$\bar{\zeta}_s = \begin{cases} \zeta_{s_*+1}(S) & \text{if } s \leq s_* \\ \zeta_s(S) & \text{if } s_* < s \leq \hat{\sigma} \\ \zeta_{\hat{\sigma}}(S) & \text{if } s > \hat{\sigma}. \end{cases}$$

By Sorting, $\bar{\zeta} \in X$ and $U_s(\bar{\zeta}) \leq U_s(\zeta(S))$ for all s. Types $s \leq s_*$ strictly prefer ζ' to $\bar{\zeta}$. Types s with $s_* < s \leq \hat{\sigma}$ strictly prefer $\bar{\zeta}$ to ζ'.

By the 'top-down' property, $\sum_{s=s_*+1}^{\hat{\sigma}} \pi_s \mathbb{E}_s(\zeta(S)) \leq 0$; by again the 'top-down' property and Monotonicity, $\mathbb{E}_s(\zeta_{\hat{\sigma}}(S)) \leq 0$ for all $s \geq \hat{\sigma}$. In fact, $\sum_{s=s_*+1}^{\hat{\sigma}} \pi_s \mathbb{E}_s(\zeta(S)) < 0$, whereas for types $s > \hat{\sigma}$, it is $\mathbb{E}_s(\bar{\zeta}) \leq 0$. Therefore, $\sum_{s:U_s(\bar{\zeta})>U_s(\zeta')} \pi_s \mathbb{E}_s(\bar{\zeta}) < 0$, concluding the argument.

For $S > 2$, we do not know if the equilibrium set coincides with $X(S)$. Instead, when $S = 2$ a stronger result holds: we can identify the equilibrium set with $X(S)$, so the equilibrium is payoff unique. This is because any incentive efficient allocation other than $\zeta(2)$ can be undercut by shutting down mechanisms that are preferred, relatively to $\zeta(2)$, by the lowest types.

Proposition 10 *When $S = 2$, ζ is an equilibrium allocation only if $\zeta = \zeta(2)$ for some $\zeta(2) \in X(2)$.*

Together with Lemma 8.v, Proposition 10 also shows that when $S = 2$ the equilibrium must be a CPO.

8 Extensions

Our results extend to similar setups of large adverse selection economies. In economies with essentially only one physical good the extension is obvious; this is the case, say, of Spence's (1973) labor market signaling model, or of Akerlof's (1976) 'rat race' analysis (as modeled for instance by Miyazaki (1977)). We illustrate this via two examples.

Labor market signaling

Consider the simplest such economy, where education does not affect workers' productivity. Here S represents the number of worker types, π_s the fraction of the type s workers in the population. Type s affects a worker's utility as well as his productivity, θ_s. Workers can supply inelastically one unit of labor yielding θ_s units of output. In addition, (prior to working) they can invest, without loss of generality, at most one unit of time in education. A 'net trade' is $z = (z_1, z_2)$, where $z_1 \geq 0$ is the salary received by the worker, and $z_2 \geq 0$ the level of education required. The worker's utility is $u(z, s) = v_s(z_1, 1 - z_2)$, and v_s is strictly increasing in its arguments. Moreover, it is assumed that single-crossing holds, and that θ_s is increasing in s.

In these economies, firms and workers match on the basis of wage-education mechanisms, and of their prices. The set X of mechanisms ζ is described as in Section 2 and so are utilities $U_s(\zeta)$. Once the map $\mathbb{E}_s(z)$ is replaced by $r_s(z) = z_1 - \theta_s$, and $\mathbb{E}_s(\zeta)$ by $r_s(\zeta) = \int r_s(z) d\zeta_s$, every definition carries verbatim: lotteries over X, feasibility, incentive efficiency, the price system and firms. For instance, feasibility is $\sum_s \pi_s \int_X r_s(\zeta) d\nu_s \leq 0$; the cost of running mechanism ζ is $r_\beta(\zeta) = \sum_s r_s(\zeta)\beta(s;\zeta)$; X^0 is defined with $r_s(\zeta) \leq 0$, all s, and the generalized RS contract has $r_1(\zeta^{RS}) = 0$. 'Full insurance' obviously coincides with the full information type-s optimal allocation,

$$\max_{\zeta \in \Delta(X)} U_s \zeta_s \quad \text{s.t. } r_s(\zeta) \leq 0,$$

here implying $z_2 = 0$ (zero education, or no signaling). Monotonicity is immediate from θ_s increasing in s, and Sorting applies unchanged. Whenever used, the pooling mechanism $-e$ is replaced with $z = (0, 0)$. Our Theorem 9 then goes through, and we obtain that, whenever constrained efficient, cross-subsidies arise at equilibrium in the market, in contrast with zero-subsidy separation typically obtained with other trading systems or equilibrium refinements.

The rat race

Here type s affects a worker's utility as well as his productivity, $f_s(\ell)$, where ℓ denotes the amount of work (or speed) provided. Workers choose labor supply as leisure affects their utility function. A 'net trade' is $z = (z_1, z_2)$, where $z_1 \geq 0$ is the salary received by the worker, and $z_2 = \ell \geq 0$ the amount of labor offered. The worker's utility again is $u(z, s) = v_s(z_1, 1 - z_2)$, assumed to satisfy standard conditions including single-crossing, and $f_s(\ell)$ is increasing in s, for all $\ell \in [0, 1]$. For these economies mechanisms represent menus of labor contracts, which are priced by the market system. Firms can be thought of as labor intermediaries. The set X and $U_s(\zeta)$, $\zeta \in X$, are described in the exact same way as in Section 2. The map $\mathbb{E}_s(z)$ is replaced now by $r_s(z) = z_1 - f_s(z_2)$. 'Full insurance' again coincides with the full information type-s optimal allocation. Again, Theorem 9 holds so that $\zeta(S)$ is the (unique, when $S = 2$) equilibrium allocation.

The extension is less obvious in economies with multiple goods such as Akerlof's (1970) lemon market, where sellers have private information and the physical good for sale is divisible, or in auctions environments, where buyers have private information and the good is indivisible. Now the firms' cost of running a mechanism depends on some input prices endogenously defining the terms of trade across goods. For such economies we only conjecture here that our price system works when immune from price cuts, and leave the formal proof to future research.

9 Discussion

We have established that a Walrasian price system can decentralize incentive efficient allocations if prices are immune from price cuts. As mentioned earlier, our result depends on the market system to price not only contracts, but mechanisms, and not only elementary, but random mechanisms. We conclude our analysis by briefly explaining why this is the case.

Let us call $MS(C)$ the market structure where trade is limited to just (lotteries over) individual contracts, while $MS(E)$ denotes the structure where trade is limited to just (lotteries over) elementary mechanisms. Finally, let MS denote our structure. The first observation is that the sets of feasible and incentive compatible allocations are equivalent in the three structures that therefore have identical CPO allocations. Equivalent here means that there exist isomorphisms across these sets preserving payoffs as well as resource consumption.

Proposition 11 *The sets of feasible and incentive compatible allocations are equivalent in the three structures.*

Notwithstanding this equivalence, the three market structures have different equilibrium outcomes, and $MS(C)$ and $MS(E)$ may fail to decentralize incentive efficient allocations. By the zero profit condition, at equilibrium $MS(E)$ does not allow transfers across mechanisms, failing therefore to achieve optimality when the latter calls for random transfers. Market structure $MS(C)$ suffers from the same problem, but in addition does not allow subsidies across types at equilibrium. To make this point as clear as possible we illustrate it with examples of $S = 2$ economies.

To study the decentralization failure of $MS(C)$ markets, consider an RS economy where $\zeta^{RS} = z^{RS}$ is not a CPO, the pooling CPO contract $\bar{z}^p$ dominates z^{RS}, and the S-CPO requires both separation of types and cross-subsidies, i.e., $\mathbb{E}_1(z(2)) > 0 > \mathbb{E}_2(z(2))$. Notice that such an economy has $z(2)$ as the unique equilibrium outcome under both market structures MS and $MS(E)$. However, $z(2)$ cannot be an equilibrium outcome of the $MS(C)$

market that, furthermore, does not decentralize any CPO allocation as equilibrium outcome.

Proposition 12 *The equilibrium allocation of such an economy with $MS(C)$ markets is not a CPO.*

We then come to the second issue: our price system evaluates *random* mechanisms. Are randomizations within a mechanism redundant when lotteries over elementary mechanisms can be chosen? Consider an $S = 2$ economy with type dependent utilities such that:

i) $\zeta(2)$ is such that $\mathbb{E}_1(\zeta(2)) > 0 > \mathbb{E}_2(\zeta(2))$ is a CPO; and

ii) $\zeta(2)$ is *essential*, that is, $(U_2(z), \mathbb{E}_2(z))$ is not constant $\zeta(2)$-a.e..

Notice that, by Proposition 10, $\zeta(2)$ is the unique equilibrium allocation for the MS market structure. Consider a candidate equilibrium allocation μ of the $MS(E)$ market. We show that μ cannot be a CPO. Let $\hat{z}(2)$ be the optimal solution to the S-th problem when X is replaced by $X^{MS(E)}$. It must be that $U_2(\mu) \geq U_2(\hat{z}(2))$, otherwise $\hat{z}(2)$ would constitute the basis for a price cut against μ. Furthermore, μ a CPO and $U_2(\mu) \geq U_2(\hat{z}(2))$ imply that μ_1 is degenerate, provides full insurance, and, by Assumption (i), $\mathbb{E}_1(\mu_1) > 0 > \mathbb{E}_2(\mu_2)$. Then, μ_2 is essential, otherwise $(U_2(z), \mathbb{E}_2(z))$ would be constant and $\mathbb{E}_2(z) < 0$, μ_2-a.e.. Then, since μ_1 provides full insurance, $\nu \equiv (\mu_1, \delta_z) \in Y^{MS(E)} \cap IC^{MS(E)}$ for some $z \in \Sigma(\mu_2)$, and ν Pareto dominates $\hat{z}(2)$, a contradiction. However, we show in Proposition 13 that such a μ is not price supportable in $MS(E)$ even if it is only a strict CPO, concluding the argument.

Proposition 13 *Let $S = 2$. Let $\tilde{\nu} \in \Delta'$ be a strict CPO such that its mean, $\tilde{\zeta} = \mathbb{E}\tilde{\nu}$, has $\tilde{\zeta}_2$ essential, provides full insurance to type 1, and $\mathbb{E}_1(\tilde{\zeta}) > 0$. Then, $\tilde{\nu}$ is not price supportable in $MS(E)$.*

10 Appendix

Auxiliary definitions and lemmas. Hereafter, we use the following certainty equivalent notion. Given a lottery $\mu \in \Delta(X)$ and its corresponding mean where $\mathbb{E}(\mu) \in X$, for each s we define the *ω-certainty equivalent* $\hat{x}_s(\omega) \in \mathbb{R}_+$ as the solution to $\int v_s(z_\omega + e_\omega) d\mathbb{E}(\mu)_{\omega,s} = v_s(\hat{x}_s(\omega))$, where $\mathbb{E}(\mu)_{\omega,s}$ is the marginal distribution of $\mathbb{E}(\mu)_s$ given ω; again, we let $\hat{z}_s$ be the corresponding net trade, i.e., $\hat{z}_{\omega,s} = \hat{x}_s(\omega) - e_\omega$, all ω.

We list the relevant properties of strict CPO and incentive compatible allocations as auxiliary lemmas whose proof is left to the reader.

The first auxiliary result states that resources are exhausted and some types are fully insured at a strict CPO allocation. Recall that a CPO allocation is a strict CPO.

Auxiliary Lemma 1 *Let ζ be a strict CPO, then $\zeta \in \bar{Y}$. Moreover, ζ provides full insurance for some type s.*

The next auxiliary lemma is concerned with incentive constraints that do not bind, under Sorting.

Auxiliary Lemma 2 *Let ζ and ζ' be elements of $\Delta(K)$. Under Sorting, if $U_{s+1}(\zeta') > U_{s+1}(\zeta)$ and $U_s(\zeta) \geq U_s(\zeta')$ for some s, then $U_j(\zeta') > U_j(\zeta)$ for all $j \geq s+1$; similarly, if $U_{s+1}(\zeta') \geq U_{s+1}(\zeta)$ and $U_s(\zeta) > U_s(\zeta')$ for some s, then $U_j(\zeta) > U_j(\zeta')$ for all $j \leq s$.*

In the analysis we also repeatedly take advantage of the order structure of incentive compatible allocations, with Sorting. For any two mechanisms ζ and ζ' in X, and for a subset S' of S, let $\zeta \vee_{S'} \zeta'$ be the mechanism defined as

$$(\zeta \vee_{S'} \zeta')_s = \begin{cases} \zeta'_s & \text{if } s \in S', \\ \zeta_s & \text{otherwise.} \end{cases}$$

Auxiliary Lemma 3 *Let $S' = \{s : s_* \leq s < s^*\}$ for $s_* \leq s^* \leq S$, and $\zeta, \zeta' \in X$ be such that $U_{s^*}(\zeta) \geq U_{s^*}(\zeta')$ and $U_{s_*}(\zeta) \geq U_{s_*}(\zeta')$, while $U_{s^*-1}(\zeta') \geq U_{s^*-1}(\zeta)$ and $U_{s_*+1}(\zeta') \geq U_{s_*+1}(\zeta)$. Then, under Sorting, $\zeta \vee_{S'} \zeta' \in X$.*

To prove Theorem 9, we will need to focus on a particular supporting price which we call *-supporting. A price q is **-supporting* for $\zeta^* \in \Delta' \cap \bar{Y}$ if $q \in \mathbb{P}(\zeta^*)$ and

$$q(\zeta) = \min[0, p_M(\zeta)], \text{ for } \zeta \in Z_-(\zeta^*),$$

where

$$Z_-(\zeta^*) = \{\zeta : U_s(\zeta) \leq U_s(\zeta^*) \text{ for all } s, \text{ and } p_m(\zeta) \leq 0\}.$$

For weak equilibria, *-supporting prices always exist.

Auxiliary Lemma 4 *Let $\zeta^* \in \Delta' \cap \bar{Y}$. Then, there exists a *-supporting price of ζ^*.*

Proof: Let q be the map $q(\zeta) = \max[\bar{p}(\zeta), \min(0, p_M(\zeta))]$, where $\bar{p}(\zeta)$ is the price function constructed in Proposition 4. Then q is continuous. It is also belief-based since if $q(\zeta) = \bar{p}(\zeta)$, then $q(\zeta) \in [p_m(\zeta), p_M(\zeta)]$, by construction of $\bar{p}$; while if $q(\zeta) = \min[0, p_M(\zeta)] \geq \bar{p}(\zeta) \geq p_m(\zeta)$ and then again $q(\zeta) \in [p_m(\zeta), p_M(\zeta)]$. At ζ^*, we have $p_M(\zeta^*) \geq 0$, while $p_m(\zeta^*) \leq 0$ and $\bar{p}(\zeta^*) = 0$, since $\zeta^* \in \bar{Y}$. Thus, $q(\zeta^*) = 0$, that is, $q \in \mathbb{P}(\zeta^*)$. Finally, let $\zeta \in Z_-(\zeta^*)$; then $\bar{p}(\zeta) \leq 0$, and therefore $q(\zeta) = \min[0, p_M(\zeta)]$. Thus, q is a *-supporting price of ζ^*.■

Preliminary properties. Proof of Lemma 3: X^0 is a nonempty, closed subset of the compact set X, while U_s are continuous maps. Therefore, X^{RS} is nonempty. By construction $\mathbb{E}_s(\zeta^{RS}) \leq 0$, all s. Thus, in order to show that $X^{RS} \subset X^0$ it suffices to show that $U_s(\zeta^{RS}) \geq U_s(\zeta^{RS}_{s'})$ for $s \neq s'$ and $\zeta^{RS} \in X^{RS}$. By definition, $\zeta^{RS}_s = \zeta^*_s(s)$, where $\zeta^*(s) \in \arg\max_{\zeta \in X^0} U_s(\zeta)$. Thus, $U_s(\zeta^{RS}) = U_s(\zeta^*(s)) \geq U_s(\zeta^*(s')) \geq U_s(\zeta^*_{s'}(s'))$, for all s, s'.■

Proof of Proposition 4: The following is an obvious but important preliminary observation.

($\diamondsuit$) For given s, let ν' be the solution to the type-s agent's optimization problem at prices p', and let ν be the solution at p. Then, if $p \leq p'$, $U_s\nu \geq U_s\nu'$. (Therefore, if ν is budget feasible at p', $\nu = \nu'$.)

[Indeed, since $p \leq p'$, $\int p d\nu' \leq \int p' d\nu' = 0 = \int p d\nu$. Hence, ν' is budget feasible at p, but ν is chosen, whence the claim, by revealed preference, proving ($\diamondsuit$).]

We are now ready to prove the proposition. By Lemma 2, we just need to show that $\mathbb{P}(\zeta^*) \neq \varnothing$. We first construct prices $\bar{p}_s \in \mathbb{P}$ supporting ζ^* for type s, and then show that $\bar{p} = \max_s \bar{p}_s$ supports ζ^*.

We claim that $\zeta^{RS} \in X^{RS}$ is an optimal solution to the individual programming problem at p_M for all types s. First, $p_M(\zeta^{RS}) = 0$ for any $\zeta^{RS} \in X^{RS}$, by Lemma 3. Second, by the convexity of the map $\max_s[\int \mathbb{E}_s z d\zeta]$ and Jensen's inequality:

$$p_M\nu = \int p_M(\zeta) d\nu = \int [\max_s \int \mathbb{E}_s z d\zeta] d\nu \geq \max_s \int \mathbb{E}_s z d\mathbb{E}\nu = p_M \delta_{\mathbb{E}\nu}.$$

Therefore, without loss of generality, for each s, the optimal solution to the individual programming problem at p_M is a mechanism $\bar{\zeta}(s) \in X$ with $p_M(\bar{\zeta}(s)) = 0$. However, $p_M(\bar{\zeta}(s)) = 0$ implies $\mathbb{E}_{s'}(\bar{\zeta}(s)) \leq 0$ for all s'. Therefore, $\bar{\zeta}(s) \in X^0$ and, by definition, $\bar{\zeta}(s) = \zeta^{RS}$.

Take any s; from Luenberger (1969, Theorem 1, p. 249, combined with Theorem 2, p. 178), the first order conditions for optimality guarantee that there exists $\lambda_s > 0$ such that

$$U_s(\zeta) - U_s(\zeta^{RS}) \leq \lambda_s p_M(\zeta) \text{ for all } \zeta \in X$$

and, since $U_s(\zeta^*) \geq U_s(\zeta^{RS})$, it is

$$U_s(\zeta) - U_s(\zeta^*) \leq \lambda_s p_M(\zeta) \text{ all } \zeta \in X. \tag{2}$$

Consider the set of contracts Z_{*s} defined as

$$Z_{*s} = \{\zeta' : U_s(\zeta') - U_s(\zeta^*) \geq \lambda_s p_m(\zeta')\}.$$

Since ζ^* is feasible, there must exist s' such that $\mathbb{E}_{s'}(\zeta^*) \leq 0$; then $p_m(\zeta^*) \leq 0$, so $\zeta^* \in Z_{*s}$. Furthermore, by the continuity of all the functions involved, Z_{*s} is closed. Define the price functional $\bar{p}_s$ as

$$\bar{p}_s(\zeta) = \max\{\frac{U_s(\zeta) - U_s(\zeta^*)}{\lambda_s}, p_m(\zeta)\}.$$

Then, $\bar{p}_s$ satisfies three key properties. First, $\bar{p}_s$ is the maximum of two continuous functions and it is therefore continuous. Second, $\bar{p}_s$ is belief-based or, equivalently, $\bar{p}_s \in \mathbb{P}$, since it satisfies the following inequalities:

$$p_m(\zeta) \leq \frac{U_s(\zeta) - U_s(\zeta^*)}{\lambda_s} = \bar{p}_s(\zeta) \leq p_M(\zeta), \text{ for } \zeta \in Z_{*s},$$

where we used (2) for the last inequality, while

$$p_M(\zeta) \geq \bar{p}_s(\zeta) = p_m(\zeta) > \frac{U_s(\zeta) - U_s(\zeta^*)}{\lambda_s}, \text{ for } \zeta \in X \backslash Z_{*s}.$$

Third, $\bar{p}_s$ supports ζ^* for type s individuals. Indeed, the price functional $\frac{U_s(\cdot) - U_s(\zeta^*)}{\lambda_s}$ supports ζ^* for type s individuals. The contract ζ^* is budget feasible at $\bar{p}_s$, that is, $\bar{p}_s(\zeta^*) = 0$; and $\bar{p}_s > \frac{U_s(\cdot) - U_s(\zeta^*)}{\lambda_s}$. Thus, the claim follows by ($\diamondsuit$).

Now let $Z_* = \cup_s Z_{*,s}$, a closed set. Define $\bar{p}$ as in the statement of the proposition. Then $\bar{p}(\zeta) = \max_s \bar{p}_s(\zeta)$, and it is therefore continuous. It is $\bar{p} \geq \bar{p}_s$, all s, and thus, by ($\diamondsuit$), $\bar{p}$ supports ζ^*. Finally since $p_m \leq \bar{p} \leq p_M$, it is $\bar{p} \in \mathbb{P}$.

Suppose now that μ is a non degenerate weak equilibrium allocation. Then, it is $U_s(\mu) \geq U_s(\zeta^{RS})$, and $\mu \in \Delta' \cap \bar{Y}$. Therefore, the mean of μ, $\mathbb{E}(\mu)$, is payoff equivalent and it is also an element of $\Delta' \cap \bar{Y}$. Importantly, it is degenerate, as desired.■

Incentive efficiency. Proof of Theorem 6: The proof of the theorem is based on the following observation:

(✠) $\forall \zeta \in Y$ and not a strict CPO, $\exists\, \zeta' \in Y$ such that $U_s(\zeta') > U_s(\zeta)$ all s, and $\sum_s \pi_s \mathbb{E}_s(\zeta') < 0$.

[To see this, observe that since ζ is not a strict CPO there exists $\zeta'' \in Y$ such that $U_s(\zeta'') > U_s(\zeta)$, all s. Let ζ' be the mechanism assigning probability α to ζ'' and $(1-\alpha)$ to the pooling mechanism $z = -e$. Now $\zeta' \in Y$, and $\sum_s \pi_s \mathbb{E}_s(\zeta') < \sum_s \pi_s \mathbb{E}_s(\zeta) \leq 0$, for all α. Moreover, for α close enough to 1, $U_s(\zeta') > U_s(\zeta)$, all s, proving (✠).]

Now let (ζ^*, p) be an equilibrium, and suppose otherwise, ζ^* is not a strict CPO. Because it is an equilibrium, $\zeta^* \in Y$, so let ζ' be the contract defined in (✠). First, observe that $p(\zeta') > 0$, otherwise we have an immediate contradiction with $U_s(\zeta^*) < U_s(\zeta')$ all s. Let (ζ', p_ϕ) be a price cut, and η the associated trading map, with $\eta(\zeta) = 1$, for all $\zeta \in X$. We claim that at (p_ϕ, η), ζ' is an optimal solution to the individual programming problem, for all s. For suppose not. Pick an s and let μ_s be an optimal solution with $U_s \mu_s > U_s(\zeta')$ and $p_\phi \mu_s = 0$. Then, by the optimality conditions and by the fact that $p_\phi(\zeta') = 0$, it is $\mu_s(\zeta') = 0$. Therefore, μ_s is also budget feasible at p (not just at p_ϕ), while $U_s \mu_s > U_s(\zeta^*)$, violating the definition of ζ^*, a contradiction.

Then, with η defined above, the array (ζ', p_ϕ, η) is a quasi equilibrium: by construction, ζ' is optimal at prices p_ϕ for all s, or $\zeta' \in M(p_\phi, \eta)$. Since $\eta = 1$ everywhere, η is obviously maximal, if it is viable. To show viability, $\Pi(\zeta; (\zeta', p_\phi, \eta)) = 0$ for all $\zeta \neq \zeta'$. Next, $p_\phi(\zeta') = 0$ and ζ' optimal for all s imply that $\Pi(\zeta'; (\zeta', p_\phi, \eta)) = 0 - \mathbb{E}_\pi(\zeta') > 0$, by the strict feasibility of ζ' —from (✠). Hence, (ζ', p_ϕ) is an effective price cut. This contradicts the fact that (ζ^*, p) was an equilibrium.■

Existence of equilibrium. **Proof of Lemma 7**: Since X^{RS} is contained in the set of strict CPO and, by definition, $\mathbb{E}_s(\zeta^{RS}) \leq 0$ for all $s \in S$, it is, by Auxiliary Lemma 1, $\mathbb{E}_s(\zeta^{RS}) = 0$ for all $s \in S$ and all $\zeta^{RS} \in X^{RS}$. First we show that if ζ^{RS} is a strict CPO, then it is an equilibrium allocation; and then that if $X^{RS} \subset CPO$, any mechanism $\zeta' \notin X^{RS}$ cannot be an equilibrium allocation.

Fix a $\zeta^{RS} \in X^{RS}$ and let $p \in \mathbb{P}(\zeta^{RS})$ be its supporting price, which exists by Proposition 4. By contradiction, suppose that there exists an effective price cut (ζ', p_ϕ) and let η be the market indicator so that (ζ', p_ϕ, η) is a quasi equilibrium. By Lemma 5, $\sum_s \pi_s \mathbb{E}_s(\zeta') < 0$. Hence, by Auxiliary Lemma 1, ζ' is not a strict CPO and therefore $U_s(\zeta') < U_s(\zeta^{RS})$ for some $s \in S$. Thus, $p(\zeta^{RS}) = p_\phi(\zeta^{RS}) = 0$ implies that $\eta(\zeta^{RS}) = 0$, or ζ' would not be optimal at (p_ϕ, η) for such s. Define $\eta' > \eta$ as $\eta'(\zeta) = \eta(\zeta)$ for all $\zeta \neq \zeta^{RS}$, while $\eta'(\zeta^{RS}) = 1$. Then, $\nu'_s \in M_s(p_\phi, \eta')$ if and only if $\nu'_s = \delta_{\zeta^{RS}}$, for all $s \in \{s' : U_{s'}(\zeta') < U_{s'}(\zeta^{RS})\}$. Therefore, since $\mathbb{E}_s(\zeta^{RS}) = 0$ for all $s \in S$, it is $\Pi(\zeta^{RS}; (\nu', p_\phi)) = 0$ for all $\nu' \in M(p_\phi, \eta')$, violating the required maximality of η, a contradiction that ends the proof.

Now suppose, again by contradiction that $\zeta' \notin X^{RS}$, but ζ' is an equilibrium allocation for some $p \in \mathbb{P}$. Since ζ^{RS} is a CPO, by Auxiliary Lemma 1, $\mathbb{E}_s(\zeta^{RS}) = 0$, for all s and $\zeta^{RS} \in X^{RS}$. Since ζ' is an equilibrium allocation, $U_s(\zeta') \geq U_s(\zeta^{RS})$ for all s. Therefore, $U_s(\zeta') = U_s(\zeta^{RS})$ for all s. Then, $\zeta' \notin X^{RS}$ means that $\mathbb{E}_\sigma(\zeta') > 0$, for some $\sigma \in S$. Let σ^* be one of them. Let $\zeta'' = \zeta' \vee_{\sigma^*} \zeta^{RS}$. The allocation ζ'' is incentive compatible because, for all $s \neq \sigma^*$, $U_s(\zeta'') = U_s(\zeta') \geq U_s(\zeta'_{\sigma^*}) = U_s(\zeta''_{\sigma^*})$, while $U_{\sigma^*}(\zeta'') = U_{\sigma^*}(\zeta^{RS}) \geq U_{\sigma^*}(\zeta^{RS}_s) = U_{\sigma^*}(\zeta''_s)$ for all s. However, $\sum_s \pi_s \mathbb{E}_s(\zeta'') < \sum_s \pi_s \mathbb{E}_s(\zeta') = \sum_s \pi_s \mathbb{E}_s(\zeta^{RS}) = 0$. The latter contradicts $X^{RS} \subset CPO$, by Auxiliary Lemma 1, thereby concluding the argument. ■

Proof of Lemma 8

Assuming a solution exists, we first prove (ii), (iii), (v), and then (i). Finally, we prove (vi) and (vii). The proof of (iv) is trivial, and therefore omitted.

(ii) Let $\zeta(s)$ solving the s-th problem be given. First, we show that $\sum_{s' \leq s} \pi_{s'} \mathbb{E}_{s'}(\zeta(s)) = 0$. For suppose not, and $\sum_{s' \leq s} \pi_{s'} \mathbb{E}_{s'}(\zeta(s)) < 0$. Let $\bar{z}$ be the pooling contract dominating any mechanism in Y. For some $\varepsilon > 0$, the mean $\mathbb{E}\nu_\varepsilon$ of the lottery assigning $\bar{z}$ with probability ε and $\zeta_{s'}(s)$ with probability $(1 - \varepsilon)$ to $s' \leq s$ has $\sum_{s' \leq s} \pi_{s'} \mathbb{E}_{s'}(\mathbb{E}\nu_\varepsilon) < 0$, $\mathbb{E}\nu_\varepsilon \in X$ and $U_{s'}(\mathbb{E}\nu_\varepsilon) > U_{s'}(\zeta(s))$ all $s' \leq s$, thereby contradicting $\zeta(s)$ optimal in the

s-th problem.

Again by contradiction, suppose now that $\sum_{s'=\sigma+1}^{s} \pi_{s'}\mathbb{E}_{s'}(\zeta(s)) > 0$ for some $\sigma < s$. Since $\sum_{s'\leq s} \pi_{s'}\mathbb{E}_{s'}(\zeta(s)) = 0$, it is $\sum_{s'\leq\sigma} \pi_{s'}\mathbb{E}_{s'}(\zeta(s)) < 0$. By the definition of $\zeta(s)$ it is $U_{s'}(\zeta(s)) \geq V_{s'}(s')$, all $s' \leq \sigma$. Then, $\zeta(s)$ belongs to the constraint set of the σ-problem and $U_\sigma(\zeta(s)) \geq V_\sigma(\sigma)$, but $\sum_{s'\leq\sigma} \pi_{s'}\mathbb{E}_{s'}(\zeta(s)) < 0$, contradicting the first part of the claim.□

(iii) We are going to show that if there exists an optimal solution $\zeta(s)$ to the s-th problem, then $V_\sigma(\sigma) \geq U_\sigma(\zeta_\sigma^{RS})$ for all $\sigma \leq s$, which together with (ii) implies (iii) for $s = S$.

Argue by induction. The statement is obviously true for $s = 2$ as ζ^{RS} is an element of the constraint set. Suppose that it is true for $s - 1 > 1$, we show that it holds true for s. Since by assumption $\zeta(s)$ exists, by the form of the constraint set of the s-th problem and by the inductive assumption we just need to show that $V_s(s) \geq U_s(\zeta^{RS})$. Equivalently, it suffices to show that there exists a point in the constraint set that satisfies such inequality. Let $S' = \{\sigma : \sigma < s\}$ and consider the two allocations $\hat{\zeta}^1(s) = (\bar{\zeta} \vee_{S'} \zeta(s-1))$, with $\bar{\zeta}_{s'} = \zeta_{s-1}(s-1)$, all s', and $\hat{\zeta}^2(s) = (\zeta^{RS} \vee_{S'} \zeta(s-1))$. Clearly, $U_\sigma(\hat{\zeta}^\kappa_\sigma(s)) \geq V_\sigma(\sigma)$ all $\sigma < s$. By part (ii) and $\hat{\zeta}^\kappa_\sigma(s) = \zeta_\sigma(s-1)$ all $\sigma < s$, it is $\sum_{\sigma<s} \pi_\sigma\mathbb{E}_\sigma(\hat{\zeta}^\kappa_\sigma(s)) = 0$, $\kappa = 1, 2$. Furthermore, by Auxiliary Lemma 3, $\hat{\zeta}^1(s) \in X$. We show that $\sum_{\sigma\leq s} \pi_\sigma\mathbb{E}_\sigma(\hat{\zeta}^1(s)) \leq 0$. By (ii), $\mathbb{E}_{s-1}(\zeta(s-1)) \leq 0$, and by the inductive assumption $U_{s-1}(\hat{\zeta}^1_{s-1}(s)) = V_{s-1}(s-1) \geq U_{s-1}(\zeta^{RS})$. Thus, Monotonicity implies $\mathbb{E}_s(\hat{\zeta}^1(s-1)) \leq 0$. Hence, it is $\sum_{\sigma\leq s} \pi_\sigma\mathbb{E}_\sigma(\hat{\zeta}^1(s)) \leq 0$ as claimed.

Instead, since $\mathbb{E}_s(\zeta_s^{RS}) \leq 0$, it is $\sum_{\sigma\leq s} \pi_\sigma\mathbb{E}_\sigma(\hat{\zeta}^2(s)) \leq 0$.

Now there are two possibilities, either $U_s(\hat{\zeta}^1(s)) = U_s(\zeta_{s-1}(s-1)) \geq U_s(\zeta^{RS})$, or vice versa. If the inequality holds true, the argument is over. Otherwise, notice that since by the inductive assumption $U_\sigma(\zeta(s-1)) \geq V_\sigma(\sigma) \geq U_\sigma(\zeta^{RS})$, all $\sigma < s$, and since $U_s(\zeta_s^{RS}) > U_s(\zeta_\sigma(s-1))$, by Auxiliary Lemma 3, $\hat{\zeta}^2(s) \in X$, once again concluding the argument.□

(v) Suppose otherwise and pick any $\zeta(S)$, solution to the the S-th problem. Thus, it must be $U_s(\zeta(S)) \geq U_s(\zeta^{RS})$ for all s, by property (iii), and $U_S(\zeta(S)) > U_S(\zeta^{RS})$. Let $s_* = \max\{s : U_s(\zeta^{RS}) = U_s(\zeta(S))\}$. Since ζ^{RS} is a strict CPO such an s_* exists and by the contradiction assumption $s_* < S$. Let $S' = \{s : s > s_*\}$. By Auxiliary Lemma 3, the mechanism $\zeta' \equiv (\zeta^{RS} \vee_{S'} \zeta(S))$ is in X. By the definition of ζ^{RS} and by property (ii), $\zeta' \in Y$. Moreover, for

$s > s_*$ and $\sigma \leq s_*$, it is

$$U_s(\zeta') = U_s(\zeta(S)) > U_s(\zeta^{RS}) > U_s(\zeta_\sigma^{RS}) = U_s(\zeta'_\sigma).$$

Consider the mechanism ζ^μ, $\mu \in (0,1)$ defined as follows:

for $s \in S'$, ζ_s^μ assigns $-e$ with probability μ, and ζ'_s with probability $(1-\mu)$;

for $s \notin S'$, $\zeta_s^\mu = \zeta'_s = \zeta_s^{RS}$.

For each $\mu \in (0,1)$, it is $\sum_s \pi_\sigma \mathbb{E}_\sigma(\zeta^\mu) < 0$. For μ arbitrarily close to 0, by the inequality above it is $\zeta^\mu \in X$, and $U_s(\zeta^\mu) \geq U_s(\zeta^{RS})$ for all s. Thus, ζ^μ is a strict CPO, contradicting Auxiliary Lemma 1.$\square$

(i) Notice that it suffices to show that the constraint set is nonempty, as continuity of the objective and weak*-compactness of the constraint set are obvious. By induction, again the statement is obviously true for $s = 2$. Suppose that it is true for $s - 1 > 1$, we show that it holds true for s. As in (iii), consider $\hat{\zeta}^\kappa(s)$, $\kappa = 1, 2$. Both $\hat{\zeta}^\kappa(s)$ satisfy the resource constraints and at least one of them is in the set X, thereby concluding the argument.

(vi) Since any $\zeta(S)$ is a strict CPO, we just have to rule out that $U_1(\zeta(S)) > U_1(\zeta'(S))$ for $\zeta(S)$ and $\zeta'(S)$, two distinct S-CPOs. Argue by contradiction. By Auxiliary Lemma 1, and the assumption that $\mathbb{E}_s(e)$ is increasing in s, type 1 must always be fully insured; then, it is $\zeta_1(S) = \hat{z}_1 > \hat{z}'_1 = \zeta'_1(S)$. Now consider the mean $\mathbb{E}\nu_{1/2}$ of the lottery assigning probability 1/2 to $\zeta(S)$ and 1/2 to $\zeta'(S)$. Obviously, $\mathbb{E}\nu_{1/2}$ is also a solution, but type 1 is not fully insured, a contradiction.

(vii) Suppose not and let $\zeta(S), \zeta'(S)$ be two solutions. As in part (vi), mechanism $\mathbb{E}\nu_{1/2}$ corresponding to the lottery $\nu_{1/2}$ assigning both $\zeta(S)$ and $\zeta'(S)$ with probability 1/2 is a solution as well. The ω-certainty equivalent of $\mathbb{E}\nu_{1/2}$, $\hat{z}$, has $\sum \pi_s \mathbb{E}_s(\hat{z}) < 0$, while by construction $U_s(\hat{z}_{s'}) = U_s(\zeta_{s'}(S))$ all s and s'. Thus, $\hat{z}$ also is a solution to the S-problem, but $\sum \pi_s \mathbb{E}_s(\hat{z}) < 0$, contradicting Auxiliary Lemma 1.$\blacksquare$

Proof of Theorem 9: We show that there cannot be any effective price cut against $(\zeta(S), p)$, where p is a *-supporting price for $\zeta(S)$, which exists by Auxiliary Lemma 4. The argument is by contradiction and is divided into steps. Suppose otherwise, and let (ζ', ϕ) be an effective price cut against the weak equilibrium $(\zeta(S), p)$, with η denoting the associated market indicator. By Lemma 5, $\phi(\zeta') = p_\phi(\zeta') = 0 \leq p(\zeta')$, and it is $\sum_s \pi_s \mathbb{E}_s(\zeta') < 0$ and $U_s(\zeta') \geq U_s(\zeta(S))$ for some s. The first step shows that the price cut must attract the highest type.

Step 1 If (ζ', p_ϕ) is an effective price cut against $\zeta(S)$, then $U_S(\zeta') \geq U_S(\zeta(S))$.

Proof. Suppose otherwise and let $s^* - 1 < S$ be the highest type satisfying $U_{s^*-1}(\zeta') \geq U_{s^*-1}(\zeta(S))$. Then, $U_{s^*}(\zeta_{s^*}(S)) > U_{s^*}(\zeta')$, while $U_{s^*-1}(\zeta(S)) \leq U_{s^*-1}(\zeta')$. Let $\hat{\zeta}$ be the pooling mechanism $\hat{\zeta}_s = \zeta_{s^*}(S)$ for all s, and let for $S' = \{s : s < s^*\}$. Notice that, by Auxiliary Lemma 3, mechanism $\bar{\zeta} = \zeta(S) \vee_{S'} \hat{\zeta}$ satisfies $\bar{\zeta} \in X$. By construction, $U_s(\bar{\zeta}) \leq U_s(\zeta(S))$ for all s.

Now consider the mean $\mathbb{E}\nu_\varepsilon$ of the lottery ν_ε assigning $\bar{\zeta}$ with probability ε and the pooling mechanism $z = -e$ with probability $(1-\varepsilon)$. For ε close enough to one, it is $U_s(\mathbb{E}\nu_\varepsilon) > U_s(\zeta')$ all $s \geq s^*$, while $U_{s^*-1}(\mathbb{E}\nu_\varepsilon) < U_{s^*-1}(\zeta')$. Then, by Auxiliary Lemma 2, $U_s(\zeta'_s) \geq U_s(\zeta_{s^*}(S)) > U_s(\mathbb{E}\nu_\varepsilon)$ for all $s < s^*$. Furthermore, by construction of $\mathbb{E}\nu_\varepsilon$, both $U_s(\mathbb{E}\nu_\varepsilon) < U_s(\zeta(S))$ all s and $\mathbb{E}_S(\mathbb{E}\nu_\varepsilon) < \mathbb{E}_S(\zeta(S)) \leq 0$.

Since p is a $*$-supporting price, it is $p_\phi(\mathbb{E}\nu_\varepsilon) = p(\mathbb{E}\nu_\varepsilon) = \min[0, p_M(\mathbb{E}\nu_\varepsilon)] \leq 0$. Since $U_s(\mathbb{E}\nu_\varepsilon) > U_s(\zeta')$ for some $s \geq s^*$ (e.g., S), it must be that $\eta(\mathbb{E}\nu_\varepsilon) = 0$. Define $\eta' > \eta$ as $\eta'(\zeta) = \eta(\zeta)$ for all $\zeta \neq \mathbb{E}\nu_\varepsilon$, while $\eta'(\mathbb{E}\nu_\varepsilon) = 1$. If $\min[0, p_M(\mathbb{E}\nu_\varepsilon)] = 0$, then $p_\phi(\mathbb{E}\nu_\varepsilon) = 0$ and $U_s(\mathbb{E}\nu_\varepsilon) < U_s(\zeta')$ imply that $\mathbb{E}\nu_\varepsilon \notin \Sigma(\nu_s)$, for all $\nu_s \in M_s(p_\phi, \eta')$ and for all $s < s^*$. By Lemma 8.ii, $\sum_{s \geq s^*} \pi_s \mathbb{E}_s(\bar{\zeta}) \leq 0$. Then,

$$\Pi(\mathbb{E}\nu_\varepsilon; (\bar{\nu}', p_\phi, \eta')) = 0 - \varepsilon \sum_{s \geq s^*} \frac{\pi_s}{\sum_{s' \geq \sigma} \pi_{s'}} \mathbb{E}_s(\bar{\zeta}) + (1-\varepsilon) \sum_{s \geq s^*} \frac{\pi_s}{\sum_{s' \geq \sigma} \pi_{s'}} \mathbb{E}_s(e) \geq 0,$$

contradicting the maximality of η. If $\min[0, p_M(\mathbb{E}\nu_\varepsilon)] = p_M(\mathbb{E}\nu_\varepsilon) < 0$, then for each $\nu \in \times_s M_s(p_\phi, \eta')$, it is $\nu_s(\mathbb{E}\nu_\varepsilon) > 0$ for $s \geq s^*$, and therefore:

$$\Pi(\mathbb{E}\nu_\varepsilon; (\bar{\nu}', p_\phi, \eta')) = p_M(\mathbb{E}\nu_\varepsilon) - \sum_s \frac{\pi_s \nu_s(\mathbb{E}\nu_\varepsilon)}{\sum_{s'} \pi_{s'} \nu_{s'}(\mathbb{E}\nu_\varepsilon)} \mathbb{E}_s(\mathbb{E}\nu_\varepsilon) \geq 0,$$

concluding the argument.□

The first step ends the proof when $S = 2$. Indeed, it must be $\zeta' \in \Delta'$, and then $U_S(\zeta') \geq U_S(\zeta(S))$ and $U_1(\zeta') \geq U_1(\zeta^{RS})$ imply necessarily that ζ' is also a Pareto optimal S-CPO. However, as in Lemma 8.vi, it cannot be $U_1(\zeta(S)) \neq U_1(\zeta')$, so ζ' cannot exist.

The second step shows that there must be a type $\hat{\sigma}$ whose 'reservation utility' is guaranteed by $\zeta(S)$, and at the price cut is making less than his 'reservation utility'. Formally, let

$$\hat{S} = \{s : U_s(\zeta') < V_s(s)\}.$$

Step 2 Let (ζ', ϕ) be an effective price cut against $\zeta(S)$. Then, there exists $\hat{\sigma} \in \hat{S}$ such that $V_{\hat{\sigma}}(S) = V_{\hat{\sigma}}(\hat{\sigma})$.

Proof. First, since by Step 1 $U_S(\zeta') \geq U_S(\zeta(S))$, it must be that $\hat{S} \neq \varnothing$, otherwise ζ' is an element of the constraint set of the S-problem, delivers a value greater than or equal to $V_S(S)$, but $\sum_s \pi_s \mathbb{E}_s(\zeta') < 0$, a contradiction to Lemma 8.ii. Now suppose otherwise: $V_s(S) > V_s(s)$ for all $s \in \hat{S}$ –since by the definition of the S-problem $V_s(S) \geq V_s(s)$ for all such s. Consider the mean $\mathbb{E}\nu_\varepsilon$ of the lottery ν_ε assigning ζ' with probability $\varepsilon > 0$ and $\zeta(S)$ with probability $(1-\varepsilon)$. For all $\varepsilon \in (0,1)$, it is $\mathbb{E}\nu_\varepsilon \in X$, and $\sum_s \pi_s \mathbb{E}_s(\mathbb{E}\nu_\varepsilon) < 0$. Also, $U_s(\mathbb{E}\nu_\varepsilon) \geq V_s(s)$ for all $s \notin \hat{S}$. For ε close enough to 0, it is as well $U_s(\mathbb{E}\nu_\varepsilon) \geq V_s(s)$ for all $s \in \hat{S}$. Hence, there exists ε' so that mechanism $\mathbb{E}\nu_{\varepsilon'}$ is an element of the constraint set of the S-th problem, delivers a value greater than or equal to $V_S(S)$, but $\sum_s \pi_s \mathbb{E}_s(\mathbb{E}\nu_{\varepsilon'}) < 0$, a contradiction to Lemma 8.ii. Thus, $V_{\hat{\sigma}}(\hat{\sigma}) = V_{\hat{\sigma}}(S)$ for some $\hat{\sigma} \in \hat{S}$.□

When $\sigma < S$, the optimal solutions $\zeta(\sigma)$ to the σ-problems pin down only the first σ components $\zeta_s(\sigma)$, $s \leq \sigma$, while they just restrict the remaining $S - \sigma$ components in order to guarantee $\zeta(\sigma) \in X$. Hereafter, we identify the last $S - \sigma$ components of $\zeta(\sigma)$ as $\zeta_s(\sigma) = \zeta_\sigma(\sigma)$, for all $s \geq \sigma$.

In the next step we show that the highest type is not transferring resources below $\hat{\sigma}$; in fact, this type is using the same aggregate resources as $\hat{\sigma}$ would use.

Step 3 If σ is such that $V_\sigma(S) = V_\sigma(\sigma)$, then:

i) $\sum_{s \leq \sigma} \pi_s \mathbb{E}_s(\zeta(\sigma)) = \sum_{s \leq \sigma} \pi_s \mathbb{E}_s(\zeta(S)) = 0$; and

ii) $\zeta(S)$ is an optimal solution to the σ-problem.

Proof. i) By Lemma 8.ii, $\sum_{s \leq \sigma} \pi_s \mathbb{E}_s(\zeta(S)) \geq \sum_{s \leq \sigma} \pi_s \mathbb{E}_s(\zeta(\sigma)) = 0$. By contradiction, suppose that $\sum_{s \leq \sigma} \pi_s \mathbb{E}_s(\zeta(S)) > \sum_{s \leq \sigma} \pi_s \mathbb{E}_s(\zeta(\sigma))$. Consider the allocation $\zeta^* = \zeta(S) \vee_{S'} \zeta(\sigma)$, for $S' = \{s : s \leq \sigma\}$. By Auxiliary Lemma 3, if $U_{\sigma+1}(\zeta(S)) \geq U_{\sigma+1}(\zeta(\sigma))$ and $U_\sigma(\zeta(S)) \leq U_\sigma(\zeta(\sigma))$, then $\zeta^* \in X$. Since $V_\sigma(\sigma) = V_\sigma(S)$, it is

$$U_\sigma(\zeta(\sigma)) = V_\sigma(\sigma) = V_\sigma(S) \geq U_\sigma(\zeta(S)).$$

On the other hand, $\mathbb{E}_\sigma(\zeta(\sigma)) \leq 0$ by Lemma 8.ii and $U_\sigma(\zeta(\sigma)) \geq U_\sigma(\zeta^{RS})$ by Lemma 8.iii, therefore $\mathbb{E}_{\sigma+1}(\zeta_\sigma(\sigma)) \leq 0$ by Monotonicity. Hence, by revealed preferences,

$$U_{\sigma+1}(\zeta(S)) = V_{\sigma+1}(S) \geq U_{\sigma+1}(\zeta(\sigma)) = U_{\sigma+1}(\zeta_\sigma(\sigma)).$$

Therefore $\zeta^* \in X$ and $U_s(\zeta^*_s) \geq V_s(s)$ for all s. However,

$$\sum_s \pi_s \mathbb{E}_s(\zeta^*) = \sum_s \pi_s[\mathbb{E}_s(\zeta^*) - \mathbb{E}_s(\zeta(S))] = \sum_{s\leq\sigma} \pi_s[\mathbb{E}_s(\zeta(\sigma)) - \mathbb{E}_s(\zeta(S))] < 0.$$

Thus, ζ^* is an element of the constraint set of the S-problem, delivers a value $V_S(S)$, but $\sum_s \pi_s \mathbb{E}_s(\zeta^*) < 0$, a contradiction to Lemma 8.ii. Thus, i) holds true.

ii) The allocation $(\zeta_s(S))_{s=1}^{\sigma}$ is an element of the constraint set of the σ-problem, delivers a value $V_\sigma(\sigma)$ to the σ-th type, and by i) it is $\sum_{s\leq\sigma} \pi_s \mathbb{E}_s(\zeta_s(S) = 0$. It is therefore an optimal solution to the σ-problem.□

Step 4 $\zeta(S)$ is an equilibrium allocation.

Proof: Let $\hat{\sigma}$ be the highest integer in $\hat{S}$ such that $V_{\hat{\sigma}}(\hat{\sigma}) = V_{\hat{\sigma}}(S)$ (such an integer exists by Step 2).

If $U_s(\zeta') \geq U_s(\zeta(S))$ for some $s < \hat{\sigma}$, let s_* be the highest such integer.

If $U_s(\zeta') < U_s(\zeta(S))$ for all $s < \hat{\sigma}$, set $s_* = 0$.

For $\hat{\zeta}$ denoting the pooling mechanism $\hat{\zeta}_s = \zeta_{s_*+1}(S)$ for all s, and $S' = \{s : s > s_*\}$, let $\zeta'' = \hat{\zeta} \vee_{S'} \zeta(S)$. By Auxiliary Lemma 3, $\zeta'' \in X$. For $\tilde{\zeta}$ denoting the pooling mechanism $\tilde{\zeta}_s = \zeta_{\hat{\sigma}}(S)$ for all s, and for $S'' = \{s : s > \hat{\sigma}\}$, let $\bar{\zeta} = \zeta'' \vee_{S''} \tilde{\zeta}$. Once again by Auxiliary Lemma 3, $\bar{\zeta} \in X$ and moreover $U_s(\bar{\zeta}) \leq U_s(\zeta(S))$ for all s, and by Step 3, $\mathbb{E}_s(\bar{\zeta}) \leq 0$ for some s.

Next, observe that by definition of s_*, $\zeta(S)$, and $\zeta' \in X$, it is

$$U_{s_*}(\zeta'_{s_*}) \geq U_{s_*}(\zeta(S)) \geq U_{s_*}(\zeta_{s_*+1}(S)),$$

while

$$U_{s_*+1}(\zeta'_{s_*}) \leq U_{s_*+1}(\zeta'_{s_*+1}) \leq U_{s_*+1}(\zeta_{s_*+1}(S)).$$

Then, $U_s(\zeta') \geq U_s(\bar{\zeta})$ all $s \leq s_*$, by Sorting. We also have that $U_s(\zeta') < U_s(\bar{\zeta})$ all s with $s_* < s < s^*$, for $s^* > \hat{\sigma}$ denoting the lowest integer above $\hat{\sigma}$ such that $U_{s^*}(\zeta') \geq U_{s^*}(\bar{\zeta}) = U_{s^*}(\zeta_{\hat{\sigma}}(S))$. Then, by Sorting again we must have $U_s(\zeta') \geq U_s(\bar{\zeta})$ for all $s \geq s^*$.

Now consider the mean $\mathbb{E}\nu_\varepsilon$ of the lottery ν_ε assigning $\bar{\zeta}$ with probability $\varepsilon \in (0, 1)$, and the pooling mechanism $-e$ with probability $(1 - \varepsilon)$. By the convexity of X, $\mathbb{E}\nu_\varepsilon \in X$ and, by construction, $U_s(\mathbb{E}\nu_\varepsilon) < U_s(\zeta(S))$ for all s. Then, since $\mathbb{E}_s(\mathbb{E}\nu_\varepsilon) < 0$, for some s, it is $p(\mathbb{E}\nu_\varepsilon) = p_\phi(\mathbb{E}\nu_\varepsilon) = \min[0, p_M(\mathbb{E}\nu_\varepsilon)] \leq 0$, as p is a $*$-supporting price. However, for ε close enough to 1, $U_s(\mathbb{E}\nu_\varepsilon) > U_s(\zeta')$, all s such that $s_* < s < s^*$. Therefore, for such ε it must be $\eta(\mathbb{E}\nu_\varepsilon) = 0$ at the associated quasi equilibrium. Define $\eta' > \eta$ as $\eta'(\zeta) = \eta(\zeta)$ for all $\zeta \neq \mathbb{E}\nu_\varepsilon$, while $\eta'(\mathbb{E}\nu_\varepsilon) = 1$.

If $\min[0, p_M(\mathbb{E}\nu_\varepsilon)] = 0$, then since $U_s(\mathbb{E}\nu_\varepsilon) > U_s(\zeta')$ for all s such that $s_* < s < s^*$, while $U_s(\mathbb{E}\nu_\varepsilon) < U_s(\zeta')$ for all $s \geq s^*$ and $s \leq s_*$, it is $M_s(p_\phi, \eta') = \delta_{\mathbb{E}\nu_\varepsilon}$ for $s_* < s < s^*$, while for $\nu_s \in M_s(p_\phi, \eta')$, it is $\mathbb{E}\nu_\varepsilon \notin \Sigma(\nu_s)$, all $s \geq s^*$ and $s \leq s_*$. Hence, the expression for $\Pi(\mathbb{E}\nu_\varepsilon; (\bar{\nu}', p_\phi, \eta'))$ is

$$0 - \varepsilon \sum_{s_*<s<s^*} \frac{\pi_s}{\sum_{s_*<s'<s^*} \pi_{s'}} \mathbb{E}_s(\bar{\zeta}) - (1-\varepsilon) \sum_{s_*<s<s^*} \frac{\pi_s}{\sum_{s_*<s'<s^*} \pi_{s'}} \mathbb{E}_s(-e).$$

Now,

$$\sum_{s_*<s<s^*} \pi_s \mathbb{E}_s(\bar{\zeta}) = \sum_{s_*<s\leq\hat{\sigma}} \pi_s \mathbb{E}_s(\zeta(S)) + \sum_{\hat{\sigma}<s<s^*} \pi_s \mathbb{E}_s(\zeta_{\hat{\sigma}}(S)) \leq 0.$$

By Step 3 and Lemma 8.ii, it is both $\sum_{s_*<s\leq\hat{\sigma}} \pi_s \mathbb{E}_s(\zeta(S)) \leq 0$ and $\mathbb{E}_{\hat{\sigma}}(\zeta_{\hat{\sigma}}(S)) \leq 0$. By Lemma 8.iii, it is $U_{\hat{\sigma}}(\zeta(S)) \geq U_{\hat{\sigma}}(\zeta^{RS}_{\hat{\sigma}})$. Therefore, Monotonicity implies $\mathbb{E}_s(\zeta_{\hat{\sigma}}(S)) \leq 0$ for all $s \geq \hat{\sigma}$. Thus, $\sum_{\hat{\sigma}<s<s^*} \pi_s \mathbb{E}_s(\zeta_{\hat{\sigma}}(S)) \leq 0$. It follows that $\Pi(\mathbb{E}\nu_\varepsilon; (\bar{\nu}', p_\phi, \eta')) \geq 0$, contradicting the maximality of η.

If $\min[0, p_M(\mathbb{E}\nu_\varepsilon)] = p_M(\mathbb{E}\nu_\varepsilon) < 0$, then for each $\nu \in \times_s M_s(p_\phi, \eta')$, it is $\nu_s(\mathbb{E}\nu_\varepsilon) > 0$ for $s_* < s < s^*$, and therefore

$$\Pi(\mathbb{E}\nu_\varepsilon; (\bar{\nu}', p_\phi, \eta')) = p_M(\mathbb{E}\nu_\varepsilon) - \sum_s \frac{\pi_s \nu_s(\mathbb{E}\nu_\varepsilon)}{\sum_{s'} \pi_{s'} \nu_{s'}(\mathbb{E}\nu_\varepsilon)} \mathbb{E}_s(\mathbb{E}\nu_\varepsilon) \leq 0,$$

concluding the argument.■

Proof of Proposition 10: We break the proof into three steps.

Step 1 If $\mathbb{E}_S(\zeta(S)) < 0$, then $U_s(\zeta(S)) > U_s(\zeta^{RS})$ for all s.

Proof: We claim that if $\mathbb{E}_S(\zeta(S)) < 0$, then $U_1(\zeta(S)) > U_1(\zeta^{RS})$, then $\zeta(S) \neq \zeta^{RS}$. This is so because by Lemma 8.iv and Auxiliary Lemma 1 type 1 must be fully insured at $\zeta(S)$ and, by Lemma 3, also at ζ^{RS}, while $\mathbb{E}_1(\zeta(S)) = -\frac{\pi_S}{\pi_1}\mathbb{E}_S(\zeta(S)) > \mathbb{E}_1(\zeta^{RS}) = 0$. Now, because ζ^{RS} is feasible for the S-problem, it must be $U_S(\zeta(S) \geq U_S(\zeta^{RS})$. If $U_S(\zeta(S)) = U_S(\zeta^{RS})$, then $\zeta^{RS} \in X(S)$, and we contradict Lemma 8.v, by the same argument.■

The next step simplifies the optimal lotteries at a quasi equilibrium.

Step 2 For each $p \in \mathbb{P}$, each $s \in S$ and each $\mu \in M_s(p)$, there exists $\nu_\alpha \in M_s(p)$ such that $\Sigma(\nu_\alpha) \subset \Sigma(\mu)$ and $\Sigma(\nu_\alpha)$ contains at most two points.

Proof: Let $\nu_s \in M_s(p)$, and $V_s = U_s\nu_s$. From Luenberger (1969), the FOCs are given by

$$\begin{aligned} U_s(\zeta) - V_s &= \lambda_s p(\zeta), \ \nu_s\text{-a.e.} \\ U_s(\zeta) - V_s &\leq \lambda_s p(\zeta), \text{ all } \zeta \in X, \end{aligned}$$

for some $\lambda_s > 0$. Let $\Sigma_+ = \{\zeta : p(\zeta) > 0\}$ and $\Sigma_- = \{\zeta : p(\zeta) < 0\}$. Since by Lemma 1 $p\nu_s = 0$, it must be that $\nu_s(\Sigma_+) = 0$ if and only if $\nu_s(\Sigma_-) = 0$. Thus, either $\nu_s(\Sigma_+) = \nu_s(\Sigma_-) = 0$ or both $\nu_s(\Sigma_+) > 0$, and $\nu_s(\Sigma_-) > 0$. In both cases, we can find $\zeta^1, \zeta^2 \in \Sigma$ (in the first case, with $\zeta^1 = \zeta^2$) such that $U_s(\zeta^\kappa) - V_s = \lambda_s p(\zeta^\kappa)$, $\kappa = 1, 2$. Thus, we can construct a lottery ν_α by putting probability α on ζ^1 and $(1-\alpha)$ on ζ^2 such that $U_s\nu_\alpha = V_s$ and $p\nu_\alpha = 0$.■

Step 3

Assume that $\zeta(S) \notin X^{RS}$, otherwise Lemma 7 concludes the argument.

Suppose that (ζ, p) is an equilibrium and by contradiction that $\zeta \neq \zeta(S)$ for any. Since $p(\zeta^{RS}) \leq 0$, it is $\zeta \in \Delta'$. Since ζ is feasible, it is $U_S(\zeta) < U_S(\zeta(S))$ –otherwise we'd contradict the definition of $\zeta(S)$.

Since $\zeta(S) \notin X^{RS}$, we have $\mathbb{E}_S(\zeta(S)) < 0$, and then by Step 1, $U_s(\zeta(S)) > U_s(\zeta^{RS})$ all s.

Next, let ζ' be the mechanism assigning $\zeta(S)$ with probability $1-\varepsilon$ and $-e$ with probability ε. Then, $\sum_s \pi_s \mathbb{E}_s(\zeta') < 0$ for all $\varepsilon \in (0,1)$, while by continuity for ε small enough we have that $U_S(\zeta') > U_S(\zeta)$ and $U_1(\zeta') > U_1(\zeta^{RS})$. Therefore, it must be that $p(\zeta') > 0$.

We now construct a market indicator η so that (ζ', p_ϕ) is an effective price cut.

Let $\hat{Z}$ be the set of contracts satisfying: for each $\zeta \in \hat{Z} \subset X \backslash \{\zeta'\}$, there exists a lottery μ_1 with $\#\Sigma(\mu_1) \leq 2$ and $\mu_1(\zeta) > 0$, such that three conditions hold:

1. $U_1(\zeta') < U_1\mu_1$;

2. $p_\phi \mu_1 \leq 0$;

3. $\mathbb{E}_1(\zeta) > p_\phi(\zeta)$.

Now set $\eta(\zeta) = 0$ if and only if $\zeta \in \hat{Z}$. To show that (ζ', p_ϕ) is an effective price cut against (ζ, p) it suffices to prove that:

a) at (p_ϕ, η), ζ' is an optimal solution for all s;

b) η is maximal.

For type S claim (a) is obvious. For type $s = 1$, suppose by contradiction that $U_1\mu_1 > U_1(\zeta')$ for $\mu_1 \in M_1(p_\phi, \eta)$. We show that this implies that $\Sigma(\mu_1) \cap \hat{Z} \neq \varnothing$, contradicting the definition of the market indicator η.

By the definition of ζ' and the properties of $\zeta^{RS} \in X^{RS}$, it is $U_1(\zeta') \geq U_1(\zeta^{RS})$, and therefore $U_1\mu_1 > U_1(\zeta^{RS})$. The latter implies $\int \mathbb{E}_1(\zeta) d\mu_1 > 0$ for all $\mu_1 \in M_1(p_\phi, \eta)$. However, by Step 2, for each $\mu_1 \in M_1(p_\phi, \eta)$ there exists a utility-equivalent lottery $\mu_1' \in M_1(p_\phi, \eta)$ with $\Sigma(\mu_1') \subset \Sigma(\mu_1)$ and with $\Sigma(\mu_1') = \{\zeta_1, \zeta_2\}$. Hence, $p_\phi \mu_1' \leq 0$ implies

$$\sum_{\kappa=1,2} \mu_1'(\zeta_\kappa)[\mathbb{E}_1(\zeta_\kappa) - p_\phi(\zeta_\kappa)] \geq \sum_{\kappa=1,2} \mu_1'(\zeta_\kappa)[\mathbb{E}_1(\zeta_\kappa)] > 0.$$

Therefore, Condition 3 above is satisfied as at least one point ζ_κ must have expected net trade for $s = 1$ higher than its price. Condition 2 is obviously satisfied by μ_1', and Condition 1 is also satisfied by assumption. Then, it is $\zeta_\kappa \in \hat{Z}$ for some κ, contradicting $\eta(\zeta_\kappa) = 0$.

Finally in order to show that η is maximal, pick $\zeta'' \in \hat{Z}$ (if $\hat{Z} = \varnothing$, we are done), and define η' as $\eta'(\zeta) = \eta(\zeta)$ for all $\zeta \neq \zeta''$, while $\eta(\zeta'') = 1$. Bear in mind that $\nu_S' \in M_S(p_\phi, \eta')$ if and only if $\nu_S' = \delta_{\zeta'}$. Then, by the definition of $\hat{Z}$, Condition 1 holds true for some μ_1 with $\mu_1(\zeta'') > 0$. Therefore, $U_1\nu_1' > U_1(\zeta')$ for all $\nu_1' \in M_1(p_\phi, \eta')$, so that by Step 2 $\Sigma(\nu_1') \cap \hat{Z} \neq \varnothing$ for all such ν_1'. Equivalently, $\nu_1'(\zeta'') > 0$ at all optimal solutions. However, since $\mathbb{E}_1(\zeta'') > p(\zeta'')$, it is $\Pi(\zeta''; (\nu'; p_\phi, \eta')) < 0$ for all $\nu' \in M(p_\phi, \eta')$, concluding the argument.■

On pricing random mechanisms. **Proof of Proposition 11:** First we argue that this is true for $MS(C)$ and $MS(E)$, then we complete the argument. The feasible and incentive compatible allocations sets are, respectively, $Y^{MS(C)} \cap IC^{MS(C)}$, i.e.,

$$\{\mu \in \Delta(K)^S : \sum_s \pi_s \int_K (\sum \pi(\omega|s) z_\omega) d\mu_s \leq 0,\, U_s\mu_s \geq U_s\mu_{s'},\text{ all } s, s'\},$$

and, denoting with $X^{MS(E)}$ the set of incentive compatible and elementary mechanisms, $Y^{MS(E)} \cap IC^{MS(E)}$, i.e.,

$$\{\mu \in \Delta(X^{MS(E)})^S : \sum \pi_s \int_K \mathbb{E}_s(z) d\mu_s \leq 0,\, U_s\mu_s \geq U_s\mu_{s'},\text{ all } s, s'\}.$$

Since an element of K can be identified with a pooling elementary mechanism in $X^{MS(E)}$, $Y^{MS(C)} \cap IC^{MS(C)}$ is equivalent to a subset of $Y^{MS(E)} \cap IC^{MS(E)}$. For the other direction, recall that by definition of X and $X^{MS(E)}$ for $z = (z_s)_{s\in S} \in X^{MS(E)}$, it is $U_s(z) = \mathbb{E}_s(v_s(z_s+e))$ as well as $\mathbb{E}_s(z) = \sum \pi(\omega|s) z_{\omega,s}$. For any probability measure ν over $X^{MS(E)}$, let $\nu(s)$ be the probability measure over K defined as $\nu(s)(B) = \int_{B\times(\times_{s'\neq s}K)} d\nu$, for all $B \in \mathcal{B}(K)$. Then $U_s\mu_s = \int_{X^{MS(E)}} U_s(z) d\mu_s = \int_K \mathbb{E}_s[v_s(z_s + e)] d\mu_s(s)$. Therefore, $(\mu_s(s))_{s\in S}$ is payoff equivalent to μ and it consumes the same resources. As already discussed in Section 5, the assumptions on individual preferences imply that an element of $\Delta(\Delta(K)^S)$ is payoff (and resource) equivalent to an element of $\Delta(K^S)$, establishing the equivalence of the three sets.■

Proof of Proposition 12: The zero-profit property of equilibrium must hold even when only contracts are priced. Hence, at equilibrium there cannot be cross-subsidies across lotteries chosen by different types. Indeed, budget feasibility requires that the value of each such lottery is zero, contradicting the existence of cross-subsidies. Therefore, an equilibrium is either separating with no cross-subsidies, or it is pooling.

Let $\bar{z}^{p,2}$ be the solution to

$$\max_{z\in K} U_2(z) \text{ subject to } \sum \pi_s \mathbb{E}_s(z) \leq 0,\, u_1(z) \geq u_1(z^{RS}).$$

First, as known, $\bar{z}^p$ provides full insurance, while under standard regularity conditions and since $\pi(\omega|2) \neq \pi(\omega|1)$, $\bar{z}^{p,2}$ does not. Thus, at the allocation $(\bar{z}^{p,2}, \bar{z}^{p,2})$ both types have ω-dependent net trades. By Auxiliary Lemma 1, $(\bar{z}^{p,2}, \bar{z}^{p,2})$ is not a (strict) CPO, but $U_2(\bar{z}^{p,2}) > U_2(\bar{z}^p)$.

Next, by mimicking the argument in Theorem 6, z^{RS} cannot be an equilibrium as it is Pareto dominated by $\bar{z}^p$. Also, $\bar{z}^p$ cannot be an equilibrium as type 2 prefers a strictly feasible pooling contract $\bar{z}'$ arbitrarily close to $\bar{z}^{p,2}$. Such $\bar{z}'$ can be used to create an effective price cut against $\bar{z}^p$. Therefore, the equilibrium allocation cannot be a CPO.■

Proof of Proposition 13: We denote $\tilde{\zeta}_1$ as z_1^*, a full insurance net trade. Suppose by contradiction that $\tilde{\nu}$ is price supportable in $MS(E)$, with p the supporting price and $\tilde{\nu}_s \in M_s(p)$. Since $p\tilde{\nu}_1 = 0$, by the optimality conditions and the definition of z_1^*, it must be that $p(z) = 0$, $\tilde{\nu}_1$-a.e.. Also, it must be that $\tilde{\nu}_1$ is absolutely continuous with respect to $\tilde{\nu}_2$. Otherwise, if $\tilde{\nu}_1(B) > 0$, but $\tilde{\nu}_2(B) = 0$ for some $B \in \mathcal{B}(X)$, then

$$\int_B \Pi(z; (\tilde{\nu}, p)) d(\sum_s \pi_s \tilde{\nu}_s) = \int_B \Pi(z; (\tilde{\nu}, p)) d(\pi_1 \tilde{\nu}_1)$$
$$= -\int \mathbb{E}_1(z) \pi_1 d\tilde{\nu}_1 = -\mathbb{E}_1(z_1^*) \pi_1 \tilde{\nu}_1(B) < 0.$$

The latter implies that $\tilde{\nu}_2(B) > 0$ for all $B \subset \Sigma(\tilde{\nu}_1)$ with $\tilde{\nu}_1(B) > 0$.

We claim that for some $B \subset \Sigma(\tilde{\nu}_1)$ with $\sum_s \pi_s \tilde{\nu}_s(B) > 0$,

$$\int_B \Pi(z; (\tilde{\nu}, p)) d(\sum_s \pi_s \tilde{\nu}_s) < 0.$$

The condition $p(z) = 0$, $\tilde{\nu}_1$-a.e., and the optimality conditions imply that

$U_2(z_2) = U_2\tilde{\nu}_2$, $\tilde{\nu}_1$-a.e.. However, $z \in \Sigma(\tilde{\nu}_1)$ is incentive compatible and $z_1 = z_1^*$, $\tilde{\nu}_1$-a.e.. Thus, if it were $\sum_s \pi_s \mathbb{E}_s(z) < 0$ in a subset of positive $\tilde{\nu}_1$ and therefore $\tilde{\nu}_2$ measure, $\tilde{\nu}$ would not be a strict CPO, a contradiction. Then, $\sum_s \pi_s \mathbb{E}_s(z) \geq 0$, $\tilde{\nu}_1$-a.e.. Therefore, if the claim were not true and by contradiction $\Pi(z; (\tilde{\nu}, p)) \geq 0$, $\tilde{\nu}_1$-a.e., then $\tilde{\nu}_2(B) \geq \tilde{\nu}_1(B)$ for all $B \subset \Sigma(\tilde{\nu}_1)$ with $\tilde{\nu}_1(B) > 0$. As a consequence, $\tilde{\nu}_2(\Sigma(\tilde{\nu}_1)) = 1$ and therefore $\tilde{\nu}_2(B) = \tilde{\nu}_1(B)$ for all $B \in \mathcal{B}(X)$, i.e., $\tilde{\nu}_1 = \tilde{\nu}_2$. It follows that $(U_2(z), \mathbb{E}_2(z))$ is constant $\tilde{\nu}_2$-a.e., hence $\tilde{\zeta}_2$-a.e., contradicting $\tilde{\zeta}_2$ essential. This establishes the claim, and therefore the statement.■

Monotonicity and Sorting are nonvacuous. The following claim proves that our assumptions hold in a nonempty set of economies, including those mostly used in analyses.

Claim *Monotonicity and Sorting hold true in Wilson economies.*

Proof: To prove Monotonicity, pick any s. Consider an individual lottery $\zeta \in \Delta(K)$ such that $U_s(\zeta) \geq U_s(\zeta^{RS})$ and $\mathbb{E}_s(\zeta) \leq 0$. Define $\bar{z} \in K$ as $\bar{z} = (\bar{z}_L, \bar{z}_H) = (\int z d\zeta_L, \int z d\zeta_H)$. By the concavity of the cardinality indexes it is $U_s(\bar{z}) \geq U_s(\zeta)$, while $\mathbb{E}_s(\zeta) = \mathbb{E}_s(\bar{z})$. We now show that

$$\bar{z}_H \leq 0 \leq \bar{z}_L,$$

which implies, since $\pi(H|s)$ is increasing in s, that $\mathbb{E}_\sigma(\zeta)$ is decreasing in σ. Since by assumption $\mathbb{E}_s(\zeta) \leq 0$, we will then get that $\mathbb{E}_\sigma(\zeta) \leq \mathbb{E}_s(\zeta) \leq 0$ for all $\sigma > s$, as desired. By contradiction, suppose that $\bar{z}_L < 0$. Since $U_s(\bar{z}) \geq U_s(\zeta) \geq U_s(\zeta^{RS})$, it is $U_s(\bar{z}) \geq U_s(0)$. This implies $\bar{z}_H > 0$. Let $t^* = -\bar{z}_L > 0$ and consider the map $F(t)$, $t > 0$ defined as:

$$F(t) = \pi(L|s)v(-t + e_L) + \pi(H|s)v(\frac{\pi(L|s)}{\pi(H|s)}t + e_H).$$

Since $\mathbb{E}_s(\zeta) \leq 0$, it is $(\bar{z}_L, \bar{z}_H) \leq (-t^*, \frac{\pi(L|s)}{\pi(H|s)}t^*)$. Since v is increasing, $F(t^*) \geq U_s(\bar{z}) \geq U_s(\zeta)$. Let $\bar{t} = e_L - \pi(H|s)e_H + \pi(L|s)e_L < 0$. Notice that $e_H + \frac{\pi(L|s)}{\pi(H|s)}\bar{t} = e_L - \bar{t} = \pi(H|s)e_H + \pi(L|s)e_L$, and that $\lambda\bar{t} + (1-\lambda)t = 0$ for $\lambda = \frac{t}{t-\bar{t}}$. By the strict concavity of v, it is both $F(\bar{t}) > \lambda F(\bar{t}) + (1-\lambda)F(t^*) > F(t^*)$, and $F(t^*) < \lambda F(\bar{t}) + (1-\lambda)F(t^*) < F(0) = U_s(0)$, a contradiction that concludes the argument.

To prove Sorting, let ζ and ζ' be as in the assumption. The inequalities in the assumption read

$$\pi(H|s)\int v(z+e_H)d\zeta_H + \pi(L|s)\int v(z+e_L)d\zeta_L \geq$$
$$\pi(H|s)\int v(z+e_H)d\zeta'_H + \pi(L|s)\int v(z+e_L)d\zeta'_L$$

and

$$-\pi(H|s+1)\int v(z+e_H)d\zeta_H - \pi(L|s+1)\int v(z+e_L)d\zeta_L \geq$$
$$-\pi(H|s+1)\int v(z+e_H)d\zeta'_H - \pi(L|s+1)\int v(z+e_L)d\zeta'_L$$

Thus, by summing these inequalities and simplifying by the negative scalar $\pi(H|s) - \pi(H|s+1)$, we get

$$\int v(z+e_H)d\zeta'_H - \int v(z+e_H)d\zeta_H \geq \int v(z+e_L)d\zeta'_L - \int v(z+e_L)d\zeta_L.$$

It cannot be that either $(\int v(z+e_L)d\zeta'_L, \int v(z+e_H)d\zeta'_H) > (\int v(z+e_L)d\zeta_L, \int v(z+e_H)d\zeta_H)$ or $(\int v(z+e_L)d\zeta_L, \int v(z+e_H)d\zeta_H) > (\int v(z+e_L)d\zeta'_L, \int v(z+e_H)d\zeta'_H)$. Then, it is $\int v(z+e_H)d\zeta'_H \geq \int v(z+e_H)d\zeta_H$, while $\int v(z+e_L)d\zeta'_L \leq \int v(z+e_L)d\zeta_L$. Then, direct computations show $U_\sigma(\zeta') - U_\sigma(\zeta)$ is weakly increasing in σ, establishing the claim.■

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