Multiple Equilibria in the Citizen-Candidate Model of Representative Democracy: A Note

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Abstract

The Besley-Coate model of representative democracy has the problem of multiple equilibria (Besley-Coate, 1997). We show that requiring the Besley-Coate political equilibria to be iteratively undominated at the voting stage refines the set of (pure strategy) political equilibrium outcomes *only for those cases where at least four candidates stand for election.* This note complements the results of De Sinopoli and Turrini (1999).

Keywords: Voting Games, Citizen-Candidate Model, Iterated Weak Dominance, Strict Nash Equilibrium.

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1 Introduction

The citizen-candidate model (Besley-Coate(1997), Osborne-Slivinski(1996)) is being increasingly used to model decision-making in environments where Condorcet winners may not exist. The model has three stages of activity; (i) citizens decide on whether to stand for office, and may incur a small cost if they do so; (ii) citizens vote for the candidates who stand, and the winner is elected by plurality rule; (iii) the candidate who is elected implements her most preferred policy from a fixed set of alternatives. Besley and Coate call the subgame perfect equilbria of this three stage game *political equilibria* (PE). Although the citizen-candidate model is a major advance over existing models, a major problem is that there are typically multiple equilibria at the voting stage, due to plurality rule (Dhillon and Lockwood(2000)). These multiple equilibria at stage (ii) generate multiple equilibria to the game as a whole. Osborne-Slivinski(1996) resolve this problem by assuming that voters vote sincerely. Sincere voting, however is an arbitrary rule for selecting strategies. In contrast, Besley-Coate (1997) impose the requirement that, conditional on any set of candidates, the voting equilibrium must be weakly undominated. Not surprisingly, this weak refinement at the voting stage still leaves many equilibria, some of them not very credible[1].

In this paper, we investigate whether imposing a stronger refinement on the (Nash) equilibrium at the voting stage, conditional on any set of candidates, eliminates any PE. Our refinement is that voting strategies be *iteratively* weakly undominated. We call PE with this refinement imposed at the voting stage *iteratively weakly undominated political equilibria* (IWUPE). Our justification for this refinement is twofold. First, that if it is common knowledge than agents will not play weakly dominated strategies, then it is "reasonable" that rational voters will not play their "second round" weakly dominated strategies, and so on. Some formal justification of this is in Rajan(1998). Second, it has been shown by De Sinopoli(2000) that more stan-

[1] Besley and Coate declare "for those who would like a clean empirical prediction, our multiple equilibria will raise a sense of dissatisfaction."

dard refinements (perfection, properness) do not have much bite in plurality voting games; in particular, requiring subgame-perfection only rules out Nash equilibria where Condorcet losers are elected.

De Sinopoli and Turrini(1999) initiated this approach to refining PE in a paper where they present an example with four candidates and one winner which has multiple equilibria. In their example, requiring voting strategies to be iteratively undominated eliminates all but one equilibrium where the Condorcet winner wins. This result raises the question of whether iterated deletion of weakly dominated strategies also refines political equilibrium outcomes in the case of one, two, and three candidate equilibria. This paper answers this question, fully and negatively. We show[2] that if there exists a PE with fewer than four candidates, and a given set of winner(s), then there also exists a IWUPE with the same candidate set and the same winner(s). So, this paper complements De Sinopoli and Turrini(1999); together, they how that iterated deletion of weakly dominated strategies also refines political equilibrium outcomes only when the number of candidates is at least four.

We describe the Besley-Coate model in greater detail in Section 2. Section 3 then discusses the main results. Section 4 concludes.

2 The Citizen-Candidate Model of Representative Democracy

Besley-Coate (1997) consider a community of n citizens, who may select a representative to implement a policy alternative. Each citizen $i \in N = \{1, ..., n\}$ has a finite action set X_i representing the policy alternatives available to him if elected. It is possible that citizens may be of different competencies i.e. $X_i \neq X_j$. If no-one is elected, a default policy $x_0 \in \cap_{i \in N} X_i$ is selected. Voters have preferences over who represents them, as well the

[2]We do *not* show that every PE is also a IWUPE, but rather that every outcome that can be achieved via a PE can also be achieved via an IWUPE.

alternatives they choose, so utility functions are defined on $X \times N \cup \{0\}$, $X = \cup_{i \in N} X_i$, i.e. $\pi_i(x, j)$ is the utility for i if j is elected and chooses action x. If no-one is elected, utilities are $\pi_i(x_0, 0)$.

The political process has three stages. At stage 1, citizens face a binary decision: to stand for election (enter) or not. At stage 2, voting takes place, and in stage 3, the elected representatives choose policy. We discuss each stage in turn.

At the final stage, once elected, a citizen i will therefore choose their own most preferred policy (assumed to be unique):

$$x_i^* = \arg\max_{x \in X_i} \pi_i(x, i)$$

Since for every citizen's most preferred point $x_i^* \in X_i$ is known, the induced preferences of citizens over *candidates* are given by $u_i(j) \equiv \pi_i(x_i^*, j),\ i, j \in N$. We assume that these induced preferences over candidates are strict: i.e. $u_i(j) \neq u_i(k)$, all $i, j, k \in N,\ j \neq k$. Also, $u_i(0) \equiv \pi_i(x_0, 0)$.

At the second stage, voting is by plurality rule: each voter has one vote, which she can cast for any one of the set $C \subset N$ of candidates who stand, and the candidate with the greatest number of votes wins. If a set of two or more candidates have the greatest number of votes, every candidate in this set is selected with equal probability. Let $W \subset C$ be the set of candidates with the most votes, which we call the *winset*. Then voter payoffs over some W are:

$$u_i(W) = \frac{1}{\#W} \sum_{j \in W} u_i(j),\ i \in N$$

Formally, let $\alpha_i = j$ if voter i votes for candidate $j \in C$.[3] Then $\alpha = (\alpha_1, ..., \alpha_n)$ denotes a *vote profile.* Let $W(\alpha, C) \subset C$ denote the winset, given the vote profile α and candidate set C. The utility to voter i from α (given C) is then $u_i(\alpha, C) \equiv u_i(W(\alpha, C))$. A *Nash equilibrium* profile α^* is

[3] We assume there is no abstention. When voting is costless and, as in our version of the Besley-Coate model preferences over candidates are strict, abstention is always weakly dominated so that ruling out abstention is without loss of generality.

defined in the usual way as a profile where α_i^* is a best response to α_{-i}^*, all $i \in N$.

Let $S \subset C^n$ be any set of voting profiles, with $S = \times_{i \in N} S_i$. Say that α_i *is undominated relative to* S if (i) $\alpha_i \in S_i$; (ii) there does not exist $\alpha_i' \in S_i$ such that $u_i(\alpha_i', \alpha_{-i}, C) \geq u_i(\alpha_i, \alpha_{-i}, C)$, all $\alpha_{-i} \in S_{-i}$, and $u_i(\alpha_i', \alpha_{-i}, C) > u_i(\alpha_i, \alpha_{-i}, C)$, some $\alpha_{-i} \in S_{-i}$. Now define the sequence of sets of vote profiles $\{A^0, A^1, A^2 ...\}$ for i as follows: $A^0 = C^n$, and $A^n = \times_{i \in N} A_i^n$, where A_i^n is the set of voting actions for i that are undominated relative to A^{n-1}, all $i \in N$. As the set of voting actions is finite, A^n converges after a finite number of steps to some A^∞, which is the set of vote profiles that are *iteratively* weakly undominated. It is always non-empty. Also, the A^n are understood to be conditional on C.

Besley and Coate define a *voting equilibrium* to be a α^* which is (i) Nash equilibrium; and (ii) weakly undominated i.e. $\alpha^* \in A^1$. We will focus on a stronger refinement of Nash equilibrium i.e. where $\alpha^* \in A^\infty$. Formally, an *iteratively weakly undominated voting equilibrium is* a α^* which is (i) Nash equilibrium; and (ii) *iteratively* weakly undominated i.e. $\alpha^* \in A^\infty$.

Finally, we turn to the entry stage. Any citizen can run for office, but if they run, they incur a small cost δ. If no-one runs for office, the default policy x_0 is implemented. In the first stage, citizens decide non-cooperatively on their entry: $\gamma_i \in \{0, 1\}$ denotes the entry[4] decision for i. When deciding upon candidacy, citizens all anticipate[5] the same voting equilibrium $\alpha^*(C)$ among the multiple equilibria at the voting stage, given any possible set of candidates C. Denote the strategy profile at the entry stage by $\gamma = \{\gamma_1, ..., \gamma_n\}$.

We can now state our equilibrium concepts. A *weakly undominated political equilibrium* (WUPE) of this game is a $(\gamma^*, \alpha^*(.))$ if (i) γ^* is an equilibrium of the entry stage, given $\alpha^*(.)$ and (ii) $\alpha(C) \in C^n$ is a weakly undominated Nash equilibrium in the voting subgame, for every $C \subset N$. Our WUPE is Besley and Coate's political equilibrium: we add the qualifier to make

[4] We do not allow citizens to randomise.

[5] This is represented as in De Sinopoli and Turrini (1999) by the function $\alpha(.) : 2^N \to (N \cup \{0\})^N$.

explicit the refinement assumed at the voting stage. An *iteratively weakly undominated political equilibrium* (IWUPE) of this game is a $(\gamma^*, \alpha^*(.))$ if (i) γ^* is an equilibrium of the entry stage, given $\alpha^*(.)$ and (ii) $\alpha(C) \in C^n$ is an iteratively weakly undominated Nash equilibrium in the voting subgame, for every $C \subset N$.

It is helpful for future reference to state the equilibrium entry conditions in either case, which are first, that $i \in C$ must prefer to enter, given α^* i.e.

$$u_i(\alpha^*(C^*/\{i\}), C^*/\{i\}) \leq u_i(\alpha^*(C^*), C^*) - \delta,\ i \in C^* \tag{1}$$

and second, that any $j \notin C$ must prefer not to enter, given α^* i.e.

$$u_j(\alpha^*(C^* \cup \{k\}), C^* \cup \{k\}) - \delta \leq u_j(\alpha^*(C^*), C^*),\ j \notin C^* \tag{2}$$

Finally, we state the assumptions we need (in addition to those made by Besley and Coate(1997)) for our analysis. First, we assume a "no indifference over lotteries" condition i.e.

NI. $u_i(W) \neq u_i(W')$, for all i and all $W \neq W',\ W, W' \subset N$.

This condition ensures that the order of deletion of weakly dominated strategies does not matter[6], and thus implies that is important to ensure that the solution concept we use is well defined. Our second assumption, already made above, and purely for convenience, is that voters cannot abstain.

3 Analysis

Let $(\gamma^*, \alpha^*(.))$ be some WUPE, and let $C(\gamma^*) = C^*$ be the equilibrium set of candidates given entry decisions γ^*. We will show that as long as $\#C^* \leq 3$, for any WUPE with equilibrium candidate set C^*, and winset $W(C^*, \alpha^*(C^*))$, there is a IWUPE $(\gamma^{**}, \alpha^{**}(.))$ with the *same* equilibrium set of candidates and the same winset - and therefore the same outcome in terms of policy chosen and political representation.

[6] That is, the calculation of A^∞ does not depend on the order in which weakly dominated strategies are deleted. See Marx and Swinkels (1997).

We proceed as follows. First, $\alpha^{**}(.)$ *must* generate the same winset as $\alpha^*(.)$ when the candidate set is the equilibrium one:

$$W(C, \alpha^{**}(C^*)) = W(C, \alpha^*(C^*)) \tag{3}$$

Second, the incentives to enter must be the same in the original WUPE and the constructed IWUPE. That is, the entry conditions (1),(2) must continue to hold when α^* is replaced *by* α^{**} i.e.

$$u_i(\alpha^{**}(C^*/\{i\}), C^*/\{i\}) \leq u_i(\alpha^{**}(C^*), C^*) - \delta,\ i \in C^* \tag{4}$$

$$u_k(\alpha^{**}(C^* \cup \{k\}), C^* \cup \{k\}) - \delta \leq u_k(\alpha^{**}(C^*), C^*),\ j \notin C^* \tag{5}$$

So, for any C^* with $\#C^* \leq 3$, we must show that we can construct some $\alpha^{**} \in A^\infty$ such that (3)-(5) hold.

Now let Ψ be the set of candidate sets comprising C^* and those sets arising from unilateral deviations from equilibrium entry decisions[7]. Note that conditions (3)-(5) impose conditions on $\alpha^{**}(C)$ when $C \in \Psi$. For candidate sets *not* in Ψ, $\alpha^{**}(.)$ can be defined arbitrarily, subject to the requirement that it is an iteratively undominated profile. That is, we can set

$$\alpha^{**}(C) \in A^\infty(C),\ \text{all}\ C \in \mathcal{N}/\Psi^* \tag{6}$$

where $\mathcal{N}$ is the power set of N. Note that (6) is always possible as $A^\infty(C)$ is always non-empty for all non-empty C.

It is helpful to break our complex task into steps by classifying political equilibria by the *number* of candidates. Following Besley and Coate, 1997, say that a political equilibrium is a $m-$*candidate* political equilibrium if m candidates stand for election in the equilibrium. We first have:

[7] Formally,

$$\Psi = \{C \subset N \,|C = C^*,\ C = C^*/\{i\},\ i \in C^*,\ C = C^* \cup \{j\},\ j \notin C^*\}$$

Proposition 1. *For any 1-candidate WUPE with equilibrium candidate set* $C^* = \{i\}$, *and winset* $W(C^*, \alpha^*(C^*)) = \{i\}$, *there is a IWUPE* $(\gamma^{**}, \alpha^{**}(.))$ *with the same equilibrium set of candidates and the same winset.*

Proof. First, $\alpha^{**}(.)$ is defined[8] on Ψ as follows. For $C = \{i\}$, or $C = \{i, j\}$, $j \in N$, set $\alpha^{**}(.) = \alpha^*(.)$. As we have set $\alpha^{**}(.) = \alpha^*(.)$, (3)-(5) hold from the fact that $\alpha^*(.)$ is part of a WUPE. To conclude, we must verify that $\alpha^*(C)$ is iteratively undominated for all $C \in \Psi$. The case $C = \{i\}$ is trivial, as every voter has only one strategy, so we must have $A^1(i) = A^\infty(i) = \{i\}^n$. In the case $C = \{i, j\}$, $j \in N$, the only undominated strategy for any voter is to vote sincerely, so $A^1(C)$ is a singleton, so again iterated deletion does not reduce it i.e. $A^1(C) = A^\infty(C)$. $\square$

To deal with two-candidate equilibria, we first need the following Lemmas. Let $\Gamma(C)$ be the voting (subgame) with candidate set C. A *strict Nash equilibrium* (Harsanyi, 1973) of $\Gamma(C)$ is a vector of voting decisions α^* where $u_i(\alpha_i^*, \alpha_{-i}^*) > u_i(\alpha_i, \alpha_{-i}^*)$ all $\alpha_i \in C$, $\alpha_i \neq \alpha_i^*$, all $i \in N$. We then have:

Lemma 1. *Any strict Nash equilibrium* α^* *is iteratively undominated i.e.* $\alpha^* \in A^\infty$.

Proof. A strict Nash equilibrium is a profile of pure strategies $(\alpha_1^*, \alpha_2^*, ..., \alpha_n^*)$, such that each α_i^* is a *unique* best response to the profile α_{-i}^*. Thus, none of these strategies can be deleted in the first round. Moreover if this profile survives for all players at any round k of iterated deletion, they must survive in round $k+1$. This is because iterated deletion means that any player has the same or fewer strategies at every round, so if a strategy was a unique best response to a profile which survived round k, it will continue to be a unique best response to this profile in round $k+1$. $\square$

For the proof of the next Lemma, the following notation will be useful. Let $\omega_{-i}(\alpha_{-i})$ be a vector recording the total votes for each candidate; given a strategy profile α_{-i} i.e. when individual i is not included. We suppress the dependence of ω_{-i} on α_{-i} except when needed and refer to ω_{-i} as a *vote*

[8]Obviously, $C/\{i\} = \emptyset$, so $\alpha(\emptyset)$ is not defined.

distribution. Clearly i's best response to ω_{-i} depends only on the information in ω_{-i}.

Lemma 2. *Any weakly undominated Nash equilibrium $\alpha^*(C)$ of a voting game $\Gamma(C)$, where $\#W(\alpha^*,\ C) > 1$ is a iteratively weakly undominated Nash equilibrium, and remains so even in $\Gamma(C \cup \{k\})$ for any $k \notin C$.*

Proof. Since all candidates in $W(\alpha^*, C)$ are tied, i.e. $\omega_i = \omega_j, \quad \forall i, j \in W(\alpha^*, C)$, it follows that every voter is pivotal between all elements of $W(\alpha, C)$. Clearly, voting for his best alternative in $W(\alpha^*, C)$ is a unique best response for any voter[9]. Hence any weakly undominated Nash equilibrium α^* with $\#W(\alpha^*, C) > 1$ must be a strict Nash equilibrium. But then by Lemma 1, $\alpha^*(C)$ is also an iteratively weakly undominated Nash equilibrium.

Now consider the game $\Gamma(C \cup \{k\})$. Assume that j is voter $i's$ most preferred candidate in $W(\alpha^*,\ C)$. The vector of votes i faces given $\alpha^*(C)$ is such that (w.l.o.g.) $\omega_j = \omega_l - 1$, $\forall l \neq j$, and $j, l \in W(C)$. It is sufficient to show that voting for candidate k is not a best response for i in $\Gamma(C \cup \{k\})$. Note that $n \geq 3$ since $\#W(C) > 1$, so $\#C > 1$. Therefore, $n \geq 4$ (otherwise we cannot have a tie between two candidates in the two candidate game). Thus, if voter i deviates to k, he would ensure that $W(C \cup \{k\}) = W(C \backslash \{j\})$. Thus, voting for j remains a unique best response. Thus, $\alpha^*(C)$ remains a strict Nash equilibrium of the game $\Gamma(C \cup \{k\})$. Again, by Lemma 1, $\alpha^*(C)$ is also an iteratively weakly undominated Nash equilibrium in $\Gamma(C \cup \{k\})$. $\square$

We now turn to two-candidate WUPE. Note that the entry condition (1) requires that if there are two candidates, both must be in the winset - otherwise, the one that does not win would withdraw (Besley and Coate(1997). So, our second result is:

Proposition 2. *For any 2-candidate WUPE with equilibrium candidate set $C^* = \{i, j\}$, and winset $W(C^*, \alpha^*(C^*)) = \{i, j\}$, there is an IWUPE $(\gamma^{**}, \alpha^{**}(.))$ with the same equilibrium set of candidates and the same winset.*

[9] This also implies, given our NI condition, that *all* voters will vote for their best alternative in $W(C)$.

Proof. First, $\alpha^{**}(.)$ is defined[10] on Ψ as follows. For $C = C^*$, or $C = C^*/\{i\}$, $i \in C^*$, set $\alpha^{**}(.) = \alpha^*(.)$. For $C = C^* \cup \{k\}$, $k \notin C^*$, set $\alpha^{**}(C^* \cup \{k\}) = \alpha^*(C^*)$. Note that by construction, (3),(4) are satisfied. Also, note that (5) is satisfied. First, note that

$$W(C^* \cup \{k\}, \alpha^{**}(C^* \cup \{k\})) = W(C^* \cup \{k\}, \alpha^*(C^*)) = W(C^*, \alpha^*(C^*)) \quad (7)$$

i.e. given α^{**}, the winner is unchanged if k enters. So, from (7),

$$u_k(\alpha^{**}(C^* \cup \{k\}), C^* \cup \{k\}) = u_k(\alpha^*(C^*), C^*)$$

and consequently (5) holds as $\delta > 0$.

Again, to conclude, we must verify that $\alpha^*(C)$ is iteratively undominated for all $C \in \Psi$. For $C = C^*$, or $C = C^*/\{i\}$, $i \in C^*$, an argument identical to the proof of Proposition 1 shows this. For $C = C^* \cup \{k\}$, Lemma 2 implies that $\alpha^*(C)$ is an iteratively undominated voting profile in the game $\Gamma(C^* \cup k)$, as required. $\square$

We now turn to the most complex case, that of 3-candidate WUPE. First, with three candidates, there may in principle, be one, two or three winners. It turns out that the case of two winners is impossible under our assumption of strict preferences. The case of three winners can be dealt with using Lemma 2, following the proof of Proposition 2. However, in the case of one winner, Lemma 2 no longer applies, and so we must find some other argument to construct an IWUPE with one winner. To illustrate our argument, we first present an example of a 3-candidate WUPE with one winner where we can find an IWUPE with the same outcome.

We need the following notation and lemma before this example. Fix some candidate set C with $\#C = 3$. Let N_i be the set of voters who rank candidate $i \in C$ as worst, with $n_i = \#N_i$. Let $q = \max_{l \in C} \{n_l/n\}$, and let w_i denote citizen $i's$ worst candidate in C, all $i \in N$. Now define a critical value of q as:

$$q_n = \begin{cases} 1 - \frac{1}{n} - \frac{1}{n}\lceil \frac{n+1}{3} \rceil, & n \text{odd} \\ 1 - \frac{1}{n}\lceil \frac{n+2}{3} \rceil, & n \text{ even} \end{cases} \quad (8)$$

[10] Obviously, $C/\{i\} = \emptyset$, so $\alpha(\emptyset)$ is not defined.

where $\lceil x \rceil$ denotes the smallest integer larger than x, and $\lfloor x \rfloor$ denotes the largest integer smaller than x. Finally, in this section, we assume that $n \geq 4$.[11] We then have the following useful result, constructed from various results of Dhillon and Lockwood, 2000:

Lemma 3. *Assume $\#C = 3$. If $q \leq q_n$, then any weakly undominated strategy profile in $\Gamma(C)$ is also iteratively weakly undominated i.e. $A^1(C) = A^\infty(C)$. Moreover, $A^1(C)$ is a subset of the set of iteratively undominated strategy profiles in $\Gamma(C \cup \{l\})$ i.e. $A^1(C) \subset A^\infty(C \cup \{l\})$.*

Proof. Every $\alpha_i \in C$ except $\alpha_i = w_i$ is weakly undominated in C, so $A^1(C) = \times_i(C/w_i)$ (Dhillon and Lockwood, Lemma 1). Moreover, by Theorem 2 of Dhillon and Lockwood, as $q \leq q_n$, $A^\infty(C) = \times_i(C/w_i)$. So,$A^1(C) = A^\infty(C)$ as required. Finally, consider $\Gamma(C \cup \{l\})$. Define the *full reduction* of $(C \cup \{l\})^n$, $V = \times_i V_i$, to be the set of strategy profiles where every $\alpha_i \in V_i$ is iteratively undominated relative to V (Marx and Swinkels(1997)). Then, by definition, $V = A^\infty(C \cup \{l\})$. We will show that $A^1(C) \subset V$. To do this, it is sufficient to show (i) that every $\alpha_i \in C/w_i$ is undominated relative to $(C \cup \{l\})^n$ and (ii)it remains undominated in every subsequent stage of deletion. In turn, it is sufficient to show that every α_i is a unique best response in $C \cup \{l\}$ to some α_{-i} in $(C \cup \{l\})^{n-1}$, and that it remains a unique best response in any subsequent stage of iterated deletion.

To prove this, let $\tilde{\alpha}_i \in C/w_i$. As $\tilde{\alpha}_i \in A^\infty(C)$, there exists $\tilde{\alpha}_{-i} \in A^\infty_{-i}(C)$ such that $\tilde{\alpha}_i$ is the unique best response in $A^\infty_i(C)$ to $\tilde{\alpha}_{-i}$. Note that the support of the set $A^\infty(C)$ consists of pure strategies that are a unique best response to some profile which is also in $A^\infty(C)$. Thus, consider the first stage of iterated deletion in the game $\Gamma(C \cup \{l\})$. We know that the set $A^\infty(C) \subset A^1(C \cup \{l\})$, because $A^\infty(C) = A^1(C) \subset A^1(C \cup \{l\})$. Thus no strategy in $A^\infty(C)$ is deleted in the first round if we can show that $\tilde{\alpha}_i$ continues to be the unique best response in $A^\infty_i(C)$ (and hence in $A^1(C \cup \{l\})$) to $\tilde{\alpha}_{-i}$, when voter i can also choose l.

To see this, let $\tilde{\alpha}_i = j$ and note that since $q \leq q_n$ for $\Gamma(C)$, $\tilde{\alpha}_i$ is a

[11] This is w.l.o.g since we only need this for the proof of the main Propositions.

unique best response to the profile $\tilde{\alpha}_{-i}$ where i is pivotal between exactly two alternatives w.l.o.g $\{i,j\} \subset C$. Thus, the vote distribution $\omega_{-i}(\tilde{\alpha}_{-i})$ must have one of the two alternatives (i,j) in C getting two or more votes when $n \geq 4$. So, voting $\alpha_i = l$ in response to $\tilde{\alpha}_{-i}$ cannot affect the outcome, and so $\tilde{\alpha}_i$ remains a unique best response in $\Gamma(C \cup \{l\})$ to $\tilde{\alpha}_{-i}$. This proves that *every* pure strategy in the support of A^{∞} remains a unique best response in $A^1(C \cup \{l\})$. In particular, the strategy profile $\tilde{\alpha}_{-i}$ also cannot be deleted in the first round.
Moreover, if $\tilde{\alpha}_i, \tilde{\alpha}_{-i}$, are not deleted in the first round they cannot be deleted in any subsequent round since no new strategies are added. $\square$

This is a powerful result which allows treatment of the 3-candidate case.

Example

There are eight citizens with preferences over N as follows:

$$
\begin{aligned}
1 &: 1 \succ 8 \succ 5 \succ 3 \succ 2 \succ 4 \succ 6 \succ 7 \\
2 &: 2 \succ 8 \succ 1 \succ 3 \succ 5 \succ 4 \succ 6 \succ 7 \\
3 &: 3 \succ 8 \succ 5 \succ 1 \succ 2 \succ 4 \succ 6 \succ 7 \\
4 &: 4 \succ 8 \succ 5 \succ 3 \succ 1 \succ 2 \succ 6 \succ 7 \\
5 &: 5 \succ 8 \succ 1 \succ 3 \succ 2 \succ 4 \succ 6 \succ 7 \\
6 &: 6 \succ 8 \succ 1 \succ 3 \succ 5 \succ 2 \succ 4 \succ 7 \\
7 &: 7 \succ 8 \succ 1 \succ 3 \succ 5 \succ 2 \succ 4 \succ 6 \\
8 &: 8 \succ 5 \succ 3 \succ 1 \succ 2 \succ 4 \succ 6 \succ 7
\end{aligned}
$$

Let $(\gamma^*, \alpha^*(.))$ represent a WUPE in this game, with an equilibrium set of 3 candidates $C^* = \{1,3,5\}$, and one winner, $W(C^*, \alpha^*(C^*)) = \{5\}$. We will first describe $\alpha^*(.)$ and verify that it does induce the equilibrium entry decisions. Then, we will show that there is an IWUPE with the same set of candidates and winset.

Description of $\alpha^*(.)$

First, $\alpha(C^*) = (5,1,5,5,5,1,3,5)$, thus candidate 5 wins. This is a Nash equilibrium, since no voter is pivotal, and moreover, the profile is undominated ($\alpha(C^*) \in A^1(C^*)$) as no-one votes for their worst candidate.
Voting profiles and winsets in all the two candidate games generated by withdrawal of one of the equilibrium candidates are as follows:

$$\begin{aligned}
\alpha(C^*/\{1\}) &= (5,3,3,5,5,3,3,5),\ W(C^*/\{1\}) = \{3,5\} \\
\alpha(C^*/\{3\}) &= (1,1,5,5,5,1,1,5),\ W(C^*/\{3\}) = \{1,5\} \\
\alpha(C^*/\{5\}) &= (1,1,3,3,1,1,1,3),\ W(C^*/\{5\}) = \{1\}
\end{aligned}$$

It is clear that withdrawal is suboptimal for all candidates. For example, if candidate 1 withdraws, he gets a lottery over 3 and 5 which is worse for him than 5. Finally, note that all these voting profiles are undominated Nash equilibrium ones, as there are only two alternatives and voting is sincere.
Next, note that if $\alpha^*(C) \in A^1(C \cup \{j\}),\ \ j = 2,4,6,7$, then j cannot win in $\Gamma(C \cup \{j\})$. This is because j is ranked worst in $C \cup \{j\} = \{1,3,5,j\}$ by all players except j himself, an is a dominated strategy to vote for one's worst alternative. So, in equilibrium, $j = 2,4,6,7$ will not enter, as required.

Finally, consider subgame $\Gamma(C^* \cup \{8\})$. We set $\alpha(C^* \cup \{8\}) = \alpha(C^*)$. This is an undominated Nash equilibrium, as shown above. Moreover, $W(C^* \cup \{8\}) = \{5\}$, so that 8 has no incentive to enter, as required.

Construction of the Equivalent IWUPE

Let $\alpha^{**}(.) \equiv \alpha^*(.)$ on Ψ. It is then obvious that α^{**} induces the same entry behavior as α^*.It remains to check that $\alpha^{**}(C)$ is iteratively undominated for all $C \in \Psi$. First, in $\Gamma(C^*)$, note from (8) that $q(C^*) = 3/8 < q_8^3 = 1/2$. Hence by Lemma 3 above, $\alpha^*(C^*) \in A^\infty(C^*)$ for all i. Next, in $\Gamma(C^*/\{i\}),\ \ \alpha^*$ is iteratively undominated, as there are only two alternatives (formally, $\alpha^*(C/\{i\}) \in A^\infty(C/\{i\}),\ \ i \in C$). Finally, in $\Gamma(C^* \cup \{l\})$, $\alpha^*(C) \in A^1(C) = A^\infty(C^* \cup \{l\})$, again by Lemma 3. $\|$
We ncan generalise the arguments used in this example to prove:

Proposition 3. *For any 3-candidate WUPE there is an IWUPE* $(\gamma^{**}, \alpha^{**}(.))$ *with the same equilibrium set of candidates and the same winset.*

Proof of Proposition 3. We divide the proof into three parts: (A) where three candidates win, (B) where two candidates win and (C) where only one candidate wins.

(A) Assume that $C^* = \{i, j, k\}$, $W(C^*) = C^*$. Now define $\alpha^{**}(.)$ on Ψ as follows. For $C = C^*$, or $C = C^*/\{l\}$, $l \in C^*$, set $\alpha^{**}(.) = \alpha^*(.)$. For $C = C^* \cup \{m\}$, $m \notin C^*$, set $\alpha^{**}(C^* \cup \{m\}) = \alpha^*(C^*)$. Then, by an identical argument to that of the proof of Proposition 2, (3),(4),(5) are satisfied.

Again, to conclude, we must verify that $\alpha^{**}(C)$ is iteratively undominated in $\Gamma(C)$ for all $C \in \Psi$. Since $W(C^*) = C^*$, $\#W(C) > 1$ so by Lemma 2, $\alpha^*(C^*)$ is an iteratively weakly undominated Nash equilibrium in $\Gamma(C^*)$ and remains so even in $\Gamma(C^* \cup \{k\})$ for any $k \notin C$. This leaves the two candidate games $\Gamma(C/\{l\})$, $l = i, j, k$: here there is nothing to prove as with two alternatives, an undominated Nash equilibrium is also iteratively undominated.

Case(B): We show that a 3-candidate WUPE where two candidates tie is impossible in our framework with strict preferences. Suppose that such a WUPE exists: $C^* = \{i, j, k\}$, $W(C^*) = \{i, j\}$. Then we must have i, j getting equal numbers of votes. Since votes are split equally between two candidates, $n \geq 4$ (since there are three citizen candidates) and n is even. Moreover, all voters are pivotal, so the unique best response for a voter is to vote for her preferred alternative in $\{i, j\}$. So, we know exactly half the voters prefer i to j and vice versa.

Also, since this is a WUPE, the entry conditions for the three candidates are satisfied. For i, j this implies that entry costs are sufficiently low, and for candidate k it must be that the outcome if he does not enter is less preferred by him to a tie between i, and j. W.l.o.g. let $W(C^*/\{k\}) = \{i\}$, and let k prefer j to i. This implies that in $\Gamma(\{i, j\})$, i wins. In turn since only equilibria with sincere voting are possible in the two candidate voting game, this implies that a majority of citizens prefer i to j, a contradiction. So, case (B) cannot arise.

Case (C). Assume that $C^* = \{i, j, k\}$, $W(C^*) = \{i\}$. We prove first that in this case there must be at least 4 voters:

Claim 1: If a three candidate WUPE in case (C) exists, then $n \geq 4$.

Proof in the Appendix

Now define $\alpha^{**}(.)$ on Ψ as follows. For $C = C^*$, or $C = C^*/\{l\}$, $l \in C^*$, set $\alpha^{**}(.) = \alpha^*(.)$. For $C = C^* \cup \{m\}$, $m \notin C^*$, set $\alpha^{**}(C^* \cup \{m\}) = \alpha^*(C^*)$. It is clear from the construction of α^{**} that α^{**} gives the "correct" entry incentives i.e. (3),(4),(5) are satisfied.

We now take all the voting sub-games $\Gamma(C)$, $C \in \Psi$ in turn, and show that $\alpha^{**}(C)$ is indeed iteratively undominated in these games.

1. $\Gamma(C^*/\{l\})$, $l \in C^*$. As $\Gamma(C^*/\{l\})$ is a two-candidate game, $\alpha^*(C^*/\{l\})$ is clearly iteratively undominated.

2. $\Gamma(C^*)$. First, we show that in this game, $q(C^*) \leq q_n$. We show this by the following two Claims:

Claim 2: $n_j(C^*) \leq \lceil n/2 \rceil$ *and* $n_k(C^*) \leq \lceil n/2 \rceil$.

Claim 3: $n_i(C^*) \leq n - \lceil \frac{n+4}{3} \rceil$

These two claims are proved in the Appendix. Now, by definition, and from the Claims,

$$q(C^*) = \max\{\frac{n_i(C^*)}{n}, \frac{n_j(C^*)}{n}, \frac{n_k(C^*)}{n}\} = \max\{\frac{n}{2}, n - \lceil \frac{n+4}{3} \rceil\} \leq q_n^3$$

where the last inequality follows by inspection of (8). So, by Lemma 3, $A^1(C^*) = A^\infty(C^*)$. So, $\alpha^*(C^*) \in A^\infty(C^*)$ as required.

3. $\Gamma(C^* \cup l)$, for any $l \notin C^*$. By Lemma 3, $\Gamma(C^* \cup \{l\})$, $\alpha^*(C) \in A^1(C) \subset A^\infty(C^* \cup \{l\})$, so again $\alpha^{**}(C^* \cup \{l\}) \in A^\infty(C^* \cup \{l\})$ as required. □

4 Conclusion

In this paper we show that we cannot refine the set of WUPE outcomes for one, two and three candidate (pure strategy) equilibria . Intuitively it is easy to see why refinements become easier with the four candidate case. Consider WUPE where one candidate wins in a four candidate race. Unlike the entry conditions in the three candidate case, the entry conditions in the four candidate case involve three candidates (and 5 candidates) and hence may involve insincere voting, thus imposing fewer restrictions on preferences (e.g. in the example we needed $n_1, n_3 \leq n/2$). In the proof of Proposition 3 above, since two candidate elections involve sincere voting only, n_j and n_k were restricted to be less than half of n. Such a restriction would not arise in the four candidate case, and it is easy to find WUPE outcomes that are not supported by any IWUPE.

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Appendix

Proof of Claim 1. It is sufficient to show that if a WUPE exists, then $n > 3$, since there are three candidates. Let the three citizen candidates be i, j, k. Let $W(C) = \{i\}$ as before. Suppose to the contrary that a WUPE exists with $n = 3$. Since a WUPE exists, i enters because he wins against the other two. Thus in $\Gamma(C)$, $n_i \leq 1$. Moreover, j (k) enters because he prefers i to k (j) and it must be that $\Gamma(C - \{j\}) = k$ $(\Gamma(C - \{k\}) = j)$: since $n = 3$ two candidates cannot tie. Thus $n_j, n_k \leq 1$. Thus $n_i = n_j = n_k = 1$. W.l.o.g let $N_i = \{j\}$ – this is a contradiction to the entry condition for j (i.e. j prefers i to k). $\square$

Proof of Claim 2. Since a WUPE exists, candidates j, k must have some incentive to enter. In particular, we cannot have $W(C^*/\{j\}) = \{i\}$ (otherwise from (1), j would not enter), so (i) $W(C^*/\{j\}) = \{k\}$ or $\{i, k\}$ and moreover, j must prefer i to k, otherwise gain, j would not enter. A symmetric argument implies that (ii) $W(C^*/\{k\}) = \{j\}$ or $\{i, j\}$.

Since the only undominated Nash equilibrium with in two candidates is sincere voting, (i) and (ii) imply that at least half the voters prefer j to i and at least half the voters prefer k to i. So, no more than half the voters can rank either j or k as worst. $\square$

Proof of Claim 3. Since there is a WUPE where i is the unique equilibrium outcome in the game with three candidates, it must be that a sufficiently large number of voters have i as a top or second ranked alternative among the three i.e. n_i must be sufficiently low. We look for the maximum n_i that is compatible with i being a unique winner. Thus, we can assume that all $\omega_i = n - n_i$ i.e. all $n - n_i$ voters vote for i. Note that in such an equilibrium, by the example we have $\max n_i > 0$.

In such an equilibrium it must be true either that (i) $\omega_i \geq \max(\omega_j, \omega_k) + 2$ or that (ii) only two candidates i, j get any votes at all. To see this, suppose first that (i) is not true. Then the vote profile is w.l.o.g $\omega_i = \omega$, $\omega_j = \omega - 1$ and $\omega_k \leq \omega - 1$. Consider any voter $m \in N_i$: It is sufficient to show that no voter would vote for k – thus assume he votes for k. Then he faces a vote profile ω_{-i} such that (a) $\omega_i = \omega$, $\omega_j = \omega - 1$ and $\omega_k \leq \omega - 2$. But then he is strictly better off voting for j since that ensures a tie.

Thus, in case (ii) the only case in which i can emerge a unique winner is if all $m \in N_i$ vote for j and the maximum votes that j can get $(= n - n_j)$ is strictly less than $\omega = n - n_i$. This implies, $n_i < n_j$. But this is impossible (Claim 1 – in the 2 candidate game between i, j, j must be in the winning set).

It remains to check case (i) i.e. the profile $\omega_i \geq \max(\omega_j, \omega_k) + 2$. Thus we have the following constraints[12]:

$$\omega_i = (n - n_i) \geq \max(\omega_j, \omega_k) + 2 \tag{9}$$

$$\max(\omega_j, \omega_k) = \lceil \frac{n_i}{2} \rceil \tag{10}$$

Combining the two, we have:

$$\lceil \frac{3n_i}{2} \rceil \leq n - 2 \tag{11}$$

From inequality (11) and the requirement that n_i is an integer we have:

$$\lceil \frac{2}{3}(n-2) \rceil \geq n_i \tag{12}$$

Thus:

$$\frac{n_i}{n} \leq \frac{1}{n} \lfloor \frac{2(n-2)}{3} \rfloor = \phi$$

Note that $\lfloor \frac{2(n-2)}{3} \rfloor$ is an integer if and only if $\lceil \frac{n+4}{3} \rceil$ is an integer. Hence $\lfloor \frac{2(n-2)}{3} \rfloor + \lceil \frac{n+4}{3} \rceil = n \ \ , \forall n$, from which we get:

$$\phi = 1 - \frac{1}{n} \lceil \frac{n+4}{3} \rceil \tag{13}$$

But then (12) and (13) yield the right result. □

[12]Thanks are due to an anonymous referee who suggested this part of the proof.