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Reconsidering the continuous time limit of the GARCH(1, 1) process

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Abstract

In this note we reconsider the continuous time limit of the GARCH(1, 1) process. Let Y_k and σ_k^2 denote, respectively, the cumulative returns and the volatility processes. We consider the continuous time approximation of the couple (Y_k, σ_k^2) . We show that, by choosing different parameterizations, as a function of the discrete interval h , we can obtain either a degenerate or a non-degenerate diffusion limit. We then show that GARCH(1, 1) processes can be obtained as Euler approximations of degenerate diffusions, while any Euler approximation of a non-degenerate diffusion is a stochastic volatility process. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

In a seminal paper, Nelson (1990) analyzed the continuous time limit of ARCH models, as the discrete interval approaches zero. He showed that different classes of GARCH processes, e.g. GARCH(1,1) and exponential ARCH, EARCH, after a proper reparameterization, as the time interval shrinks, converge in distribution to a two-dimensional non-degenerate diffusion; i.e. to a

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diffusion in $R \times R^+$ driven by two Brownian motions, whose covariance matrix is non-singular. More recently, Fornari and Mele (1996) analyze the continuous time limit for a class of nonlinear ARCH models proposed by Ding et al. (1993). Duan (1997) introduces a new general class of volatility models, called augmented GARCH, where the augmentation plays the role of a Box–Cox transformation of the conditional variance and analyzes their continuous time limit. Also Fornari and Mele, as well as Duan obtain as a continuous time limit a non-degenerate diffusion. Such a (correct) result is somewhat surprising. In fact, discrete time GARCH models are characterized by only one source of noise, nevertheless their continuous time counterpart is a diffusion process driven by two independent (or at least non perfectly correlated) Brownian motions. In this sense, the continuous time limit of GARCH processes is the same as the continuous time limit of stochastic volatility processes, that are instead characterized by two non-perfectly correlated sources of noise. On the other hand, Kallsen and Taqqu (1998) via a continuous time embedding technique, show that the continuous time counterpart of a GARCH process is a process driven by only one Brownian motion. Their methodology is somewhat different from the diffusion approximation technique followed in the papers cited above, in that they allow asset prices to move continuously, while they allow volatility to jump only at integer values of time.

The purpose of this note is to reconsider the continuous time approximation of the GARCH(1,1) process and to show that, depending on the specific continuous approximation we consider, as the time interval shrinks, either a non-degenerate or a degenerate diffusion limit may arise. Furthermore, we shall show that GARCH processes can be obtained as Euler approximations of degenerate diffusions, while any Euler approximation of a non-degenerate diffusion results in a stochastic volatility process.

Non-degenerate and degenerate diffusion limits have very different implications for option pricing. In fact, if volatility is not a tradeable asset, then the degeneracy of the diffusion preserves market completeness, thus allowing for unique preference independent prices for contingent claims (e.g. Hobson and Rogers, 1998). On the other hand, non-degenerate diffusions in general do not preserve market completeness, so that the pricing of options requires additional assumptions on, e.g. risk premia and/or investors preferences (see e.g. Hull and White, 1987; Melino and Turnbull, 1990). From an empirical point of view, market completeness and so the existence of an exact option pricing formula based only on the assumption of no arbitrage opportunities, is not a very big issue, given the rapidly growing literature on nonparametric option pricing (e.g. see the recent survey by Ghysels et al., 1998 and the nonparametric test for the relevance of ARCH effect in option pricing by Christoffersen and Hahn, 1997).

The rest of this note is organized as follows. Section 2 considers the continuous approximation of the couple (Y_k, σ_k^2) and show that, depending on which

parameterization we choose, as a function of the time interval h , either a degenerate or a non-degenerate limit may arise. Section 3 analyzes the volatility models arising as Euler approximations to, respectively, non-degenerate and degenerate diffusions.

2. Continuous time limits for (Y_k, σ_k^2)

This section considers two different continuous time approximations of the GARCH(1, 1) process, one leading to a degenerate diffusion and another leading to a non-degenerate diffusion (Nelson’s result).

Let $Y_k - Y_{k-1}$, $k = 1, 2, \dots$ be returns on a generic financial asset and so let Y_k denote the cumulative returns. We begin by considering the following discrete time GARCH(1, 1) process, written as in Bollerslev et al. (1994),

$$Y_k - Y_{k-1} = \sigma_{k-1}\varepsilon_k, \tag{1}$$

$$\sigma_k^2 = \omega_0 + \omega_1\sigma_{k-1}^2 + \omega_2\sigma_{k-1}^2\varepsilon_k^2 \tag{2}$$

with $\omega_0, \omega_1, \omega_2 > 0$ and $\omega_1 + \omega_2 < 1$, $\varepsilon_k \sim iidN(0, 1)$. Let $F_k = \sigma(Y_0, \sigma_0^2, Y_1, \dots, Y_k)$, now σ_k^2 is F_k -measurable; however, $E((Y_k - Y_{k-1})^2 | F_{k-1}) = \sigma_{k-1}^2$, so that the conditional variance from time $k - 1$ to k is F_{k-1} -measurable, thus preserving the predictability of volatility; simply σ_k^2 is the conditional variance from time k to $k + 1$, thus σ_k^2 is F_k , but not F_{k-1} -measurable. Let h be the discrete time interval, and consider the following approximation scheme:

$$Y_{kh} - Y_{(k-1)h} = \sigma_{(k-1)h}\varepsilon_{kh} \tag{3}$$

$$\sigma_{kh}^2 - \sigma_{(k-1)h}^2 = \omega_0h + (\omega_1h - 1)\sigma_{(k-1)h}^2 + h^{-1}\omega_2h\sigma_{(k-1)h}^2\varepsilon_{kh}^2, \tag{4}$$

where $\varepsilon_{kh} \sim iidN(0, h)$.

We shall compute the first two conditional moments, and then, after few mild technical conditions, appeal to the theorems for weak convergence of Markov chains to diffusion processes by Strook and Varadhan (1979, Chapter 11) or by Ethier and Kurtz (1986, Chapter 8). Now, let $F_{(k-1)h} = \sigma(Y_{(k-1)h}, \sigma_{(k-1)h}^2)$, note that $\forall k$ and for any given h , $((Y_k - Y_{(k-1)h}), (\sigma_{kh}^2 - \sigma_{(k-1)h}^2))$ are first-order Markov chains. Now,

$$h^{-1}E((Y_{kh} - Y_{(k-1)h})|F_{(k-1)h}) = 0, \tag{5}$$

$$h^{-1}E((Y_{kh} - Y_{(k-1)h})^2|F_{(k-1)h}) = \sigma_{(k-1)h}^2 \tag{6}$$

$$h^{-1}E((\sigma_{kh}^2 - \sigma_{(k-1)h}^2)|F_{(k-1)h}) = h^{-1}\omega_0h + h^{-1}(\omega_1h + \omega_2h - 1)\sigma_{(k-1)h}^2, \tag{7}$$

$$\begin{aligned} h^{-1}E((\sigma_{kh}^2 - \sigma_{(k-1)h}^2)^2|F_{(k-1)h}) &= h^{-1}\omega_0^2h + h^{-1}(\omega_1h + \omega_2h - 1)^2\sigma_{(k-1)h}^4 \\ &\quad + 2h^{-1}\omega_2h\sigma_{(k-1)h}^4 \\ &\quad + 2h^{-1}\omega_0h(\omega_1h + \omega_2h - 1)\sigma_{(k-1)h}^2. \end{aligned} \tag{8}$$

We shall consider the following two sets of parameterizations, as a function of h . In both the cases:

$$\lim_{h \rightarrow 0} h^{-1} \omega_{0h} = \omega_0, \tag{9}$$

$$\lim_{h \rightarrow 0} h^{-1} (\omega_{1h} + \omega_{2h} - 1) = \theta < 0. \tag{10}$$

As for the first parameterization, we assume that ω_{2h} is at most of order h , so that

$$\lim_{h \rightarrow 0} h^{-\delta} \omega_{2h} = 0, \quad \forall \delta < 1. \tag{11}$$

As for the second (Nelson’s) parameterization, let

$$\lim_{h \rightarrow 0} 2h^{-1} \omega_{2h}^2 = \alpha^2. \tag{12}$$

Note that (10) and (12) are both satisfied if

$$\omega_{2h} = h^{1/2} \sqrt{\frac{\alpha^2}{2}} + \theta_2 h + o(h), \tag{13}$$

$$\omega_{1h} = 1 - h^{1/2} \sqrt{\frac{\alpha^2}{2}} + \theta_1 h + o(h) \tag{14}$$

with $\theta_1 + \theta_2 = \theta$.

We note that (11), for $\delta = \frac{1}{2}$, can be seen as a special case of (12), obtained by putting $\alpha = 0$. Nevertheless, it is not immediate whether $\alpha = 0$ or $\alpha \neq 0$ is a more general specification. Basically, if $\alpha \neq 0$, then ω_{2h} is of order $h^{1/2}$, so that the last term on the right-hand side (RHS) of (4) is of order $h^{1/2}$. Thus, the second conditional moment, scaled by h^{-1} , does not vanish as $h \rightarrow 0$ and we obtain a non-degenerate diffusion limit, i.e. a two-dimensional diffusion driven by two independent Brownian motions. On the other hand, in order to avoid the divergence, as $h \rightarrow 0$, of the first conditional moment, we need that ω_{1h} includes a $h^{1/2}$ -term that completely offsets the $h^{1/2}$ -term in ω_{2h} (see (13), (14)); in this sense a choice of $\alpha = 0$ is not necessarily less general. Now both the parameterizations (10)(11) and (10)(12), imply that $\omega_{1h} + \omega_{2h} = 1 + \theta h$, this is consistent with the aggregation result of Drost and Nijman (1993), according to which $\omega_{1h} + \omega_{2h}$ can be expressed as the h th power of $\omega_1 + \omega_2$, and so for small h we indeed have $\omega_{1h} + \omega_{2h} = 1 + \theta h$. Finally, it is immediate to see that, if ω_{2h} is at most of order h , as in (11), then the last term on the RHS of (4) is at most of order h , so that the conditional second moment, scaled by h^{-1} , vanishes as $h \rightarrow 0$, thus leading to

a degenerate diffusion, i.e. a two-dimensional diffusion driven by only one Brownian motion.

Now let us consider the following right continuous with left limit (CADLAG) processes, $Y_t^h = Y_{kh}$, $\sigma_t^{2,h} = \sigma_{kh}^2$, $kh \leq t < (k + 1)h$, where Y_{kh} , σ_{kh}^2 are defined as in (3)(4). Hereafter \Rightarrow denotes weak convergence, P_0^h , P_0 denotes the probability measure of $(Y_0^h, \sigma_0^{2,h})$ and of (Y_0, σ_0^2) , respectively.

Proposition 2.1. (i) *If as $h \rightarrow 0$, $(Y_0^h, \sigma_0^{2,h}) \rightarrow (Y_0, \sigma_0^2)$ or $P_0^h \rightarrow P_0$, then, under the parameterization in (9)–(11), as $h \rightarrow 0$, $(Y_t^h, \sigma_t^{2,h}) \Rightarrow (Y_t, \sigma_t^2)$, where (Y_t, σ_t^2) is a diffusion process solution to*

$$dY_t = \sigma_t dW_{1t}, \tag{15}$$

$$d\sigma_t^2 = (\omega_0 + \theta\sigma_t^2) dt. \tag{16}$$

(ii) *If as $h \rightarrow 0$, $(Y_0^h, \sigma_0^{2,h}) \rightarrow (Y_0, \sigma_0^2)$, or $P_0^h \rightarrow P_0$, then under the parameterization in (9), (10) and (12), as $h \rightarrow 0$, $(Y_t^h, \sigma_t^{2,h}) \Rightarrow (Y_t, \sigma_t^2)$, where (Y_t, σ_t^2) is a diffusion process solution to*

$$dY_t = \sigma_t dW_{1t}, \tag{17}$$

$$d\sigma_t^2 = (\omega_0 + \theta\sigma_t^2) dt + \alpha\sigma_t^2 dW_{2t}, \tag{18}$$

where $\theta < 0$, and (W_{1t}, W_{2t}) are two standard independent Brownian motions. Note that (16) can be seen as a special case of (18), putting $\alpha = 0$.

Proof. See the appendix. \square

It is immediate to see that the volatility process in (16) has smooth (differentiable) paths, is F_0 -measurable and, for given initial condition, is the solution to the following ordinary differentiable equation:

$$\sigma_t^2 = (\sigma_0^2 + \omega_0\theta^{-1})e^{\theta t} - \omega_0\theta^{-1}$$

as $\theta < 0$, we see that, as $t \rightarrow \infty$, σ_t^2 , for given initial condition, has a well-defined deterministic limit, in fact $\sigma_t^2 \rightarrow -\omega_0\theta^{-1}$. We also note that $Y_t = Y_0 + \int_0^t \sigma_s dW_s$ is a Gaussian process with time varying, but deterministic variance, for given σ_0^2 , and so with deterministic quadratic variation process. On the other hand the volatility in (18) is a diffusion process, and thus has nowhere differentiable paths. Further, as $t \rightarrow \infty$, as shown by Nelson (1990), σ_t^2 is distributed according to an inverted gamma. Unless volatility is a tradeable asset, the non-degeneracy of (18) leads to market incompleteness, thus option pricing requires additional assumptions on investors preferences and behavior.

3. GARCH as diffusion approximations

In the previous section we started from a discrete time GARCH(1, 1) process and we derived two different diffusion approximation results, one leading to a degenerate diffusion and another one leading to a non-degenerate diffusion. We now move in the opposite direction, by considering an Euler approximation (it is well known that Euler approximations are not unique) of the two diffusion limits defined in (15) and (16), degenerate case, and in (17) and (18) non-degenerate case. We shall see that Euler approximations of degenerate diffusions are indeed GARCH(1, 1) processes, while Euler approximations of non-degenerate diffusions are instead stochastic volatility processes. Let us consider again

$$dY_t = \sigma_t dW_t,$$

$$d\sigma_t^2 = (\omega_0 + \theta\sigma_t^2) dt = (\omega_0 + \theta_1\sigma_t^2 + \theta_2\sigma_t^2) dt,$$

where $\theta = \theta_1 + \theta_2$. Needless to say, there are infinitely many couples (θ_1, θ_2) for which $\theta = \theta_1 + \theta_2$. Nevertheless, no matter which values we choose for θ_1 and θ_2 , we can show that the corresponding Euler approximation is either a GARCH(1, 1) process or a process with deterministic volatility. Now, because of the conventional multiplication tables of stochastic calculus (see e.g. Karatzas and Shreve, 1991, p. 153),

$$(dY_t)^2 = \sigma_t^2(dW_t)^2 = \sigma_t^2 dt.$$

So we can rewrite the degenerate diffusion in (15) and (16) as,

$$dY_t = \sigma_t dW_t, \tag{19}$$

$$d\sigma_t^2 = (\omega_0 + \theta_1\sigma_t^2) dt + \theta_2(dY_t)^2. \tag{20}$$

We now consider an Euler discrete approximation to (19) and (20),

$$Y_{kh} - Y_{(k-1)h} = \sigma_{(k-1)h}\varepsilon_{kh},$$

$$\sigma_{kh}^2 - \sigma_{(k-1)h}^2 = h\omega_0 + \theta_1 h\sigma_{(k-1)h}^2 + \theta_2 \sigma_{(k-1)h}^2 \varepsilon_{kh}^2$$

with $\varepsilon_{kh} \sim iidN(0, h)$. Let $\omega_{0h} = h\omega_0$, $\omega_{1h} - 1 = \theta_1 h$, and $\omega_{2h} = \theta_2 h$, $\forall h$, we have

$$Y_{kh} - Y_{(k-1)h} = \sigma_{(k-1)h}\varepsilon_{kh} \tag{21}$$

$$\sigma_{kh}^2 = \omega_{0h} + (\omega_{1h} - 1)\sigma_{(k-1)h}^2 + h^{-1}\omega_{2h}\sigma_{(k-1)h}^2\varepsilon_{kh}^2. \tag{22}$$

We now observe that (21)–(22) is the same as (3)–(4), the continuous approximation scheme of the discrete GARCH(1, 1) processes. The same outcome holds even if we allow for a drift component in the return equation. Suppose

$$dY_t = (c + \gamma\sigma_t^2) dt + \sigma_t dW_t.$$

Now

$$(dY_t)^2 = (c + \gamma\sigma_t^2)(dt)^2 + \sigma_t^2(dW_t)^2 + 2\sigma_t(c + \gamma\sigma_t^2)dt dW_t = \sigma_t^2 dt,$$

in fact, again from the conventional multiplication table of stochastic calculus (Karatzas and Shreve, p. 153), $dW_t dt = (dt)^2 = 0$. The same outcome will follow.

We should also note that another Euler approximation of (15)–(16) leads to the following:

$$Y_{kh} - Y_{(k-1)h} = \sigma_{(k-1)h}^2 \varepsilon_{kh},$$

$$\sigma_{kh}^2 - \sigma_{(k-1)h}^2 = \omega_{0h} + \theta h \sigma_{(k-1)h}^2.$$

Thus, both the GARCH(1, 1) process and a process characterized by F_0 -measurable (deterministic given σ_0^2) volatility can be seen as possible Euler approximations to the degenerate diffusion limit obtained in Proposition 2.1(i). The intuition behind is that the continuous time counterpart of $\varepsilon_{kh}^2 = (W_{kh} - W_{(k-1)h})^2$ is (dW_t^2) which is equal to dt by the conventional rule of stochastic calculus.

We now consider the non-degenerate diffusion limit in (17)–(18). Drost and Werker (1996, p. 34–35) have shown that (17) and (18) is a GARCH-diffusion, in the sense that the implied discrete differences $(Y_{kh} - Y_{(k-1)h})$, for any fixed h , are weak GARCH with parameter $\xi_h = (\psi_h, \alpha_h, \beta_h, c_h)$. For weak GARCH $\forall h$, they mean that (see Definition 2.1 in Drost and Werker, 1996), there exists a covariance stationary process,

$$\sigma_{kh}^2 = \psi_h + \alpha_h(Y_{kh} - Y_{(k-1)h})^2 + \beta_h \sigma_{(k-1)h}^2,$$

such that, $\forall k$ and for any given time interval h , $\sigma_{(k-1)h}^2$ is the best linear predictor of $(Y_{kh} - Y_{(k-1)h})^2$. The fourth parameter c_h denotes the kurtosis of the implied discrete differences and is relevant only for the case of jump-diffusions. As (17)–(18) do not allow for leverage effect, in this case also the continuous $SR - SARV$ processes of Meddahi and Renault (1996) are indeed weak GARCH diffusion. Now both weak GARCH and $SR - SARV$ processes display stochastic volatility. In fact, it is immediate to see that Euler approximations to the diffusion in (17)–(18) are not (strong or semi-strong) GARCH processes, but stochastic volatility processes, such as

$$Y_{kh} - Y_{(k-1)h} = \sigma_{(k-1)h} \varepsilon_{1,kh},$$

$$\sigma_{kh}^2 - \sigma_{(k-1)h}^2 = \omega_{0h} + \theta h \sigma_{(k-1)h}^2 + \alpha \sigma_{(k-1)h}^2 \varepsilon_{2,kh},$$

where $\varepsilon_{kh} = (\varepsilon_{1,kh}, \varepsilon_{2,kh})'$ and $\varepsilon_{kh} \sim N(0, Ih)$.

Thus, while there exists an Euler approximation to a degenerate diffusion that leads to a GARCH(1, 1) process, any Euler approximation of a non-degenerate diffusion leads to a stochastic volatility process, rather than to a process characterized by only one source of noise.

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Appendix

Proof of Proposition 2.1. (i) The conditions for the existence of a unique strong solution (e.g. conditions 10.6 and 10.7 in Chung and Williams, 1990) are satisfied. This implies that there is a unique weak solution. We note that $\forall y, \sigma^2$

$$h^{-1}E((Y_{kh} - Y_{(k-1)h})^4 | Y_{(k-1)h} = y, \sigma_{(k-1)h}^2 = \sigma^2) = o(h),$$

and

$$h^{-1}E((\sigma_{kh}^2 - \sigma_{(k-1)h}^2)^4 | Y_{kh} = y, \sigma_{(k-1)h}^2 = \sigma^2) = o(h).$$

Thus, from Arnold (1974, p. 40), we know that it will suffice to check conditions (2.4)–(2.6) in Strook and Varadhan (Chapter 11, 1979) over the entire state space. Given the first two conditional moments in (5)–(8) and the parameterization in (9)–(11), and given that the size of the jumps approaches zero as $h \rightarrow 0$, the result then follows from Theorem 11.23 in Strook and Varadhan (1979).

(ii) Note that

$$E((Y_{kh} - Y_{(k-1)h})(\sigma_{kh}^2 - \sigma_{(k-1)h}^2) | F_{(k-1)h}) = 0.$$

Then the result follows by the same argument as the proof of Part (i).

References

- Arnold, L., 1974. Stochastic Differential Equations: Theory and Applications. Wiley, New York.
- Bollerslev, T., Engle, R.F., Nelson, D.B., 1994. ARCH models. In: R.F. Engle, D.L. McFadden (Eds.), Handbook of Econometrics, vol. iv. Elsevier, Amsterdam.
- Christoffersen, P., Hahn, J., 1997. Nonparametric testing of ARCH for option pricing. Mimeo, University of Pennsylvania.
- Chung, K.L., Williams, R.J., 1990. Introduction to Stochastic Integration, 2nd Edition. Birkhauser, Boston.

- Ding, Z., Granger, C.W.J., Engle, R.F., 1993. A long memory property of stock market return and a new model. *Journal of Empirical Finance* 1, 83–106.
- Drost, F.C., Nijman, T.E., 1993. Temporal aggregation of GARCH processes. *Econometrica* 61, 909–927.
- Drost, F.C., Werker, J.C., 1996. Closing the GARCH gap: continuous GARCH modelling. *Journal of Econometrics* 74, 31–57.
- Duan, J.C., 1997. Augmented GARCH(p, q) process and its diffusion limit. *Journal of Econometrics* 79, 97–127.
- Ethier, S.N., Kurtz, T.G., 1986. *Markov Processes: Characterization and Convergence*. Wiley, New York.
- Fornari, F., Mele, A., 1996. Weak convergence and distributional assumptions for a general class of nonlinear ARCH models. *Econometric Reviews*, forthcoming.
- Ghysels, E., Patilea, V., Renault, E., Torres, O., 1998. Nonparametric methods and option pricing, in: *Statistical Finance*, Ch. 13, D. Hand and S. Jacka (Eds.), Arnold, London.
- Hobson, D.G., Rogers, L.C.G., 1998. Complete models with stochastic volatility, *Mathematical Finance* 8, 27–48.
- Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatility. *Journal of Finance* 47, 1259–1282.
- Kallsen, J., Taqqu, M.S., 1998. Option pricing in ARCH-type, *Mathematical Finance* 8, 13–26.
- Karatzas, I., Shreve, S.E., 1991. *Brownian Motion and Stochastic Calculus*, 2nd Edition. Springer, New York.
- Meddahi, N., Renault, E., 1996. Aggregation and marginalization of GARCH and stochastic volatility models. Working Paper, GREMAQ and Université de Toulouse.
- Melino, A., Turnbull, S., 1990. Pricing foreign currency options with stochastic volatility. *Journal of Econometrics* 45, 239–266.
- Nelson, D.B., 1990. ARCH models as diffusion approximations. *Journal of Econometrics* 45, 7–38.
- Ströok, D.W., Varadhan, S.R.S., 1979. *Multidimensional Diffusion Processes*. Springer, Berlin.