Evolutionarily Stable Strategies: who does the mutant have to beat?*

by

Norman J Ireland

Department of Economics
University of Warwick
Coventry CV4 7AL
UK

Email: N.J.Ireland@warwick.ac.uk

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Abstract

This paper develops the notion of weights for different classes of player in assessing a mutant's ability to invade the status quo strategy. Different weights give rise to a continuum of ESS equilibria, ranging from "competitive" to Nash. The emphasis is on applications to oligopoly and to complete information bidding contests. The theory and applications are extended to include a condition that the mutant has to have some chance of survival in an economic sense in order to invade: in the oligopoly model, the constraint leads to a zero-profit equilibrium, while the contest model reveals different distributions of bids but the same expected loss for contestants.

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1 Introduction

The concept of an Evolutionarily Stable Strategy (henceforth ESS) (eg Maynard Smith, 1982) is usually applied to evolutionary processes where agents adopt a strategy and then learn of its comparative success. An ESS is such that any player adopting any different (mutant) strategy does no better than the average of all other players, all of whom are playing the same ESS strategy. Since it is a choice of strategy that becomes and remains conventional, the adoption of the ESS requires mimicking convention rather than fully rational calculation. Behaviour which evolves and which is economical on information is an important element in any economic process. The adaptation of ESS to the case of finite populations by Schaffer (1988, 1989) creates a clearer distinction between a Nash equilibrium and an ESS equilibrium. This can be explained because players in an ESS equilibrium are essentially playing a Nash equilibrium in an artificially constructed and different (and zero-sum) game: a game where individual players have an objective of the difference between the individual's "real" payoff and the average "real" payoff of other players (Leininger, 2002). If, given this objective, each player chooses the same strategy (pure or mixed) as simultaneous best responses, then the Nash equilibrium in this constructed game signals an ESS equilibrium in the "real" game. Each player has taken into account the effect of her action on other players' "real" payoffs: reducing other players' payoffs is part of the objective (the notion of spitefulness).

This paper uses two applications from the field of industrial organisation to develop a simple trick proposed in Schaffer (1988) and incorporated elsewhere, for example in Hehenkamp et al (2004), pp1055-6. This approach generalises the properties of an ESS by focussing on, and offering re-interpretations of, the set of games and their players, that agents will use to assess a mutant strategy relative to the established strategy. This set might consist of the whole population, or it might consist only of those close to the mutant, playing a game involving the mutant, and thus directly affected by the mutant's behaviour. In our applications we will distinguish between the set of internal players who are playing in the game with the mutant, and the set of external players who are not directly affected by the mutant but who nevertheless observe the mutant's performance and consider it relevant to their own choice of strategy. The relative numbers of internal and external players can change the ESS significantly by changing the balance of the mutant's payoff and the effect of the mutant on the average player.

Our applications illustrate this point. The first application relates to quantity-setting oligopoly equilibria. Qin and Stuart (1997) investigate a standard model with a large number (approaching infinity) of n-firm oligopolies. In this case quantity-setting Cournot-Nash behaviour is found to yield an ESS (Theorem 1, p44). Passajennikov (2003) shows that with a finite population the ESS instead coincides with the Walrasian equilibrium (if this exists) in the quantity setting game (Proposition 4, p925). Related arguments are contained in Soytas and Becker (2003), Vega-Redondo (1997) and Reichman (2002). Our approach here yields an equilibrium with the ESS intermediate between a Walrasian and Cournot-Nash outcome. In contrast to Vega-Redondo (1997), the Cournot-Nash equilibrium does not tend to the Walrasian equilibrium as the total number of firms tends to infinity. This is because the Cournot-Nash equilibrium in our paper is an output-setting equilibrium in each of a number of oligopolies. The total number of firms can increase without the size of any one oligopoly increasing, simply by having a larger number of oligopolies. In particular, we will be able to describe the ESS in terms of the average price-cost margin in the market as a function of the demand elasticity in that market and the numbers of firms in that oligopoly and in other relevant oligopolies.

The range of outcomes for an ESS also allows us to consider outcomes of two-stage oligopoly games in terms of whether such economies can be invaded by mutant strategies. The particular case of interest is that of the Vickers (1984) model of strategic delegation. Here, a firm's output decision is delegated to a manager, and the manager may have size or market share objectives as well as a profit objective. In equilibrium, firms supply more than the Cournot-Nash outputs due to the strategic externality for each firm in appointing an aggressive manager. Could such a prisoners' dilemma outcome have ESS properties? Could an ESS which has evolved be similar to the outcome of the two-stage game of selecting managers and then managers selecting outputs. We find that the Vickers' two-stage perfect equilibrium can always be invaded by a more aggressive manager type, except when internal players are insignificant in number relative to external players. This is particularly interesting since selecting an agent type, or equivalently designing appropriate agent incentives, may be a key evolutionary strategy within business behaviour.

Our second application considers an ESS that takes the form of a mixed strategy. Here the application is to contests with exogenous prizes. For example groups of firms can each hold a bidding contest for a supply contract. Again we find that we can have an ESS ranging from a Walrasian (competitive) case to a (mixed strategy) Nash equilibrium, according to the composition of the set of players. The simple contests have the form of what is commonly termed a "dollar all-pay auction". These are a special case of a Tullock (1980) rent-seeking contest with the parameter (r) set to infinity. ESS equilibria in Tullock contests with finite r are considered in Hehenkamp et al (2004), Leininger (2003), Possajennikov (2003) and Reichman (2002), but the limiting case we consider has only mixed strategy equilibria in both Nash and ESS constructions. This is of particular interest since many contests take this form: the player who tries hardest, or spends most, gains the prize. Many economic phenomena including patent and contract chasing have this feature. We obtain strong results about the "over-dissipation" in the ESS outcomes to such sets of contests. Thus the ESS predicts more expenditure on average than the value of the prize.

In both applications, two further extensions are made. The first extension is made relevant by our approach. Due to the division in the population of players between those directly affected by an invasion and those playing in other games and thus not directly affected, the actual survival of the mutant might be viewed as a condition for the mutation to gain a hold in the population. We define a Constrained ESS (CESS) as a strategy proof against invasion by mutants requiring a payoff permitting survival. This is a fairly minimal constraint but has significant impact. In the case of the oligopoly model, survival is equated with a non-negative profit, a condition not guaranteed by the ESS solution. It seems reasonable to investigate the situation where an external firm, in an unaffected oligopoly, would hesitate before exchanging her profitable strategy for the loss-making strategy of the bankrupt mutant! We thus extend the model to incorporate an explicit condition that the ESS only has to be proof against invading mutations that make non-negative profit when they first appear. A CESS is then found. The important result here is that the CESS involves a zero-profit outcome for all firms, whatever the internal / external player mix. This outcome is like a contestable market where the potential "entrant" is a strategy mutation of a current firm.

In the contest application, the mixed strategy solution to the unconstrained ESS does not rule out the case where behaviour might be chosen by one player such that she will lose money in the contest with probability one. We again extend the analysis to consider that invasion may require a side condition of (economic) survival to rule out this case. We equate survival here with some non-zero probability of the mutant's strategy leading to a non-negative surplus of prize over expenditure. Thus we impose an upper bound on expenditures so that there is at least some chance than a player will not make a loss and will survive “to tell the tale”. A key outcome in this application is that the players’ expected pay-offs (but not their strategies) in the CESS and ESS equilibria are the same.

A second extension relates to dropping the symmetry assumption used in the simple quantification of internal and external players. Although our analysis is mostly carried out in terms of a model with full symmetry, asymmetries across games can be included. For example we use a simple version of the oligopoly model to adopt strategies as output-setting rules rather than levels. The rules are dependent on the number of firms in the oligopoly and specific parameters describing the oligopoly. A mutation is then the adoption of an output level different to that given by the rule.

Our extensions expand the use of the evolutionary concept of the ESS to specific industrial organisation issues such as strategic instruments, zero-profit equilibria and mixed strategies, and our key focus is always on the balance of effect of those players involved in the particular game that the mutant enters and those players not directly involved but observing and learning from relative performance. The paper is organised with the model outlined in section 2 followed by the applications in sections 3 and 4. The extensions are considered separately in each of these two sections. Section 5 draws conclusions and discusses the way the interpretation of balance of effect can include more general considerations than numbers of players.

2. The Model

Our model is the simplest to make our argument, and follows the approach of Schaffer (1988). Consider an economy composed of N agents, who each act as a player in one n-person symmetric game. Thus there are N/n such games and we assume they are identical in structure. Extensions to asymmetric games are left to sections 3 and 4. We consider a candidate ESS strategy as s and a mutant strategy m. The mutant strategy may be any strategy but the chance of a mutation is small and so only one mutant can appear at any one time. The payoff of any player i in any game j is $\pi_{ij}(x_{1j}, x_{2j}, x_{3j}, \ldots, x_{nj})$, where x_{ij} is the strategy chosen by the i^{th} player (i = 1,2,…,n) in the j^{th} game (j=1,2,…,N/n). Suppose all players apart from player 1 in game 1 play $x_{ij}=s$. Then s is the ESS if no mutant strategy m can invade: that is if player 1 in game 1 chooses $x_{11}=m$, then, for any m:

$$\pi_{11}(m, s, s,\ldots, s) \leq [(n-1)\, \pi_{i1}(m, s, s,\ldots, s) + (N-n)\, \pi_{ij}(s, s, s,\ldots, s)]/(N-1) \quad \text{for } j>1,\ i \neq 1 \qquad (1)$$

The right hand side of (1) is simply the average payoff over all players other than player 1 in game 1. Symmetry means that the choice of player 1 in game 1 as the mutant is without loss of generality. Any other player could be chosen to play m without any real change. For there to be no mutation m contradicting (1) we need that the strategy m maximising

$$V(m,s) = \pi_{11}(m,s,s,...,s) - \frac{(n-1)\pi_{i1}(m,s,s,...,s) + (N-n)\pi_{ij}(s,s,s,...,s)}{N-1} \qquad (2)$$

should be s and yield $V(s,s)$=0. The ESS strategy s is thus found by maximising (2) with respect to m, for arbitrary s, and then setting m=s and solving for s. The derivative of V with respect to m will involve two terms. The presence of the second term, the derivative of the ratio term in (2), means that the strategy will not simply maximise π_{11}: that is it will not generally be the Nash strategy. The ratio term in (2) represents two kinds of player. There are the other (n-1) *internal* players in the game in which the mutant player is participating, and there are (N-n) *external* players in symmetric games where the mutant is not participating. The coefficient on π_{i1} in (2) is thus the proportionate weight of other internal to all other players, (n-1)/(N-1). As $N \to \infty$, for fixed n, those actually affected by a mutant become relatively few and any equilibrium will be the Nash equilibrium. As $n \to N$, all weight is on the internal players and we obtain Schaffer's evolutionary equilibrium with full "spite".

Our approach is essentially comparative static, asking the question whether a strategy *s* will not be infected by any mutant *m*. Very arbitrary dynamics could be incorporated (for example, if a mutation *m* performs better than *s*, all players adopt *m* in the next period) and these may be used for intuition. More credible dynamics could be constructed (for example, Taylor et al (2004)) but would add to the complexity of considering the continuous strategy spaces that our applications involve. The applications that we consider have properties that might require a refinement of the ESS equilibrium, and we consider this below.

2.1 *A Constrained ESS*

The distinctive features of learning across industrial organisation institutions are that such learning is most likely to be based on some degree of mimicry, together with translation of actions or strategies from one setting to another. Infection is the willing acceptance of new ideas rather than involuntary change. Suppose n=N=2 in the above, and that all 4 players were playing a strategy *s*, with non-negative payoffs. Let player 1 in game 1 deviate to a mutation *m* such that the payoffs for *both* players in game 1 were negative, or, more generally, payoffs were below some threshold for player survival. The issue is then whether non-surviving players' experience and performance would indeed have any effect on the continuing players in game 2. A numerical example might be as follows, with the players' payoffs $\{(\pi_{11}(m, s), \pi_{21}(m, s)), (\pi_{12}(s, s), \pi_{22}(s, s))\}$ given as Outcome (a): {(-1,-10), (2,2)}.

Now -1 is the payoff from *m* and (-10+2+2)/3=-2 is the average payoff from *s* played by the second player in game 1 (payoff of -10) and the two players in game 2. Hence by applying these numbers to (2), *m* could invade since -1>-2. Thus *s* is not an ESS. However, it is asking a lot for the players in game 2 to take notice of the outcomes in game 1 when *all* the players in game 1 have not survived. If we assume that the players in the no-survivor game are replaced by new players (for example, firms could be taken over by their creditors) then we could reconsider continuing (and new) players' strategy choice as the best performing strategy in other games. Note that any *m* such that $\pi_{11}(m, s, s,\ldots, s)<0$ and $\pi_{i1}(m, s, s,\ldots, s) \geq 0$ cannot yield $V(m,s)>V(s,s)$ (from (2)), so that any *m* with $\pi_{11}(m, s, s,\ldots, s)<0$ such that $V(m,s)>V(s,s)$ must have $\pi_{i1}(m, s, s,\ldots, s)<0$. Thus ruling out any *m* with $\pi_{11}(m, s, s,\ldots, s)<0$ is equivalent, in terms of potential invasion, to deleting no-survivor games. We therefore state the following definition of a Constrained ESS, termed a CESS :

Definition: The strategy *s* with $\pi_{ij}(s, s, s,\ldots, s) \geq 0$, is a CESS if $V(m,s)\leq V(s,s)$ for all *m* such that $\pi_{11}(m, s, s,\ldots, s) \geq 0$.

The numbers in outcome (a) prevent *s* being an ESS but do not prevent it being a CESS. In contrast, the outcome {(1,-10), (2, 2)} will allow invasion by the mutant, whether the constraint is applied or not (therefore *s* can be neither ESS nor CESS), while the outcome {(-1,-1), (2, 2)} will not allow invasion whether the constraint is applied or not (*s* can be either ESS or CESS). Although other constraints could be applied to the ESS criterion, the non-negative mutant payoff constraint is the most intuitive in terms of mutations likely to challenge other strategies. Note that the mutant's payoff being non-negative does not mean that the mutant's payoff will continue to be non-negative, if the mutation is adopted by other players. Also, our definition of a CESS implies that all players will survive in equilibrium. The formal solution for a CESS is from the Lagrangian

$$L = V(m,s) + \lambda(\pi_{11}(m,s,s,..,s))$$

The Lagrangian is assumed concave in *m* (note it is concave if *V* is concave). Then Kuhn-Tucker conditions for an optimal *m* subject to $\pi_{11} \geq 0$ are:

$$\frac{\partial V}{\partial m} + \lambda \frac{\partial \pi_{11}}{\partial m} = \frac{\partial V}{\partial m}(1+\lambda) + \lambda \frac{n-1}{N-1}\frac{\partial \pi_{i1}}{\partial m} = 0 \quad (3)$$
$$\lambda \geq 0 \quad complementary \quad with \quad \pi_{11} \geq 0$$

Solving for λ and *m*, and setting *m*=*s*, we have that *s* is an ESS if λ=0 and a CESS if λ>0. In the latter case, we have the result from (3) that $\frac{\partial V}{\partial m}$ and $\frac{\partial \pi_{i1}}{\partial m}$ are of opposite signs: the mutant would want to increase *m* further than allowed if this was "spiteful". We apply the CESS concept in extensions to both our applications below with very different results.

3.Application 1: Quantity setting oligopolies

In this application, players are quantity-setting firms and we assume that the payoffs, ignoring any fixed costs, are given by

$$\pi_{11}(m,s,s,...,s) = [f(m-s+ns)-c]m$$
$$\pi_{i1}(m,s,s,...,s) = [f(m-s+ns)-c]s$$
$$\pi_{ij}(s,s,s,...,s) = [f(ns)-c]s$$

The interpretation is that firms face (continuous and differentiable) inverse demands of *f*(.) and have constant per unit cost of *c*.

3.1. Analysis of the ESS

The mutant with the best chance of invading, when *s* is otherwise played, is that *m* which maximises (2), given these special forms of payoff. The relevant first-order condition is

$$f(m-s+ns) - c + f'(m-s+ns)m - f'(m-s+ns)s\,\frac{n-1}{N-1} = 0$$

set *m*=*s* to find the strategy *s* which is immune to invasion by any mutant *m*:

$$f(ns) - c + f'(ns)s - f'(ns)s\,\frac{n-1}{N-1} = 0 \tag{4}$$

Denote $f(ns)$ as the price p when ns is the total supply in a market and ε as the price elasticity of demand at that total supply. Then $f'(ns) = -f(ns)/(\varepsilon ns)$ and (4) can be expressed as

$$\frac{p-c}{p} = \frac{(N-n)}{\varepsilon n(N-1)} \tag{5}$$

Increasing the total number of firms N for a fixed oligopoly size n (that is, increasing the number of symmetric oligopolies) moves the equilibrium price-cost mark-up towards the well-known Cournot-Nash result of $1/(\varepsilon n)$. Decreasing the total number of firms N to n reduces the mark-up to zero, where price equals marginal cost. We have:

Result 1: Varying N for given n yields ESS outcomes ranging from the (spiteful) Walrasian (Passajennikov, 2003) to the (less spiteful) Cournot-Nash (Qin and Stuart, (1997).[1]

Result 2: Increasing n for a fixed N (that is moving towards fewer but larger oligopolies) has two effects: there is the normal reduced monopoly power effect of increasing n in the denominator of (5), but there is also the increased mutant power effect of increasing n in the numerator. Both effects tend to reduce the market price by increasing output levels.

The price-cost mark-up has been expressed as a function of a "conjectural variation" parameter (we denote this by ω), having the interpretation of a firm's anticipated change in other firms' outputs to a change in her output. Waterson (1984) p19 equation 2.4, for example, is the same as (5) but with $\omega = (1-n)/(N-1)$. Thus for different (n, N), ω varies from 0 (when $N=\infty$) to -1 (when $N=n$). It is sometimes argued that the only value of ω that is

[1] The literature that specifically considers adapting firm behaviour over time, for example Huang (2003), also predicts a Walrasian outcome if all players are supplying to the same market.

consistent with actual behaviour in a Nash equilibrium, if a firm did change its output, is -1 (the Walrasian outcome). We have:

Result 3: Any ω in the range $-1<\omega<0$ is consistent with an ESS for some particular (n, N).

3.2. Extension 1: A zero-profit constraint and the CESS

The equilibrium identified above may be associated with loss-making production. A fixed cost (not sunk) may exist which is greater than a firm's revenue net of variable cost. Suppose that the ESS has this property, while firms make non-negative profit in the Cournot-Nash equilibrium. The outputs in the ESS are thus such that a firm has a current supply level above the zero-profit contour in Figure 1 (and is hence loss-making). The zero-profit contour shows the values of *q* and *m* such that the firm, with output *m* measured on the horizontal axis, makes zero profit. Also in Figure 1, the Cournot-Nash equilibrium is shown as below the zero profit contour, and hence profitable. Now consider the zero-profit symmetric outputs (*s*, *s*) in Figure 1. Let *s* be the strategy adopted by all other firms, and consider that the potential mutant selects an output different from *s*. Assume that a condition on invasion is that the mutant firm must survive by making non-negative profit. Thus, if a mutant firm raised *m* above *s*, it would make a loss by going above the zero-profit contour and break the constraint. Now consider a cut in output. In the output-setting model this would increase other internal firms' profits and so $\frac{\partial \pi_{i1}}{\partial m} < 0$. Then from (3) we have that $\frac{\partial V}{\partial m} > 0$: thus decreasing *m* will reduce the mutant's payoff, and the mutation will not invade. Hence zero-profit *s* is a CESS. Thus, if "spitefulness" drives profit to zero, but more spite is self-defeating (a loss for the mutant), the CESS is at the zero-profit point with equal outputs of all firms. No other point in output space has the properties that (i) firms continuing the current strategy and not meeting a mutant do not make losses and can continue; (ii) a mutation can only do better than its direct competitors by itself making a loss. We have thus found:

Result 4: A CESS, with a constraint that mutants who make losses do not displace the status quo strategy, can replicate the zero-profit character of contestable markets, without potential entrants actually existing.

3.3 Extension 2: the ESS and the Vickers' model of strategic delegation

Next consider the Vickers' (1984) two-stage game where each firm's profit-maximising shareholders appoint a manager in a simultaneous first stage, and then each manager sets her firm's output in a simultaneous second-stage output-setting game. A manager of type t_i managing firm *i* in oligopoly 1 has the objective of maximising $z_{i1} = \pi_{i1} + t_i q_{i1}$. Hence a type 0 is concerned only with maximising profit. A more aggressive manager (positive t_i) can be appointed, or she can be converted by an incentive contract rewarding sales level as well as profit. Suppose market demand is linear so that the price is $p_1 = a - \sum_i q_{i1}$ and, for simplicity, there are no costs other than fixed costs. A manager of type t_i in the output-setting Nash game with payoffs z_i would have a best response function of $q_i = \dfrac{a + t_i - \sum_{j \neq i} q_j}{2}$ and the resulting oligopoly market price, firm *i*'s output and firm *i*'s profit would be:

$$p = \frac{a - \sum_j t_j}{n+1} \qquad q_i = \frac{a + (n+1)t_i - \sum_j t_j}{n+1} \qquad \pi_i = \frac{(a - \sum_j t_j)(a + (n+1)t_i - \sum_j t_j)}{(n+1)^2} \tag{6}$$

First consider the case where a particular type of manager, denoted t^s, has been adopted by all firms as the conventional choice. The managers then play a Nash game, each setting output to maximise their objective z. Thus, the higher the types of manager (the more aggressive the managers) in an oligopoly, the higher are the outputs and the lower the price (and profit). Committing to an aggressive manager leads other firms to reduce their outputs. If all firms appoint aggressive managers then profits fall. We can compare this outcome with an outcome which would be an ESS. The profit levels when a mutant plays t^m and other internal and external players use t^s are

$$\pi_{11} = \frac{a-(n-1)t^s - t^m}{n+1}\frac{a+nt^m-(n-1)t^s}{n+1}$$

$$\pi_{i1} = \frac{a-(n-1)t^s - t^m}{n+1}\frac{a+2t^s - t^m}{n+1} \qquad i \neq 1 \qquad (7)$$

$$\pi_{ij} = \frac{a-nt^s}{n+1}\frac{a+t^s}{n+1} \qquad j \neq 1$$

Substituting the expressions (7) into (2), maximising with respect to t^m and then setting t^m=t^s, yields the ESS as a manager type:

$$t^s = \frac{(N+1)(n-1)a}{N(1+n^2)-2n} \qquad (8)$$

The type defined by (8) is the unique common manager type that would be proof against invasion by any different type. Clearly $t^s > 0$. Also, if N=n and there are no external players then t^s =a/n and price is zero in the Walrasian case. The Vickers'solution is a perfect equilibrium where the choice of manager type by each firm is taken in a simultaneous first stage, given the payoffs of profit defined in (6)[2]. The symmetric perfect equilibrium has types

$$t^v = \frac{(n-1)a}{1+n^2} \qquad (9)$$

Comparison of (8) and (9) shows that $t^s > t^v$, and that $t^s \to t^v$ as $N \to \infty$. We thus have:

[2] See also Vickers (1984).

Result 5: Only in the case where there is no spite, because internal players are so relatively insignificant, do we have that the perfect equilibrium and ESS coincide. For all finite N we have that the ESS is even more "aggressive" than Vickers' strategic perfect equilibrium.

This example has shown that the strategy of setting a manager's type to condition a market game again leads to a range of ESS equilibria according to the balance between internal and external players. At one extreme (N=n) we have the Walrasian outcome of zero profit, and at the other ($N\rightarrow\infty$) the Perfect Equilibrium outcome.

3.4 Extension 3: Asymmetric oligopolies

Extension 2 has demonstrated that one strategy within the definition of the model may be the adoption of a rule (hiring a type of manager or setting a managerial incentive scheme). If our set of oligopolies are different, because either they have differing numbers of firms or there are market differences, then we can consider a rule as a mapping from those characteristics to the firm's decision making. Provided the same rule is applied across all oligopolies the analysis of an ESS can proceed, although we will retain symmetry within each oligopoly for simplicity, and we will also have to adjust the payoff function in the constructed game to allow for across-oligopoly payoff variations.

Formally, we will rewrite (2) as

$$V(m,s) = \pi_{11}(m,s,s,...,s;a_1) - \frac{(n_1 - 1)\pi_{i1}(m,s,s,...,s;a_1) + \sum_{j\neq 1} n_j \pi_{ij}(s,s,s,...,s;a_j)}{N-1} \qquad (2')$$

where $N = n_1 + \sum_{j\neq 1} n_j$ and the profit expressions are the same apart from each game having differing numbers of internal players and some parameter(s) a_j, which vary across games but not across players in the same game. We think of $s(n_j, a_j)$ as the strategy rule. We then consider the payoff functions normalised for other games' payoffs by taking

$$V(m,s)-V(s,s)=\pi_{11}(m,s,s,...,s;a_1)-\pi_{11}(s,s,s,...,s;a_1)- \frac{(n_1-1)\pi_{i1}(m,s,s,...,s;a_1)-(n_1-1)\pi_{i1}(s,s,s,...,s;a_1)}{N-1} \qquad (10)$$

In the simple linear demand case, with $p_j = a_j - \sum_j q_j = a_j - \sum_j s(n_j, a_j)$, and no variable costs, the profit expressions are

$$\pi_{11} = (a_1 - m - (n_1-1)s(n_1,a_1))\, m$$

$$\pi_{i1} = (a_1 - m - (n_1-1)s(n_1,a_1))\, s(n_1,a_1) \qquad (11)$$

Using (11) in (10) we can find the ESS rule for setting output by maximising (10) with respect to m, and then setting $m = s(n_1,a_1)$ yields a solution for s which requires that outputs are set according to

$$s(n_1,a_1) = q^s{}_1 = \frac{a_1}{n_1+1-\frac{n_1-1}{N-1}}$$

and as the choice of oligopoly 1 is arbitrary, we have

$$s(n_j,a_j) = q^s{}_j = \frac{a_j}{n_j+1-\frac{n_j-1}{N-1}} \qquad \forall j \qquad (12)$$

Result 6: The rule (12) that maps (a_j, n_j)$\to q_j$ is the ESS when games are differentiated by the number of players and the demand parameter a_j.

The set of external players have to be in a position to note variations of strategy and the performance of internal players. This may limit the set to those oligopolies with similar markets and features. External players who observe (a_j, n_j) and q_j as well as the profit from any mutant m need to be able to adapt the mutant rule to their own oligopoly markets. If the identification of the kind of mutation is not perfect then one might expect a longer period of adjustment and experimentation. Dynamics would have to involve learning the nature and transfer of the mutation as well as its relative success. Our analysis really only considers sufficient conditions for an ESS and not the question of whether that ESS will have global stability.

4. Application 2: Mixed strategy equilibrium in simple contests.

An ESS, just like any other form of strategy, might be a mixed strategy where different actions are adopted according to a probability distribution. One situation where this is almost certain to be the case is where the payoff functions are discontinuous. Then we need to deal with expected payoffs with opponents randomising over their actions. Suppose N/n sets of agents play contests of the following form. In each contest j, each of n players make a non-negative expenditure x_{ij}, $i=1,2,\ldots,n$. The agent making the highest expenditure wins W; the other agents in n win $L\geq 0$. We normalise so that W-L = 1. The payoffs are thus discontinuous and we take an ESS strategy as the following: all players playing ESS use a distribution function $F^s(x)$ to independently and randomly select their individual expenditures. Any mutant requires that she obtains a better expected outcome than the average expected outcomes of all agents playing $F^s(x)$. We analyse this case below.

4.1 Analysis of the ESS

The mutant strategy cannot be better than a choice of a particular value of x, and we again denote this as m. (Note that any mixed strategy must be an average over a set of m-values and so cannot do better than the best m value.) We again select player 1 in contest 1 to be the potential mutant and thus have

$$\pi_{11}(m,s) = [F^s(m)]^{n-1} + L - m$$
$$\pi_{i1}(m,s) = \frac{1-[F^s(m)]^{n-1}}{n-1} + L - E^s(x) \qquad (13)$$

$$\pi_{ij}(s, s) = 1/n + L - E^s(x)$$

These payoffs are expectations over other players' actions. $\pi_{11}(m, s)$ is thus composed of a chance $[F^s(m)]^{n-1}$ that the mutant's opponents all play less than m (then the mutant gains 1) plus the baseline quantity L-m. $\pi_{i1}(m, s)$ has the same interpretation where L- $E^s(x)$ is the baseline, with $E^s(x)$ the expected expenditure associated with $F^s(x)$, while the ESS player may win 1/(n-1) on average if the mutant does not win (the mutant does not win with probability $\{1 - [F^s(m)]^{n-1}\}$). Players in other contests all gain $1/n + L - E^s(x)$ on average. The analogy to a Nash equilibrium of a game of relative outcomes (Leininger, 2002) allows us to see a number of properties of an ESS in mixed strategies. For example, the value 0 must be played with positive density (else the lowest value played with positive density could be reduced, saving expenditure but with no loss of a chance of the prize and no change in opponents pay-off). Also, any spikes or mass points would lead to invasion by a player with a slightly higher x value. To derive the solution, write $V(m)$ analogous to (2) using the expected payoffs (13):

$$V(m) = L - m + [F^s(m)]^{n-1} - \left(\frac{1-[F^s(m)]^{n-1}}{n-1} + L - E^s(x)\right)\frac{n-1}{N-1} - \left(\frac{1}{n} + L - E^s(x)\right)\frac{N-n}{N-1} \qquad (14)$$

The first-order condition for a maximum of $V(m)$ (defining the best mutation) gives

$$V'(m) = \frac{d[F^s(m)]^{n-1}}{dm}\frac{N}{N-1} - 1 = 0$$

Given $m \geq 0$, the absence of spikes and the positive density at m=0, we have

$$F^s(x) = \left(\frac{N-1}{N}x\right)^{\frac{1}{n-1}} \quad for \quad 0 < x < \frac{N}{N-1} \qquad (15)$$

From (15), we have that

$$E^s(x) = \frac{N}{n(N-1)}$$

so that

$$nE^s(x) = \frac{N}{N-1} \tag{16}$$

and the (expected) total expenditure of the n players in each game is $N/(N\text{-}1)$. We have:

Result 7: (i) aggregate expenditure is greater than the available prize for all finite N. (ii) If N approaches infinity then the total expenditure approaches 1 and the mixed strategies approach the mixed strategy Nash equilibrium (where $F(x)=1$ for $0<x<1$), for any n. (iii) If N = n (a single game), then the expected total expenditure increases to n/(n-1). Further if n is 2, expected total expenditure is 2; if n increases this again tends to one.

This application can be related to the Tullock model of contests for rents, with the Tullock parameter (r) set at infinity. Mixed strategy Nash equilibria are found in Baye et al (1994) to involve full (but not over) dissipation when the strategy space is continuous. Hehenkamp et al (2004) consider ESS in pure strategies in the Tullock model and find they exist for relatively low values of r ($r \leq n/(n\text{-}1)$). Then a single value of x exists and satisfies $nE^s(x) \leq N/(N\text{-}1)$. The cases where pure strategies failed were those where a mutant could invade with an expenditure of zero. Our application here finds a mixed strategy ESS for the infinite r case, and finds that the mixed strategies have the same property in terms of expected expenditure (and thus expected "overdissipation") as holds when r is at its maximum value to permit a pure strategy ESS. Thus (16) states that, on average, contestants together spend more than 1 for a prize of 1. Of course, the baseline prize of L can be set to ensure participation.

4.2. Extension: Living to tell the tale and the CESS

We again take a variation such that the mutant has to survive to invade. We assume that no individual player can make expenditure higher than the prize on offer (again standardised to 1). We also set the consolation prize of L to zero to remove the contestants' protection against overdissipation. The case is of interest since legal or institutional restrictions may prevent players from making an expenditure that will result in a loss with probability 1. Alternatively, economic survival may be ruled out if an agent makes a loss, and survival for some length of time may be necessary for the agent's strategy to become known. We again take N players in total, with n players engaging in each of N/n parallel contests. Note that the constraint here means only that the contestant will not make a loss for sure: she may lose *almost* always!

Given that meaningful mutations cannot take place with expenditures greater than one, since this now implies non-survival, spikes may exist at expenditure levels of 1. This is because they cannot be "smoothed" by players going slightly above 1. Also, values of x slightly less than 1 must be inferior to playing x=1 if spikes exist at 1 since the latter give positive probabilities of sharing the winning prize, and we will assume for simplicity that these are *equal* probabilities for all contestants making the same expenditure. To find the size of the spike, denoted e^s, and the top of the range of x<1 played with positive density, denoted x^*, we note the following. First, the analysis of $x<x^*$ will largely follow that of the last section. Any value of x played must give the same $V = 0$, and $F^s(1) = F^s(x^*) = 1 - e^s$, and so from (15):

$$x^* = [(1 - e^s]^{n-1}N/(N-1) \tag{17}$$

Then at m=x^*:

$$V(x^*) = (1-e^s)^{n-1} - x^* - \left(\frac{1}{N-1}\right)[1-(1-e^s)^{n-1}] - \left(\frac{N-n}{N-1}\right)\frac{1}{n} + E^s(x)$$

or, substituting for x^*:

$$V(x^*) = -\left(\frac{1}{N-1}\right) - \left(\frac{N-n}{N-1}\right)\frac{1}{n} + E^s(x) = 0 \text{ iff } E^s(x) = \frac{N}{n(N-1)} \tag{18}$$

Result 8: Expected profit of a CESS player when all play CESS is

$$\frac{1}{n} - E^s(x) = \frac{1}{n} - \frac{N}{(N-1)n} = \frac{-1}{(N-1)n} \tag{19}$$

and all players make a loss on average due to the spitefulness of the strategy, even though bids are capped at 1. The imposition of the upper bound of 1 on the mutation possibilities has had no effect on the expected expenditure and thus no effect on the over-dissipation result

Generally, e^s and x^*can be shown to be unique and this is relegated to the Appendix. When n=2, e^s and x^*can be found explicitly and it is useful to consider this case in some detail below. The CESS solution is a mixed strategy with

$$F^s(x) = \frac{N-1}{N}x \quad for\ 0 \le x \le \frac{N-2}{N-1}$$

$$F^s(x) = \frac{N-2}{N} \quad for\ \frac{N-2}{N-1} \le x \tag{20}$$

$$e^s = \frac{2}{N}$$

It is clear from (20) that the CESS strategy has all the right properties. Thus $V(m)<0$ for any mutant playing an m in the interval $(N-2)/(N-1)<m<1$, since there is no greater chance of defeating opponents than with playing $m = (N-2)/(N-1)$, and there is more expenditure to be made. $V(m) = 0$ for all other $m \le 1$, and so there exists no m that can invade the CESS solution. Now the point of the exercise is again to see how the CESS strategy changes as N changes, and also to see how the overdissipation result of application 2 is affected. If N=n=2 so that only a single duopoly exists, then x^*=0, and e^s=1 from (20). Thus both players play the most aggressive expenditure equal to 1 with probability 1. This is the most competitive (spiteful) outcome. Each player then receives on average a payoff of ½ -1 = -½. Note that m=0 is not a mutation that will successfully invade since that mutant's opponent, playing CESS, will win 1 with expenditure 1 to obtain a payoff of *0*, which is as much as the mutant obtains with m=0.

Now consider the case of n=2 when N is very large. Here $e^s \to 0$, $x^* \to 1$, and $F^s(x) \to x$, as $N \to \infty$. This limit is exactly the mixed strategy Nash equilibrium in the contests. Expected payoffs from the CESS tend to 0 (again the Nash equilibrium outcome). Intermediate cases are well behaved and include the following, calculated directly from (20): when N=4, e^s = ½; when N=10, e^s = 0.2; when N=100, e^s = 2/100. In all these cases payoffs are ½ - $N/(2(N\text{-}1))$ < 0, and again have the property of overdissipation.

On average the expenditure by a player is $e^s(1) + (1\text{-}\ e^s)\ x^*/2$. This is calculated from (20) as $\frac{N}{2(N-1)}$, which is exactly the average expenditure in the absence of a survival constraint. We still have a CESS where expected expenditure per player is higher than expected winnings.

4.3. *Extension* 2*: asymmetric contests*.

We will follow the same argument as in section 3.4, but limit differences across contests to different numbers of contestants. The equivalent payoff to (10) for contestant 1 in contest 1 is

$$V(m)-V(s)=[F^s(m)]^{n_1-1}-[F^s(s)]^{n_1-1}+E^s(x)-m+\left(\frac{[F^s(m)]^{n_1-1}-[F^s(s)]^{n_1-1}}{n_1-1}\right)\frac{n_1-1}{N-1}=$$

$$([F^s(m)]^{n_1-1}-[F^s(s)]^{n_1-1})\frac{N}{N-1}-m+E^s(x)$$

Since the choice of the first contest was arbitrary, an ESS is identified as

$$F_j^s(x) = \left(\frac{N-1}{N} x \right)^{\frac{1}{n_j - 1}} \quad for \quad 0 < x < \frac{N}{N-1} \tag{13′}$$

From (13′), we have that $E_j^s(x) = \frac{N}{n_j(N-1)}$ so that

$$n_j E_j^s(x) = \frac{N}{N-1} \tag{14′}$$

Thus we have:

Result 9: (i) The over-dissipation result holds for all contests. (ii) The total amount bid in each contest has the same expectation so that the over-dissipation is the same, even though some contests have many players and others have few.

Note however that the value of N could be extremely large: this is the sum of all participants in all contests where cross observation takes place. The RHS of (14′) tends to 1 as $N \to \infty$.

5. Discussion

This paper has used recent contributions to the concept of the ESS in finite populations in order to stress that the ESS is dependent on the definition of the set of players. This feature can be used to generalise the implications of evolutionary processes in economics. An invading mutation can be compared to those affected by the mutation and to those not directly affected. Such a division has two clear effects. First, the relative weights determine the extent of departure of the ESS from the Nash equilibrium. If most agents are unaffected by the mutant, the ESS is close to the Nash equilibrium; if most are affected then the ESS is at its most spiteful. Thus we have found that the price-cost mark-up in the ESS in the quantity-setting oligopoly model can be related to a conjectural variation coefficient in the

Nash equilibrium. Also the output levels are higher in the ESS, than in the perfect equilibrium, in a two-stage strategic delegation oligopoly model, but the extent depends on how small is N. In the contest model, an ESS has total expected bids greater than the offered prize, but over-dissipation disappears as $N \rightarrow \infty$.

The second implication of the fact that some agents are "external" to the mutant and are unaffected is the possibility that a successful invasion might require the mutant itself surviving the invasion. This possibility can be studied by imposing a "survival constraint" on eligible mutations and can act to reinforce the lower level of incentive for spitefulness that comes from the N-n "other" agents. Our two applications here have contrasting outcomes from a survival constraint. The quantity-setting oligopoly can have the outputs limited by such a constraint and the CESS can take the form of a zero-profit equilibrium with the same characteristics as a contestable market equilibrium, but without actual potential entrants. However, the mixed strategies in the contest application are only constrained from above in their range of expenditures: the expected expenditure of participants is unaffected by the survival constraint as applied. Densities of expenditures both above and below 1 are removed to form the spike at 1. Bidding distributions change but any gains to players from the bidding cap are offset by higher densities at high bids.

We have extended our symmetric analysis by showing how asymmetries across games might be included with some added complexity. However, serious caveats have to be emphasised about how players in one game translate a mutation introduced into another game for their own application. The understanding of rules or principles underlying detailed policy and ability to transfer between similar but different games is necessary. There may be an asymmetry between infection rates within and between games. Also, in our contest application with its mixed strategy ESS, we need to acknowledge that a number of plays of the game may be necessary to identify mutant behaviour. Nevertheless, such factors must be part of the evolutionary process within industrial organisation applications, since management practices and strategies are expected to apply across many types of business and market.

We might in fact wish to speculate further. Instead of a simple division of players into internal and external groups, we could consider probabilities of observation of strategy or payoff combinations. Comparison of strategy success is then bounded by observations that occur. If only a finite number of players can be observed with positive probability then we are in a finite population world from the perspective of an individual player, and the analysis here can be applied even if the actual population is (virtually) infinite. As well as using such probabilities to weight different players, differential accuracy of observation could also be a reason for discounting some information more than others. One reason might be that there is more or less confidence in the validity of a particular agent as someone whose example is of relevance. For example, if n-1 players are directly involved with a mutant, but the weight given to this group is $\rho<1$, then, setting ρ = (n-1)/(N-1), this can be represented by a virtual population size of $N = n/\rho - (1-\rho)/\rho$. If there is a history of one or more previous periods without mutations occurring then the historical performance may be given some discounted weight: in this case $\rho = 1/(1+D)$ where D is a discount factor, and $N = n + D(n-1)$ recreates the desired weights. The discount factor may reflect uncertainty about the presence of (unsuccessful) mutations in earlier periods, or indeed whether the game structure has changed. The application to firm performance in oligopolies is an obvious case: profit levels may be aggregated across several markets and products, and historical records taken into account. In all these situations the simple N used here can be re-interpreted as the level which reflects the weight put on external relative to internal players. The conclusion that different relative weights yield a continuum of ESS from "competitive" to Nash reflects the relative impact of behaviour on the set of players, and our applications demonstrate the usefulness and easy applicability of this result.

The approach in this paper has been static. The division of players into internal and external players, possible asymmetries across games, and the issues raised above, would all have implications for any dynamic model of changing adoption of strategies. For example, observation of mutations and their effects may take time, calculating the translation of the mutation to a player's own strategy choice may involve errors. Analysis would be complex and would have to be focussed on particular goals.

Finally, the question arises as to whether the evolution of strategies in economic processes such as oligopoly supply and contests can indeed be viewed as a sequence of "tests of strength" where participants who do better than others, then replace the status quo strategy with their own. Managerial performance in particular, and agent performance more generally, can be considered as being judged and rewarded increasingly on the basis of relative performance. In any such judgement, the choice of relevant alternatives is crucial. To the extent that agents can weaken the competition and gain better judgements, so the spitefulness of the ESS equilibrium becomes appropriate. The approach here has drawn out two key factors. The first factor was the balance of players between those directly affected by an agent and those not so affected. The second factor was that the agent might have constraints that limit the extent of spitefulness and hence might change the ESS equilibrium.

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Appendix: Unique CESS when *n*>2.

When $n>2$, e^s cannot be found explicitly, but we can establish its uniqueness and hence characterise the solution and confirm Result 8. Now at the spike we have

$$V(1) = E_t \frac{1}{t+1} - 1 - \left(\frac{1}{N-1}\right)\left(1 - E_t \frac{1}{t+1}\right) - \left(\frac{N-n}{N-1}\right)\frac{1}{n} + E^s(x) \qquad \text{(A1)}$$

where E_t is the expected number of other players in the group who also choose to play 1. $1/(t+1)$ is then the share of the prize received if t other of the n-1 players make this choice, where t has a binomial distribution with e^s as the probability of a "success" and n-1 as the number of trials. First principles yield that

$$E_t \frac{1}{t+1} = \frac{1}{e^s n}[1-(1-e^s)^n] \qquad \text{(A2)}$$

Substitution of (A2) into (A1) thus yields

$$V(1) = \frac{1}{e^s n}[1-(1-e^s)^n]\frac{N}{N-1} - \frac{N}{N-1} - \frac{N-n}{(N-1)n} + E^s(x) = \frac{1}{e^s n}[1-(1-e^s)^n]\frac{N}{N-1} - 1$$

For $V(1)=0$ we need e^s to solve

$$\frac{1}{e^s n}[1-(1-e^s)^n] = \frac{N-1}{N} \qquad \text{(A3)}$$

Now as $e^s \to 0$, the limit of the left-hand-side of (A3) is $1 \neq (N-1)/N$. Thus $e^s=0$ is not a solution. For $e^s > 0$, multiply (A3) through by e^s to obtain

$$(1-e^s)^n = 1 - e^s n \frac{N-1}{N} \qquad \text{(A4)}$$

The left-hand-side (LHS) of (A4) is strictly convex while the right-hand-side (RHS) is linear. Further, the gradient of the LHS is more negative at $e^s=0$, and the LHS<RHS at $e^s=1$. Thus the RHS and LHS only cross once in the interior of the unit interval. This proves that there is a unique value e^s strictly between zero and one, which solves (A4). Given the unique value e^s, $x^* = [(1-e^s]^{n-1}N/(N-1)$ (from (17)) and $E^s(x) = \frac{N}{(N-1)n}$ (from (18)) confirms Result 8.

Figure 1: (*s*, *s*) is a zero-profit CESS. An increase in output *m* by the mutant from (*s*, *s*) leads to non-survival (negative profit); a decrease increases other competitors' average profit more than the mutant's.