

# Oblivious Equilibrium with Dominant Firms

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## 1 Introduction

In a recent paper, Weintraub, Benkard, and Van Roy (2008b) propose an approximation method for analyzing Ericson and Pakes (1995)-style dynamic models of imperfect competition. In that paper, we defined a new notion of equilibrium, *oblivious equilibrium* (henceforth, OE), in which each firm is assumed to make decisions based only on its own state and knowledge of the long run average industry state, but where firms ignore current information about competitors' states. The great advantage of OE is that they are much easier to compute than are Markov perfect equilibria (henceforth, MPE). Moreover, we showed that an OE provides meaningful approximations of *long-run* Markov perfect dynamics of an industry with many firms if, alongside some technical requirements, the equilibrium distribution of firm states obeys a *light-tail condition*.

To facilitate using OE in practice, in Weintraub, Benkard, and Van Roy (2008a) we provide a computational algorithm for solving for OE, and approximation bounds that can be computed to provide researchers with a numerical measure of how close OE is to MPE in their particular application. We also provided computational evidence supporting the conclusion that OE often yields good approximations of MPE behavior for industries like those that empirical researchers would like to study.

While our computational results suggest that OE will be useful in many applications on its own, we believe that a major contribution of OE will be as a starting point with which to build even better approximations. As a matter of fact, in Weintraub, Benkard, and Van Roy (2008a) we extend our base model as well as algorithms for computing OE and error bounds to incorporate aggregate shocks common to all firms. Such an extension is important, for example, when analyzing the dynamic effects of industry-wide business cycles.

In this note we introduce another important extension to OE. We develop an extended notion of oblivious equilibrium that allows for there to be a set of “dominant firms”, whose firm states are always monitored by every other firm. Our hope is that the dominant firm OE will provide better approximations for more concentrated industries. This extension trade offs increased computation time for a better behavioral model and a better approximation to MPE behavior.

The rest of the note is organized as follows. First, in Section 2 we introduce our dynamic industry model. In Section ?? we introduce nonstationary oblivious equilibrium as a way to approximate short-run industry dynamics. Section 5 introduces a new equilibrium concept in which firms keep track of few dominant firms. We conclude in Section 6. All proofs and mathematical arguments can be found in Section ??.

## 2 A Dynamic Model of Imperfect Competition

In this section we formulate a model of an industry in which firms compete in a single-good market. Our model is based on Weintraub, Benkard, and Van Roy (2008b) and our base model includes only idiosyncratic shocks.

### 2.1 Model and Notation

The industry evolves over discrete time periods and an infinite horizon. We index time periods with non-negative integers  $t \in \mathbb{N}$  ( $\mathbb{N} = \{0, 1, 2, \dots\}$ ). All random variables are defined on a probability space  $(\Omega, \mathcal{F}, \mathcal{P})$  equipped with a filtration  $\{\mathcal{F}_t : t \geq 0\}$ . We adopt a convention of indexing by  $t$  variables that are  $\mathcal{F}_t$ -measurable.

Each firm that enters the industry is assigned a unique positive integer-valued index. The set of indices of incumbent firms at time  $t$  is denoted by  $S_t$ . At each time  $t \in \mathbb{N}$ , we denote the number of incumbent firms as  $n_t$ .

Firm heterogeneity is reflected through firm states. To fix an interpretation, we will refer to a firm’s state as its quality level. However, firm states might more generally reflect productivity, capacity, the size of its consumer network, or any other aspect of the firm that affects its profits. At time  $t$ , the quality level of firm  $i \in S_t$  is denoted by  $x_{it} \in \mathbb{N}$ .

We define the *industry state*  $s_t$  to be a vector over quality levels that specifies, for each quality level  $x \in \mathbb{N}$ , the number of incumbent firms at quality level  $x$  in period  $t$ . We define the state space  $\bar{\mathcal{S}} = \left\{ s \in \mathbb{N}^\infty \mid \sum_{x=0}^\infty s(x) < \infty \right\}$ . Though in principle there are a countable number of industry states, we will

also consider an extended state space  $\mathcal{S} = \left\{ s \in \mathfrak{R}_+^\infty \mid \sum_{x=0}^\infty s(x) < \infty \right\}$ . This will allow us, for example, to consider derivatives of functions with respect to the industry state. For each  $i \in S_t$ , we define  $s_{-i,t} \in \mathcal{S}$  to be the state of the *competitors* of firm  $i$ ; that is,  $s_{-i,t}(x) = s_t(x) - 1$  if  $x_{it} = x$ , and  $s_{-i,t}(x) = s_t(x)$ , otherwise. Similarly,  $n_{-i,t}$  denotes to the number of competitors of firm  $i$ .

In each period, each incumbent firm earns profits on a spot market. A firm's single period expected profit  $\pi(x_{it}, s_{-i,t})$  depends on its quality level  $x_{it}$  and its competitors' state  $s_{-i,t}$ .

The model also allows for entry and exit. In each period, each incumbent firm  $i \in S_t$  observes a positive real-valued sell-off value  $\phi_{it}$  that is private information to the firm. If the sell-off value exceeds the value of continuing in the industry then the firm may choose to exit, in which case it earns the sell-off value and then ceases operations permanently.

If the firm instead decides to remain in the industry, then it can invest to improve its quality level. If a firm invests  $\iota_{it} \in \mathfrak{R}_+$ , then the firm's state at time  $t + 1$  is given by,

$$x_{i,t+1} = x_{it} + h(\iota_{it}, \zeta_{i,t+1}),$$

where the function  $h$  captures the impact of investment on quality and  $\zeta_{i,t+1}$  reflects uncertainty in the outcome of investment. Uncertainty may arise, for example, due to the risk associated with a research and development endeavor or a marketing campaign. Note that this specification is very general as  $h$  may take on either positive or negative values (e.g., allowing for positive depreciation). We denote the unit cost of investment by  $d$ .

In each period new firms can enter the industry by paying a setup cost  $\kappa$ . Entrants do not earn profits in the period that they enter. They appear in the following period at state  $x^e \in \mathbb{N}$  and can earn profits thereafter.<sup>1</sup>

Each firm aims to maximize expected net present value. The interest rate is assumed to be positive and constant over time, resulting in a constant discount factor of  $\beta \in (0, 1)$  per time period.

In each period, events occur in the following order:

1. Each incumbent firms observes its sell-off value and then makes exit and investment decisions.
2. The number of entering firms is determined and each entrant pays an entry cost of  $\kappa$ .
3. Incumbent firms compete in the spot market and receive profits.
4. Exiting firms exit and receive their sell-off values.
5. Investment outcomes are determined, new entrants enter, and the industry takes on a new state  $s_{t+1}$ .

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<sup>1</sup>Note that it would not change any of our results to assume that the entry state was a random variable.

## 2.2 Model Primitives

The model as specified is general enough to encompass numerous applied problems in economics. Indeed, a blossoming recent literature on EP-type models has applied similar models to advertising, auctions, collusion, consumer learning, environmental policy, international trade policy, learning-by-doing, limit order markets, mergers, network externalities, and other applied problems. To study any particular applied problem it is necessary to further specify the primitives of the model, including:

profit function	$\pi$
sell-off value distribution	$\sim \phi_{it}$
investment impact function	$h$
investment uncertainty distribution	$\sim \zeta_{it}$
unit investment cost	$d$
entry cost	$\kappa$
discount factor	$\beta$

Note that in most applications the profit function would not be specified directly, but would instead result from a deeper set of primitives that specify a demand function, a cost function, and a static equilibrium concept.

## 2.3 Assumptions

We make several assumptions about the model primitives, beginning with the profit function.

### Assumption 2.1.

1. For all  $s \in \mathcal{S}$ ,  $\pi(x, s)$  is increasing in  $x$ .
2. For all  $x \in \mathbb{N}$  and  $s \in \mathcal{S}$ ,  $\pi(x, s) > 0$ , and  $\sup_{x,s} \pi(x, s) < \infty$ .

The assumptions are natural. Assumption 2.1.1 ensures that increases in quality lead to increases in profit. Assumption 2.1.2 ensures that profits are positive and bounded. We also make assumptions about investment and the distributions of the private shocks:

### Assumption 2.2.

1. The variables  $\{\phi_{it} | t \geq 0, i \geq 1\}$  are i.i.d. and have finite expectations and well-defined density functions with support  $\mathbb{R}_+$ .
2. The random variables  $\{\zeta_{it} | t \geq 0, i \geq 1\}$  are i.i.d. and independent of  $\{\phi_{it} | t \geq 0, i \geq 1\}$ .
3. For all  $\zeta$ ,  $h(\iota, \zeta)$  is nondecreasing in  $\iota$ .
4. For all  $\iota > 0$ ,  $\mathcal{P}[h(\iota, \zeta_{i,t+1}) > 0] > 0$ .

5. *There exists a positive constant  $\bar{h} \in \mathbb{N}$  such that  $|h(\iota, \zeta)| \leq \bar{h}$ , for all  $(\iota, \zeta)$ . There exists a positive constant  $\bar{\iota}$  such that  $\iota_{it} < \bar{\iota}$ ,  $\forall i, \forall t$ .*
6. *For all  $k \in \{-\bar{h}, \dots, \bar{h}\}$ ,  $\mathcal{P}[h(\iota, \zeta_{i,t+1}) = k]$  is continuous in  $\iota$ .*
7. *The transitions generated by  $h(\iota, \zeta)$  are unique investment choice admissible .*

Again the assumptions are natural and fairly weak. Assumptions 2.2.1 and 2.2.2 imply that investment and exit outcomes are idiosyncratic conditional on the state. Assumption 2.2.3 and 2.2.4 imply that investment is productive. Note that positive depreciation is neither required nor ruled out. Assumption 2.2.5 places a finite bound on how much progress can be made or lost in a single period through investment. Assumption 2.2.6 ensures that the impact of investment on transition probabilities is continuous. Assumption 2.2.7 is an assumption introduced by Doraszelski and Satterthwaite (2007) that ensures a unique solution to the firms' investment decision problem. It is used to guarantee existence of an equilibrium in pure strategies, and is satisfied by many of the commonly used specifications in the literature.

We assume that there are a large number of potential entrants who play a symmetric mixed entry strategy. In that case the number of actual entrants is well approximated by the Poisson distribution (see Weintraub, Benkart, and Van Roy (2008b) for a derivation of this result). This leads to the following assumptions:

**Assumption 2.3.**

1. *The number of firms entering during period  $t$  is a Poisson random variable that is conditionally independent of  $\{\phi_{it}, \zeta_{it} | t \geq 0, i \geq 1\}$ , conditioned on  $s_t$ .*
2.  *$\kappa > \beta \cdot \bar{\phi}$ , where  $\bar{\phi}$  is the expected net present value of entering the market, investing zero and earning zero profits each period, and then exiting at an optimal stopping time.*

We denote the expected number of firms entering at industry state  $s_t$ , by  $\lambda(s_t)$ . This state-dependent entry rate will be endogenously determined, and our solution concept will require that it satisfies a zero expected profit condition. Modeling the number of entrants as a Poisson random variable has the advantage that it leads to simpler dynamics. However, our results can accommodate other entry processes as well. Assumption 2.3.2 ensures that the sell-off value by itself is not sufficient reason to enter the industry.

**2.4 Equilibrium**

As a model of industry behavior we focus on pure strategy Markov perfect equilibrium (MPE), in the sense of Maskin and Tirole (1988). We further assume that equilibrium is symmetric, such that all firms use a common stationary investment/exit strategy. In particular, there is a function  $\iota$  such that at each time  $t$ , each incumbent firm  $i \in S_t$  invests an amount  $\iota_{it} = \iota(x_{it}, s_{-i,t})$ . Similarly, each firm follows an exit strategy that takes the form of a cutoff rule: there is a real-valued function  $\rho$  such that an incumbent firm  $i \in S_t$  exits

at time  $t$  if and only if  $\phi_{it} \geq \rho(x_{it}, s_{-i,t})$ . In Weintraub, Benkard, and Van Roy (2008b) we show that there always exists an optimal exit strategy of this form even among very general classes of exit strategies. Let  $\mathcal{M}$  denote the set of exit/investment strategies such that an element  $\mu \in \mathcal{M}$  is a pair of functions  $\mu = (\iota, \rho)$ , where  $\iota : \mathbb{N} \times \mathcal{S} \rightarrow \mathfrak{R}_+$  is an investment strategy and  $\rho : \mathbb{N} \times \mathcal{S} \rightarrow \mathfrak{R}_+$  is an exit strategy. Similarly, we denote the set of entry rate functions by  $\Lambda$ , where an element of  $\Lambda$  is a function  $\lambda : \mathcal{S} \rightarrow \mathfrak{R}_+$ .

We define the value function  $V(x, s|\mu', \mu, \lambda)$  to be the expected net present value for a firm at state  $x$  when its competitors' state is  $s$ , given that its competitors each follows a common strategy  $\mu \in \mathcal{M}$ , the entry rate function is  $\lambda \in \Lambda$ , and the firm itself follows strategy  $\mu' \in \mathcal{M}$ . In particular,

$$V(x, s|\mu', \mu, \lambda) = E_{\mu', \mu, \lambda} \left[ \sum_{k=t}^{\tau_i} \beta^{k-t} (\pi(x_{ik}, s_{-i,k}) - d_{ik}) + \beta^{\tau_i-t} \phi_{i, \tau_i} \mid x_{it} = x, s_{-i,t} = s \right],$$

where  $i$  is taken to be the index of a firm at quality level  $x$  at time  $t$ ,  $\tau_i$  is a random variable representing the time at which firm  $i$  exits the industry, and the subscripts of the expectation indicate the strategy followed by firm  $i$ , the strategy followed by its competitors, and the entry rate function. In an abuse of notation, we will use the shorthand,  $V(x, s|\mu, \lambda) \equiv V(x, s|\mu, \mu, \lambda)$ , to refer to the expected discounted value of profits when firm  $i$  follows the same strategy  $\mu$  as its competitors.

An equilibrium to our model comprises an investment/exit strategy  $\mu = (\iota, \rho) \in \mathcal{M}$ , and an entry rate function  $\lambda \in \Lambda$  that satisfy the following conditions:

1. Incumbent firm strategies represent a MPE:

$$(2.1) \quad \sup_{\mu' \in \mathcal{M}} V(x, s|\mu', \mu, \lambda) = V(x, s|\mu, \lambda) \quad \forall x \in \mathbb{N}, \forall s \in \bar{\mathcal{S}}.$$

2. At each state, either entrants have zero expected profits or the entry rate is zero (or both):

$$\begin{aligned} \sum_{s \in \bar{\mathcal{S}}} \lambda(s) (\beta E_{\mu, \lambda} [V(x^e, s_{-i, t+1} | \mu, \lambda) | s_t = s] - \kappa) &= 0 \\ \beta E_{\mu, \lambda} [V(x^e, s_{-i, t+1} | \mu, \lambda) | s_t = s] - \kappa &\leq 0 \quad \forall s \in \bar{\mathcal{S}} \\ \lambda(s) &\geq 0 \quad \forall s \in \bar{\mathcal{S}}. \end{aligned}$$

In Weintraub, Benkard, and Van Roy (2008b), we show that the supremum in part 1 of the definition above can always be attained simultaneously for all  $x$  and  $s$  by a common strategy  $\mu'$ .

Doraszelski and Satterthwaite (2007) establish existence of an equilibrium in pure strategies for a closely related model. We do not provide an existence proof here because it is long and cumbersome and would replicate this previous work. With respect to uniqueness, in general we presume that our model may have

multiple equilibria.<sup>2</sup>

Dynamic programming algorithms can be used to optimize firm strategies, and equilibria to our model can be computed via their iterative application. However, these algorithms require compute time and memory that grow proportionately with the number of relevant industry states, which is often intractable in contexts of practical interest. This difficulty motivates our alternative approach.

### 3 Stationary Dominant Firms OE

In Weintraub, Benkard, and Van Roy (2008b) we argued that approximations based on oblivious strategies do not offer accurate representations of MPE behavior if there are few dominant firms in the industry, even if the number of other firms is large. A strategy that does not keep track of the dominant firms will not perform well. To overcome this limitation, in this section we extend the notion of OE and introduce an approximation method where each firm keeps track of its own state and the state of few dominant firms. In Section 3.1 we define the new equilibrium concept; we derive bounds to assess approximation error in Section 3.2, and we introduce an algorithm in Section 3.3.

#### 3.1 Partially Oblivious Equilibrium

In this section we extend the concept of oblivious equilibrium and let firms keep track of the state of few firms (which we call *dominant firms*) and make an estimate based on averages for the state of all other firms (which we call *fringe firms*). We call this new concept *partially oblivious equilibrium*, hereafter POE.

Let  $\bar{S} = \{i_1, i_2, \dots, i_n\}$  be the set of indices associated to the dominant firms. We assume the identity of the  $n$  dominant firms does not change over time; firms always keep track of the same set of firms  $\bar{S}$  and all new entrants become part of the fringe. Dominant firms never exit the industry.

We extend the state of firms to include a binary variable that identifies dominant firms. Hence, the state of firm  $i$  at time  $t$  is  $\bar{x}_{it} = (x_{it}, 1)$  if  $i \in \bar{S}$ , and  $\bar{x}_{it} = (x_{it}, 0)$  if  $i \notin \bar{S}$ . This specification will allow for equilibrium strategies to differ between dominant firms and fringe firms.

##### 3.1.1 Partially Oblivious Strategies

A partially oblivious strategy is a function of the firm's state (which in this case also indicates whether the firm is dominant or not) and the state of firms in  $\bar{S}$ . Let  $y_t$  be a vector that represents the state of the

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<sup>2</sup>Doraszelski and Satterthwaite (2007) also provide an example of multiple equilibria in their closely related model.

dominant firms at time  $t$ . Formally,  $y_t = (x_{(i_1)t}, x_{(i_2)t}, \dots, x_{(i_n)t})$ , where  $x_{(i_k)t}$  is the  $k$ -th order statistic of  $(x_{i_1t}, x_{i_2t}, \dots, x_{i_nt})$ . Our equilibrium concept will assume symmetric strategies that are a function of the order statistics of  $(x_{i_1t}, x_{i_2t}, \dots, x_{i_nt})$  and will not depend on the identities of the dominant firms.

While there may be few dominant firms on a given industry, so it is computationally feasible to optimize over strategies that are a function of the dominant firms' state, it is likely that the number of other firms (fringe firms) may be large. In this case, it will not be computationally feasible to optimize over strategies that are a function of the fringe firms' state. Instead, we will assume firms make estimates of the fringe firms' state based on averages. Actually, if there are many firms, because of averaging effects, firms should be able to accurately predict the fringe firms' state for a given time period based on the entire history of dominant firms' state.<sup>3</sup> This is computationally impractical; instead, we will allow firms to predict the fringe firms' state based on a finite set of statistics that depend on the entire evolution of the dominant firms' state.<sup>4</sup>

Based on this motivation, we will restrict firms' strategies so that each firm's decisions depend only on the firm's state, the current state of the dominant firms, and a finite set of statistics that depend on the history of realizations of the dominant firms' state. We call such restricted strategies *partially oblivious strategies*. To convey this dependence, we define the sequence  $\{w_t \in \mathcal{W} = \mathcal{W}_1 \times \dots \times \mathcal{W}_K : t \geq 0\}$  where  $w_t(1) = y_t$ , for all  $t \geq 0$ , and  $\mathcal{W}_j$  are countable sets.

We define  $\tilde{\mathcal{M}}_p$  and  $\tilde{\Lambda}_p$  as the set of partially oblivious strategies and the set of partially oblivious entry rate functions, respectively. If firm  $i$  uses strategy  $\mu \in \tilde{\mathcal{M}}_p$ , then firm  $i$  takes action  $\mu(\bar{x}_{it}, w_t)$  at time period  $t$ , where  $\bar{x}_{it}$  is the state of firm  $i$  at time  $t$  (which indicates the firm's quality level and whether it's dominant or not). Similarly, if the entry rate function is  $\lambda \in \tilde{\Lambda}_p$ , the entry rate is equal to  $\lambda(w_t)$  at time period  $t$ . Since  $w_t(1) = y_t$ , firms keep track of the current dominant firms' state when making decisions with partially oblivious strategies. The state variables  $w_t(2), \dots, w_t(K)$  allow firms to incorporate additional information about the history of realizations of the dominant firms' state into the strategies. This information could be useful to better predict the average fringe firm state conditional on observing  $w_t$ . In general, accounting for past information will generally improve a firms decisions. It is worth mentioning here, though, that past information is not payoff-relevant (hence, it does not influence MPE strategies), so allowing partially oblivious strategies to depend on that may give rise to POE that are poor approximations to MPE.

We make the following assumption over the state statistics  $w_t$ .

**Assumption 3.1.** *For any set of partially oblivious strategies that dominant firms use, we assume that  $\{w_t : t \geq 0\}$  is a finite *irreducible and aperiodic* Markov chain adapted to the filtration generated by*

<sup>3</sup>Because firms keep track of the state of the dominant firms, even if there are a large number of firms, the fringe firms' state will not necessarily be close to a constant state like in the case of OE; the fringe firm state will depend on the dominant firms' evolution.

<sup>4</sup>This idea is similar to OE with aggregate shocks; see Weintraub, Benkard, and Van Roy (2008a).



$\{y_t : t \geq 0\}$ . [DELETED GW:  $\{w_t : t \geq 0\}$  has a single recurrent class  $\tilde{\mathcal{W}} \subseteq \mathcal{W}$  and admits a unique invariant distribution.] For all  $t \geq 0$ ,  $w_t(1) = y_t$ .

The assumption implies that the Markov chain  $\{w_t : t \geq 0\}$  admits a unique invariant distribution that assigns positive mass to all states  $w \in \mathcal{W}$ .

Different partially oblivious strategies can be defined depending on the specification of  $w_t$ . For example, suppose that for  $j \in \{1, \dots, K\}$ ,  $w_t(j) = y_{t-j+1}$ . Hence,  $w_t = \{y_t, y_{t-1}, \dots, y_{t-K+1}\}$ ; the statistics correspond to the last realizations of the dominant firms' state. In this case  $\mathcal{W}$  is the set of feasible  $K$ -tuples of consecutive dominant firms' state. If, in addition, for any set of partially oblivious strategies that dominant firms use,  $\{y_t : t \geq 0\}$  is a finite irreducible and aperiodic Markov chain, then the specification satisfies Assumption 5.1.

### 3.1.2 Fringe Firms' Expected State

In a POE firms will make decisions assuming that the fringe firms' state is the *expected* fringe firms' state conditional on the current value of the dominant firm state statistics  $w_t$ . Formally, suppose that firms use strategy  $\mu \in \tilde{\mathcal{M}}_p$  and enter according to  $\lambda \in \tilde{\Lambda}_p$ . Let  $z_t$  be a vector over quality levels that specifies, for each quality level  $x \in \mathbb{N}$ , the *number of fringe firms* that are at quality level  $x$  in period  $t$ . Note that Assumption 3.1 together with Assumptions 2.1, 2.2, and 2.3, imply that  $\{(z_t, w_t) : t \geq 0\}$  is a Markov chain that admits a unique invariant distribution. We assume that  $(z_0, w_0)$  is distributed according to the invariant distribution of  $\{(z_t, w_t) : t \geq 0\}$ . Hence,  $(z_t, w_t)$  is a stationary process.

Firms predict the fringe firm state based on the current realization of the dominant firm statistics,  $w_t$ . Accordingly, for all [DELETED GW:  $w \in \tilde{\mathcal{W}}$ ]  $w \in \mathcal{W}$ , we define  $\tilde{z}_{\mu, \lambda}(w) = E[z_t | w_t = w]$ . In words,  $\tilde{z}_{\mu, \lambda}(w)$  is the long-run expected fringe firm state when dynamics are governed by partially oblivious strategy  $\mu$  and partially oblivious entry rate function  $\lambda$ , conditional on the current realization of  $w_t$  being  $w$ .

### 3.1.3 Partially Oblivious Value Function

With some abuse of notation, let  $\pi(\bar{x}_{it}, y_t, z_t)$  be the single-period profits for a firm in state  $\bar{x}_{it}$ , if the dominant firms' state is  $y_t$ , and the fringe firms' state is  $z_t$ . Because  $y_t$  represents the dominant firms' state, note that if  $i \in \bar{S}$ , then  $x_{it} = x$  only if  $y_t(k) = x$ , for some  $k = 1, \dots, n$ . Consequently, for all  $w \in \mathcal{W}$ , we define the set  $\bar{X}(w) = \{(x, 1) : x = w(1, k) \text{ for some } k = 1, \dots, n\} \cup \{(x, 0) : x \in \mathbb{N}\}$ , where  $w(1, k)$  is the  $k$ -th component of  $w(1)$ .<sup>5</sup> If firm  $i$  is dominant it subtracts itself from  $y_t$ . Because there are likely to

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<sup>5</sup>Recall that for all  $t \geq 0$ ,  $w_t(1) = y_t$ .

be many fringe firms, we do not subtract a firm from the fringe firm expected state when evaluating profits for fringe firms.

We define a *partially oblivious value function* for all  $(\bar{x}, w) \in \{(\bar{x}, w) : w \in \mathcal{W}, \bar{x} \in \bar{X}(w)\}$ :

(3.1)

$$\tilde{V}(\bar{x}, w | \mu', \mu, \lambda) = E_{\mu', \mu} \left[ \sum_{k=t}^{\tau_i} \beta^{k-t} (\pi(\bar{x}_{ik}, y_k, \tilde{z}_{\mu, \lambda}(w_k)) - d_{ik}) + \beta^{\tau_i - t} \phi_{i, \tau_i} \Big| \bar{x}_{it} = \bar{x}, w_t = w \right],$$

where  $i$  is the index of a firm in state  $\bar{x}$  at time period  $t$ . This value function should be interpreted as the expected net present value of firm  $i$  at state  $\bar{x}$  when the dominant firms' statistics have value  $w$ , and firm  $i$  follows partially oblivious strategy  $\mu'$ . Competitors use strategy  $\mu$  and enter according to  $\lambda$ . Note that even if firm  $i$  is dominant and uses strategy  $\mu'$  which may be different to  $\mu$ , the firm assumes that the fringe firms' state will be  $\tilde{z}_{\mu, \lambda}(w_k)$  in time period  $k$ , for all  $k \geq 0$ . That is, when computing a partially oblivious value function the dominant firm ignores that its deviation will change the fringe firms' evolution and, hence,  $\tilde{z}$ . This restriction over the set of deviations for dominant firms greatly simplifies computation. Note, however, that in equilibrium the optimal dominant firm's strategy and the expected fringe firm state will be consistent. Finally, in an abuse of notation, we define  $\tilde{V}(\bar{x}, w | \mu, \lambda) = \tilde{V}(\bar{x}, w | \mu, \mu, \lambda)$ .

### 3.1.4 Partially Oblivious Equilibrium

We now define a new solution concept: a *partially oblivious equilibrium* consists of a strategy  $\mu \in \tilde{\mathcal{M}}_p$  and an entry rate function  $\lambda \in \tilde{\Lambda}_p$  that satisfy the following conditions:

1. Firm strategies optimize a partially oblivious value function.

$$(3.2) \quad \sup_{\mu' \in \tilde{\mathcal{M}}_p} \tilde{V}(\bar{x}, w | \mu', \mu, \lambda) = \tilde{V}(\bar{x}, w | \mu, \lambda), \quad \forall (\bar{x}, w) \in \{(\bar{x}, w) : w \in \mathcal{W}, \bar{x} \in \bar{X}(w)\}.$$

2. The partially oblivious expected value of entry is zero or the entry rate is zero (or both).

$$\begin{aligned} \sum_{w \in \mathcal{W}} \lambda(w) \left( \beta E \left[ \tilde{V}((x^e, 0), w_{t+1} | \mu, \lambda) \Big| w_t = w \right] - \kappa \right) &= 0, \\ \beta E \left[ \tilde{V}((x^e, 0), w_{t+1} | \mu, \lambda) \Big| w_t = w \right] - \kappa &\leq 0, \quad \forall w \in \mathcal{W}, \\ \lambda(w) &\geq 0, \quad \forall w \in \mathcal{W}. \end{aligned}$$

Note that to derive a POE it is enough to consider one dominant firm and one fringe firm. New entrants become part of the fringe. Finally, if  $n = 0$ , a POE is an OE.

In Section 3.3 we provide an algorithm for computing a POE. Note that the state space of the firm's dynamic programming problem scales with the number of firm states and with the size of  $\mathcal{W}$ , the feasible set for the dominant firm statistics process. As the set  $\mathcal{W}$  becomes richer, more computation time and memory is needed.

### 3.2 Error Bounds

We derive error bounds for fringe firms in this model. Approximation error is the amount by which a fringe firm at state  $\bar{x} = (x, 0)$ ,  $x \in \mathbb{N}$  can improve its expected net present value by unilaterally deviating from the partial OE strategy, and instead following an optimal (non-oblivious) best response.

We define  $\mathcal{M}_p$  and  $\Lambda_p$  as the set of Markov strategies and entry rate functions for the fringe, keeping the oblivious strategy for the leader. A Markov strategy is a function of the firm's own state, the industry state (including the leaders' state), and the leaders' state statistics. If a fringe firm  $i$  uses strategy  $\mu \in \mathcal{M}_p$  then at time period  $t$ , fringe firm  $i$  takes action  $\mu(\bar{x}_{it}, s_{-i,t}, w_t)$  and dominant firm takes action  $\tilde{\mu}(\bar{x}_{it}, w_t)$ . Note that we do not extend the information set of the dominant firm, still restricting its strategy to depend only on its own state, other leaders' state, and the leaders' state statistics. Similarly, if  $\lambda \in \Lambda_p$ , then the entry rate at time  $t$  is  $\lambda(s_t, w_t)$ <sup>6</sup>.

For Markov strategy  $\mu'$ ,  $\mu \in \mathcal{M}_p$  and entry rate function  $\lambda \in \Lambda_p$ , with some abuse of notation, we define for  $\bar{x} = (x, 0)$  the fringe extended value function,

$$(3.3) \quad V(\bar{x}, s, w | \mu', \mu, \lambda) = E_{\mu', \mu, \lambda} \left[ \sum_{k=t}^{\tau_i} \beta^{k-t} (\pi(x_{ik}, s_{-i,k}, z_k) - dl_{ik}) + \beta^{\tau_i-t} \phi_{i, \tau_i} \Big| x_{it} = x, s_{-i,t} = s, w_t = w \right],$$

where  $i$  is taken to be the index of a firm at quality level  $x$  at time  $t$ . The extended value function generalizes the value function defined in Section 2.4 allowing for dependence on extended strategies. We use this value function to evaluate the *actual* expected discounted profits garnered by a firm that uses an extended Markov strategy.

Consider an partial OE strategy and entry rate  $(\tilde{\mu}, \tilde{\lambda}) \in \tilde{\mathcal{M}}_p \times \tilde{\Lambda}_p$ . We assume the initial state  $(s_0, w_0)$  is sampled from the invariant distribution of  $\{(s_t, w_t) : t \geq 0\}$ . Hence,  $(s_t, w_t)$  is a stationary process, it is distributed according to its invariant distribution for all  $t \geq 0$ . To abbreviate, let  $\tilde{s} = \tilde{s}_{\tilde{\mu}, \tilde{\lambda}}$ . With some abuse of notation, let  $\Delta_A(s, w) = \sup_{y \in A} (\pi(y, s, w(1)) - \pi(y, \tilde{s}(w), w(1)))$  and let  $\Delta(y, s, w) =$

<sup>6</sup>Recall that  $w_t(1) = z_t$ , hence, strategies are a function of  $z_t$ .

$\pi(y, s, w(1)) - \pi(y, \tilde{s}(w), w(1))$ . We have the following result that we prove in the Appendix.

**Theorem 3.1.** *Let Assumptions 2.1, 5.1, 2.2, and 2.3 hold. Then, for any partial OE  $(\tilde{\mu}, \tilde{\lambda})$ , and fringe firm state  $\bar{x} = (x, 0), x \in \mathbb{N}$ ,*

$$(3.4) \quad E \left[ \sup_{\mu' \in \mathcal{M}_p} V(\bar{x}, s_t, w_t | \mu', \tilde{\mu}, \tilde{\lambda}) - V(\bar{x}, s_t, w_t | \tilde{\mu}, \tilde{\lambda}) \right] \\ \leq \sum_{k=t}^{\infty} \beta^{k-t} E \left[ \left[ \Delta_{\{\underline{x}(k,t), \dots, x+(k-t)\bar{h}\}}(s_k, w_k) \right]^+ \right] \\ + E \left[ E_{\tilde{\mu}, \tilde{\lambda}} \left[ \sum_{k=t}^{\tau_i} \beta^{k-t} (\pi(x_{ik}, \tilde{s}(w_k), z_k) - \pi(x_{ik}, s_{-i,k}, z_k)) \mid x_{it} = x, s_{-i,t} = s_t, w_t \right] \right].$$

Suppose that, for all  $s \in \bar{\mathcal{S}}$  and  $w \in \mathcal{W}$ , the function  $\Delta(y, s, w)^+$  is nondecreasing in  $y$ . Then, for any partial OE  $(\tilde{\mu}, \tilde{\lambda})$ , and fringe firm state  $\bar{x} = (x, 0), x \in \mathbb{N}$ ,

$$(3.5) \quad E \left[ \sup_{\mu' \in \mathcal{M}_p} V(\bar{x}, s_t, w_t | \mu', \tilde{\mu}, \tilde{\lambda}) - V(\bar{x}, s_t, w_t | \tilde{\mu}, \tilde{\lambda}) \right] \\ \leq \sum_{k=t}^{\infty} \beta^{k-t} E_{\tilde{\mu}, \tilde{\lambda}} \left[ \Delta(\hat{x}_k, s_k, w_k)^+ \mid \hat{x}_t = x \right] \\ + E \left[ E_{\tilde{\mu}, \tilde{\lambda}} \left[ \sum_{k=t}^{\tau_i} \beta^{k-t} (\pi(x_{ik}, \tilde{s}(w_k), z_k) - \pi(x_{ik}, s_{-i,k}, z_k)) \mid x_{it} = x, s_{-i,t} = s_t, w_t \right] \right].$$

Note that  $\hat{x}_k$  in the second bound is controlled by strategy  $\hat{\mu}$ , that is, a strategy in which the fringe firm never exits the industry and invests an infinite amount at every state. Recall that  $(s_k, w_k)$  is distributed according to the invariant distribution for all  $k \geq 0$ . The bound can be computed using simulation. As before, these bounds are quite general and do not rely on many of the detailed modeling assumptions.

### 3.3 Algorithms and Computations

In this section we introduce an algorithm to compute POE. Throughout this section we only consider states  $(\bar{x}, w) \in \{(\bar{x}, w) : w \in \mathcal{W}, \bar{x} \in \bar{\mathcal{X}}(w)\}$ .

We introduce the following algorithm to compute a POE. At each iteration of the algorithm, we (1) compute the expected fringe firm state conditional on the dominant firm statistics,  $\tilde{z}_{\mu, \lambda}(w)$  (step 5); (2) we compute the strategy that maximizes the partially oblivious value function (step 6)<sup>7</sup>; and (3) we compute new entry rates depending on the extent of the violation of the zero-profit conditions (step 8). Strategies

<sup>7</sup>We implement this with a Gauss-Seidel version of value iteration.

and entry rates are updated “smoothly” (steps 12 and 13). The parameters  $N_1$ ,  $N_2$ ,  $\gamma_1$ , and  $\gamma_2$  are set after some experimentation to speed up convergence. If the termination condition of the outer loop is satisfied with  $\epsilon_1 = \epsilon_2 = 0$ , we have an partial OE. Small values of  $\epsilon_1$  and  $\epsilon_2$  allow for small errors associated with limitations of numerical precision.

For initialization, let  $(\tilde{\mu}, \tilde{\lambda})$  be an OE. Let  $\tilde{V}$  be the respective value function.

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**Algorithm 1** Partially Oblivious Equilibrium Solver

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- 1:  $\lambda(w) := \tilde{\lambda}$ , for all  $w$
  - 2:  $\mu(\bar{x}, w) := \tilde{\mu}(x)$ , for all  $\bar{x}, w$
  - 3:  $n := 0$
  - 4: **repeat**
  - 5:   Compute  $\tilde{z}_{\mu, \lambda}(w)$ , for all  $w$
  - 6:   Choose  $\mu^* \in \mathcal{M}_p$  to maximize  $\tilde{V}(\bar{x}, w | \mu^*, \mu, \lambda)$  simultaneously for all  $\bar{x}, w$
  - 7:   **for all**  $w$  **do**
  - 8:      $\lambda^*(w) := \lambda(w) \left( \beta E \left[ \tilde{V}((x^e, 0), w_{t+1} | \mu^*, \mu, \lambda) \middle| w_t = w \right] / \kappa \right)$
  - 9:   **end for**
  - 10:    $\Delta_1 := \|\mu - \mu^*\|_\infty$ ,  $\Delta_2 := \|\lambda - \lambda^*\|_\infty$
  - 11:    $n := n + 1$
  - 12:    $\mu := \mu + (\mu^* - \mu) / (n^{\gamma_1} + N_1)$
  - 13:    $\lambda := \lambda + (\lambda^* - \lambda) / (n^{\gamma_2} + N_2)$
  - 14: **until**  $\Delta_1 \leq \epsilon_1$  and  $\Delta_2 \leq \epsilon_2$
- 

We finish by suggesting a way of computing  $\tilde{z}_{\mu, \lambda}(w)$  (step 5 in the algorithm). Let  $p(x, w, y, w') = P_{\mu, \lambda}[x_{i, t+1} = y, w_{t+1} = w' \mid x_{it} = x, w_t = w]$ , where  $i \notin \bar{S}$ . The probability that the firm exits from a state  $(x, w)$  is one minus the sum of transition probabilities from that state. Let  $\tilde{z}(x, w)$  be the  $x$  component of  $\tilde{z}_{\mu, \lambda}(w)$ . Let  $r(x, w)$  be the product of  $\tilde{z}(x, w)$  and the steady state probability that the dominant firm statistics process is in state  $w$ ,  $q(w)$ . Then,  $r(x, w)$  satisfies the balance equations:

$$r(x, w) = \sum_{(y, w')} r(y, w') p(y, w', x, w) + \mathbf{1}(x = x^e) \sum_{w'} \lambda(w') q(w') p(w', w),$$

where  $p(w', w) = P[w_{t+1} = w \mid w_t = w']$  and  $\mathbf{1}$  is the indicator function. We can obtain  $r(x, w)$  by solving this set of balance equations. We can also obtain steady state probabilities of the dominant firm process by solving another set of balance equations. From these two objects, we obtain  $\tilde{z}(x, w)$ .

## 4 Computational experiments

In this section we provide results of computational experiments that compare an OE with a one dominant firm to a pure OE. The purpose of the experiments is to evaluate the usefulness of a dominant firms extension in practice. For example we argue that adding a dominant firms helps to capture such things like: entry deterrence, investment deterrence or fat tail distributions of firms' state.

### 4.1 The computational model

**SINGLE-PERIOD PROFIT FUNCTION.** We consider an industry with differentiated products, where each firm's state variable represents the quality of its product. There are  $m$  consumers in the market. In period  $t$ , consumer  $j$  receives utility  $u_{ijt}$  from consuming the good produced by firm  $i$  given by:

$$u_{ijt} = \theta_1 \ln\left(\frac{x_{it}}{\psi} + 1\right) + \theta_2 \ln(Y - p_{it}) + \nu_{ijt}, \quad i \in S_t, \quad j = 1, \dots, m,$$

where  $Y$  is the consumer's income,  $p_{it}$  is the price of the good produced by firm  $i$ , and  $\psi$  is a scaling factor.  $\nu_{ijt}$  are i.i.d. random variables distributed Gumbel that represent unobserved characteristics for each consumer-good pair. There is also an outside good that provides consumers zero utility. We assume consumers buy at most one product each period and that they choose the product that maximizes utility. Under these assumptions our demand system is a classical logit model.

Let  $N(x_{it}, p_{it}) = \exp(\theta_1 \ln(\frac{x_{it}}{\psi} + 1) + \theta_2 \ln(Y - p_{it}))$ . Then, the expected market share of each firm is given by:

$$\sigma(x_{it}, s_{-i,t}, p_t) = \frac{N(x_{it}, p_{it})}{1 + \sum_{j \in S_t} N(x_{jt}, p_{jt})}, \quad \forall i \in S_t.$$

We assume that firms set prices in the spot market. If there is a constant marginal cost  $c$ , the Nash equilibrium of the pricing game satisfies the first-order conditions,

$$(4.1) \quad Y - p_{it} + \theta_2(p_{it} - c)(\sigma(x_{it}, s_{-i,t}, p_t) - 1) = 0, \quad \forall i \in S_t.$$

There is a unique Nash equilibrium in pure strategies, denoted  $p_t^*$  (Caplin and Nalebuff (1991)). Expected profits are given by:

$$\pi_m(x_{it}, s_{-i,t}) = m\sigma(x_{it}, s_{-i,t}, p_t^*)(p_{it}^* - c), \quad \forall i \in S_t.$$

**SELL-OFF PRICE.**  $\phi_{it}$  are i.i.d. exponential random variables with mean  $K$ .

**TRANSITION DYNAMICS.** A firm's investment is successful with probability  $\frac{at}{1+at}$ , in which case the quality

of its product increases by one level. The firm's product depreciates one quality level with probability  $\delta$ , independently each period. Note that our model differs from Pakes and McGuire (1994) here because the depreciation shocks in our model are idiosyncratic. Combining the investment and depreciation processes, it follows that the transition probabilities for a firm in state  $x$  that does not exit and invests  $\iota$  are given by:

$$\mathcal{P} \left[ x_{i,t+1} = y \mid x_{it} = x, \iota \right] = \begin{cases} \frac{(1-\delta)a\iota}{1+a\iota} & \text{if } y = x + 1 \\ \frac{(1-\delta)+\delta a\iota}{1+a\iota} & \text{if } y = x \\ \frac{\delta}{1+a\iota} & \text{if } y = x - 1 . \end{cases}$$

INVESTMENT COST. Cost of investing  $\iota$  by the firm in state  $x$  is given by  $\iota(d_1 + d_2x^2)$ .

## 4.2 Numerical results

We provide the results for three sets of parameters of the model that represent two distinct types of markets. In the first experiment adding a dominant firm does not change the equilibrium in a substantial way. In the latter two cases an extra dominant firm enriches results obtained with a pure OE.

The first experiment represents a market with a technology that is cheap to obtain at the beginning and gets more expensive later on (the parameters can be found in Table 1). We set low fixed cost of investment  $d_1$ , but we add a cost component  $d_2$  that is a function of the square of current technological advancement. Such convex investment cost makes it unprofitable to invest when reaching a medium technology level. Since the consumer quality sensitivity parameter  $\theta_1$  is small, being a technologically advanced firm does not bring much competitive advantage. It makes it hard for a dominant firm to deter entry. From the Figure 2 we can see that expected number of fringe firms does not depend much on the state of a dominant firm. Moreover, the number of firms in the OE with a dominant firm is almost the same as in the pure OE. Convex cost and lack of entry deterrence create similar investment incentives for a dominant firm. Distribution of firms' technology state is depicted at Figure 1. As one can expect there is also no impact of a dominant firm on the bounds (Table 2) and industry statistics (Table 3).

Parameter values for the second experiment are presented in the first row of the Table 1. The market is characterized by a low state depreciation rate  $\delta$  and a high yield from investment  $a$ . In the same time, investment is very expensive with a fixed cost equal to 2 and is highly valued by consumers ( $\theta_1 = 0.7$ ). Those parameter values represent the situation in which technology is very profitable by in the same time very costly. The distribution of firms in the pure OE is presented on the Figure 3. In this case firms concentrate in low states. The benefit from investment that is captured by a pure OE specification does not

cover an investment cost. If we enrich the model by adding a dominant firm, things change. It turns out that the state distribution of a dominant firm is concentrated at high values. Extra incentives to invest for a leader are coming from an entry and investment deterrence. The Figure 4 depicts an expected number of fringe firms conditional on the state of a dominant firm. We observe a significant ability of a dominant firm to deter entry and induce exit. A total number of firms in the industry drops by nearly 50% when a dominant moves from the lowest to the highest state. Moreover, an unconditional expected number of firms drops by 50% if we compare a pure OE with the extension. Investment deterrence is smaller but still significant. Average fringe investment drops by about 10% if a dominant firm advances from the lowest to the highest state. The comparison of economic indicators is presented in the Table 3. The market with a dominant firm is significantly more concentrated with  $C1$  of 24.5% compared to 12% in the pure OE. The presence of a dominant firm decreases both consumer and producer surplus. Impact of adding a leader on error bounds is very significant. As we can see from the Table 2 average bounds as a percentage of an expected value function drop by nearly 50%. It suggests that a change in the equilibrium behavior that resulted from introducing a dominant brings us closer to classical MPE.

The market in the third experiment is characterized by a cheap investment and a high depreciation rate (see second column of the Table 1). A pure OE model results in a bimodal state distribution, a low number of firms and high bounds (about 10% of the expected value function for the tightest OE bound). We believe that the dominant firm OE might be a better choice to model such markets. The distribution of firms in both models is depicted at Figure 5. A dominant firm spends most of the time in the high state. It causes the left tail of the fringe firms to become thinner. The amount of an entry deterrence is much smaller than in the previous case as we can see from the Figure 6.

The numerical results show that adding a dominant firm expands the number of problems that could be computationally analyzed using an OE framework. The proposed extension is appealing because it keeps the model simple and computation times short. In the same time it proves useful in capturing such activities as entry deterrence and enriches an analysis of markets with a fat-tailed firms' state distribution. In all presented cases introducing a dominant firm decreased the number of expected firms in the industry. According to the intuition given by the asymptotic theorems, it can result in higher errors bounds because with lower number of firms OE should be a worse approximation of MPE. However, as we can observe in the Table 2, presence of a dominant firm counteracts this effect and brings us closer to MPE. It suggests that if one is worried that OE might not be the best model to approximate a given industry (like in Experiment 2), one should consider using a dominant firm extension.



## 5 Nonstationary Dominant Firms

In this section we introduce a version of NPOE that overcomes the difficulties that the version introduced in the previous section suffers from. In Section 5.1 we define the new equilibrium concept; we derive bounds to assess approximation error in Section 5.2, and we introduce an algorithm in Section 5.3.

### 5.1 Nonstationary Partially Oblivious Equilibrium

In this section we extend the concept of nonstationary oblivious equilibrium and let firms keep track of the state of few firms (which we call *dominant firms*) and make an estimate based on averages for the state of all other firms (which we call *fringe firms*). We call this new concept *nonstationary partially oblivious equilibrium*, hereafter NPOE.

Let  $\bar{S}_0 = \{i_1, i_2, \dots, i_n\} \subseteq S_0$  be the set of indices associated to the dominant firms. A natural way of selecting  $\bar{S}_0$  may be to choose the  $n$  largest firms from the initial state at  $t = 0$ . In this approximation, the identity of the  $n$  dominant firms does not change over time; firms always keep track of the same set of firms indexed by  $\bar{S}_0$  and all new entrants become part of the fringe. This is consistent with the fact that in many industries the identity of the dominant firms is unlikely to change in the short-term.

We extend the state of firms to include a binary variable that identifies dominant firms. Hence, the state of firm  $i$  at time  $t$  is  $\bar{x}_{it} = (x_{it}, 1)$  if  $i \in \bar{S}_0$ , and  $\bar{x}_{it} = (x_{it}, 0)$  if  $i \notin \bar{S}_0$ . This specification will allow for equilibrium strategies to differ between dominant firms and fringe firms.

#### 5.1.1 Nonstationary Partially Oblivious Strategies

Similarly to a nonstationary oblivious strategy, a nonstationary partially oblivious strategy is a function of the firm's state (which in this case also indicates whether the firm is dominant or not) and the time period. However, firms also keep track of the state of firms in  $\bar{S}_0$ . Let  $y_t$  be a vector that represents the state of the dominant firms at time  $t$ . Formally,  $y_t = (x_{(i_1)t}, x_{(i_2)t}, \dots, x_{(i_n)t})$ , where  $x_{(i_k)t}$  is the  $k$ -th order statistic of  $(x_{i_1t}, x_{i_2t}, \dots, x_{i_nt})$ . Our equilibrium concept will assume symmetric strategies that are a function of the order statistics of  $(x_{i_1t}, x_{i_2t}, \dots, x_{i_nt})$  and will not depend on the identities of the dominant firms. If during the period of analysis, firm  $i \in \bar{S}_0$  exits the industry, then  $x_{it} = -1$  from there on.

While there may be few dominant firms on a given industry, so it is computationally feasible to optimize over strategies that are a function of the dominant firms' state, it is likely that the number of other firms (fringe firms) may be large. In this case, it will not be computationally feasible to optimize over strategies that are a function of the fringe firms' state. Instead, we will assume firms make estimates of the fringe firms'

state based on averages. Actually, if there are many firms, because of averaging effects, firms should be able to accurately predict the fringe firms' state for a given time period based on the entire history of dominant firms' state.<sup>8</sup> This is computationally impractical; instead, we will allow firms to predict the fringe firms' state based on a finite set of statistics that depend on the entire evolution of the dominant firms' state.<sup>9</sup>

Based on this motivation, we will restrict firms' strategies so that each firm's decisions depend only on the firm's state, the time period, the current state of the dominant firms, and a finite set of statistics that depend on the history of realizations of the dominant firms' state. We call such restricted strategies *nonstationary partially oblivious strategies*. To convey this dependence, we define the sequence  $\{w_t \in \mathcal{W} = \mathcal{W}_1 \times \dots \times \mathcal{W}_K : t \geq 0\}$  where  $w_t(1) = y_t$ , for all  $t \geq 0$ , and  $\mathcal{W}_j$  are countable sets.

Recall that  $\tilde{\mathcal{M}}_p$  and  $\tilde{\Lambda}_p$  are the set of partially oblivious strategies and the set of partially oblivious entry rate functions, respectively. If firm  $i$  uses strategy  $\mu_t \in \tilde{\mathcal{M}}_p$  at time period  $t$ , then firm  $i$  takes action  $\mu_t(\bar{x}_{it}, w_t)$ , where  $\bar{x}_{it}$  is the state of firm  $i$  at time  $t$  (which indicates the firm's quality level and if it's dominant or not). Similarly, if for time period  $t$ , the entry rate function is  $\lambda_t \in \tilde{\Lambda}_p$ , the entry rate is equal to  $\lambda_t(w_t)$ . Since  $w_t(1) = y_t$ , firms keep track of the current dominant firms' state when making decisions with partially oblivious strategies. The state variables  $w_t(2), \dots, w_t(K)$  allow firms to incorporate additional information about the history of realizations of the dominant firms' state into the strategies. This information could be useful to better predict the average fringe firm state conditional on observing  $w_t$ . We provide some examples below. In general, accounting for past information will generally improve a firms decisions. It is worth mentioning here, though, that past information is not payoff-relevant (hence, it does not influence MPE strategies), so allowing partially oblivious strategies to depend on that may give rise to NPOE that are poor approximations to MPE.

Now, let  $\tilde{\mathcal{M}}_{pns} = \tilde{\mathcal{M}}_p^\infty$  and  $\tilde{\Lambda}_{pns} = \tilde{\Lambda}_p^\infty$  denote the set of nonstationary partially oblivious strategies and the set of nonstationary partially oblivious entry rate functions. A nonstationary partially oblivious strategy is a sequence of partially oblivious strategies. Hence, if  $\mu \in \tilde{\mathcal{M}}_{pns}$  is a nonstationary partially oblivious strategy, then  $\mu = \{\mu_0, \mu_1, \dots\}$ , where for each time period  $t \geq 0$ ,  $\mu_t \in \tilde{\mathcal{M}}_p$  is a partially oblivious strategy. For example, if firm  $i$  uses strategy  $\mu \in \tilde{\mathcal{M}}_{pns}$  then at time period  $t$ , firm  $i$  takes action  $\mu_t(\bar{x}_{it}, w_t)$ . A nonstationary partially oblivious entry rate function is a sequence of partially oblivious entry rate functions. Hence, if  $\lambda \in \tilde{\Lambda}_{pns}$  is a nonstationary oblivious entry rate function, then  $\lambda = \{\lambda_0, \lambda_1, \dots\}$  where for every period  $t \geq 0$ ,  $\lambda_t \in \tilde{\Lambda}_p$ , and the entry rate is  $\lambda_t(w_t)$ .

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<sup>8</sup>Because firms keep track of the state of the dominant firms, even if there are a large number of firms, the fringe firms' state will not necessarily follow a deterministic trajectory like in the case of NOE; the trajectory will depend on the dominant firms' evolution.

<sup>9</sup>This idea is similar to OE with aggregate shocks; see Weintraub, Benkard, and Van Roy (2008a).

We make the following assumption over the state statistics  $w_t$ .

**Assumption 5.1.** *For any set of partially oblivious strategies that dominant firms use, we assume  $\{w_t : t \geq 0\}$  is a non-homogeneous finite Markov chain adapted to the filtration generated by  $\{y_t : t \geq 0\}$ . For all  $t \geq 0$ ,  $w_t(1) = y_t$ .*

Different partially oblivious strategies can be defined depending on the specification of  $w_t$ . We provide a few examples below. Both examples satisfy Assumption 5.1.

**Example 5.1.** *Suppose that for  $j \in \{1, \dots, K\}$ ,  $w_t(j) = y_{t-j+1}$ . Hence,  $w_t = \{y_t, y_{t-1}, \dots, y_{t-K+1}\}$ ; the statistics correspond to the last realizations of the dominant firms' state. In this case  $\mathcal{W}$  is the set of feasible  $K$ -tuples of consecutive dominant firms' state.*

One disadvantage of the previous scheme is that realizations of the dominant firms' state that appear in a certain window of time influence the strategy, but if a realization occurs even slightly outside this window, it has no influence. With this motivation we introduce an alternative scheme based on exponentially weighted averages of past states.

**Example 5.2.** *Suppose that  $w_t(1) = y_t$  and that for  $j \in \{2, \dots, K\}$ ,  $w_{t+1}(j) = \alpha_j g_j(y_t) + (1 - \alpha_j)w_t(j)$  and  $w_0(j) = 0$ , where  $\alpha_j \in [0, 1]$  and  $g_j : \mathbb{N}^n \rightarrow \mathfrak{R}$ .<sup>10</sup>*

### 5.1.2 Fringe Firms' Expected State

In a NPOE firms will make decisions assuming that the fringe firms' state at time period  $t$  is the *expected* fringe firms' state after  $t$  time periods of evolution given firms' strategy, starting from the industry state of interest, and conditional on the current value of the state statistics  $w_t$ . Formally, suppose that firms use strategy  $\mu \in \tilde{\mathcal{M}}_{pns}$  and enter according to  $\lambda \in \tilde{\Lambda}_{nps}$ . Let  $z_t$  be a vector over quality levels that specifies, for each quality level  $x \in \mathbb{N}$ , the *number of fringe firms* that are at quality level  $x$  in period  $t$ . Suppose that the initial state of the industry at  $t = 0$  is  $s = (y_0, z_0)$ . Note that under our assumptions,  $\{(w_t, z_t) : t \geq 0\}$  is a non-homogeneous Markov chain.

Firms predict the fringe firm state based on the current realization of the dominant firm statistics,  $w_t$ . Accordingly, we define  $\tilde{z}_{(\mu, \lambda, s), t}(w) = E[z_t | w_t = w]$ , for all  $w \in \mathcal{W}$ . In words,  $\tilde{z}_{(\mu, \lambda, s), t}(w)$  is the expected fringe firm state in time period  $t$  when dynamics are governed by nonstationary partially oblivious strategy  $\mu$  and nonstationary partially oblivious entry rate function  $\lambda$ , the initial state is  $s$ , and conditional on the current realization of  $w_t$  being  $w$ .

<sup>10</sup>In principle,  $\mathcal{W}_j$  is an uncountable set that takes values between  $\underline{a}_j = \min_{a \in A} g_j(a)$  and  $\bar{a}_j = \max_{a \in A} g_j(a)$ . However, for computational purposes we could assume that  $\mathcal{W}_j$  is a finite grid contained in  $[\underline{a}_j, \bar{a}_j]$  and we could approximate the values of  $w_t(j)$  with its closest element in the grid.

### 5.1.3 Nonstationary Partially Oblivious Value Function

With some abuse of notation, let  $\pi(\bar{x}_{it}, y_t, z_t)$  be the single-period profits for a firm in state  $\bar{x}_{it}$ , if the dominant firms' state is  $y_t$ , and the fringe firms' state is  $z_t$ .<sup>11</sup> We define a *nonstationary partially oblivious value function*:

$$(5.1) \quad \tilde{V}_t(\bar{x}, w | \mu', \mu, \lambda, s) = E_{\mu', \mu} \left[ \sum_{k=t}^{\tau_i} \beta^{k-t} (\pi(\bar{x}_{ik}, y_k, \tilde{z}_{(\mu, \lambda, s), k}(w_k)) - d_{ik}) + \beta^{\tau_i-t} \phi_{i, \tau_i} \Big| \bar{x}_{it} = \bar{x}, w_t = w \right],$$

where  $i$  is the index of a firm in state  $\bar{x}$  at time period  $t$ . This value function should be interpreted as the expected net present value of firm  $i$  in state  $\bar{x}$  at time period  $t$  when the dominant firms' statistics have value  $w$ , and firm  $i$  follow nonstationary partially oblivious strategies  $\mu'$ . Competitors use strategy  $\mu$  and enter according to  $\lambda$  and the initial state of the industry is  $s$ . Note that even if firm  $i$  is dominant and uses strategy  $\mu'$  which may be different to  $\mu$ , the firm assumes that the fringe firms' state will be  $\tilde{z}_{(\mu, \lambda, s), k}(w_k)$  in time period  $k$ , for all  $k \geq 0$ . That is, when computing a nonstationary partially oblivious value function the dominant firm ignores that its deviation will change the fringe firms' evolution and, hence,  $\tilde{z}$ . This restriction over the set of deviations for dominant firms greatly simplifies computation. Note, however, that in equilibrium the optimal dominant firm's strategy and the expected fringe firm state will be consistent. Finally, in an abuse of notation, we define  $\tilde{V}_t(\bar{x}, w | \mu, \lambda, s) = \tilde{V}_t(\bar{x}, w | \mu, \mu, \lambda, s)$ .

### 5.1.4 Nonstationary Partially Oblivious Equilibrium

We now define a new solution concept: an *s-nonstationary partially oblivious equilibrium* consists of a strategy  $\mu \in \tilde{\mathcal{M}}_{pns}$  and an entry rate function  $\lambda \in \tilde{\Lambda}_{pns}$  that satisfy the following conditions:

1. Firm strategies optimize a nonstationary partially oblivious value function.

$$(5.2) \quad \sup_{\mu' \in \tilde{\mathcal{M}}_{pns}} \tilde{V}_0(\bar{x}, w_0 | \mu', \mu, \lambda, s) = \tilde{V}_0(\bar{x}, w_0 | \mu, \lambda, s), \quad \forall \bar{x} \in \mathbb{N} \times \{0, 1\}$$

2. At every period of time, either the nonstationary partially oblivious expected value of entry is zero or

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<sup>11</sup>Note that if firm  $i$  is dominant, it needs to subtract itself from  $y_t$ . Because there are likely to be many fringe firms, we do not subtract a firm from the fringe firm expected state when evaluating profits for fringe firms.

the entry rate is zero (or both) . For all  $t \geq 0$ ,

$$\begin{aligned} \sum_{w \in \mathcal{W}} \lambda_t(w) \left( \beta E \left[ \tilde{V}_{t+1}((x^e, 0), w_{t+1} | \mu, \lambda, s) \middle| w_t = w \right] - \kappa \right) &= 0, \\ \beta E \left[ \tilde{V}_{t+1}((x^e, 0), w_{t+1} | \mu, \lambda, s) \middle| w_t = w \right] - \kappa &\leq 0, \quad \forall w \in \mathcal{W}, \\ \lambda_t(w) &\geq 0, \quad \forall w \in \mathcal{W}. \end{aligned}$$

Note that the optimization of  $\tilde{V}_0$  implies, by dynamic programming principles, that firms optimize  $\tilde{V}_t$  for all  $t \geq 0$ . However, like in NOE, for computational purposes we consider a finite time horizon. Note that to derive a NPOE it is enough to consider one dominant firm and one fringe firm. New entrants become part of the fringe. Finally, if  $n = 0$ , a NPOE is a NOE.

## 5.2 Error Bounds

TO BE COMPLETED.

## 5.3 Algorithms and Computations

We introduce an algorithm for computing an  $s$ -nonstationary partially oblivious equilibrium. Suppose we are mostly interested in the behavior of the industry in the interval between time periods  $t = 0$  and  $t = \bar{T}$ . Let  $\tilde{V}, \tilde{\mu}, \tilde{\lambda}$  be a POE value function, strategy and entry rate, respectively. Let  $\bar{T} := \min\{t | \beta^{t-\bar{T}} E_{\tilde{\mu}, \tilde{\lambda}} \tilde{V}(x^e + \bar{t}h, w_t) < \delta\}$ , where  $\delta$  is a predetermined precision and the expectation is taken with respect to the invariant distribution of  $\{w_t : t \geq 0\}$  when dynamics are governed by  $(\tilde{\mu}, \tilde{\lambda})$ . We assume there is a finite time horizon of length  $\bar{T} + 1$  and that at  $\bar{T} + 1$  firms garner profits according to a POE value function (step 3). Additionally, we define  $x_{max}$  as the largest quality level a firm can ever reach. We let  $x_{max} := x^e + \bar{T}h$ .

At each iteration of the algorithm, we (1) compute the strategies that maximize the nonstationary partially oblivious value functions (step 10); and (2) we compute new entry rates depending on the extent of the violation of the zero-profit conditions for fringe firms (step 17). Strategies and entry rates are updated ‘‘smoothly’’ (steps 22 and 23). The parameters  $N_1, N_2, \gamma_1$ , and  $\gamma_2$  are set after some experimentation to speed up convergence.

If the termination condition of the outer loop is satisfied with  $\epsilon_1 = \epsilon_2 = 0$ , we have an  $s$ -nonstationary partially oblivious equilibrium. Small values of  $\epsilon_1$  and  $\epsilon_2$  allow for small errors associated with limitations of numerical precision.

We finish by suggesting a way of computing  $\tilde{z}_{(\mu, \lambda, s), t}(w)$  (step 6 in the algorithm). Let  $p_t(x, w, y, w') = P_{\mu, \lambda}[x_{i, t+1} = y, w_{t+1} = w' | x_{it} = x, w_t = w]$ , where  $i \notin \bar{S}_0$ . The probability that the firm exits from a

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**Algorithm 2**  $s$ –Nonstationary Partially Oblivious Equilibrium Solver
 

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- 1:  $\lambda_t(w) := \tilde{\lambda}(w)$ , for all  $t, w$ .
  - 2:  $\mu_t(\bar{x}, w) := \tilde{\mu}(\bar{x}, w)$ , for all  $t, \bar{x}, w$ .
  - 3: Define  $\tilde{V}_{\bar{T}+1}(\bar{x}, w | \mu^*, \mu, \lambda, s) := \tilde{V}(\bar{x}, w)$ , for all  $\bar{x}, w; \mu, \mu^* \in \tilde{\mathcal{M}}_{pns}$ , and  $\lambda \in \tilde{\Lambda}_{pns}$ .
  - 4:  $n := 1$ .
  - 5: **repeat**
  - 6:   Compute  $\tilde{z}_{(\mu, \lambda, s), t}(w)$  for  $t \in \{0, \dots, \bar{T}\}$  and for all  $w$ .
  - 7:    $\Delta_0 := 0; \Delta_1 := 0$ .
  - 8:    $t := \bar{T}$ .
  - 9:   **repeat**
  - 10:     Choose  $\mu_t^* \in \tilde{\mathcal{M}}_p$  to maximize  $\tilde{V}_t(\bar{x}, w | \mu_t^*, \mu, \lambda, s)$  simultaneously for all  $\bar{x}, w$ .
  - 11:     **for all w do**
  - 12:        $\psi_t(w) = \beta \tilde{V}_{t+1}((x^e, 0), w | \mu_t^*, \mu, \lambda, s) - \kappa$ .
  - 13:        $\Delta_0 = \max(\Delta_0, \psi_t(w))$ .
  - 14:       **if**  $\lambda_t(w) > \epsilon_0$  **then**
  - 15:          $\Delta_1 = \max(\Delta_1, -\psi_t(w))$ .
  - 16:       **end if**
  - 17:        $\lambda_t^*(w) := \lambda_t(w) (\beta \tilde{V}_{t+1}((x^e, 0), w | \mu_t^*, \mu, \lambda, s)) / \kappa$ .
  - 18:     **end for**
  - 19:     Let  $t := t - 1$ .
  - 20:   **until**  $t = 0$ .
  - 21:    $\Delta_2 := \|\mu - \mu^*\|_\infty$ .
  - 22:    $\mu := \mu + (\mu^* - \mu) / (n^{\gamma_1} + N_1)$ .
  - 23:    $\lambda := \lambda + (\lambda^* - \lambda) / (n^{\gamma_2} + N_2)$ .
  - 24:    $n := n + 1$ .
  - 25: **until**  $\Delta_0 \leq \epsilon_1$  and  $\Delta_1 \leq \epsilon_1$  and  $\Delta_2 \leq \epsilon_2$ .
-

state  $(x, w)$  is one minus the sum of transition probabilities from that state. Let  $\tilde{z}_t(x, w)$  be the  $x$  component of  $\tilde{z}_{(\mu, \lambda, s), t}(w)$ . Let  $r_t(x, w)$  be the product of  $\tilde{z}_t(x, w)$  and the probability that the dominant firm statistics process is in state  $w$  at time period  $t$ ,  $q_t(w)$ . Then,  $r_t(x, w)$  satisfies the following recursive equations:

$$r_{t+1}(x, w) = \sum_{(y, w')} r_t(y, w') p_t(y, w', x, w) + \mathbf{1}(x = x^e) \sum_{w'} \lambda_t(w') q_t(w') p_t(w', w),$$

where  $p_t(w', w) = P[w_{t+1} = w \mid w_t = w']$  and  $\mathbf{1}$  is the indicator function. In addition,  $q_0(w) = 1$  if  $w = w_0$  and zero otherwise. Similarly,  $r_0(x, w) = z_0(x)$  if  $w = w_0$  and zero otherwise. We can obtain  $r_t(x, w)$  by solving this set of recursive equations. We can also obtain probabilities of the dominant firm process by solving another set of recursive equations. From these two objects, we obtain  $\tilde{z}_t(x, w)$ .

## 6 Conclusions

In this note we studied several important extensions of oblivious equilibrium, including an approximation method to analyze the short-run dynamic behavior of an industry and more sophisticated approximation methods where each firm keeps track of the state of few dominant firms.

Several theoretical and computational issues will be studied in future research:

1. We will explore bounds to assess approximation error for the dominant firms approximation.
2. Additionally, we plan to combine the ideas of Weintraub, Benkard, and Van Roy (2008a) and Section 5 to incorporate an aggregate shock to an approximation method where each firm keeps track of the state of few dominant firms.
3. In a partially oblivious equilibrium, firms always keep track of the same (dominant) firms. In the future we plan to explore approximation methods where each firm keeps track of the current dominant firms in the industry (whose identity might change over time). Additionally, we will study how to build more sophisticated approximations where firms keep track of additional industry statistics, such as the total number of firms in the industry. One complication of these approximations, though, is that the processes that describe the evolution of many of these industry statistics are not Markov.
4. We will do a large set of computational experiments, considering industries with different levels of market concentration, to study how our approximation methods perform in practice.

## A Proof of error bound

**Proof of Theorem 3.1.** Let

$$\mu^*(\bar{x}) = \begin{cases} \tilde{\mu}(\bar{x}) & \text{if } \bar{x} = (x, 1) \\ \bar{\mu}^*(\bar{x}) & \text{if } \bar{x} = (x, 0) \end{cases}$$

be an optimal Markovian (non-oblivious) best response  $\bar{\mu}^*$  to partial OE  $(\tilde{\mu}, \tilde{\lambda})$  for a fringe firms, keeping the partial OE strategy of the dominant firm unchanged.

Hence,  $\mu^* \in \mathcal{M}_p$  is such that

$$\sup_{\mu' \in \mathcal{M}_p} V(\bar{x}, s, w | \mu', \tilde{\mu}, \tilde{\lambda}) = V(\bar{x}, s, w | \mu^*, \tilde{\mu}, \tilde{\lambda}), \quad \forall \bar{x} = (x, 0), x \in \mathbb{N}, s, w$$

where  $\mathcal{M}_p$  was defined in Section 3.2 as a set of Markov strategies for the fringe, and partial OE strategies for the dominant firms.

Take any state of the fringe  $\bar{x} = (x, 0), x \in \mathbb{N}$ . We have that:

$$(A.1) \quad \begin{aligned} E[V(\bar{x}, s_t, w_t | \mu^*, \tilde{\mu}, \tilde{\lambda}) - V(\bar{x}, s_t, w_t | \bar{\mu}, \tilde{\lambda})] &= E[V(\bar{x}, s_t, w_t | \mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(\bar{x}, w_t | \tilde{\mu}, \tilde{\lambda})] \\ &+ E[\tilde{V}(\bar{x}, w_t | \tilde{\mu}, \tilde{\lambda}) - V(\bar{x}, s_t, w_t | \bar{\mu}, \tilde{\lambda})] \end{aligned}$$

First, let us bound the first term in the right hand side of the previous equation.

Because  $\tilde{\mu}$  and  $\tilde{\lambda}$  attain a partial OE, for all  $\bar{x} = (x, 0), w$ ,

$$\tilde{V}(\bar{x}, w | \tilde{\mu}, \tilde{\lambda}) = \sup_{\mu' \in \tilde{\mathcal{M}}_z} \tilde{V}(\bar{x}, w | \mu', \tilde{\mu}, \tilde{\lambda}) = \sup_{\mu' \in \mathcal{M}_p} \tilde{V}(\bar{x}, w | \mu', \tilde{\mu}, \tilde{\lambda}),$$

where the last equation follows because there will always be an optimal partial OE strategy when optimizing an partial oblivious value function even if we consider Markovian strategies that keep track of the full industry state. It follows that,

$$\begin{aligned} V(\bar{x}, s, w | \mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(\bar{x}, w | \tilde{\mu}, \tilde{\lambda}) &\leq \\ &E_{\mu^*, \tilde{\mu}, \tilde{\lambda}} \left[ \sum_{k=t}^{\tau_i} \beta^{k-t} (\pi(x_{ik}, s_{-i,k}, z_k) - \pi(x_{ik}, \tilde{s}(w_k), z_k)) \Big| x_{it} = x, s_{-i,t} = s, w_t = w \right] \\ &= \sum_{k=t}^{\infty} \beta^{k-t} \sum_{\substack{y \in \mathbb{N} \\ s' \in \tilde{\mathcal{S}}, w' \in \tilde{\mathcal{W}}} } P_{\mu^*, \tilde{\mu}, \tilde{\lambda}}[x_{ik} = y, s_{-i,k} = s', w_k = w' \mid x_{it} = x, s_{-i,t} = s, w_t = w] \\ &\quad \times (\pi(y, s', w'(1)) - \pi(y, \tilde{s}(w'), w'(1))), \end{aligned}$$



where we abbreviate  $\tilde{s} = \tilde{s}_{\mu, \lambda}$ . We can write:

$$\begin{aligned} P_{\mu^*, \tilde{\mu}, \tilde{\lambda}}[x_{ik} = y, s_{-i, k} = s', w_k = w' \mid x_{it} = x, s_{-i, t} = s, w_t = w] \\ = P_{\mu^*, \tilde{\mu}, \tilde{\lambda}}[x_{ik} = y \mid s_{-i, k} = s', w_k = w', x_{it} = x, s_{-i, t} = s, w_t = w] \\ \times P_{\mu^*, \tilde{\mu}, \tilde{\lambda}}[s_{-i, k} = s', w_k = w' \mid x_{it} = x, s_{-i, t} = s, w_t = w] \end{aligned}$$

Additionally,

$$\begin{aligned} P_{\mu^*, \tilde{\mu}, \tilde{\lambda}}[s_{-i, k} = s', w_k = w' \mid x_{it} = x, s_{-i, t} = s, w_t = w] \\ = P_{\tilde{\mu}, \tilde{\lambda}}[s_{-i, k} = s', w_k = w' \mid s_{-i, t} = s, w_t = w], \end{aligned}$$

because under extended OE strategies,  $(s_{-i, k}, w_k)$  is independent of  $x_{it}$ , conditional on  $(s_{-i, t}, w_t)$ . Replacing and using Fubini's theorem we obtain:

$$\begin{aligned} V(\bar{x}, s, w \mid \mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(\bar{x}, w \mid \tilde{\mu}, \tilde{\lambda}) \leq \\ \sum_{k=t}^{\infty} \beta^{k-t} \sum_{\substack{s' \in \bar{S} \\ w' \in \mathcal{W}}} P_{\tilde{\mu}, \tilde{\lambda}}[s_{-i, k} = s', w_k = w' \mid s_{-i, t} = s, w_t = w] \\ \times \left[ \max_{y \in \{\underline{x}(k, t), \dots, x + (k-t)\bar{h}\}} (\pi(y, s', w'(1)) - \pi(y, \tilde{s}(w'), w'(1))) \right]^+. \end{aligned}$$

Finally, multiplying by  $q(s, w)$ , the invariant distribution of  $\{(s_t, w_t) : t \geq 0\}$ , summing over all  $(s, w)$ , and using Fubini we get:

$$(A.2) \quad E[V(\bar{x}, s_t, w_t \mid \mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(\bar{x}, w_t \mid \tilde{\mu}, \tilde{\lambda})] \leq \sum_{k=t}^{\infty} \beta^{k-t} E \left[ \left[ \Delta_{\{\underline{x}(k, t), \dots, x + (k-t)\bar{h}\}}(s_k, w_k) \right]^+ \right].$$

Now, let us bound the second term in equation (A.1). We have that,

$$\begin{aligned} \tilde{V}(\bar{x}, w \mid \tilde{\mu}, \tilde{\lambda}) - V(\bar{x}, s, w \mid \tilde{\mu}, \tilde{\lambda}) \\ = E_{\tilde{\mu}, \tilde{\lambda}} \left[ \sum_{k=t}^{\tau_i} \beta^{k-t} (\pi(x_{ik}, \tilde{s}(w_k), z_k) - \pi(x_{ik}, s_{-i, k}, z_k)) \mid x_{it} = x, s_{-i, t} = s, w_t = w \right] \end{aligned}$$

Hence,

$$(A.3) \quad E[\tilde{V}(\bar{x}, w_t | \tilde{\mu}, \tilde{\lambda}) - V(\bar{x}, s_t, w_t | \tilde{\mu}, \tilde{\lambda})] \\ = E \left[ E_{\tilde{\mu}, \tilde{\lambda}} \left[ \sum_{k=t}^{\tau_i} \beta^{k-t} (\pi(x_{ik}, \tilde{s}(w_k), z_k) - \pi(x_{ik}, s_{-i,k}, z_k)) \mid x_{it} = x, s_{-i,t} = s_t, w_t \right] \right].$$

The result follows by equations (A.1), (A.2), and (A.3).  $\square$

## B Tables and Figures

	Experiment 1	Experiment 2	Experiment 3
$a$	9	9	5
$\delta$	0.6	0.05	0.3
$\theta_1$	0.15	0.7	0.5
$\theta_2$	0.5	0.5	0.5
$c$	0.5	0.5	0.5
$Y$	1	1	1
$\psi$	1	2	1
$\beta$	0.95	0.95	0.95
$x^e$	5	1	1
$x^{max}$	15	25	35
$K$	10	10	10
$d_1$	0.3	2	1
$d_2$	0.0051	0	0

Table 1: Model parameters for computational experiments in Section 4

## References

- Caplin, A. and B. Nalebuff (1991). Aggregation and imperfect competition - on the existence of equilibrium. *Econometrica* 59(1), 25 – 59.
- Doraszelski, U. and M. Satterthwaite (2007). Computable markov-perfect industry dynamics: Existence, purification, and multiplicity. Working Paper, Harvard University.

	Experiment 1	Experiment 2	Experiment 3
Pure OE	0.0216	0.0649	0.0967
Dominant Firm OE	0.0216	0.0352	0.0940

Table 2: Average relative bounds in Pure OE and Dominant Firm OE

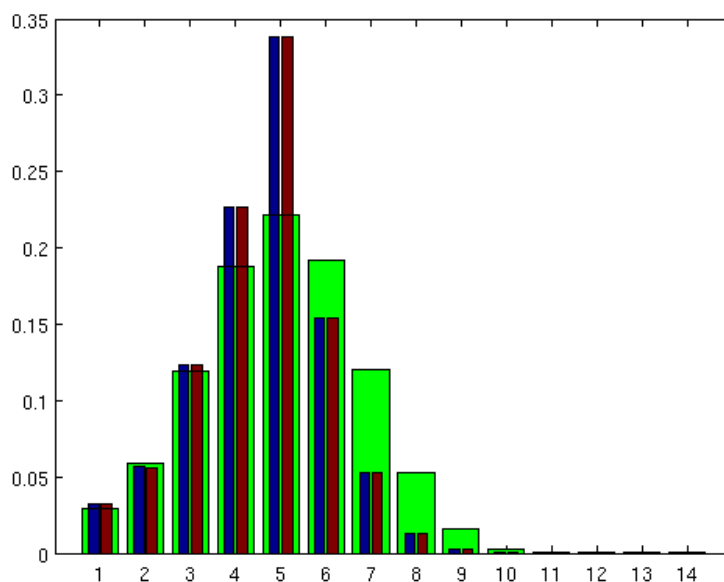


Figure 1: Distribution of firms in Experiment 1: blue - pure OE, red - dom. firm OE fringe, green - dom. firm OE leader

Ericson, R. and A. Pakes (1995). Markov-perfect industry dynamics: A framework for empirical work. *Review of Economic Studies* 62(1), 53 – 82.

Maskin, E. and J. Tirole (1988). A theory of dynamic oligopoly, I and II. *Econometrica* 56(3), 549 – 570.

Pakes, A. and P. McGuire (1994). Computing Markov-perfect Nash equilibria: Numerical implications of a dynamic differentiated product model. *RAND Journal of Economics* 25(4), 555 – 589.

Weintraub, G. Y., C. L. Benkard, and B. Van Roy (2008a). Computational methods for oblivious equilibrium. Working Paper, Stanford University.

	Experiment 1		Experiment 2		Experiment 3	
	Pure OE	Dom. firms	Pure OE	Dom. firms OE	Pure OE	Dom. firms OE
Cons. sur.	541.9381	542.8009	44.2822	41.7780	321.5376	315.0227
Prod. sur.	37.2859	37.2180	5.0211	4.4113	20.0222	19.7387
Total sur.	579.2240	580.0189	49.3033	46.1893	341.5598	334.7614
C1	0.0150	0.0150	0.1234	0.2427	0.0884	0.0946
C4	0.0592	0.0591	0.4251	0.5815	0.3318	0.3567
C10	0.1460	0.1457	0.8547	0.9523	0.6870	0.7238
C20	0.2882	0.2879	0.9993	1.0000	0.9620	0.9791
HHI	100.3318	99.7328	763.0432	1170.1994	587.7590	631.4629

Table 3: Some economic indicators in Pure OE and Dominant Firm OE

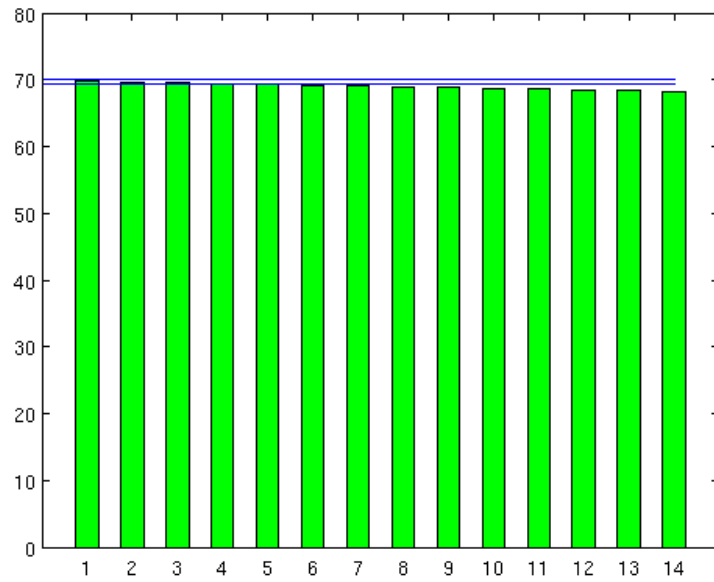


Figure 2: Expected number of fringe firms in Experiment 1

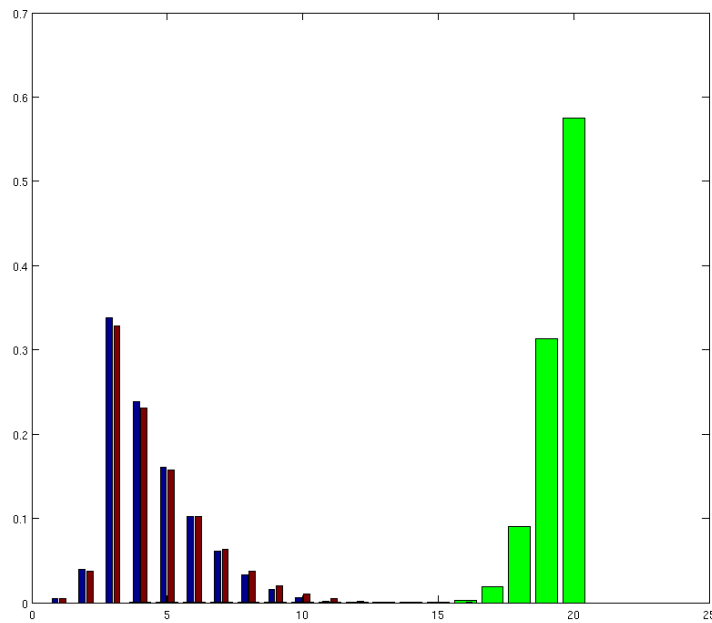


Figure 3: Distribution of firms in Experiment 2: blue - pure OE, red - dom. firm OE fringe, green - dom. firm OE leader

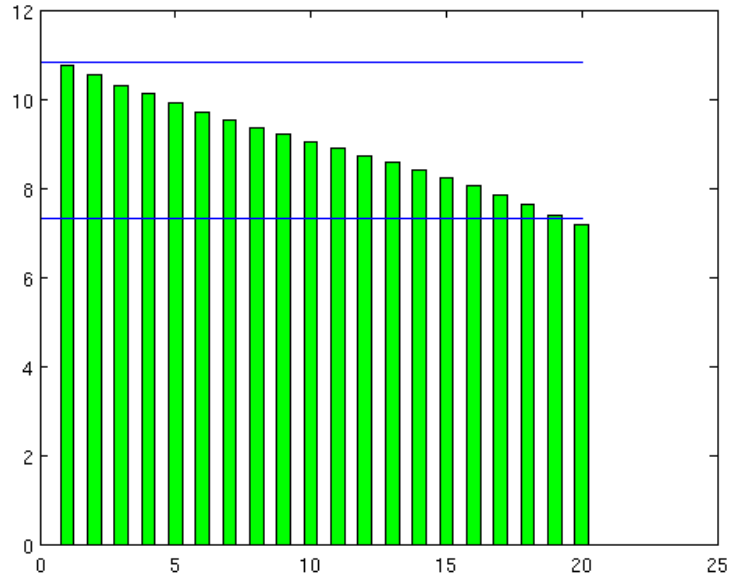


Figure 4: Expected number of fringe firms in Experiment 2

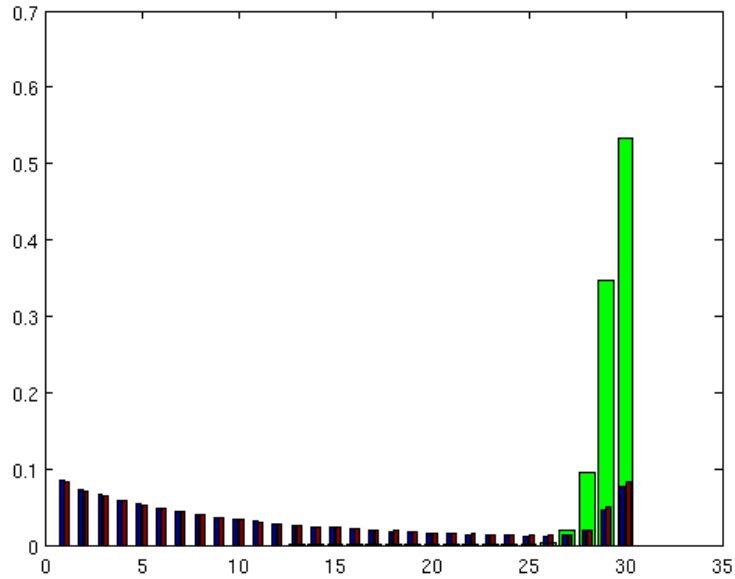


Figure 5: Distribution of firms in Experiment 3: blue - pure OE, red - dom. firm OE fringe, green - dom. firm OE leader

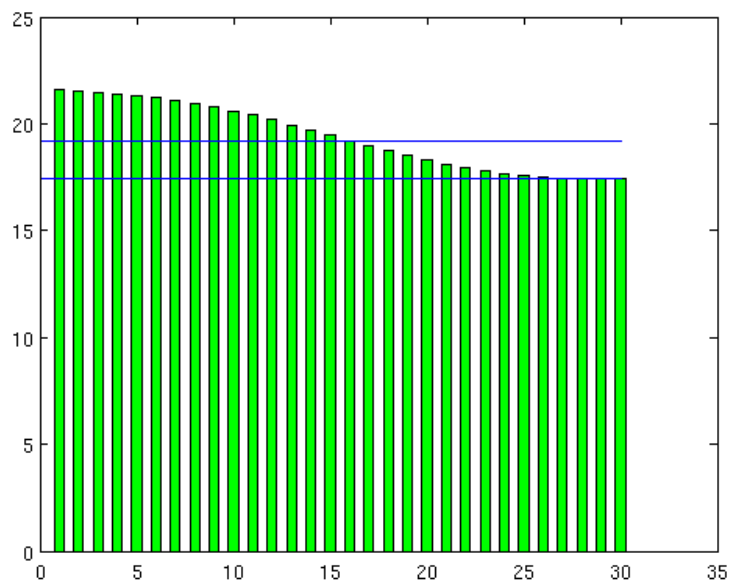


Figure 6: Expected number of fringe firms in Experiment 3

Weintraub, G. Y., C. L. Benkard, and B. Van Roy (2008b). Markov perfect industry dynamics with many firms. *Econometrica* 76(6), 1375–1411.