Reallocation Costs and Efficiency*

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Abstract

We study the efficient allocation of a divisible asset when reallocation is costly. Each of two individuals is initially allocated a share of the asset. At the time of this initial division, the individuals' valuations for the asset are uncertain. After the uncertainty resolves, the asset may be reallocated. Reallocation is costly, and the reallocation costs may depend on the amount reallocated and on the individuals' valuations. We first show that given the initial division of the asset an optimal contract maximizes the expected surplus that can generated by reallocating the asset. We then show that this maximal expected surplus is monotonic in the larger share of the initial division for a wide range of reallocation cost specifications.

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1 Introduction

The efficient allocation of an asset in environments with uncertainty presents several challenges. One challenge arises when the uncertainty takes the form of private information about the best use of the asset. In this case, the initial allocation of the asset determines parties' outside options and hence whether a subsidy is needed to implement ex-post efficiency. This is the focus of the literature started by Myerson and Satterthwaite (1983) and Cramton, Gibbons, and Klemperer (1987). Another challenge arises when it is possible to invest in improving the asset, but investment is not fully contractible. In this case, the allocation of the asset affects parties' investment incentives, and hence the surplus from the asset. This is the focus of the literature started by Grossman and Hart (1986). A third challenge arises when costly reallocation of the asset may take place after the uncertainty regarding its best use is resolved. In this case, the initial allocation of the asset determines whether and how much of the asset will be reallocated ex-post, and hence the surplus from the asset. This is the focus of our paper.

We begin with two examples. The first deals with the allocation of capital in a non-strategic setting, and the second deals with the allocation of control rights in a strategic setting. In both examples, the best use of the asset (capital or control rights) is unknown when the asset is initially allocated, and reallocating it after the uncertainty resolves is costly.

Scenario 1: Capital. When a firm allocates capital among different product lines to determine their production capacity, there may be uncertainty about the market price of each product. This uncertainty may result from possible changes in consumer demand or other market conditions. Reallocating capital after the uncertainty resolves is costly, as it requires adjusting machines and retraining employees.

Anticipating the costly reallocation, the firm's management may wish to delay the allocation of capital until the uncertainty resolves. This would be the case, for example, when the uncertainty is expected to resolve quickly. But when the uncertainty takes time to resolve or may not resolve until the products are put on the market, or when postponing the allocation of capital, and hence production, puts the firm in a competitive disadvantage, it may be better to incur the costs of possible reallocation than to postpone the allocation of capital altogether.$\Diamond$

Scenario 2: Control rights. When a law firm contracts with a new business client, the partners divide the client's cases between them. This roughly corresponds to their control rights in the account in the sense that if the partnership dissolves, each partner can walk away with his share of the cases. At the time of the initial allocation of the cases, there may be aggregate uncertainty about the fit of each partner with the customer or with the cases. This uncertainty may resolve privately or publicly. After it resolves, shifting cases between the partners is costly: Handling additional cases requires learning what the other partner did and possibly reneging on prior commitments in order to free time. Delaying the allocation of the cases is infeasible, because if they are not handled promptly, the client will go elsewhere.$\Diamond$

The goals of the participating parties in scenarios 1 and 2 differ: a firm maximizes its profits while each self-interested partner would like to maximize his utility. And yet, in both scenarios, risk-neutral parties are likely to choose an initial allocation of the asset that maximizes the expected surplus from the asset net of any reallocation costs. This is clearly the case in the context of a risk-neutral firm. This is also the case in strategic environments in which players' valuations are realized publicly. Two self-interested players who can contract on the ex-ante and ex-post allocation of the asset, are likely to choose an allocation that maximizes the surplus from the asset net of reallocation costs. Otherwise, there exists a contract that generates a higher surplus, which can be shared by the players via ex-ante transfers. This outcome is also likely in environments with private information, as long as players can be induced to reveal their valuations (we comment on this issue below). Our analysis will thus focus on how the initial allocation influences the expected surplus from the asset.

The reallocation costs in scenarios 1 and 2 increase in the amount reallocated: it is more costly to adjust more machines and retrain more employees, and it more costly to renege on more commitments or learn more cases. In both scenarios, it may also be that an additional fixed overhead is incurred in any reallocation. The variable reallocation cost in scenario 1 may be thought of as convex in the amount reallocated.[1] This follows from the standard argument that parties reallocate the least costly units first if they have the flexibility to do so, which may be the case if reallocation involves adjusting machines and retraining employees, and also when reneging on prior commitments. But parties do not always have this flexibility. Consider, for example, the costs associated with learning additional cases in scenario 2. If the cases are roughly homogenous and learning amounts to understanding how the other partner handled them, then the cost of learning an additional case decreases in the number of cases that have already been learned. This leads to concave reallocation costs. Here is another example in which reallocation costs are concave.

Scenario 3: Ownership. When parents want to transfer ownership of a plot of land to their children (preserving equality by allocating monetary assets as well), it may not be clear how each child will use his share of the land. Delaying the transfer may not be feasible, because it may have unfavorable tax implications or lead to family feuds. After the initial division of ownership has been determined, a later reallocation may cause the sibling whose share decreases to experience an emotional loss. In the spirit of Kahneman and Tversky's (1991) model of reference-dependent preferences, with the initial division as the reference point, the marginal sensitivity to losses decreases in the size of the loss, and hence the variable reallocation cost is concave. Reallocating ownership may also require legal, accounting, and other costly services.

[1] Throughout the paper, by "increasing," "decreasing," "concave," "convex," "positive," "negative," etc. we mean "weakly increasing," "weakly decreasing," "weakly concave," "weakly convex," "weakly positive," and "weakly negative."

Such overhead may be roughly independent of the amount reallocated.◊

In addition to their possible dependency on the amount reallocated, reallocation costs may depend on the parties' valuations for the asset. In scenario 1, decreasing the production capacity of one product line in order to increase the capacity of another line may imply that certain commitments for producing the first product cannot be met. Breaching such commitments may entail fines whose magnitude depends on the realized market price for the product. In scenario 2 the effort involved in learning may depend on both partners' fit with the account. In scenario 3 the mental cost of losing shares in a family asset may well depend on the asset's value. We thus allow for reallocation costs that depend on players' valuations as well as on the amount reallocated.

We study an environment with one divisible asset, two players (or "uses" in the capital example), an initial division, and a final allocation. At the time of the initial division the asset is homogenous. Examples of such an asset include capital, land, control rights, workload, authority, and so on. Each player generates a surplus of xv from a share $x \leq 1$ of the asset, where v denotes the player's valuation for the entire asset. At the time of the initial division, players' valuations for the asset are unknown and are expected to be drawn from a symmetric distribution. After players' valuations are realized – either publicly or privately – the asset may be reallocated. Reallocation is costly, and reallocation costs may depend on the amount reallocated and on players' valuations. A final allocation specifies players' final shares for every realization of players' valuations. Given such a realization, the ex-post surplus is the aggregate surplus the players generate with the shares specified by the final allocation, minus the reallocation costs. Given an initial division, an efficient final allocation maximizes the ex-post surplus for every realization, and therefore maximizes the expected surplus from the asset.

Note that we assume the asset is allocated before the uncertainty about its best use is resolved. This fits situations in which the uncertainty takes a long time to resolve or in which postponing the allocation of the asset is associated with a substantial cost, as illustrated in the above scenarios. Analyzing other situations, in which the tradeoff between early and delayed allocation is less clear, also involves studying the value of allocating early, which corresponds to the expected surplus in our model.

We study how the initial division of the asset affects the expected surplus from the asset in an efficient final allocation. As mentioned above, choosing an initial division that maximizes the expected surplus is in the best interest of a single risk-neutral decision maker who owns the asset. Section 2 discusses several strategic environments in which the players are likely to reach an efficient final allocation. As mentioned above, one such environment is where a complete contract is signed and valuations are publicly revealed. Even if the contract is incomplete, the same outcome should arise provided that the players can bargain efficiently ex-post. Now suppose that valuations are revealed privately. If the size of each player's share of the reallocation costs does not depend on the other player's valuation, then a Groves (1973) mechanism can be used to induce players to reveal their valuations. Otherwise, we observe that truthful revelation can be induced when each player's share of the reallocation costs satisfies increasing differences in the losing player's valuation and the amount reallocated and decreasing differences in the gaining player's valuation and the amount reallocated. In scenario 2, this would correspond to assuming that increasing the gaining partner's fit with the cases makes it easier for him to study cases previously handled by the losing partner, and increasing the losing partner's fit with the cases makes it more difficult for the gaining partner to study the cases he receives from the losing partner (because the losing partner has done more with the cases). Once the informational problem is overcome, any funding necessary for the mechanism can be obtained from the players ex-ante. Moreover, the mechanism can be designed in a way that balances the budget ex-post, thus eliminating the need for a budget breaker. We thus abstract from how valuations are revealed when we characterize the relationship between the initial division of the asset and the maximal expected surplus.

This relationship hinges on the assumption that reallocation is costly. To see why, note that when reallocation is costless the maximal expected surplus does not depend on the initial division. This is because in this case the efficient final allocation awards the entire asset to the party that will generate the most surplus from it, independently of the initial division. But when reallocation is costly, this is no longer the case. The initial division determines the maximal amount that can be transferred, and hence influences the cost and benefit of reallocation. This maximal amount differs between the players when the initial division is not equal. As a result, an efficient final allocation changes with the initial division, and may be asymmetric. Consequently, the maximal expected surplus depends on the initial division.

Section 3 relates the maximal expected surplus to the concentration of the initial division, which is the larger share in the division. For a large class of reallocation cost functions, we show that the maximal expected surplus is constant in the initial concentration as long as the initial concentration is smaller than some threshold, and is then strictly monotonic. It is monotonically increasing if the cost function is amount-insensitive or concave in the amount reallocated, and is monotonically decreasing in the initial concentration if the cost function is convex in the amount reallocated. This is true regardless of how the reallocation costs depend on players' valuations and regardless of how the valuations are distributed. These parameters only influence the threshold above which the monotonicity becomes strict.

To illustrate the result, consider the simple case of a reallocation cost that is insensitive to the amount reallocated and does not depend on players' valuations. Because the cost is constant, for any realization of players' valuations it is efficient either to reallocate the asset to the player with the higher valuation or to maintain the initial division. Suppose that player 1 has the larger initial share, and consider how increasing the concentration of the initial division, that is, increasing the initial share of player 1, affects the efficient final allocation. First, it changes the set of realizations for which reallocation is optimal. Second, it increases the ex-post surplus generated when the initial division is maintained and player 1 has the higher valuation. Third, it decreases the ex-post surplus generated when the initial division is maintained and player 2 has the higher valuation. In terms of expected surplus, the second effect dominates the third because the likelihood that the initial division is maintained is larger when player 1 has the larger valuation than when player 2 has the larger valuation. This is because the benefit of reallocating the asset to a player decreases in the player's initial share. Thus, there is an increase in expected surplus that results from situations in which the initial division is maintained. The first effect further increases the expected surplus, by optimizing ex-post whether to reallocate the asset or maintain the initial division. In contrast, these effects do not arise in the final allocation that always allocates the asset to the player with the higher valuation.

Our monotonicity results do not generally apply to reallocation costs that have an amount-insensitive component and a convex variable component. This is because the amount-insensitive component pushes toward a more concentrated initial division, whereas the convex variable component pushes toward a less concentrated initial division. Nevertheless, we show that if the amount-insensitive component is "relatively large" and the variable cost is not "too convex," then the fully concentrated initial division yields the maximal expected surplus, whereas if the amount insensitive cost is relatively small and the variable cost is sufficiently convex, then the equal initial division is optimal. We also provide examples in which the optimal initial division is neither the equal nor the fully concentrated division.

Our results demonstrate how the nature of the reallocation costs affects the optimal allocation of a divisible asset. In environments in which players can choose which shares to transfer first and the cost of reallocating units varies substantially across units relative to the amount-insensitive cost component, we expect to see relatively equal initial asset divisions. Other forces, such as risk aversion, may push in the same direction. In environments in which the reallocation costs are either mostly amount-insensitive or concave in the amount reallocated, as is the case with mental losses, our results predict that the initial division will be highly concentrated. Other forces, such as increasing returns to scale, may push in the same direction.

The monotonicity results extend to the case of N players as follows. Given two initial divisions, we say that the first is more concentrated than the second if the first majorizes the second. That is, the sum of the largest k initial shares in the first initial division is larger than the sum of the largest k initial shares in the second initial division, for any $k \leq N$. We show that the maximal expected surplus increases in the concentration of the initial division when the reallocation cost is amount insensitive or concave in the amount reallocated, and decreases in the initial concentration when the reallocation costs are convex.

Our paper is related to several literatures. In the literature on implementation of ex-post efficiency, a typical model has a divisible asset and agents with statistically independent private information who each initially own a non-negative share of the asset. The question is whether there exists an interim incentive-compatible mechanism that awards the asset to the agent with the highest valuation for the asset, while satisfying agents' interim participation constraints and without incurring a deficit. Under mild assumptions, in the private-value setting Myerson and Satterthwaite (1983) showed that the answer is "no" when the asset is initially owned by one of the agents, and Cramton, Gibbons, and Klemperer (1987) showed that the answer is "yes" if the agents are ex-ante symmetric and agents' initial shares are sufficiently close to being equal. Environments with interdependent valuations have been studied by Fiesler, Kittsteiner, and Moldovanu (2003), Jehiel and Pauzner (2006), and Segal and Whinston (2011).

Our paper differs from this literature in two respects. First, because of the reallocation costs, what is ex-post efficient in our model and how much surplus is generated by an efficient final allocation depends not only on players' valuations but also on the initial division. Understanding this dependency is the focus of our paper. Second, the issue of avoiding a deficit, which is central to this literature, can be overcome in our model. Because the players have no private information at the time of the initial division, they are willing to initially fund a mechanism that is executed after they obtain their private information. Therefore, if an incentive compatible mechanism exists, it can be made to satisfy agents' interim participation constraints. Moreover, the techniques of Arrow (1979) and d'Aspremont and Gérard-Varet (1979) and Kosenok and Severinov (2008) can be used make sure that the mechanism is budget-balanced ex-post. Whether such a mechanism exists, however, is not immediate in our model, because players' utilities need not be continuously differentiable, and may depend on both players' valuations through their share of the reallocation costs. We provide sufficient conditions that are suitable for our setting in the spirit of those given by Bergemann Välimäki (2002).

Another related literature is the literature on the property rights approach to the theory of the firm, pioneered by Grossman and Hart (1986) and Hart and Moore (1990). This literature studies how to allocate asset ownership between several parties when it is possible to invest in improving the asset, but investment is not fully contractible. An optimal ownership structure is one that induces the most efficient investments. In our model, by way of contrast, there are no explicit investments. The initial division of the asset affects the ex-post surplus from the asset because reallocation is costly. Our analysis can therefore be viewed as an investigation of another channel through which the ownership structure affect the ex-post surplus from the asset.

There is also a finance literature that studies costly reallocation in the context of rebalancing a portfolio. Typical models in this literature study dynamic settings in which an investor can frequently rebalance his investments in a risk-free asset and a risky asset whose value is determined by a random process. The transaction costs involved in rebalancing may be proportional (linear), constant, or a fraction of the portfolio value.[2] Our work differs from this literature in two aspects. First, because we are interested in understanding the effect of reallocation costs

[2] Cadenillas (2000) surveys infinite horizon, continuous time models of portfolio rebalancing in which a risky asset's value is determined by Brownian motion. There are also two-period models in which the portfolio can be

in general, rather than in a particular application, our environment is simpler than is typical in this literature except for our specification of reallocation costs, which is richer. Second, we study how different forms of reallocation costs influence the connection between the initial division of the asset and the surplus generated by optimally reallocating it, rather than taking the initial division as given (if it is at all relevant) and focusing only on optimal reallocation.

Our paper is also related to the literature on reference-dependent preferences. As noted by Kahneman and Tversky (1991), individuals often demand higher payments to give up an asset they own than what they are willing to pay to acquire the same asset.[3] This aversion to losing an asset can be thought of as a mental cost associated with reallocating an asset, as in our ownership example. Under this interpretation, the initial division of the asset corresponds to individuals' reference point, which they compare to the final allocation. This reference point differs from Koszegi and Rabin's (2006) reference point, which is the individual's expectation about the final allocation. Which reference point is more appropriate may depend on the particular application. As Kahneman and Tversky (2006) noted on page 1046, "Although the reference state usually corresponds to the decision maker's current position, it can also be influenced by aspirations, expectations, norms, and social comparisons [...]."

2 Environment

Two risk-neutral players with quasi-linear utilities need to allocate a divisible asset of size 1 between them. The players first allocate the asset at stage 0. At this time, their valuations for the asset are unknown, and are expected to be drawn from a symmetric cumulative distribution F. Between stage 0 and stage 1 the valuations are realized – either publicly or privately. In stage 1 costly reallocation of the asset may take place. We assume no discounting between periods.[4]

For ease of exposition, we begin by describing a simple complete-contracting complete-information environment. In this environment, players' valuations for the asset are realized publicly (between stage 0 and stage 1), and players can contract at the beginning of stage 0 on how the asset will be reallocated in stage 1 for any realization of the valuations. We then relax the complete contracting and the complete information assumptions. We observe that in all these environments, for any initial allocation of the asset players can achieve an ex-post optimal

rebalanced between the two periods. Mitchell and Braun (2004) consider this problem in the presence of convex transaction costs, and characterize for a risk-averse investor the efficient frontier that trades off risk and expected return. Dybvig (2005) considers a similar problem with linear or fixed costs when the investor has mean-variance preferences.

[3]Kahneman and Tversky (1991) also argue that the same phenomenon does not arise with respect to monetary payments made in the process of acquiring the object (page 1055).

[4]This assumption does not affect any of our results.

allocation of the asset that maximizes the sum of their utilities net of the reallocation costs.

2.1 Environments with complete information

We first describe an environment with complete contracting.

Contract. At the beginning of stage 0, the players sign a contract (S, x, t) that specifies the following:

1. A stage 0 *initial division* $S = (s, 1-s)$ of the asset, where $s \in [0,1]$ is player 1's initial share and $1-s$ is player 2's initial share.

2. A stage 0 transfer $t \in \mathbb{R}$ from player 1 to player 2 (if t is negative, then the transfer is from player 2 to player 1).

3. A stage 1 *final allocation* that specifies, for each realization (v_1, v_2) of players' valuations, the final shares $x(v_1, v_2)$ and $1 - x(v_1, v_2)$ of players 1 and 2 respectively.

Reallocation costs. Players may incur a reallocation cost whenever the final allocation is different from the initial division. These costs may depend on the amount reallocated and on players' valuations, but not on players' identities. The cost of reallocating an amount $\Delta > 0$ from a player with valuation v_L (the Losing player) to a player with valuation v_G (the Gaining player) is $C^L(v_L, v_G, \Delta) \geq 0$ to the losing player and $C^G(v_L, v_G, \Delta) \geq 0$ to the gaining player. The total cost $C(v_L, v_G, \Delta) = C^L(v_L, v_G, \Delta) + C^G(v_L, v_G, \Delta)$ is strictly positive.

Players' utilities. A player's per-period utility from a share y of the asset when his valuation is v is yv. Player 1's expected utility from the initial division and the transfer specified in the contract is $E(sv_1) - t$, where the expectation is taken with respect to F. Given a realization (v_1, v_2) of players' valuations, player 1's stage 1 utility from the contract is

$$u_1(s, x, v_1, v_2) = x(v_1, v_2)\, v_1 - \left(1_{x(v_1,v_2)<s} C^L(v_1, v_2, s - x(v_1, v_2)) + 1_{x(v_1,v_2)>s} C^G(v_2, v_1, x(v_1, v_2) - s)\right),$$

where the first term is the player's benefit from the final allocation x, and the other two terms are the costs incurred by player 1 when he loses $s - x(v_1, v_2)$ or gains $x(v_1, v_2) - s$. Player 1's total expected utility from the contract is $E(sv_1 + u_1(s, x, v_1, v_2)) - t$. Player 2's expected utility from the contract is defined similarly.[5]

Expected surplus. Because players are ex-ante symmetric, the sum of players' utilities from the initial division and the transfer specified in the contract is $E[sv_1 + (1-s)\, v_2]$, which does not depend on the contract. We thus ignore this term, and refer to the sum of players' expected utilities from the contract without this term as the *expected surplus* from the asset:

$$E\left[x(v_1, v_2)\, v_1 + (1 - x(v_1, v_2))v_2 - \left(1_{x(v_1,v_2)<s} C(v_1, v_2, s - x(v_1, v_2)) + 1_{x(v_1,v_2)>s} C(v_2, v_1, x(v_1, v_2) - s\right.\right.$$

[5] It is $E[(1-s)v_2 + (1-x(v_1, v_2))v_2 - (1_{x(v_1,v_2)<s} C^G(v_1, v_2, s-x(v_1, v_2)) + 1_{x(v_1,v_2)>s} C^L(v_2, v_1, x(v_1, v_2)-s))] + t$.

The following example illustrates how the expected surplus is calculated in a *naive* contract that assigns the asset ex-post to the player with the higher valuation.

Example 1. Consider a contract that specifies a final allocation that assigns the asset to player 1 unless player 2 has a strictly higher valuation. That is, $x(v_1, v_2) = 1$ if $v_1 \geq v_2$, and $x(v_1, v_2) = 0$ otherwise. To calculate the expected surplus, suppose that valuations are distributed independently and uniformly on $[0, 1]$, and that the total reallocation cost is independent of players' valuations and the amount reallocated, so $C(v_L, v_G, \Delta) = C$. If the initial division is not fully concentrated, i.e., if $S \notin \{(0, 1), (1, 0)\}$, then for any realization of players' valuations some of the asset is reallocated, which gives an expected reallocation cost of C. If the initial division is fully concentrated, then for half of the realizations some of the asset is reallocated, for an expected reallocation cost of $C/2$. The expected benefit associated with the final allocation x is the expected value of the higher of the two players' valuations, which is $2/3$. Therefore, the expected surplus is $2/3 - C$ if the initial division is not fully concentrated and $2/3 - 1/2C$ otherwise.$\diamondsuit$

The contract in Example 1 is not ex-post optimal: for any $C > 0$ and for any initial division, players can increase the surplus from the asset by specifying a final allocation that maintains the initial division when the difference in valuations is small enough. This increase in surplus can be shared by the players using the stage 0 transfer. Thus,

Ex-post optimal contracts. Given an initial division S, players will specify an S-*efficient* final allocation x^S, which for any realization of players' valuations maximizes the sum of players' stage 1 utilities, or the ex-post surplus:[6]

$$x^S(v_1, v_2) \in \arg\max_{x \in [0,1]} xv_1 + (1 - x)v_2 - C(\min\{v_1, v_2\}, \max\{v_1, v_2\}, \max\{x - s, s - x\}).$$

An S-efficient final allocation also maximizes the expected surplus from the asset. Example 2 describes an S-efficient final allocation in the setting of example 1.

Example 2. Suppose, as in Example 1, that valuations are distributed independently and uniformly on $[0, 1]$, and that the total reallocation cost is $C < 1/2$. For any initial division S, an S-efficient final allocation has a "bang-bang" form. It allocates the entire asset to player 1 if the average benefit of doing so, $v_1 - v_2$, is larger than the average cost, $C/(1 - s)$.[7] Similarly, it allocates the entire asset to player 2 if $v_2 - v_1$ is larger than C/s. Otherwise, it maintains the initial division. Because $C < 1/2$, we have that $\min\{C/s, C/(1 - s)\} < 1$, so with positive probability the final allocation differs from the initial division.$\diamondsuit$

[6] To guarantee the existence of a maximizer, we require that $C(v_1, v_2, \cdot)$ be lower semi-continuous in its third argument.

[7] If the benefit and cost are equal, both allocating the entire asset to player 1 and maintaining the initial division are optimal.

Example 2 demonstrates several properties of an S-efficient final allocation. First, an S-efficient final allocation may not be symmetric when players' initial shares differ ($s \neq 1/2$). In Example 2, when player 1's initial share is larger ($s > 1/2$), the probability that player 1 is allocated the entire asset is strictly smaller than the probability that player 2 is allocated the entire asset. This is because the average cost of transferring shares to player 1, $C/(1-s)$, is larger than the average cost of transferring shares to player 2, C/s. This also shows that an S-efficient final allocation may change with S. Finally, the expected surplus in an S-efficient final allocation may be strictly larger than in a naive final allocation that always awards the entire asset to the player with the higher valuation. Going back to Examples 1 and 2, the expected surplus in the S-efficient final allocation in Example 2 is strictly larger than the expected surplus in Example 1.

Thus, given an initial division S, an ex-post optimal contract maximizes the expected surplus from the asset by specifying (1) an S-efficient final allocation that maximizes the ex-post surplus and (2) an ex-ante transfer that appropriately divides the expected surplus between the players. Note that achieving S-efficiency is also possible when the contract is incomplete in the sense that each player can walk away with his share of the asset in stage 1. In this case, because the S-efficient final allocation creates surplus, the contract can specify stage 1 transfers to the players (that sum to zero) that incentivize the players to implement the S-efficient final allocation.[8]

2.2 Incomplete information

When players' valuations are realized privately, in addition to the initial division and final allocation the contract also specifies:

1. Stage 0 payments $t_1^0, t_2^0 \in \mathbb{R}$, made by players 1 and 2 respectively.
2. Stage 1 payment schedules that specify, for each pair of reports (v_1, v_2) of players' valuations, the payments $t_1^1(v_1, v_2)$ and $t_2^1(v_1, v_2)$ made by players 1 and 2 respectively.

The payments can be thought of as being made to a third party (or received from that party if they are negative). Below we comment on how to overcome the need for a third party. Implementing an S-efficient final allocation requires that:

(IC) t_1^1 and t_2^1 induce players to reveal their valuations for the asset truthfully,

[8] In fact, S-efficiency is also achievable when the contract is incomplete in the sense that it does not specify a final allocation. If players can bargain efficiently and make transfers in stage 1, then they will likely maximize the ex-post surplus from the asset and divide the surplus between them. The distribution of this surplus would depend on the initial division of the asset, but anticipating this the players can use the initial transfer t to compensate the player with the weaker ex-post bargaining position.

(IR) t_1^1 and t_2^1 provide players with incentives, if needed, to participate in the mechanism (after they learn their valuations), and

(BB) the stage 0 payments and the stage 1 expected payments sum up to zero.

Incentive Compatibility (IC) is required to implement any final allocation; Individual Rationality (IR) is relevant in settings in which players can walk away with their initial share of the asset at the beginning of stage 1; Budget Balance (BB) is required to guarantee that the players capture the entire surplus generated from the asset.

The difficulty in guaranteeing (IC) arises because each player's utility may depend on the other player's valuation through the reallocation costs.[9] The following result identifies sufficient conditions on the reallocation cost functions for implementing an S-efficient allocation as an ex-post equilibrium. In an ex-post equilibrium, given that the other player reports truthfully, a player prefers to report his valuation truthfully for any valuation of the other player. The result requires that the cost functions be regular,[10] and also uses the following definition.

Definition 1 *Let V and X be two sets of real numbers. A function $f : V \times X \to \mathbb{R}$ satisfies (strict) increasing differences in (v, x) if $g(v) = f(v, x') - f(v, x)$ (strictly) increases in v for any $x' > x$. If $g(v)$ (strictly) decreases in v, then f satisfies (strict) decreasing differences in (v, x).*

Proposition 1 *For any initial division S, an S-efficient final allocation is implementable as an ex-post equilibrium if the reallocation cost functions C^G and C^L are regular and satisfy decreasing differences in (v_G, Δ) and increasing differences in (v_L, Δ).*

The conditions in the statement of the proposition say that the marginal cost of reallocation increases in the losing player's valuation and decreases in the gaining player's valuation, both

[9] When this is not the case, players have "private values" and a Groves (1973) mechanism can be used to implement an S-efficient allocation in dominant strategies without imposing any additional conditions on the cost functions.

[10] The reallocation cost functions C^L and C^G are *regular* if (1) the support of each player's possible valuations is an interval I, (2) C^L and C^G are continuously differentiable in v^L and v^G, and (3) their derivatives are bounded uniformly in Δ by a function that is integrable on I. Condition (1) can be weakened provided that conditions (2) and (3) hold on an interval that contains the support of a player's valuations.
This regularity condition is weaker than the compactness and continuous differentiability conditions on players' utilities specified in Section 5.2 of Bergemann and Välimäki (2002), who provide sufficient conditions for ex-post implementation. Note that even requiring that players' valuations be drawn from a bounded interval and that C^L and C^G be twice continuously differentiable would not be enough to satisfy their conditions. This is because for a player's utility to be twice continuously differentiable, the "switch" from C^L to C^G that occurs when a player's final allocation increases to more than his initial share must be done in a sufficiently smooth manner. For this, additional conditions on C^L and C^G must be specified. We replace these conditions with our requirements below for increasing and decreasing differences.

for the gaining player and for the losing player. The proof of Proposition 1 is in the Appendix.

Given t_1^1 and t_2^1 that guarantee (IC), these payment schedules can be increased by a constant (if necessary) to guarantee (IR), without affecting players' incentives for truthful revelation. The payments t_1^0 and t_2^0 can then be chosen to simultaneously guarantee (BB) and divide the surplus between the players. Thus, $t_1^0 + t_2^0$ equals the sum of the expected payments specified by $t_1^1 + t_2^1$. If this expectation is positive, then the players can be thought of as being paid an amount "in advance" in stage 0, and repaying this amount, in expectation, in stage 1. If this expectation is negative, then the players can be thought of as funding the expected stage 1 payments that they will receive. This allows the players to overcome the the expected stage 1 deficit that arises for some initial divisions when (IR) holds.[11] If (IC) is required to hold as a Bayesian equilibrium, instead of an ex-post post equilibrium, then, under mild conditions, t_1^1 and t_2^1 can be chosen so that for any realization (v_1, v_2) of players' valuations, $t_1^1(v_1, v_2) + t_1^2(v_1, v_2) = t_1^0 + t_2^0$.[12] In this case, there is no need for a third party to balance the budget ex-post: positive payments t_i^0 are put in a "box" in stage 0, and payments t_i^1 are put in or taken from the box at stage 1; negative payments t_i^0 are taken from the box at the end of stage 1, after payments t_i^1 have been made.

To summarize, whether players' valuations are realized publicly or privately, for any initial division S the players are likely to reach an S-efficient final allocation that maximizes the expected surplus from the asset. We therefore proceed to study how the maximal expected surplus changes with the initial division without specifying the exact details of the environment.

Comment. Studying the relationship between the initial division of the asset and the maximal expected surplus from the asset is also relevant in single-person settings, such as the capital allocation example described in the Introduction. In such settings, the asset can be initially allocated between two uses, and xv_i is the surplus generated by allocating x shares of the asset to use i. The reallocation costs capture the loss of surplus that arises from reallocating the asset. Maximizing the expected surplus then corresponds to maximizing the single-person's utility from the asset.

3 Results

We study how the maximal expected surplus changes with the concentration of the initial division. This initial concentration $s_{\max}$ is the larger share in the initial division. Because the

[11] Such a deficit arises in the standard setting with no reallocation costs when the initial division is sufficiently concentrated, as shown by Myerson and Satterthwaite (1983) and Cramton, Gibbons, and Klemperer (1987).

[12] If players' valuations are stochastically independent, then this can be achieved with payment schedules similar to that of Arrow (1979) and d'Aspremont and Gérard-Varet (1979). If players' valuations are not stochastically independent but meet certain (generic) distributional assumptions, then the payment schemes of Kosenock and Severinov (2008) can be used without imposing on players' utilities the restrictions in Proposition 1.

distribution of valuations is symmetric, the maximal expected surplus is identical in the two initial divisions with the same concentration.

We begin by analyzing how amount-insensitive reallocation costs $AIC(v_L, v_G)$, which may depend on players' valuations but are independent of the amount reallocated, influence the relationship between the initial concentration and the maximal expected surplus. We then perform a similar analysis for variable reallocation costs $VC(v_L, v_G, \Delta)$[13], which may depend on players' valuations and on the amount reallocated. Finally, we examine reallocation costs that have an amount-insensitive component and a variable component.

3.1 Amount-insensitive reallocation costs

Some reallocation costs do not depend on the amount reallocated. One example is the overhead associated with transferring ownership of a real-estate property. The overhead incurred in the form of meetings, legal documents, and fees may essentially be the same regardless of how much of the property changes ownership. Nevertheless, this overhead may depend on how the property is used (commercial, residential, etc.). We denote by $AIC(v_L, v_G)$ the amount-insensitive cost involved in transferring any amount $\Delta > 0$ from a player with valuation v_L to a player with valuation v_G. For reallocation to be efficient we must have $v_G > v_L$.

Because the cost of reallocating is independent of the amount reallocated, and the benefit of reallocating increases in the amount reallocated, it is optimal either to allocate the entire asset to the player with valuation v_G or to maintain the initial division. Which of these is optimal depends on the initial division, since it may be that the benefit exceeds the cost when the amount reallocated is large, but not when it is small. We thus denote by $s_{change}(v_L, v_G)$ the minimal amount for which the benefit of reallocation is larger than the cost. That is,

$$s_{change}(v_L, v_G) = \inf \{ s \leq 1 : s(v_G - v_L) \geq AIC(v_L, v_G) \},$$

with $s_{change}(v_L, v_G) = 1$ if no such s exists. We omit the dependence of s_{change} on the valuations when this does not introduce any ambiguity.

Let S be some initial division. The S-efficient final allocation x^S allocates the entire asset to the player with the higher valuation if the other player's initial share is sufficiently large, and otherwise maintains the initial division S:

$$x^S(v_1, v_2) = \begin{cases} 1 & v_1 > v_2 \text{ and } 1 - s > s_{change}(v_1, v_2) \\ 0 & v_2 > v_1 \text{ and } s > s_{change}(v_2, v_1) \\ s & \text{otherwise.} \end{cases} \quad . \tag{1}$$

Changes in the initial division affect the probabilities that players are allocated the entire asset in this efficient final allocation. The probability that a player, say player 1, is allocated

[13] Variable costs are assumed to converge to 0 as Δ converges to 0

the entire asset ex-post decreases in player 1's initial share, and the probability that player 2 is allocated the entire asset increases in player 1's initial share. To see the former, choose a realization of players' valuations in which player 1's valuation is larger than player 2's valuation. As the initial share of player 1 increases, the benefit of allocating him the entire asset decreases, whereas the cost of doing so remains constant. Hence, the set of realizations for which player 1 is allocated the entire asset ex-post shrinks in his initial share, so the probability he is allocated the entire asset decreases.[14]

Suppose that the initial division S is not fully concentrated, that is, $s_{\max} < 1$. To understand how an increase in the initial concentration $s_{\max}$ affects the maximal expected surplus, we compare the expected surplus in x^S when the initial division is S with the expected surplus in a "modified" final allocation when the initial division is more concentrated. The modified final allocation is the allocation that maintains the new, more concentrated initial division for those realizations for which x^S maintains S, and allocates the entire asset to the player with the higher valuation for all other realizations (as does x^S).

Relative to x^S with the initial division S, the effect on the expected surplus of increasing the initial concentration and using the modified final allocation is the aggregate effect on all the realizations of players' valuations. To discern this effect, choose two valuations $v_G > v_L$ and consider changes in the *sum* of the ex-post surpluses of the two realizations (v_G, v_L) and (v_L, v_G). There are three cases to consider, which depend on the relative magnitudes of $s_{\max}$ and $s_{change}(v_L, v_G)$:

1. If $s_{\max} \leq s_{change}$, then x^S maintains initial division for both realizations, and the sum of the ex-post surpluses of the two realizations is

$$s_{\max} v_L + (1 - s_{\max}) v_G + (1 - s_{\max}) v_G + s_{\max} v_L = v_L + v_G.$$

   This is also the sum in the modified final allocation if $s_{\max}$ is increased.

2. If $s_{\max} < 1 - s_{change}$, then x^S allocates the asset to the player with v_G both when his initial share is $s_{\max}$ and when it is $1 - s_{\max}$. The benefit of doing so (summed over the two realizations) is $2v_G$, and the cost is $2AIC(v_L, v_G)$. The benefit and cost are the same in the modified final allocation if $s_{\max}$ is increased so that the initial share of each player remains strictly positive. If $s_{\max}$ is increased to the fully concentrated initial division, then the benefit is the same and the cost is reduced to $AIC(v_L, v_G)$.

3. Otherwise, x^S allocates the asset to the player with valuation v_G if his initial share is $1 - s_{\max}$, because $s_{\max} > s_{change}$, and maintains the initial division if his initial share

[14] This does not imply that the expected ex-post share of a player decreases in his initial share. Whether this happens depends on the distribution of players' valuations. This is because as a player's initial share increases, his ex-post share increases for those realizations for which the initial division is maintained.

is $s_{\max}$, because $1 - s_{\max} \leq s_{change}$. In the former case, moving to the modified final allocation and increasing the initial concentration has no effect on the ex-post surplus because the reallocation costs are amount insensitive. But in the latter case, moving to the modified final allocation and increasing the initial concentration strictly increases the ex-post surplus. This is because in this case the initial division with the higher concentration is maintained, and in this division strictly more than $s_{\max}$ of the asset is given to the player with the higher valuation.

Therefore, increasing the initial concentration and using the modified final allocation increases the sum of the ex-post surpluses of (v_L, v_G) and (v_G, v_L) relative to using x^S with the initial division S.

Moreover, if $s_{\max} \geq s_{const}(v_L, v_G) = \max\{s_{change}(v_L, v_G), 1 - s_{change}(v_L, v_G)\}$ then the increase in the sum is strict even if the initial concentration is increased slightly. The change is even higher if we move to an efficient final allocation for the more concentrated initial division. If $s_{\max} \leq s_{const}(v_L, v_G)$, then an argument similar to the one developed in cases 1 and 2 shows that as we decrease the initial concentration and use the modified final allocation, the sum of the ex-post surpluses does not change, and must therefore be constant.[15] Applying this analysis to all realizations, we obtain the following result.[16]

Let $s_{AIC} = \inf\{s : s_{const}(v_L, v_G) \leq s \text{ for a positive measure of pairs } (v_L, v_G)\}$.

Proposition 2 *The maximal expected surplus increases in the concentration of the initial division. The increase is strict if and only if the initial concentration is larger than s_{AIC}.*

In Example 2, if $C < 1/2$ then $s_{AIC} = 1/2$, because by definition $s_{const}(v_L, v_G) \geq 1/2$ for any v_L and v_G, and $s_{const}(v_L, v_G) = s_{change}(v_L, v_G) = 1/2$ when $v_G - v_L = 2C$. Therefore, the expected surplus strictly increases in the initial concentration. If $1/2 \leq C < 1$ then $s_{AIC} = C$, because $s_{const}(v_L, v_G) \geq s_{change}(v_L, v_G) \geq C$ for any v_L and v_G (so $s_{AIC} \geq C$) and $s_{const}(0, 1) = s_{change}(0, 1) = C$ (so $s_{AIC} \leq C$). The maximal expected surplus is thus constant if the initial concentration is less than C, and otherwise strictly increases in the initial concentration. If $C \geq 1$, then $s_{AIC} = 1$, because $s_{const}(v_L, v_G) = 1$ for any v_L and v_G. Therefore, the maximal expected surplus is constant in the initial concentration (indeed, when $C \geq 1$ maintaining the initial division is optimal). In summary,

$$s_{AIC} = \begin{cases} \frac{1}{2} & \text{if } C < \frac{1}{2} \\ C & \text{if } \frac{1}{2} \leq C < 1 \\ 1 & \text{if } C \geq 1 \end{cases}.$$

[15] This includes the case $s_{\max} = 1$, because then Case 2 does not arise.

[16] For $v_L = v_G$ case 1 applies and gives an ex-post surplus of $v_L = v_G$ both in x^S and in the modified final allocation.

In Example 2, the threshold s_{AIC} is monotone in the reallocation costs C. This is not always the case, as the following example demonstrates.

Example 3. Suppose that player 1's valuation is distributed uniformly on $[0, 1/3] \cup [2/3, 1]$, that player 2's valuation is $v_2 = 1 - v_1$, and that reallocation costs are amount-insensitive at C. If $C < 1/6$ then $s_{AIC} = 1 - 3C$, because $s_{change}(v_L, v_G) \leq 3C \leq 1/2$ for any v_L and v_G, so $s_{const}(v_L, v_G) = 1 - s_{change}(v_L, v_G) \geq 1 - 3C$ (and $s_{AIC} \geq 1 - 3C$), and $s_{const}(1/3, 2/3) = 1 - s_{change}(1/3, 2/3) = 1 - 3C$ (so $s_{AIC} \leq 1/3C$). If $1/6 \leq C < 1/2$ then $s_{AIC} = 1/2$, because by definition $s_{const}(v_L, v_G) \geq 1/2$ for any v_L and v_G, and $s_{const}(v_L, v_G) = s_{change}(v_L, v_G) = 1/2$ when $v_G - v_L = 2C$. As in Example 2, if $1/2 \leq C < 1$ then $s_{AIC} = C$, and if $C \geq 1$ then $s_{AIC} = 1$. In summary,

$$s_{AIC} = \begin{cases} 1 - 3C & \text{if } C < \frac{1}{6} \\ \frac{1}{2} & \text{if } \frac{1}{6} \leq C < \frac{1}{2} \\ C & \text{if } \frac{1}{2} \leq C < 1 \\ 1 & \text{if } C \geq 1 \end{cases}.$$

Figure 1 depicts the threshold S_{AIC} as a function of C in Examples 2 and 3.

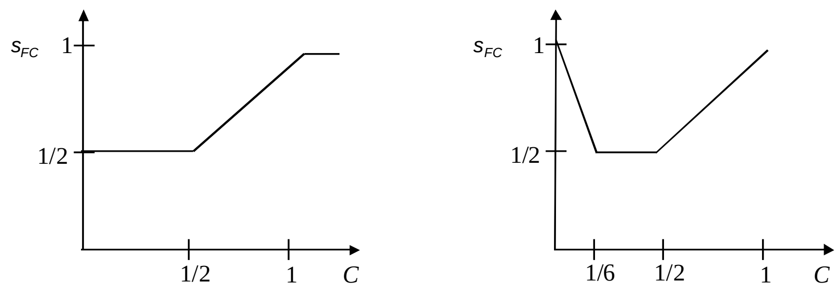

Figure 1: The strict monotonicity threshold S_{AIC} as a function of the amount-insensitive reallocation cost C in Example 2 (left) and Example 3 (right)

Proposition 2 implies that whenever $s_{AIC} < 1$ the only two initial divisions that maximize efficiency are the fully concentrated ones, in which one player's initial share is the entire asset. A necessary and sufficient condition for $s_{AIC} < 1$ is that there is a positive measure of realizations for which the difference between the two valuations is strictly larger than the reallocation cost associated with the two valuations. If this condition holds, then there is an $\varepsilon > 0$ such that there is a positive measure of realizations for which $s_{const} < 1 - \varepsilon$, so $s_{AIC} < 1 - \varepsilon$. If the condition does not hold, then for a measure 1 of realizations the initial division is maintained. As we vary the initial division the maximal expected surplus then remains constant, because the distribution of valuations is symmetric and there is no reallocation. In Examples 2 and 3, the maximal difference between the valuations of the two players is 1, so $s_{AIC} < 1$ if and only if $C < 1$.

Because efficiency considerations push toward a fully concentrated initial division and a final allocation that either maintains the initial division or reallocates the entire asset, ex-post one of the players is allocated the entire asset. This is similar to the final allocation in the setting of Cramton, Gibbons, and Klemperer (1987), in which there are no reallocation costs. But in contrast to their setting, in which the player who ends up with the asset is the one with the highest valuation, in our setting with amount-insensitive reallocation costs this player may sometimes be the one with the lower valuation.

Proposition 2 shows that increasing the concentration of the initial division increases the maximal expected surplus, regardless of the magnitude of the amount-insensitive cost or how the cost depends on the players' valuations. This observation tells us how to allocate the asset when the initial share of each player i must be at least ε_i, and ex-post the asset can be reallocated fully. In such cases, the initial division should assign the remaining $1 - \varepsilon_1 - \varepsilon_2$ shares to the player with the larger ε_i. The observation also tells us that when one can initially assign only part of the asset and wait with the rest until valuations are realized, it is efficient to assign that part to a single player.

3.2 Concave reallocation costs

Reallocation costs may have a variable component that is concave in the amount reallocated. This may be the case if reallocation requires learning by the gainer. This may also be the case if reallocation creates psychological discomfort for the loser, as discussed in the Introduction. The total reallocation cost is then concave, whether or not there are also amount-insensitive costs.

In this case, as with amount-insensitive reallocation costs, for any initial division and any realization of players' valuations it is efficient ex-post either to allocate the entire asset to the player with the higher valuation or to maintain the initial division. The reason for this is that the marginal benefit of reallocating is constant in the amount reallocated, whereas the marginal cost of reallocating decreases in the amount reallocated.

The relationship between the initial division and the maximal expected surplus is also similar to the one in the case of amount-insensitive costs. The maximal surplus remains the same for initial divisions that are close to equal, and strictly increases as the initial division becomes more concentrated. The proof of this result has the same structure of the proof of Proposition 2 – it compares the sum of the ex-post surpluses of two realizations, (v_L, v_G) and (v_G, v_L), in an efficient final allocation and in a modified final allocation with a slightly more concentrated initial division. The modified final allocation is the same as in the proof of Proposition 2, and as in that proof the comparison involves three cases.

In case 1, the initial allocation is maintained for both realizations, so the sum of the ex-post surpluses does not change, as in the proof of Proposition 2. In case 2, the asset is reallocated for both realizations. If we slightly increase the concentration of the initial division and use the modified final allocation, the cost of reallocating shares to the player with the larger initial share decreases, and the cost of reallocating shares to the player with the smaller share increases. Concavity implies that the decrease is larger than the increase, so the sum of the ex-post surpluses increases. The increase is strict if the cost function is not linear in this region. In case 3, the asset is reallocated to the player with the higher valuation if he has the smaller initial share, and the initial division is maintained if the player with the higher valuation has the larger initial share. If we slightly increase the concentration of the initial division and use the modified final allocation, then the reallocation costs increase when the asset is reallocated, and the ex-post surplus increases when the asset is not reallocated. The former increase is smaller than the latter one. This is because by concavity the marginal increase in the reallocation cost is smaller than the average reallocation cost. And by the optimality of the final allocation, the average reallocation cost is smaller than the average benefit of reallocating, which equals $v_G - v_L$. This average benefit equals the marginal increase in the ex-post surplus when the initial division is maintained.

To state the result formally, consider a total reallocation cost function

$$C(v_L, v_G, \Delta) = AIC(v_L, v_G) + VC(v_L, v_G, \Delta),$$

where VC is concave in Δ. For every two valuations $v_G > v_L$, let

$$s_{change}\left(v_L, v_G\right) = \inf\left\{s : s\left(v_G - v_L\right) \geq C\left(v_L, v_G, s\right)\right\},$$

with $s_{change}\left(v_L, v_G\right) = 1$ if no such s exists, and let

$$s_{lin}\left(v_L, v_G\right) = \sup\left\{s : C\left(v_L, v_G, \Delta\right) \text{ is linear in } \Delta \text{ on } [1-s, s]\right\}. \tag{2}$$

Let

$$s_{const}\left(v_L, v_G\right) = \begin{cases} \min\left\{s_{lin}\left(v_L, v_G\right), 1 - s_{change}\left(v_L, v_G\right)\right\} & \text{if } s_{change}\left(v_L, v_G\right) < 1/2 \\ s_{change}\left(v_L, v_G\right) & \text{Otherwise.} \end{cases}$$

This definition of s_{const} extends the definition of s_{const} for amount-insensitive reallocation costs, because for such costs $s_{lin}\left(v_L, v_G\right) = 1$. Let

$$s_{concave} = \inf\left\{s : s_{const}\left(v_L, v_G\right) \leq s \text{ for a positive measure of realizations } \left(v_L, v_G\right)\right\}.$$

We can now state the result for concave reallocation costs. The proof is in the Appendix.

Proposition 3 *The maximal expected surplus increases in the concentration of the initial division. The increase is strict if and only if the initial concentration is larger than* $s_{concave}$.

Proposition 3 extends to a dynamic environment in which players' valuations in a given period are statistically independent from their valuations in previous periods (but need not be identically distributed across periods or independently distributed within a period). The asset can be reallocated each period, after that period's valuations are realized. Efficiency amounts to maximizing the (possibly discounted) sum of the expected surpluses across periods. The maximal surplus is obtained by initially allocating the entire asset to a single player, and then, in each period after valuations are revealed, either letting the player keep the entire asset or reallocating the entire asset to the other player. To see why this maximizes the sum of expected surpluses, note that the sum cannot exceed what is obtained by maximizing the expected surplus in each period separately. And per-period maximization is obtained by allocating the entire asset to a single player at the beginning of the period, before the valuations for that period are known, which is exactly the final allocation from the previous period, for any realization of players' valuations.

3.3 Convex variable reallocation costs

Suppose that reallocation costs have a variable component that is convex in the amount reallocated. In this case, it is generally not efficient ex-post either to reallocate the entire asset to one player or to maintain the initial division. In addition, unlike with concave costs, the qualitative relationship between the initial allocation and the maximal expected surplus depends on whether there are also amount-insensitive costs. We begin with the case of no amount-insensitive costs, and then consider the combination of amount-insensitive costs and convex variable costs.

With no amount-insensitive costs, the maximal surplus remains the same for initial divisions that are close to equal, and strictly decreases as the initial division becomes more concentrated. The proof of this result compares the sum of the ex-post surpluses of two realizations, (v_L, v_G) and (v_G, v_L) with $v_G > v_L$, in two efficient final allocations that correspond to two different initial divisions, with the second initial division being slightly less concentrated (more equal) than the first. There are three cases to consider, which differ in how the first initial division relates to the optimal amount that would be reallocated from a player with valuation v_L to a player with valuation v_G if the initial share of the player with valuation v_L were 1.

In case 1, this optimal unconstrained amount is smaller than both players' initial shares, so this amount is reallocated for both pairs of realized valuations. This amount is also reallocated when the initial division is slightly less concentrated. Therefore, the sum of the ex-post surpluses does not change between the two final allocations. In case 2, the optimal unconstrained amount is larger than both players' initial shares, so the entire asset is allocated to the player with the higher valuation for both realizations. The same is true when the initial division is slightly less concentrated. Therefore, the total benefit of reallocation is the same in both sums of ex-post surpluses. But the total cost is smaller when the initial division is less concentrated, because the variable costs are convex. The increase in the sum of the ex-surpluses is strict if the cost function is not linear in this region.

In case 3, the optimal unconstrained amount is smaller than the larger initial share and larger than the smaller initial share. Therefore, this optimal amount is reallocated if the player with the higher valuation has the smaller initial share, and the entire asset is allocated to the player with the higher valuation if he has the larger initial share. The same is true when the initial division is slightly less concentrated. Therefore, the benefit of reallocation when the player with the higher valuation has the smaller initial share is the same in both final allocations, and the same is true for the cost of reallocation. But when the player with the higher valuation has the larger initial share, making the initial division less concentrated increases the amount reallocated. Because the optimal unconstrained amount is even larger, the benefit of the additional amount reallocated is larger than the cost. These three cases show that regardless of how the initial division relates to the optimal unconstrained amount, decreasing the concentration of the initial division increases the maximal expected surplus.

To state the result formally, consider variable costs $VC(v_L, v_G, \Delta)$ that are convex in Δ. For every two valuations $v_G > v_L$, let

$$s_{opt}(v_L, v_G) = \inf \left\{ \arg\max_{s \leq 1} \{ s(v_G - v_L) - VC(v_L, v_G, s) \} \right\}$$

be the optimal unconstrained amount, and define $s_{lin}(v_L, v_G)$ as in (2) with $VC(v_L, v_G, s)$ instead of $C(v_L, v_G, s)$. Let

$$s_{const}(v_L, v_G) = \begin{cases} 1 - s_{opt}(v_L, v_G) & \text{if } s_{opt}(v_L, v_G) < 1/2 \\ \min\{s_{lin}(v_L, v_G), s_{opt}(v_L, v_G)\} & \text{otherwise.} \end{cases}$$

With linear reallocation costs, $s_{lin}(v_L, v_G) = 1$, so

$$s_{const}(v_L, v_G) = \max\{s_{opt}(v_L, v_G), 1 - s_{opt}(v_L, v_G)\},$$

and because the benefit of reallocating is also linear in the amount reallocated, $s_{opt}(v_L, v_G)$ is either 0 or 1, so $s_{const}(v_L, v_G) = 1$. Let

$$s_{convex} = \inf\{s : s_{const}(v_L, v_G) \leq s \text{ for a positive measure of realizations } (v_L, v_G)\}.$$

We can now state the result for convex variable costs with no amount-insensitive costs. The proof is in the Appendix.

Proposition 4 *The maximal expected surplus decreases in the concentration of the initial division. The decrease is strict if and only if the initial concentration is larger than* s_{convex}.

When the reallocation costs include an amount-insensitive cost component in addition to convex variable costs, so that

$$C(v_L, v_G, \Delta) = AIC(v_L, v_G) + VC(v_L, v_G, \Delta),$$

the maximal expected surplus is not always monotone in the initial concentration. This is because, as Propositions 2 and 4 show, amount-insensitive costs and convex variable costs push in opposite directions. When one component dominates, a fully concentrated or an equal initial division is optimal. When neither dominates, other initial divisions may be strictly better. This latter possibility is shown in the following example in which neither the amount-insensitive cost nor the variable cost depend on players' valuations.

Example 4. Suppose there are three possible valuations, 2, 5/4, and 0, and that the distribution of valuations is uniform over the four realizations $(0, 5/4), (5/4, 0), (0, 2)$, and $(2, 0)$. Suppose that $VC(v_L, v_G, \Delta) = \Delta^2$, so the convex variable component does not depend on the valuations. The optimal unconstrained amount to reallocate (ignoring amount-insensitive costs) is the amount at which the difference between the valuations (which is either 2 or 5/4) equals the marginal cost. Thus, $s_{opt}(0, 2) = 1$ and $s_{opt}(0, 5/4) = 5/8$. Suppose that $AIC(v_L, v_G) = 3/8$, so the amount-insensitive cost also does not depend on valuations. Reallocation increases the surplus if the amount reallocated is such that the benefit minus the variable cost exceeds the amount-insensitive cost. This is what happens in (1) and (2) below.

Consider the following optimal final allocations and resulting expected surpluses for the initial divisions $(1, 0)$, $(1/2, 1/2)$, and $(9/16, 7/16)$.

1. If $s_{\max} = 1$, then the sum of the ex-post surpluses for the realizations $(0, 2)$ and $(2, 0)$ is

$$2 + 2 - 1 - \frac{3}{8} = 2\frac{5}{8},$$

   because the entire asset is allocated to the player with the higher valuation when his initial share is 0. The sum of the ex-post surpluses for the pairs $(0, 5/4)$ and $(5/4, 0)$ is

$$\frac{5}{4} + \frac{5}{8}\frac{5}{4} - \left(\frac{5}{8}\right)^2 - \frac{3}{8} = 1\frac{17}{64},$$

   because 5/8 of the asset is reallocated to the player with the higher valuation when his initial share is 0. The maximal expected surplus is therefore

$$\frac{1}{2}\left(2\frac{5}{8}\right) + \frac{1}{2}\left(1\frac{17}{64}\right) = 1\frac{121}{128}.$$

2. If $s_{\max} = 1/2$, then the entire asset is allocated to the player with the higher valuation. The sum of the ex-post surpluses for the realizations $(0, 2)$ and $(2, 0)$ is

$$2\left(2 - \frac{1}{4} - \frac{3}{8}\right) = 2\frac{3}{4},$$

   and for the pairs $(0, 5/4)$ and $(5/4, 0)$ the sum is

$$2\left(\frac{5}{4} - \frac{1}{4} - \frac{3}{8}\right) = 1\frac{1}{4}.$$

The maximal expected surplus is therefore

$$\frac{1}{2}\left(2\frac{3}{4}\right)+\frac{1}{2}\left(1\frac{1}{4}\right)=2.$$

Note that the sum of the ex-post surpluses of the latter two pairs is the same as if the initial division were maintained - the benefit of reallocation exactly equals the cost (amount-insensitive plus variable).

3. If $s_{\max}=9/16$, then the sum of the ex-post surpluses for the realizations $(0,2)$ and $(2,0)$ is

$$2-\left(\frac{9}{16}\right)^2-\frac{3}{8}+2-\left(\frac{7}{16}\right)^2-\frac{3}{8}=2\frac{95}{128},$$

because the entire asset is allocated to the player with the higher valuation. The sum of the ex-post surpluses for the realizations$(0,5/4)$ and $(5/4,0)$ is

$$\frac{5}{4}-\left(\frac{9}{16}\right)^2-\frac{3}{8}+\frac{9}{16}\frac{5}{4}=1\frac{67}{256},$$

because the entire asset is allocated to the player with the higher valuation when his initial share is $7/16$, and otherwise the initial division is maintained, because the amount-insensitive cost makes reallocation non-beneficial. The maximal expected surplus is therefore

$$\frac{1}{2}\left(2\frac{95}{128}\right)+\frac{1}{2}\left(1\frac{67}{256}\right)=2\frac{1}{512},$$

which is larger than those obtained in the fully concentrated and equal initial divisions. $\diamondsuit$

We now provide sufficient conditions on the amount-insensitive and variable cost that guarantee that either the equal initial division or the fully concentrated one are optimal. For every two valuations $v_G>v_L$, denote by $s_0(v_L,v_G)$ the minimal unconstrained share for which reallocation is beneficial, that is,

$$s_0\left(v_L,v_G\right)=\inf\left\{s:s\left(v_G-v_L\right)-VC\left(v_L,v_G,s\right)\geq AIC\left(v_L,v_G\right)\right\},$$

and $s_0(v_L,v_G)=1$ if this set is empty.

We first observe that for every two valuations $v_G>v_L$ the sum of the ex-post surpluses for the two realizations (v_L,v_G) and (v_G,v_L) in an efficient final allocation is maximized either in the equal initial division or in a fully concentrated one. To see why, fix some initial division S with concentration $s_{\max}$. Suppose first that $1-s_{\max}>s_0$. In this case, the amount-insensitive cost is not "binding" in the sense that reallocating at least some of the asset to the player with the higher valuation is beneficial whether his initial share is $s_{\max}$ or $1-s_{\max}$. Because the variable reallocation cost is convex, the arguments developed for Proposition 4 now show that the sum of the ex-post surpluses weakly decreases in $s_{\max}$. Therefore, $s_{\max}=1/2$ is better than any $s_{\max}$

for which $1 - s_{\max} > s_0$. Now suppose that $1 - s_{\max} \leq s_0$. In this case, the amount-insensitive cost component is binding when the player with the higher valuation has the larger initial share $s_{\max}$, so the initial division is then maintained. As $s_{\max}$ increases, this remains true. If $s_0 = 1$, then the initial division is also maintained when the player with the higher valuation has the smaller share (no matter how small that share is), and the sum of the maximal ex-post surpluses is constant at $v_L + v_G$. Otherwise, the sum of the the ex-post surpluses is $v_L + v_G$ plus any surplus generated by reallocation. The benefit of reallocation when the player with the higher valuation has the smaller initial share is

$$\max_{y \leq s_{\max}} \{y (v_G - v_L) - C (v_L, v_G, y)\}.$$

This benefit increases in $s_{\max}$, because the domain of maximization increases with $s_{\max}$. Therefore, $s_{\max} = 1$ is better than any $s_{\max}$ for which $1 - s_{\max} \leq s_0$.

We now provide a condition that ensures that the equal initial division maximizes the sum of the maximal ex-post surpluses for any two realizations (v_L, v_G) and (v_G, v_L), and a condition that assures the same is true for the fully concentrated initial division. Denote by $v^s_{opt}(v_L, v_G)$ the maximal surplus that can be achieved by transferring no more than s shares from a player with valuation v_L to a player with valuation v_G when ignoring amount-insensitive costs, that is,

$$v^s_{opt} (v_L, v_G) = \max_{y \leq s} \{y (v_G - v_L) - VC (v_L, v_G, y)\}.$$

Proposition 5 *If for a measure* 1 *of realizations the amount-insensitive cost is higher than* $2v^{1/2}_{opt} - v^1_{opt}$ *, then the expected surplus is maximized at the fully concentrated initial division. If for a measure* 1 *of realizations the amount-insensitive cost is lower than* $2v^{1/2}_{opt} - v^1_{opt}$*, then the expected surplus is maximized at the equal initial division.*

The proof of Proposition 5 is in the Appendix. Because $v^1_{opt} \geq v^{1/2}_{opt}$ for any realization of players' valuations, a sufficient condition for the first part of Proposition 5 to hold is that the amount-insensitive cost is always larger than $v^{1/2}_{opt}$. It is also instructive to consider variable costs that are linear in the amount reallocated. In this case, $2v^{1/2}_{opt} = v^1_{opt}$ so the fully concentrated initial division is strictly optimal whenever the amount-insensitive cost is strictly positive. Note that this case is covered by Proposition 3, because the total cost function is then concave.

4 More than two players

Our results extend to settings with more than two players. Consider such a setting with $n > 2$ players, and suppose that the total cost of reallocating Δ from a player with valuation v_L to a player with valuation v_G is $C (\Delta, v_L, v_G)$. That is, costs are additively separable reallocation by reallocation, and the cost of each reallocation depends only on the amount reallocated and the valuations of the losing and gaining players.

To generalize Propositions 2, 3, and 4, we introduce a partial ordering of the initial divisions. This ordering generalizes the complete ordering we used for initial divisions with two players, which was induced by the concentration of each initial division. Consider two initial divisions $\bar{s} = (s_1, \ldots, s_n)$ and $\bar{t} = (t_1, \ldots, t_n)$. We say that $\bar{s}$ *majorizes* $\bar{t}$ if when the coordinates of $\bar{s}$ and $\bar{t}$ are ordered so that $i < j$ implies $s_i \geq s_j$ and $t_i \geq t_j$, we have that

$$\sum_{i=1}^{k} s_i \geq \sum_{i=1}^{k} t_i \text{ for every } k = 1, \ldots, n.$$

We then have the following result, whose proof is in the Appendix.

Proposition 6 *Suppose that $\bar{s}$ majorizes $\bar{t}$. If the reallocation costs are amount-insensitive or concave in the amount reallocated, then the maximal expected surplus associated with $\bar{s}$ is higher than the maximal expected surplus associated with $\bar{t}$. If the reallocation costs are convex in the amount reallocated, then the maximal expected surplus associated with $\bar{s}$ is higher than the maximal expected surplus associated with $\bar{t}$.*

5 Appendix

5.1 Proof of Proposition 1

For truthful reporting by player 1 to be part of an ex-post equilibrium that implements a final allocation x, the following ex-post incentive compatibility condition must hold for every pair of valuations (v_1, v_2) and report $\hat{v}_1$ of player 1:

$$u(v_1, v_2, x(v_1, v_2), s) + t_1(v_1, v_2) \geq u(v_1, v_2, x(\hat{v}_1, v_2), s) + t_1(\hat{v}_1, v_2), \tag{EXIC}$$

and similarly for player 2. In what follows, we focus on demonstrating (EXIC) for player 1; the treatment of player 2 is analogous. Lemma 1 below characterizes transfers $t_1(\cdot, \cdot)$ such that if

$$x(v_1, v_2) \text{ increases in } v_1 \text{ for every } v_2 \tag{EXM}$$

and

$$u(v_1, v_2, x, s) \text{ satisfies increasing differences in } (v_1, x) \text{ (for every } v_2 \text{ and } s), \tag{ID}$$

then (EXIC) holds. The conditions (EXM) and (ID) are similar to the ones in Proposition 4 of Bergemann and Välimäki (2002). That result does not apply here, however, because for a fixed s it requires u to be twice continuously differentiable. By Lemma 1 below, to prove Proposition 1 it suffices to show that for an S-efficient allocation the conditions in the statement of Proposition 1 imply (EXM) and (ID) . To see that (EXM) holds, fix v_2. First, consider valuations $v_1 \geq v_2$. For such valuations we have $x(v_1, v_2) \geq s$, because in an S-efficient final allocation none of the asset is reallocated to a player with the lower valuation. Therefore, player 1 is the gaining player and player 2 is the losing player. Because $x(v_1, v_2)$ maximizes the sum of the players' utilities (surplus), we can apply Topkis's (1998) theorem 2.8.7 to show that $x(v_1, v_2)$ increases in v_1. For this it suffices to show that

$$f(v_1, x) \equiv u_1(v_1, v_2, x, s) + u_2(v_1, v_2, 1-x, 1-s) = xv_1 - C^G(v_2, v_1, x-s) + (1-x)v_2 - C^L(v_2, v_1, x-s)$$

$$= v_1 x + (1-x)\, v_2 - C(v_2, v_1, x-s)$$

satisfies strict increasing differences in (v_1, x) when $v_1 \geq v_2$ and x is restricted to $[s, 1]$.[17] Since $v_1 = v_G$ and C^G and C^L satisfy decreasing differences in $(v_G, x-s)$, we have that $-C = -C^G - C^L$ satisfies increasing differences in (v_1, x). In addition, $v_1 x$ satisfies strict increasing differences in (v_1, x) and $(1-x)\, v_2$ satisfies increasing difference in (v_1, x). Therefore, f satisfies strict increasing differences in (v_1, x), so $x(v_1, v_2)$ increases in v_1. Now consider valuations $v_1 \leq v_2$. For such valuations we have $x(v_1, v_2) \leq s$, so player 1 is the losing player and player 2 is

[17] This would satisfy the conditions of Topkis's theorem because f is trivially quasisupermodular in x, and strict increasing differences in (v_1, x) implies the strict single crossing property in (v_1, x).

the gaining player. Because $v_1 = v_L$ and C^G and C^L satisfy increasing differences in $(v_L, s - x)$, we have that the sum of players' utilities,

$$v_1 x + (1 - x)\, v_2 - C(v_1, v_2, s - x),$$

satisfies strict increasing differences in (v_1, x) when $v_1 \leq v_2$ and x is restricted to $[0, s]$ (note that $s - x$ decreases as x increases). Therefore, $x\,(v_1, v_2)$ increases in v_1. Finally, because for valuations v_1 and v_1' such that $v_1' \geq v_2 \geq v_1$ we have $x(v_1', v_2) \geq s \geq x\,(v_1, v_2)$, we conclude that $x\,(v_1, v_2)$ increases in v_1.

To show that (ID) holds, first consider allocations x restricted to $[s, 1]$. This implies that $v_1 = v_G$ and

$$u(v_1, v_2, x, s) = xv_1 - C^G(v_2, v_1, x - s).$$

Since xv_1 satisfies increasing differences in (v_1, x) and C^G satisfies decreasing differences in $(v_1, x - s)$, we are done. Now consider allocations x restricted to $[0, s]$. This implies that $v_1 = v_L$ and

$$u(v_1, v_2, x, s) = xv_1 - C^L(v_1, v_2, s - x).$$

Since xv_1 satisfies increasing differences in (v_1, x) and C^L satisfies increasing differences in $(v_1, s - x)$, we are done. Finally, consider allocations x' and x such that $x' \geq s \geq x$. We have that

$$\begin{aligned} u\,(v_1, v_2, x', s) - u\,(v_1, v_2, x, s) &= x'v_1 - C^G\,(v_2, v_1, x' - s) - \big(xv_1 - C^L\,(v_1, v_2, s - x)\big) \\ &= (x' - x)\, v_1 - C^G\,(v_2, v_1, x' - s) + C^L\,(v_1, v_2, s - x)\,. \end{aligned}$$

To show that the expression increases in v_1, it suffices to show that $C^G\,(v_2, v_1, x' - s)$ decreases in v_1 and $C^L\,(v_2, v_1, s - x)$ increases in v_1. Because C^G satisfies decreasing differences in (v_G, Δ), for $v_1' > v_1$ we have that

$$C^G\,(v_2, v_1', x' - s) - C^G\,(v_2, v_1', 0) \leq C^G\,(v_2, v_1, x' - s) - C^G\,(v_2, v_1, 0)\,,$$

and because $C^G\,(\cdot, \cdot, 0) = 0$, we have that

$$C^G\,(v_2, v_1', x' - s) \leq C^G\,(v_2, v_1, x' - s)\,.$$

Similarly, because C^L satisfies increasing differences in (v_L, Δ), for $v_1' > v_1$ we have that

$$C^L\,(v_2, v_1', s - x) - C^L\,(v_2, v_1', 0) \geq C^L\,(v_2, v_1, s - x) - C^L\,(v_2, v_1, 0)\,,$$

and because $C^L\,(\cdot, \cdot, 0) = 0$, we have that

$$C^L\,(v_2, v_1', s - x) \geq C^L\,(v_2, v_1, s - x)\,.$$

Lemma 1 *If*

$$t_1(\hat{v}_1, \hat{v}_2) = u(\hat{v}_1, \hat{v}_2, x(\hat{v}_1, \hat{v}_2), s) - \int_{-\infty}^{\hat{v}_1} \frac{\partial u(y, \hat{v}_2, x(y, \hat{v}_2), s)}{\partial v_1} 1_{y \in I} dy - K(\hat{v}_2) \tag{3}$$

for the interval I in our regularity condition and some $K(\hat{v}_2)$, (EXM) holds, and (ID) holds, then (EXIC) holds.

Proof. Fix some truthful report $\hat{v}_2$ of player 2, and denote by

$$U(\hat{v}_1, v_1) = u(v_1, \hat{v}_2, x(\hat{v}_1, \hat{v}_2), s) - t_1(\hat{v}_1, \hat{v}_2)$$

player 1's utility in the mechanism if he reports $\hat{v}_1$ when his valuation is v_1. Let $U(v_1) \equiv U(v_1, v_1)$ and denote by $H(\hat{v}_1, v_1) \equiv U(\hat{v}_1, v_1) - U(v_1)$ player 1's gain from reporting $\hat{v}_1$ instead of v_1 when his valuation is v_1. To show (EXIC) it suffices to show that for any $\hat{v}_1$ we have $U(\hat{v}_1, v_1) \leq U(v_1)$ or, equivalently, $H(\hat{v}_1, v_1) \leq 0$. To begin, regularity implies that $\partial u(v_1, v_2, x, s)/\partial v_1$ exists for v_1 in I, is continuous, and is uniformly bounded by a function that is integrable on I. Therefore, $H(\hat{v}_1, \cdot)$ is absolutely continuous, because $U(\cdot)$ and $U(\hat{v}_1, \cdot)$ are (by (3) and the properties of $\partial u(v_1, v_2, x, s)/\partial v_1$). Denote by H_{v_1} and u_{v_1} the derivatives of H and u with respect to their second argument. Because of (3), almost surely

$$H_{v_1}(\hat{v}_1, v_1) = u_{v_1}(v_1, \hat{v}_2, x(\hat{v}_1, \hat{v}_2), s) - u_{v_1}(v_1, \hat{v}_2, x(v_1, \hat{v}_2), s).$$

Suppose $\hat{v}_1 > v_1$. Then, by (EXM), $x(\hat{v}_1, v_2) \geq x(v_1, v_2)$. Now, (ID) implies that for any $v_1' > v_1$ we have

$$\int_{v_1}^{v_1'} u_{v_1}(y, \hat{v}_2, x(\hat{v}_1, \hat{v}_2), s) dy = u(v_1', \hat{v}_2, x(\hat{v}_1, \hat{v}_2), s) - u(v_1, \hat{v}_2, x(\hat{v}_1, \hat{v}_2), s)$$

$$\geq u(v_1', \hat{v}_2, x(v_1, \hat{v}_2), s) - u(v_1, \hat{v}_2, x(v_1, \hat{v}_2), s) = \int_{v_1}^{v_1'} u_{v_1}(y, \hat{v}_2, x(v_1, \hat{v}_2), s) dy.$$

By continuity of $u_{v_1}(\cdot, \hat{v}_2, x(v_1, \hat{v}_2), s)$, we have $u_{v_1}(v_1, \hat{v}_2, x(\hat{v}_1, \hat{v}_2), s) \geq u_{v_1}(v_1, \hat{v}_2, x(v_1, \hat{v}_2), s)$. Thus, $H_{v_1}(\hat{v}_1, v_1) \geq 0$ for $\hat{v}_1 > v_1$, and similarly $H_{v_1}(\hat{v}_1, v_1) \leq 0$ for $\hat{v}_1 < v_1$. Therefore, $H(\hat{v}_1, v_1) \leq H(v_1, v_1)$ for every $\hat{v}_1$ and v_1, so $H(\cdot, v_1)$ is maximized at $H(v_1, v_1) = 0$. ■

5.2 Proof of Proposition 3

Suppose that the initial division S is not fully concentrated, that is, $s_{\max} < 1$. Consider the S-efficient final allocation x^S given by (1). To understand how an increase in $s_{\max}$ affects the maximal expected surplus, compare the expected surplus in x^S when the initial division is S with the expected surplus in a final allocation that is a modification of x^S when $s_{\max}$ is increased. The modified final allocation maintains the new, more concentrated initial division for those realizations for which x^S maintains S, and allocates the entire asset to the player with the higher valuation for those realizations for which x^S does.

Relative to x^S with the initial division S, the effect on the expected surplus of increasing $s_{\max}$ and using the modified final allocation is the aggregate effect on all the realizations of players' valuations. To discern this effect, choose two valuations $v_G > v_L$ and consider changes in the sum of the ex-post surpluses of the two realizations (v_G, v_L) and (v_L, v_G). There are three cases to consider:

1. If $s_{\max} \leq s_{change}$, then the initial division is maintained in x^S for both realizations, and the sum of the ex-post surpluses of the two realizations is

$$s_{\max} v_L + (1 - s_{\max}) v_G + (1 - s_{\max}) v_G + s_{\max} v_L = v_L + v_G.$$

   This is also the sum in the modified final allocation if the initial concentration is increased.

2. If $s_{\max} < 1 - s_{change}$, then x^S allocates the asset to the player with v_G both when his initial share is $s_{\max}$ and when it is $1 - s_{\max}$. The sum of the ex-post surpluses of the two realizations is

$$v_G - C(v_L, v_G, s_{\max}) + v_G - C(v_L, v_G, 1 - s_{\max}).$$

   In the modified final allocation if $s_{\max}$ is increased to $s_{\max} + \varepsilon$ the sum becomes

$$v_G - C(v_L, v_G, s_{\max} + \varepsilon) + v_G - C(v_L, v_G, 1 - s_{\max} - \varepsilon).$$

   The second sum is larger than the first if and only if

$$C(v_L, v_G, s_{\max} + \varepsilon) - C(v_L, v_G, s_{\max}) \leq C(v_L, v_G, 1 - s_{\max}) - (v_L, v_G, 1 - s_{\max} - \varepsilon),$$

   which holds by concavity of $C(v_L, v_G, \cdot)$. Moreover, the inequality is an equality for small $\varepsilon > 0$ if and only if $s_{\max} < s_{lin}(v_L, v_G)$.

3. Otherwise, x^S allocates the asset to the player with valuation v_G if his initial share is $1 - s_{\max}$, because $s_{\max} > s_{change}$, and maintains the initial division if the player's initial share is $s_{\max}$, because $1 - s_{\max} \leq s_{change}$. The sum of the ex-post surpluses of the two realizations is

$$v_G - C(v_L, v_G, s_{\max}) + s_{\max} v_G + (1 - s_{\max}) v_L.$$

   In the modified allocation if $s_{\max}$ is increased to $s_{\max} + \varepsilon$ the sum becomes

$$v_G - C(v_L, v_G, s_{\max} + \varepsilon) + (s_{\max} + \varepsilon) v_G + (1 - s_{\max} - \varepsilon) v_L.$$

   The second sum is larger than the first if and only if

$$\varepsilon (v_G - v_L) - (C(v_L, v_G, s_{\max} + \varepsilon) - C(v_L, v_G, s_{\max})) \geq 0$$

$$\iff v_G - v_L \geq \frac{C(v_L, v_G, s_{\max} + \varepsilon) - C(v_L, v_G, s_{\max})}{\varepsilon}.$$

This inequality holds because

$$v_G - v_L \geq \frac{C(v_L, v_G, s_{\max})}{s_{\max}} \geq \frac{C(v_L, v_G, s_{\max} + \varepsilon) - C(v_L, v_G, s_{\max})}{\varepsilon},$$

where the first inequality follows from $s_{\max} \geq s_{change}(v_L, v_G)$ and the second inequality follows from the concavity of $C(v_L, v_G, \cdot)$. One of the inequalities holds strictly. Otherwise, if the second inequality is an equality then $C(v_L, v_G, \cdot)$ is linear on $[0, s_{\max}]$. If, in addition, the first inequality holds as an equality, then by definition $s_{change} = 0$. But this is impossible, because $s_{\max} < 1$ and $0 < 1 - s_{\max} \leq s_{change}$.

This shows that for the two realizations the sum of the ex-post surpluses in some S-efficient final allocation is lower than in the modified final allocation if the initial share of the player whose initial share is $s_{\max}$ increases to some $s_{\max} + \varepsilon$. Moreover, if $s_{change}(v_L, v_G) \geq 1/2$ and $s_{\max} \geq s_{change}(v_L, v_G)$, then $1 - s_{\max} \leq s_{change}(v_L, v_G)$, which means we are in case (3) and the increase is strict. If $s_{change}(v_L, v_G) \geq 1/2$ and $s_{\max} < s_{change}(v_L, v_G)$, then we are in case (1) and the increase is not strict. So if $s_{change}(v_L, v_G) \geq 1/2$ the increases is strict if and only if $s_{\max} \geq s_{change}(v_L, v_G)$. If $s_{change}(v_L, v_G) < 1/2$, then $s_{\max} > s_{change}(v_L, v_G)$. In this case, the increase is strict for small $\varepsilon > 0$ if and only if (a) $1 - s_{\max} > s_{change}(v_L, v_G)$ and $s_{\max} \geq s_{lin}(v_L, v_G)$ so we are in case (2) or (b) $1 - s_{\max} \leq s_{change}(v_L, v_G)$ so we are in case (3). By definition of $s_{const}(v_L, v_G)$, this means that regardless of the value of $s_{change}(v_L, v_G)$ the increase is strict for small $\varepsilon > 0$ if and only if $s_{\max} \geq s_{const}(v_L, v_G)$. The sum of the ex-post surpluses in an $(s_{\max} + \varepsilon)$-efficient final allocation is even higher.

If $s_{\max} \leq s_{const}(v_L, v_G)$, then an argument similar to the one developed in cases 1 and 2 shows that as we decrease the initial concentration and use the modified final allocation, the sum of the ex-post surpluses does not change, and must therefore be constant.[18] Therefore, for the two realizations the sum of the ex-post surpluses in an $s_{\max}$-efficient allocation as a function of $s_{\max}$ is constant on $[1/2, s_{const}(v_L, v_G)]$ and strictly increases on $[s_{const}(v_L, v_G), 1]$. The result now follows from the definition of $s_{concave}$.

5.3 Proof of Proposition 4

Consider some initial division S with an initial concentration $s_{\max} > 1/2$. Let $S^{-\varepsilon}$ be another initial division in which the share of the player with with share $s_{\max}$ in S is decreased to $s_{\max} - \varepsilon$ for a small $\varepsilon > 0$. For any $v_L < v_G$, because $VC(v_L, v_G, \cdot)$ is increasing and convex, it is easy to see that if the initial share of the player with the lower valuation is s, then it is optimal to

[18] This includes the case $s_{\max} = 1$, because then Case 2 does not arise.

reallocate $\min\{s, s_{opt}(v_L, v_G)\}$ of the asset to the player with the higher valuation. Therefore, the following final allocation x^S is S-efficient:

$$x^S(v_1, v_2) = \begin{cases} 1 & v_1 > v_2 \text{ and } 1 - s \leq s_{opt}(v_2, v_1) \\ s + s_{opt}(v_2, v_1) & v_1 > v_2 \text{ and } 1 - s > s_{opt}(v_2, v_1) \\ 0 & v_2 > v_1 \text{ and } s \leq s_{opt}(v_1, v_2) \\ s - s_{opt}(v_2, v_1) & v_2 > v_1 \text{ and } s > s_{opt}(v_2, v_1) \end{cases}.$$

Compare the expected surplus in x^S when the initial division is S with the expected surplus in $x^{S^{-\varepsilon}}$ when the initial division is $S^{-\varepsilon}$. To this end, choose two valuations $v_G > v_L$ and consider changes in the sum of the ex-post surpluses of the two realizations (v_G, v_L) and (v_L, v_G). There are three cases to consider:

1. If $s_{\max} \leq s_{opt}(v_L, v_G)$, then for both realizations x^S allocates the entire asset to the player with valuation v_G. The sum of the ex-post surplus of the two realizations is

$$v_G - VC(v_L, v_G, s_{\max}) + v_G - VC(v_L, v_G, 1 - s_{\max}).$$

When the initial division is $S^{-\varepsilon}$ and the asset is reallocated according to $x^{S^{-\varepsilon}}$, the sum is

$$v_G - VC(v_L, v_G, s_{\max} + \varepsilon) + v_G - VC(v_L, v_G, 1 - s_{\max} - \varepsilon).$$

Therefore, the sum of the surpluses is larger when we move to $S^{-\varepsilon}$ if and only if

$$VC(v_L, v_G, s_{\max}) - VC(v_L, v_G, s_{\max} - \varepsilon) \geq VC(v_L, v_G, 1 - s_{\max} + \varepsilon) - (v_L, v_G, 1 - s_{\max}),$$

which holds by the convexity of $VC(v_L, v_G, \cdot)$. Moreover, the inequality is strict if and only if $s_{\max} > s_{lin}(v_L, v_G)$.

2. If $s_{\max} \leq 1 - s_{opt}(v_G, v_L)$, then for both realizations x^S reallocates $s_{opt}(v_L, v_G)$ shares to the player with valuation v_G. The sum of the ex-post surpluses of the two realizations is

$$v_L + v_G + 2s_{opt}(v_L, v_G)(v_G - v_L) - 2VC(v_L, v_G, s_{opt}(v_L, v_G)),$$

and is independent of $s_{\max}$. When the initial division is $S^{-\varepsilon}$ and the asset is reallocated according to $x^{S^{-\varepsilon}}$, the same amount is reallocated and the sum of the ex-post surpluses remains the same.

3. Otherwise, x^S allocates the entire asset to the player with valuation v_G if his initial share is $s_{\max}$, and reallocates $s_{opt}(v_L, v_G)$ to the player with valuation v_G if his initial share is $1 - s_{\max}$. The sum of the ex-post surpluses is therefore

$$v_G - VC(v_L, v_G, 1 - s_{\max})$$

$$+\left(1-s_{\max}+s_{opt}\left(v_L,v_G\right)\right)v_G+\left(s_{\max}-s_{opt}\left(v_L,v_G\right)\right)v_L-VC\left(v_L,v_G,s_{opt}\left(v_L,v_G\right)\right).$$

When the initial division is $S^{-\varepsilon}$ for a small $\varepsilon>0$, $x^{S^{-\varepsilon}}$ also allocates the entire asset to the player with valuation v_G if his initial share is $s_{\max}-\varepsilon$. The cost of doing so is larger than before, and the difference in the cost is

$$VC\left(v_L,v_G,1-s_{\max}+\varepsilon\right)-VC\left(v_L,v_G,1-s_{\max}\right)).$$

$x^{s_{\max}-\varepsilon}$ also reallocates $s_{opt}\left(v_L,v_G\right)$ to the player with valuation v_G if his initial share is $1-s_{\max}+\varepsilon$, and the cost of doing so is the same as before. But the benefit of doing so is larger by $\varepsilon(v_G-v_L)$. The sum of surpluses is therefore larger if and only if

$$\varepsilon\left(v_G-v_L\right)\geq VC\left(v_L,v_G,1-s_{\max}+\varepsilon\right)-VC\left(v_L,v_G,1-s_{\max}\right).$$

This inequality holds strictly, because $s_{opt}>1-s_{\max}$ implies that after reallocating $1-s_{\max}$ shares to the player with valuation v_G, the benefit of reallocating an additional ε is strictly larger than the cost of doing so.

This shows that for some $\varepsilon>0$ the sum of the maximal ex-post surpluses for the two realizations (v_L,v_G) and (v_G,v_L) when the initial concentration is in $(s-\varepsilon,\varepsilon)$ is larger than when the initial concentration is s. Because the sum of the maximal ex-post surpluses for the two realizations is continuous in the initial concentration,[19] this implies that the sum decreases in the initial concentration on the interval $[1/2,1]$.[20] It remains to identify where the decrease is strict. First suppose that $s_{opt}\left(v_L,v_G\right)\geq 1/2$. If $s_{\max}>s_{opt}\left(v_L,v_G\right)\geq 1/2$, then $s_{\max}>1-s_{opt}\left(v_L,v_G\right)$, which means we are in case (3) above and the sum strictly decreases in the initial concentration. If $s_{opt}\left(v_L,v_G\right)\geq s_{\max}\geq 1/2$, then we are in case (1) so the sum strictly decreases if and only if $s_{\max}>s_{lin}\left(v_L,v_G\right)$. Thus, if $s_{opt}\left(v_L,v_G\right)\geq 1/2$ then the sum strictly decreases if and only if $s_{\max}>\min\left\{s_{opt}\left(v_L,v_G\right),s_{lin}\left(v_L,v_G\right)\right\}$. Now suppose that $s_{opt}\left(v_L,v_G\right)<1/2$, so $s_{\max}>s_{opt}\left(v_L,v_G\right)$. In this case, if $1-s_{\max}\geq s_{opt}\left(v_L,v_G\right)$ then we are in case (2) and the sum is constant in the initial concentration. If $1-s_{\max}<s_{opt}\left(v_L,v_G\right)$, then we are in case (3) and the sum strictly decreases. Thus, if $s_{opt}\left(v_L,v_G\right)<1/2$ then the sum strictly decreases if and only if $s_{\max}>1-s_{opt}\left(v_L,v_G\right)$. By definition of $s_{const}\left(v_L,v_G\right)$, this means that the sum strictly decreases in the initial concentration if and only if $s_{\max}>s_{const}\left(v_L,v_G\right)$, and is otherwise constant. The result now follows from the definition of s_{convex}.

[19]Convexity implies that $C\left(v_L,v_G,\cdot\right)$ is continuous on $[0,1]$, so it is equicontinuous there, as is v_G-v_L. This implies that the maximal ex-post surplus is continuous in the initial concentration.

[20]For any $s>1/2$, consider the lowest $s'<s$ such that for any initial concentration x in (s',s) the sum of the maximal ex-post surpluses with initial concentration x is larger than the sum with initial concentration s. By continuity of the sum of the maximal ex-post surpluses, the sum with initial concentration s' is also larger than the sum with initial concentration s. And if $s'>1/2$, then our results show that (s',s) can be extended downward by some $\varepsilon>0$, a contradiction.

5.4 Proof of Proposition 5

Consider some initial division S with concentration $s_{\max}$, and suppose that $s_{\max}$ is in $(0,1)$. Choose two valuations $v_G > v_L$, and consider the two realizations (v_L, v_G) and (v_G, v_L). We already know that the sum of the ex-post surpluses for the two pairs is maximized in either a fully concentrated initial division or an equal one.

To prove the first statement in the proposition, suppose that

$$AIC\left(v_L, v_G\right) \geq 2v_{opt}^{1/2}\left(v_L, v_G\right) - v_{opt}^{1}\left(v_L, v_G\right).$$

If the initial division is maintained, the sum of the ex-post surpluses for the two realizations is $v_L + v_G$. Consider the following cases:

1. If $s_0\left(v_L, v_G\right) > 1/2$, then an efficient final allocation for an equal initial division is to maintain the initial division. In this case, the sum of ex-post surpluses is $v_L + v_G$. If the initial division is fully concentrated, then the sum of the ex-post surpluses in an efficient final allocation is

$$v_G + v_L + \max_{y \leq 1}\left\{y\left(v_G - v_L\right) - C\left(v_L, v_G, y\right)\right\},$$

   which is larger than $v_L + v_G$.

2. If $s_0\left(v_L, v_G\right) \leq 1/2$, then the sum of the ex-post surpluses in an efficient final allocation for the equal initial division is

$$v_L + v_G + 2v_{opt}^{1/2}\left(v_L, v_G\right) - 2AIC\left(v_L, v_G\right),$$

   and for a fully concentrated one is

$$v_L + v_G + v_{opt}^{1}\left(v_L, v_G\right) - AIC\left(v_L, v_G\right).$$

   The latter sum is larger by assumption.

To prove the second statement in the proposition, suppose that

$$AIC\left(v_L, v_G\right) \leq 2v_{opt}^{1/2}\left(v_L, v_G\right) - v_{opt}^{1}\left(v_L, v_G\right).$$

This implies that $s_0\left(v_L, v_G\right) \leq 1/2$. Otherwise, $s_0\left(v_L, v_G\right) > 1/2$ implies that $v_{opt}^{1/2}\left(v_L, v_G\right) < AIC\left(v_L, v_G\right)$, which together with $v_{opt}^{1/2}\left(v_L, v_G\right) \leq v_{opt}^{1}\left(v_L, v_G\right)$ implies that

$$AIC\left(v_L, v_G\right) > 2v_{opt}^{1/2}\left(v_L, v_G\right) - v_{opt}^{1}\left(v_L, v_G\right).$$

The result now follows as in (2) in the proof of the first statement of the proposition.

5.5 Proof of Proposition 6

The proof requires the following definitions and lemmas. For any two players, we say that the concentration of the two players increases (decreases), if some of the initial share of the player with the lower (higher) initial share is added to the initial share of the player with the higher (lower) initial share, and refer to this process as increasing (decreasing) the concentration of the two players. Then, we have the following two results.

Lemma 2 *If the reallocation costs are amount insensitive or concave in the amount reallocated (with or without an amount-insensitive component), then increasing the concentration of any two players increases the maximal expected surplus.*

Proof. Because the marginal costs of reallocating are decreasing in the amount reallocated, there is an efficient final allocation with the property that if for a given realization of valuations some amount is transferred from player i to player j, then player i's entire initial share is transferred to player j. We consider such an efficient final allocation. Without loss of generality consider players 1 and 2, some realization of all players' valuations, and its "reflection" along the diagonal in which v_1 and v_2 are reversed. Suppose that $s_1 \geq s_2 > 0$, where s_i is player i's initial share. Let $v_L = \min\{v_1, v_2\}$ and $v_H = \max\{v_1, v_2\}$, and for simplicity suppose that $v_L < v_H$.[21] Consider the sum of the ex-post surpluses associated with the two realizations of players' valuations in the efficient final allocation. In this sum, consider the sum of surpluses associated with shares s_1 and s_2 that are initially assigned to the player with valuation v_L. Similarly to the proof of Propositions 2 and 3, we now show that this sum increases if the concentration of players 1 and 2 increases and a "modified" final allocation is implemented (the same argument applies to the sum of the surpluses associate with shares s_1 and s_2 that are initially assigned to the player with valuation v_H, which concludes the proof). There are three cases to consider:

1. The initial share of the player with valuation v_L is not reallocated whether it is s_1 or s_2. Then, the sum of the ex-post surpluses associated with these shares when they are initially assigned to the player with valuation v_L is $(s_1 + s_2)\, v_L$, and this is also the sum if the concentration of the players 1 and 2 increases.

2. The initial share of the player with valuation v_L is reallocated to a player with valuation $\bar{v} > v_L$ when the initial share is s_1, but not when it is s_2. Then, the sum is

$$s_1\bar{v} - c\,(v_L, \bar{v}, s_1) + s_2 v_L.$$

Because the initial share s_1 is reallocated, we have

$$s_1\bar{v} - c\,(v_L, \bar{v}, s_1) \geq s_1 v_L, \text{ or equivalently } s_1\,(\bar{v} - v_L) \geq c\,(v_L, \bar{v}, s_1)\,,$$

[21] An almost identical proof applies when $v_L = v_H$.

which together with the concavity of $c(v_L, \bar{v}, \cdot)$ implies that for positive $\varepsilon \leq s_2$ we have

$$\varepsilon(\bar{v} - v_L) \geq c(v_L, \bar{v}, s_1 + \varepsilon) - c(v_L, \bar{v}, s_1).$$

This last inequality implies that

$$(s_1 + \varepsilon)\bar{v} - c(v_L, \bar{v}, s_1 + \varepsilon) + (s_2 - \varepsilon)v_L \geq s_1\bar{v} - c(v_L, \bar{v}, s_1) + s_2 v_L.$$

3. The initial share of the player with valuation v_L is reallocated to a player with valuation $\bar{v} > v_L$ when the initial share is s_1, and to a player with valuation $\tilde{v} > v_L$ when the initial share is s_2 ($\tilde{v}$ may or may not equal $\bar{v}$). Then, the sum is

$$\underbrace{s_1\bar{v} - c(v_L, \bar{v}, s_1)}_{f(s_1)} + \underbrace{s_2\tilde{v} - c(v_L, \tilde{v}, s_2)}_{g(s_2)}.$$

To show that this sum increases if the concentration of players 1 and 2 increases, it suffices to show that the marginal value of f is higher than that of g. Suppose this is not the case. First note that $f(s_2) \leq g(s_2)$, otherwise it would have been better to reallocate the initial share s_2 to the player with valuation $\bar{v}$ instead of to the player with valuation $\tilde{v}$. Now, because the marginal values of f and g increase from s_2 to s_1 (reallocation costs are concave), and the marginal value of f at s_1 is lower than than of g at s_2, we have that the marginal value of f at x is lower than that of g at x for every x in $[s_2, s_1]$. Together with $f(s_2) \leq g(s_2)$ this implies that $f(s_1) < g(s_1)$. But then it would have been better to reallocate the initial share s_1 to the player with valuation $\tilde{v}$ instead of to the player with valuation $\bar{v}$.

Note that there is no "Case 4," because if s_2 is reallocated when it is the initial share of the player with valuation v_L, then by concavity of the costs s_1 is reallocated when it is the initial share of the player with valuation v_L. ■

Lemma 3 *If the reallocation costs are convex in the amount reallocated, then decreasing the concentration of any two players increases the maximal expected surplus.*

Proof. Without loss of generality consider players 1 and 2, some realization of all players' valuations, and its "reflection" along the diagonal in which v_1 and v_2 are reversed. Suppose that $s_1 > s_2$, where s_i is player i's initial share. Let $v_L = \min\{v_1, v_2\}$ and $v_H = \max\{v_1, v_2\}$, and for simplicity suppose that $v_L < v_H$.[22]. Let $x_{opt}(s)$ be the highest ranked sequence $(x_1, \ldots, x_n)$ within the set of sequences

$$\arg\max_{x_1, x_2, \ldots, x_n} \left\{ x_1 v_L + x_2 v_H - c(v_L, v_H, x_2) + \sum_{i=3}^{n} (x_i v_i - c(v_L, v_i, x_i)) : \sum_{i=1}^{n} x_i = s \right\}. \quad (4)$$

[22] An almost identical proof applies when $v_L = v_H$.

according to the lexicographic order (that is, for any other sequence $(y_1, \ldots, y_n)$ in the set, if there is some $i \leq n$ such that $y_i > x_i$, then there is some $j < i$ such that $y_j < x_j$). Consider the sum of the ex-post surpluses associated with the two realizations of players' valuations in some efficient final allocation. In this sum, consider the sum of surpluses associated with shares s_1 and s_2 that are initially assigned to the player with valuation v_L. This sum is given by the expression in the curly brackets in (4) with $(x_1, \ldots, x_n)$ given by $x_{opt}(s_1)$ plus the same expression with $(x_1, \ldots, x_n)$ given by $x_{opt}(s_2)$. We now show that this sum increases if the concentration of players 1 and 2 decreases, so that $x_{opt}(s_1)$ is replaced with $x_{opt}(s_1 + \varepsilon)$ and $x_{opt}(s_2)$ is replaced with $x_{opt}(s_2 - \varepsilon)$ for some small $\varepsilon > 0$ (an almost identical same argument applies to the sum of the surpluses associate with shares s_1 and s_2 that are initially assigned to the player with valuation v_H, which concludes the proof).

Fix some $\varepsilon < s_1 - s_2$. By the definition of x_{opt} and the convexity of the reallocation costs, for every i coordinate i of $x_{opt}(s_2 + \varepsilon)$ is $\varepsilon_i \geq 0$ larger than coordinate i of $x_{opt}(s_2)$, with $\sum_{i=1}^{n} \varepsilon_i = \varepsilon$. The increase in the surplus associated with share s_2 that is initially assigned to the player with valuation v_L resulting from changing $x_{opt}(s_2)$ to $x_{opt}(s_2 + \varepsilon)$ is

$$\varepsilon_1 v_L + \varepsilon_2 v_H - (c(v_L, v_H, x_2 + \varepsilon_2) - c(v_L, v_H, x_2)) + \sum_{i=3}^{n} (\varepsilon_i v_i - (c(v_L, v_i, x_i + \varepsilon_i) - c(v_L, v_i, x_i))), \tag{5}$$

where x_i is the i^{th} coordinate of $x_{opt}(s_2)$. The expression (5) is larger than

$$\varepsilon_1 v_L + \varepsilon_2 v_H - (c(v_L, v_H, y_2) - c(v_L, v_H, y_2 - \varepsilon_2)) + \sum_{i=3}^{n} (\varepsilon_i v_i + (c(v_L, v_i, y_i) - c(v_L, v_i, y_i - \varepsilon_i))), \tag{6}$$

where y_i is the i^{th} coordinate of $x_{opt}(s_1)$, because the reallocation costs are convex and (by definition of x_{opt}) $y_i \geq x_i + \varepsilon_i$. The expression (6) is an upper bound on the decrease in the surplus associated with share s_1 that is initially assigned to the player with valuation v_L resulting from changing $x_{opt}(s_1)$ to $x_{opt}(s_1 - \varepsilon)$. ■

Now, if $\bar{s}$ majorizes $\bar{t}$, then by Lemma 2 on page 47 of Hardy, Littlewood, and Pólya (1952) there exists a finite sequence of initial divisions $\bar{s}^1, \ldots, \bar{s}^m$, (where $\bar{s}^i = (\bar{s}^i_1, \ldots, \bar{s}^i_n)$) such that (1) $\bar{s}^1 = \bar{t}$, (2) $\bar{s}^m = \bar{s}$, and (3) for every $i < m$, there are players j_i and k_i such that $\bar{s}^{i+1}$ is derived from $\bar{s}^i$ by increasing the concentration of players j_i and k_i. For the first part of the proposition, Lemma 2 shows that for every $i < m$ the maximal expected surplus associated with $\bar{s}^{i+1}$ is higher than the maximal expected surplus associated with $\bar{s}^i$, which implies the result. The second part of the proposition follows from applying Lemma 3 to the same sequence of initial divisions.

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