Risk Aversion and Aggression in Tournaments *

by

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Abstract

Risk aversion is introduced into a traditional model of a contest where there is a prize for the most aggressive player and where there is complete information. Players only differ by their aversion to risk. Aggression is defined as expenditure on attempting to win the prize. It is shown that total aggression is less the more the players are risk averse, but that in a game with heterogeneous risk preferences, there may be an equilibrium where the more risk averse players play more aggressively. The analysis is extended to consider a sequence of contests, and it is argued that differences in numbers of wins in such a sequence are reduced by risk aversion.

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1. Introduction

Games in Economics frequently have Nash equilibria only in mixed strategies. Such MSNE either introduce or add to risk for the players in the game. This paper considers a common version of such a game with many applications. The point of the paper is to consider how risk aversion of the game's players affect the equilibrium mixed strategies. In particular, whether more risk aversion of all players leads to less aggressive equilibria, and how players with more risk aversion fare when playing against less risk averse opponents. The particular game considered will be of the form of a contest with complete information: hence more aggression translates into spending more resources on winning the contest. The risk is that if the contest is not won, then the resources have still been spent. Thus risk is less if few resources are spent or if so many are spent that there is little chance of not winning. If players are more risk averse, then will they be more or less aggressive? A second question relates to when players with different attitudes to risk play each other: then is it better to play against more risk averse opponents?

The game is of the arms race variety (each player's strategic variable (denoted x) being their expenditure on arms). Suitable redefinitions yield many interpretations in terms of patent races (x as R&D), for example Dasgupta (1986), Amaldoss and Jain (2002), Waterson and Ireland (1998); status races (x as purchase of status goods); and all-pay auctions, both bidding and client competition (x being the bid), including procurement auctions (Taylor, 1995) and lobbying for rents (as in Hillman and Riley, 1989). It also fits the comparison shopping models of competition between firms (eg Wilde and Schwartz (1979) and Burdett and Judd (1983)), where the x can be defined as negatively related to the price which is offered to customers. In all these applications we are assuming a model of complete information in the same way as Baye et al (1996), but the difference is that the financial pay-off (denoted by a winning prize W or a losing prize L) is common to all players and is common knowledge. While Baye et al assume players are risk neutral but may value the winning prizes differently, we assume that players may be risk averse and that they know the risk preferences of their opponents. Each player seeks to maximise expected utility of his surplus in the contest. In common with Baye et al (1996), uncertainty only arises due to the Nash equilibrium of the game being in mixed strategies. Thus the approach differs from the more common approach to tournaments where either complete information is tempered by exogenous shocks (for example in measurement of worker effort) or where there is incomplete information (competitors' preferences or skills unknown). A further key difference compared to many models of tournaments is that the prizes are exogenous. Thus the prizes do not depend on the level of x. To some extent, this implies that our emphasis is on equilibrium and behaviour rather than on efficiency and incentives. Expenditures to gain prizes have no welfare value in our analysis; in particular there is no transfer of expenditures from loser to winner.

To make the analysis simple we assume in section 2 that there are only two players in the contest. The form of the game is as follows. Each player simultaneously selects an action $x \in R^+$. Then the player has a money pay-off of W-x if her x-choice is the higher, while she receives L-x otherwise. We look for a symmetric equilibrium if both players have the same risk aversion (same utility function and are thus ex ante identical). We examine how the equilibrium responds to differences in risk aversion between the players.

In section 3 we briefly extend our analysis to the case of n players. For simplicity we assume that there are just two types of preferences so that some players have preferences that exhibit higher risk aversion than those of other players. In particular n_A players have utility functions U_A and n_B utility functions U_B, and type A players are assumed to be more risk averse than type B. We find that when the number of players is large then the competition faced by typical players of each type is insufficiently different to accommodate the different risk attitudes within such an equilibrium. Instead, the more risk averse players essentially drop out of the game by playing x=0 with probability 1. We thus have two kinds of equilibria, and at least one kind will be present in any game. In one kind, all players have mixed strategies and the difference in their risk preferences is accommodated by the different population of opponents each faces (since the population of opponents excludes the player herself). In the other, only the less risk averse players take active part, and behavior is thus specialised according to risk aversion.

In section 4 we consider a finite sequence of repeated contests (between the same two players). We argue that the equilibrium can be found by a simple backwards-induction method, and that the perfect equilibrium has an important property in the case of approximately constant relative risk aversion. If one player wins the early contests then that player is less (absolute) risk averse in later contests and then the other player plays more aggressively in equilibrium. This provides an automatic convergence mechanism so that the share of wins in the total sequence is nearer ½ the more risk averse the players (the higher the coefficient of relative risk aversion). We provide a simulation of this sequence.

Section 5 considers some of the implications of the analysis and draws conclusions. An underlying theme is the general principle that aggression varies with risk aversion of competitors and dissipates funds. When contestants have more wealth, they may be less risk averse and their competitors have to act more aggressively. Thus all players having lower risk aversion means that all play more aggressively and financial pay-offs are lower. If this leads to contestants becoming poorer and hence more risk averse, their competitors will then play less aggressively and financial pay-offs become higher. An endogenous cycle in financial pay-offs and average aggression thus occurs. In a sequence of contests, the same effect results in reduced variance of results.

The analysis and conclusions reflect a MSNE. In terms of technicalities we will see that we have similar non-standard comparative static results to those noted for models with MSNE but without risk aversion. Cheng and Zhu (1992) discuss the “three inherent difficulties” to MSNE, as reflecting the fact that the equilibrium strategy is defined to ensure the other player’s behavior is mixed. We have the result that the more risk averse player acts more aggressively because she faces the less risk averse competitor.

2. The Model

We will consider only two types of player. Player i is of type A or B and maximizes the expected value of $U_i(p_i)$ where $U_i'(p_i) > 0$, $U_i''(p_i) < 0$ and $U_i(0) = 0$. Profit is p_i defined as

$$p_i = (1- S)\, L + S\, W - x \qquad (1)$$

where S is the result of the contest (S= 1 if player i wins and 0 if player i loses). In this section we will assume that there are two players, 1 and 2. We assume that L>0, so that x is in effect bounded from above by W-L (choosing higher values of x will lead to lower pay-offs than would be achieved by setting x=0). Also it is clear that x=0 has to be played with positive density in equilibrium. (Suppose it isn't: then, if the lowest x played with positive density is b then any player would do better by playing 0 rather than b.) Finally note that any equilibrium mixed strategy will not have holes (where density is zero) since then density would be shifted from the top of the hole to the bottom. Nor would there be spikes (with measurable probability of playing a particular x), since then an opponent would have a best response of just capping that x which would make the spike an inferior action for its player. This of course is the reason why pure strategy Nash equilibria are not present in this game. It is also the reason why we can ignore the case where both players play exactly the same value of x and thus have to share the prizes.

Let $F_i(x)$ be the probability of player i winning. In the two-player contest (but of course not more generally) this is also the probability of player i's opponent choosing an action less than x. Then

$$\mathrm{E}U_i(x) = U_i(L\text{-}x) + (U_i(W\text{-}x) - U_i(L\text{-}x))\, F_i(x) \qquad (2)$$

Players are constrained by $x \geq 0$, and any $x > W\text{-}L$ will be inferior to $x=0$. A MSNE will have both players choosing x from a density function with supports [0, W-L]. A player can obtain $U_i(L)$ by playing $x=0$ or by playing $x = W\text{-}L$. At $x = 0$ or W-L the outcome is non-stochastic since the action will surely lose or surely win. For $0<x<W\text{-}L$, we have a stochastic outcome and the evaluation of this lottery will depend on the chance of winning ($F_i(x)$) and the risk preference of the player. Thus the equilibrium will imply that for players 1 and 2 respectively:

$$U_1(L\text{-}x)\,(1\text{-}\,F_1(x)) + (U_1(W\text{-}x)\, F_1(x) = U_1(L) \qquad (2)$$

$$U_2(L\text{-}x)\,(1\text{-}\,F_2(x)) + (U_2(W\text{-}x)\, F_2(x) = U_2(L) \qquad (3)$$

for all x in [0, W-L].

Proposition 1 *(existence). In the two player game there exists a unique equilibrium pair of distribution functions $F_1(x)$, $F_2(x)$ of winning probabilities.*

Proof: The linearity of (2) and (3) in F_1 and F_2, and $W>L$, ensures uniqueness. Also, clearly $F_1(0) = F_2(0) = 0$, and $F_1(W\text{-}L) = F_2(W\text{-}L) = 1$. Next consider what happens to the left-hand-side of (2) if x increases: strict concavity of U_1 implies that $U_1(L\text{-}x)$ decreases more than $U_1(W\text{-}x)$, and so the solution value for $F_1(x)$ must increase to compensate. Thus

$F_1(x)$ is increasing in x for all x in [0, W-L]; similarly for $F_2(x)$. Thus $F_1(x)$, $F_2(x)$ are distribution functions.

These equations can be interpreted in the following way**: *L* is the certainty equivalent for player i to the lottery {*W*-*x*, *L*-*x*; *F*i(*x*)} , i = 1,2.** We immediately have the following result.

***Result**: If player* 1 *plays a mixed strategy using the distribution function* $f_1(x)$ º $F_2(x)$ *and player* 2 *plays a mixed strategy using the distribution function* $f_2(x)$ º $F_1(x)$*, then both players are in equilibrium since neither can gain from changing strategies.*

***Proposition 2**. If both players become more risk averse they both become less aggressive.*

***Proof* :** Suppose player 1 became more risk averse. Then the certainty equivalent for her of any lottery would decrease. Thus, for any particular x and $F_1(x)$, L would no longer be the certainty equivalent for player 1 to the lottery {W-x, L-x; $F_1(x)$}. In order to restore the equilibrium condition, the only accommodation possible (since L and W are fixed parameters) is that $F_1(x)$ increases, so that the chance of the "good" outcome increases to counteract the increased aversion to risk. This occurs at all x values except 0 and W-L (where the outcome involves no risk). Thus $F_1(x)$ shifts to a new function which is first-order stochastically dominated by the original function, and player 2 becomes less aggressive and easier to "beat" since it adjusts to the new $f_2(x) \equiv F_1(x)$. Suppose 2 also becomes more risk averse, then by a similar argument, $F_2(x)$ has to increase for all x in (0, W-L), and player 1 also becomes less aggressive. Both players become less aggressive (and as a result both gain in terms of expected wealth since on average they spend less but obtain the same prizes).

Proposition 3 *If player* 1 *is more risk averse than player* 2 *then player* 1 *is more aggressive.*

Proof: Suppose both players are equally risk averse ($U_1(.) \equiv U_2(.)$). Now let 1's preferences change to be more risk averse. By the argument in Proposition 2, player 2 becomes less aggressive to reflect the changed $F_1(x)$, and player 1's aggression is unchanged. Hence player 1, the more risk averse player, plays more aggressively.

Note that mixed strategies in games characterized by other asymmetries also give rise to apparently perverse outcomes. Amaldoss and Jain (2002) show that a firm that values a patent less will bid more aggressively within a MSNE. They also provide experimental evidence to support their result. In our analysis the level of aggression is determined by the need to keep the certainty equivalent of the opponent's lotteries the same, and the need is for more aggressive behavior to reduce the expected utility of high expenditure lotteries for less risk averse opponents.

In this section considering two players, it is clear that any Nash equilibrium would require both players choosing distribution functions which are strictly increasing on [0, W-L]. Thus any difference between the two players' risk preferences are accommodated by their opponent's mixed strategies. There could be no equilibrium with player 1 playing L with probability 1 (for example) since then player 2 would respond by playing just above L with probability 1, and then player 1 would not be in equilibrium, etc. In the next section we consider the case of n players, where other equilibria are possible.

3. The *n*-player game

Let there be $n > 2$ players, with n_A players having preferences U_A and n_B having preferences U_B. Type A are more risk averse than type B. The equations (2) and (3) can be re-interpreted as typical conditions for MSNE. Players 1 and 2 could be typical players of types A and B respectively. In any "symmetric" equilibrium to this game, all players of type A play distribution function f_A and all players of type B play distribution function f_B. The re-interpretation requires that f_A and f_B combine (given the distribution of opponents) to yield the same probabilities of winning and losing (same $F_1(x)$ and $F_2(x)$) as in the 2-player case. We define an equilibrium where all players have mixed strategies involving positive densities for all x in $[0, W\text{-}L]$ as an *accommodation equilibrium*. This term reflects the fact that the less risk averse types B makes room for the more risk averse types A by being less aggressive. We first consider when such an accommodation equilibrium exists, and then consider the existence and nature of some other forms of equilibrium. If $F_1(x)$ and $F_2(x)$ solve (2) and (3) for a particular x, and player 1 is a type A while player 2 is a type B, then f_A and f_B must satisfy

$$F_1(x) = F_A(x) = H(x)/\, f_A(x) \qquad (4)$$

$$F_2(x) = F_B(x) = H(x)/\, f_B(x) \qquad (5)$$

where

$$H(x) = [f_A(x)]^{n_A}[f_B(x)]^{n_B}$$

is the probability that no action greater than x is played. Further the solution for $f_A(x)$ and $f_B(x)$ must be non-decreasing in x on [0, W-L]. Solving for $f_A(x)$ and $f_B(x)$ from (4) and (5) yields

$$\log f_A(x) = \frac{n_B}{n-1} \log F_B(x) - \frac{n_B - 1}{n-1} \log F_A(x) \tag{6}$$

$$\log f_B(x) = \frac{n_A}{n-1} \log F_A(x) - \frac{n_A - 1}{n-1} \log F_B(x) \tag{7}$$

We see from (6) and (7) that if $n_A = n_B = 1$, then $f_A(x) = F_B(x)$, $f_B(x) = F_A(x)$, as in section 2. If the preference types are the same (and so $F_A(x) \equiv F_B(x) = F(x)$) then $f_A(x) = f_B(x) = F(x)^{1/(n-1)}$. In both these cases $F(x)$ increasing in x implies $f(x)$ increasing in x and hence the functions $f(x)$ are indeed distribution functions and an accommodation equilibrium exists. More generally the existence of such an equilibrium requires that the right hand sides of (6) and (7) are non-decreasing. To see that this is not always the case, suppose that n, n_A and n_B are very large. Then these right hand sides are approximately $\frac{n_B}{n-1}\log\frac{F_B(x)}{F_A(x)}$ *and* $-\frac{n_A}{n-1}\log\frac{F_B(x)}{F_A(x)}$ respectively. These cannot both be increasing in x, and so no accommodation equilibrium can occur in this case. We can thus provide a formal requirement for the existence of an accommodation equilibrium.

Proposition 4 *(accommodation equilibrium): A MSNE will exist with players of each type choosing the same distribution with positive densities for all x in [*0*, W-L] if and only if*

$$\frac{n_B - 1}{n_B} < \frac{\dfrac{d\,\log F_B(x)}{dx}}{\dfrac{d\,\log F_A(x)}{dx}} < \frac{n_A}{n_A - 1}$$

No accommodation equilibrium will exist with players of the same type choosing different mixed strategies.

Proof: For the first part of the Proposition, set the derivatives of (6) and (7) to be positive and solve for the conditions on the ratio of derivatives. To prove the second part of the Proposition, suppose that instead of symmetric behavior across all players of a certain type, mixed strategies vary across such players. Then interpret players 1 and 2 as being of the same type. If their mixed strategies differ then $f_1(x) \neq f_2(x)$ but $F_1(x) \equiv F_2(x)$, and this is ruled out by (6) and (7) having to hold. This completes the proof.

Other kinds of equilibria would involve (some) players not playing some range of x with positive density. To identify the nature of such equilibria we can make the following points.

i. If any player k played x with a distribution with lower support $x_L > 0$ then no other player would play any x in $(0, x_L)$ since such play would always lose and hence be inferior to playing 0. Then player k would not be in equilibrium since x_L could be reduced without penalty. Thus equilibria cannot have any player playing with a lower support $x_L > 0$.

ii. Since all players have a lower support at 0, or play x=0 with positive probability, all players have expected utility for any choice of x played with positive density equal to $U_i(L)$, i = A,B.

iii. Any equilibrium thus satisfies

$$U_1(L-x)(1-\frac{H(x)}{f_1(x)})+U_1(W-x)\frac{H(x)}{f_1(x)}\leq U_1(L) \quad and U_1(L)=0 \; if f_1'(x)>0 \qquad (2')$$

$$U_2(L-x)(1-\frac{H(x)}{f_2(x)})+U_2(W-x)\frac{H(x)}{f_2(x)}\leq U_2(L) \quad and U_2(L)=0 \; if f_2'(x)>0 \qquad (3')$$

for typical players 1 and 2.

iv. If there are a large number of both types of players then the only equilibrium is for all type A to play $f_A(x)$ = 1 for all x in [0, W-L], and for all type B to play

$$f_B(x) = F_B(x)^{1/(n_B-1)}.$$

In this equilibrium, all type A players play safe and opt for the risk-free x=0 - thus basically not participating in the contest, while the type B players participate in a symmetric game among themselves. Players thus specialise their behavior according to type.

Our conclusion is that when there is a small number of players both types can play mixed strategies, whereas only the less risk averse can play mixed strategies if there are large numbers of each type. In the latter case players of different types cannot be faced simultaneously with probabilities of winning that equalise the certainty equivalent across different values of x.[1]

4. A sequence of contests

Now consider a sequence of T contests between two players. Since there are only two players we know that an accommodation equilibrium exists in each contest. Each player wishes to maximise the expected value of utility defined by

$$\mathrm{E}\ U(TL + S_T(W\text{-}L) - X_T) \tag{8}$$

Where W and L are the winning and losing prizes in each contest; S_T is the total number of successes (ie number of wins) this player scores in the T contests, and X_T is the aggregate expenditure in the T contests ($X_T = x_1 + x_2 + x_3 + \ldots + x_T$). We assume that the two players have the same utility function but at any stage in the sequence the history will be different. The successes to date and the expenditure to date are common knowledge for the two players, and we seek a Markov perfect equilibrium. Thus at the end of the t^{th} contest, players A and B would have

$$\mathrm{E}U^{A} = \mathrm{E}\ U(TL + S_t^{A} (W\text{-}L) - X_t^{A} + (W\text{-}L)\ S_{t<t \pounds T}\, Q_t^{A} - S_{t<t \pounds T}\, x_t^{A}) \tag{9A}$$

$$\mathrm{E}U^{B} = \mathrm{E}\ U(TL + S_t^{B} (W\text{-}L) - X_t^{B} + (W\text{-}L)\ S_{t<t \pounds T}\, Q_t^{B} - S_{t<t \pounds T}\, x_t^{B}) \tag{9B}$$

[1] An application to price-setting games among labor-managed firms with a similar conclusion is given in Ireland (2003).

where Q_t ^A^ is the binary random variable {0,1} indicating a future win or loss for A in period t, and x_t^{A} is the future random draw for expenditure by A in period t. $S_t^{A} + S_t^{B} = t$, and $Q_t^{A} + Q_t^{B} = 1$ all t.

We are primarily interested in the distribution function of player i 's expenditure x_t, given the history of the game up until contest t. From this we can see if that player, who has been more successful (in terms of winning) up to t, will then play more or less aggressively in t. If the latter is the case, then this behaviour would suggest that there is an in-built tendency for players who (in terms of winning) do better (worse) early, to do worse (better) later and thus tend to equalise winning performance overall.

Proposition 5. *The Markov perfect equilibrium in the finite sequence of contests is for each player A and B to adopt in contest t a choice from distribution functions dependent on their opponent's summary history of the form:*

$$f_t^{B}(x_t^{B} \mid S_{t-1}^{A}, X_{t-1}^{A}) = \frac{U(TL + S_{t-1}^{A}(W-L) - X_{t-1}^{A}) - U(TL + S_{t-1}^{A}(W-L) + 0 - X_{t-1}^{A} - x_t^{B})}{U(TL + S_{t-1}^{A}(W-L) + (W-L) - X_{t-1}^{A} - x_t^{B}) - U(TL + S_{t-1}^{A}(W-L) + 0 - X_{t-1}^{A} - x_t^{B})} \tag{10B}$$

$$f_t^{A}(x_t^{A} \mid S_{t-1}^{B}, X_{t-1}^{B}) = \frac{U(TL + S_{t-1}^{B}(W-L) - X_{t-1}^{B}) - U(TL + S_{t-1}^{B}(W-L) + 0 - X_{t-1}^{B} - x_t^{A})}{U(TL + S_{t-1}^{B}(W-L) + (W-L) - X_{t-1}^{B} - x_t^{A}) - U(TL + S_{t-1}^{B}(W-L) + 0 - X_{t-1}^{B} - x_t^{A})} \tag{10A}$$

Proof: see Appendix 1.

For purposes of analysis we can take second order expansions of both the numerator and denominator of equations (10A) and (10B) around the respective argument values

$$P_t^{\mathrm{B}} = TL + S_{t\text{-}1}^{\mathrm{B}}(W\text{-}L) + 0 - X_{t\text{-}1}^{\mathrm{B}} - x_t^{\mathrm{A}} \tag{11A}$$

and

$$P_t^{\mathrm{A}} = TL + S_{t\text{-}1}^{\mathrm{A}}(W\text{-}L) + 0 - X_{t\text{-}1}^{\mathrm{A}} - x_t^{\mathrm{B}} \tag{11B}$$

to obtain using (11B in (10B)

$$f_t^{B}(x_t^{B} \,|\, S_{t-1}^{A}, X_{t-1}^{A}) = \frac{x_t^{B}}{W-L}\,\frac{U'(\Pi_t^{A}) + \tfrac{1}{2}U''(\Pi_t^{A})x_t^{B}}{U'(\Pi_t^{A}) + \tfrac{1}{2}U''(\Pi_t^{A})(W-L)} \tag{12}$$

or

$$f_t^{\mathrm{B}}(x_t^{\mathrm{B}}) = f_t^{B}(x_t^{B} \,|\, S_{t-1}^{A}, X_{t-1}^{A}) = \frac{x_t^{B}}{W-L}\,\frac{\Pi_t^{A} - \tfrac{1}{2}Rx_t^{B}}{\Pi_t^{A} - \tfrac{1}{2}R(W-L)} \tag{13B}$$

while similarly

$$f_t^{\mathrm{A}}(x_t^{\mathrm{A}}) = f_t^{A}(x_t^{A} \,|\, S_{t-1}^{B}, X_{t-1}^{B}) = \frac{x_t^{A}}{W-L}\,\frac{\Pi_t^{B} - \tfrac{1}{2}Rx_t^{A}}{\Pi_t^{B} - \tfrac{1}{2}R(W-L)} \tag{13A}$$

where R is the coefficient of relative risk aversion evaluated at P_t^{A} or P_t^{B} respectively. We will assume that R is the same at both values, or more generally that utility has constant relative risk aversion, for simplicity. We can see from (13B) how the distribution $f_t^{\mathrm{B}}(x_t^{\mathrm{B}})$ shifts with the history of A's successes. Higher $S_{t\text{-}1}^{\mathrm{A}}$ or lower $X_{t\text{-}1}^{\mathrm{A}}$ will change $f_t^{\mathrm{B}}(x_t^{\mathrm{B}})$ for a fixed x_t^{B} by the same sign as $\partial f_t^{\mathrm{B}}(x_t^{\mathrm{B}}) / \partial P_t^{\mathrm{A}} = (x_t^{\mathrm{B}}/(W\text{-}L))(-\tfrac{1}{2}R)(W - L - x_t^{\mathrm{B}}) / [P_t^{\mathrm{A}} - \tfrac{1}{2}R(W\text{-}L)]^2 < 0$. Thus more success for A will mean more aggressive expenditure for B as B shifts to a stochastically-dominating distribution. In exactly the same way less success for B will mean that A will shift to a lower distribution of expenditure. Hence the balance of likely success in the t^{th} contest is shifted towards the player with less success in the past. The fact that a player takes increased absolute risk aversion of the other, due to a poor history, as a reason for economising on expenditure means that there is a natural tendency to balance wins and losses in a sequence of contests.

Simulation Exercise: the improvement and convergence of outcomes due to risk aversion.

The sequence of contests above can be easily simulated. Given numerical values for T, L, and W, we can see how a different R changes the distribution of outcomes. Here we report the outcome of a sequence of 20 contests (i.e. T= 20), repeated 50 times, for when L = 0.3, W = 1.3, and for each of a set of values of R ranging from 0.1 to 5. To remove unnecessary sampling variation we use the same 2000 random numbers (2 players choosing from a mixed strategy for each of 20 contests and these repeated 50 times) for each value of R. We thus find 2000 random numbers from a uniform distribution on the unit interval. We set $S_0^A = S_0^B = X_0^A = X_0^B = 0$. We then use the first pair of random numbers as f_t^A, f_t^B, for t = 1. By inverting the (quadratic) definitions of the distribution functions (13B) and (13A) using the usual formula for solving quadratics to find x_t^B and x_t^A, we find the x values associated with the randomly drawn f values. Thus we calculate, for t=1

$$x_t^{B} = \{ [TL + S_{t-1}^{A} (W\text{-}L) – X_{t-1}^{A} + f_t^{B}(W\text{-}L)] – [(TL + S_{t-1}^{A} (W\text{-}L) – X_{t-1}^{A} + f_t^{B}(W\text{-}L))^2 – 4(1+ R/2)(W\text{-}L)\, f_t^{B}(TL + S_{t-1}^{A} (W\text{-}L) – X_{t-1}^{A} – (W\text{-}L)R/2)]^{1/2} \}/ (2+R) \quad (14)$$

$$x_t^{A} = \{ [TL + S_{t-1}^{B} (W\text{-}L) – X_{t-1}^{B} + f_t^{A}(W\text{-}L)] – [(TL + S_{t-1}^{B} (W\text{-}L) – X_{t-1}^{B} + f_t^{A}(W\text{-}L))^2 – 4(1+ R/2)(W\text{-}L)\, f_t^{A}(TL + S_{t-1}^{B} (W\text{-}L) – X_{t-1}^{B} – (W\text{-}L)R/2)]^{1/2} \}/ (2+R) \quad (15)$$

We record the outcomes by updating the number of successes and the amount of expenditure so far:

If $x_t^{A} > x_t^{B}$ then $S_t^{A} = S_{t-1}^{A} + 1$, $S_t^{B} = S_{t-1}^{B}$

If $x_t^{A} < x_t^{B}$ then $S_t^{A} = S_{t-1}^{A}$, $S_t^{B} = S_{t-1}^{B} + 1$

$X_t^{j} = X_{t-1}^{j} + x_t^{j}$ for j = A, B

Now we repeat for $t = 2$, etc. When $t = T$, we find summary outcomes: S_T^{j}, X_T^{j}, and profit outcomes: $TL + S_T^{j} (W\text{-}L) - X_T^{j}$ for j=A,B. We find the absolute value of the difference in success: $|S_T^{A} - S_T^{B}|$; in aggression: $|X_T^{A} - X_T^{B}|$; and in profit outcome: $|S_T^{A} (W\text{-}L) - X_T^{A} - (S_T^{B} (W\text{-}L) - X_T^{B})|$. We store these as results from trial 1, and repeat and store the same outcomes for a further 49 trials. The mean and variance of all these outcomes are presented in Table 1 for each of 5 values of *R*.

There are two implications of the theory developed in the last section. One is that if players are more risk averse (higher *R*) then they will bid from a “lower” distribution and hence be less aggressive. Since aggression is mutually destructive this will increase the average profits earned by the players. This is clearly shown by the first 6 columns of the

Table: as R increases, expenditure (aggression) over a contest sequence is less and profit is more. When R is very small profit is near 6 (just LT). When R is large profit is much higher: profits will have increased by about a third if R is 5.

The second implication is that a player who has been doing badly in his sequence will become more (absolute) risk averse since her expected end-sequence wealth has decreased. This asymmetry is enhanced by the opposite result for the other player. The effect of this asymmetry is that the successful player will reduce aggression and the unsuccessful player will increase aggression. This will lead to higher bids on average by the player who has won fewer contests so far, and the impact of this is that the outcomes at the end of the sequence of contests will be less divergent on average. To see this important effect, consider the last 3 columns of Table 1. The absolute differences in outcomes (expenditure, success and profits) have been calculated since it is immaterial which of the ex ante identical players is the more successful. The mean absolute difference in number of successes is seen to reduce from 3.2 when R=0.1 to less than 2.4 when R=5. There are similar reductions in the difference in aggression, since early success will reflect relatively high aggression and will be followed by a relative reduction in aggression if the opponent is risk averse. Also the outcomes in terms of profits are appreciably less diverse.

The simulation has thus confirmed two characteristics of a sequence of contests when the players are risk averse. First, aggression is lower and profits are higher when risk aversion is present. Second the difference between winners and losers is less when risk aversion is present. The results also show that these effects are small when risk aversion is small (R less that 1) but considerable when risk aversion is high (R>2). Of course the size of the effect depends on the impact of early outcomes on expected end-sequence wealth, since this determines local risk aversion. If the parameters change over the sequence (for example R is not a constant) the results change to reflect risk preference changes at different wealth levels.

Table 1: Simulations of sequence of 20 contests – average of 50 simulations; players A and B.

		Expenditure		Success		Profits		Absolute Differences (A-B)		
R	**0.1**	**A**	**B**	**A**	**B**	**A**	**B**	**Exp**	**Success**	**Profits**
	Mean	9.995715	10.18458	9.8	10.2	5.804285	6.015417	1.270285	3.2	2.56845
	SD	1.397112	1.116738	2.05	2.049	1.902862	1.774582	1.01989	2.592	1.81267
R	**0.5**	**A**	**B**	**A**	**B**	**A**	**B**	**Exp**	**Success**	**Profits**
	Mean	9.85996	10.05239	9.76	10.24	5.90004	6.187609	1.255832	3.12	2.46388
	SD	1.380631	1.096348	2.03	2.025	1.866943	1.737472	1.000203	2.628	1.84960
R	**1**	**A**	**B**	**A**	**B**	**A**	**B**	**Exp**	**Success**	**Profits**
	Mean	9.678522	9.875885	9.78	10.22	6.101478	6.344115	1.239113	3.08	2.46313
	SD	1.356299	1.068079	2.02	2.023	1.851135	1.760422	0.966115	2.66	1.89371
R	**2**	**A**	**B**	**A**	**B**	**A**	**B**	**Exp**	**Success**	**Profits**
	Mean	9.286498	9.475962	9.74	10.26	6.453502	6.784038	1.181949	2.84	2.17073
	SD	1.299016	0.999031	1.9	1.895	1.749562	1.672423	0.895536	2.564	1.99315
R	**5**	**A**	**B**	**A**	**B**	**A**	**B**	**Exp**	**Succ**	**Profits**
	Mean	7.798564	7.997237	9.76	10.24	7.961436	8.242763	1.045092	2.4	1.87227
	SD	1.103726	0.810441	1.54	1.544	1.532539	1.280456	0.754859	2	1.57209

5. Welfare and Conclusions

The full information games we have used imply mixed strategy equilibria. In such equilibria in a risk-aversion setting it is the opponent's risk aversion which determines the distribution that randomizes your choice. The more risk averse the opponent, the lower the equilibrium distribution of aggression that you apply. Profits (as distinct from utilities) are thus higher when contests take place between players who are more risk averse. Aggression is also higher for the more risk-averse player in an asymmetric contest. One consequence is that a sequence of contests tends to turn out more evenly when players are risk averse, as well as yielding a larger cash profit. A possible application of such results relates to the business cycle. If business is considered to consist of (broadly) a series of contests, then we might think that players are less risk averse in the upswing of the cycle than the downswing. Then aggression is high in the upswing, and aggression is low in the downswing. One manifestation of aggression could be low prices set by competing suppliers, and then prices would move counter-cyclically. Further, the tendency for profits to be high in good times when many contest opportunities were present would be offset increasingly by high aggression reducing the profit from individual contests. A cycle is suggested composed of high profits leading to low risk aversion, high aggression and then low profits, in turn followed by high risk aversion, low aggression and then back to high profits again. Clearly, at this level of abstraction such applications of the analysis are merely suggestive. However, if x is interpreted as aggressive promotion of a firm's products then high x would be associated with low prices, and then prices would be observed to move in a counter-cyclical pattern relative to the business cycle. Obviously there are many supply side reasons for prices to vary pro-cyclically, including rising costs as inputs become scarce. The argument here might therefore be thought to explain some of the mixed evidence on the relationship of prices and the business cycle. Certainly in some classes of product and oligopoly structures prices do move counter-cyclically.[2]

The basis for our analysis has been that the expenditure to win prizes forms a prisoners' dilemma game: if the opponents could agree to limit their expenditure and take turns at winning, both would be better off. We have assumed that no such commitment is possible and so the Nash equilibrium of the game (Markov perfect equilibrium in the finite sequence of games) is the relevant model. "Second prizes" of value L provide the incentive for participation; on average gains from winning are competed away by the aggressive levels of expenditure.

One variation of the model would be to reject the MSNE concept as an appropriate equilibrium strategy. For example, it might be thought that the information content required is just too high. In this case, we might turn to an equilibrium based on bounded rationality such as the evolutionarily stable strategy (ESS). However it is well known that

[2] See for instance the summary paper by Domowitz in Norman and La Manna (ed) (1992). Chadha et al (2000) claim that UK consumer prices have become countercyclical post 1945.

such equilibria are “spiteful”, and they would be likely to lead to still lower profits than are present in the MSNE case.[3]

[3] See Cheng and Zhu (1995) and Holler (1990) for discussions of the arguments for replacing MSNE in general, and Leininger (2002) for the application of ESS in contests.

Appendix 1: Proof of Proposition 5

We will approach the problem by first considering the T^{th} contest distributions (essentially already discussed in the one-period model). Then we consider the $(T-1)^{th}$ contest distributions, and hence then infer all contest distributions conditional on the (summary) history of outcomes.

At the end of the $(T-1)^{th}$ contest we have random utility for player A of

$$U(TL + S_{T-1}^{A}(W-L) - X_{T-1}^{A} + (W-L)\,Q_T^{A} - x_T^{A})$$

Now x_T^A and x_T^B will have well-defined distribution functions $f_T^A(x_T^A)$, $f_T^B(x_T^B)$, and these will have positive derivatives for all $0 \leq x_T^A \leq W\text{-}L$. $f_T^B(x_T^B)$ must be such as to make player A indifferent among all the x_T^A possibilities, including $x_T^A=0$. Thus:

$$\mathrm{E}\,U(TL + S_{T-1}^{A}(W-L) - X_{T-1}^{A} + (W-L)\,Q_T^{A} - x_T^{A}) = U(TL + S_{T-1}^{A}(W-L) - X_{T-1}^{A}) \text{ for all } 0 \leq x_T^A \leq W\text{-}L. \quad \text{(A1)}$$

Given that the higher x_T wins the contest, we can describe the expected utility in terms of B's mixed strategy:

$$U(TL + S_{T-1}^{A}(W-L) - X_{T-1}^{A} + (W-L) - x_T^{A})\, f_T^B(x_T^A) + U(TL + S_{T-1}^{A}(W-L) - X_{T-1}^{A} - x_T^{A})\,(1 - f_T^B(x_T^A)) = U(TL + S_{T-1}^{A}(W-L) - X_{T-1}^{A}) \quad \text{(A2)}$$

$f_T^{B}(x_T^{A})$ is the probability that a choice of x_T^{A} by A will beat x_T^{B} and hence the distribution function $f_T^{B}(x_T^{B})$ takes the form, directly from (A2):

$$f_T^{B}(x_T^{B}) = [U(TL + S_{T\text{-}1}^{A}(W\text{-}L) - X_{T\text{-}1}^{A}) - U(TL + S_{T\text{-}1}^{A}(W\text{-}L) - X_{T\text{-}1}^{A} - x_T^{B})] / [U(TL + S_{T\text{-}1}^{A}(W\text{-}L) - X_{T\text{-}1}^{A} + (W\text{-}L) - x_T^{B}) - U(TL + S_{T\text{-}1}^{A}(W\text{-}L) - X_{T\text{-}1}^{A} - x_T^{B})] \quad \text{(A3)}$$

Notice that f_T^{B} will depend on the summary of the history: $S_{T\text{-}1}^{A}$ and $X_{T\text{-}1}^{A}$. Thus the T subscript will determine the distribution and be interpreted as $f_T^{B}(x_T^{B} \mid S_{T\text{-}1}^{A}, X_{T\text{-}1}^{A})$, and the function f_T^{B} will generally differ for differences in A's history. Similarly the distribution function $f_T^{A}(x_T^{A})$ is found as

$$f_T^{A}(x_T^{A} \mid S_{T\text{-}1}^{B}, X_{T\text{-}1}^{B}) = [U(TL + S_{T\text{-}1}^{B}(W\text{-}L) - X_{T\text{-}1}^{B}) - U(TL + S_{T\text{-}1}^{B}(W\text{-}L) - X_{T\text{-}1}^{B} - x_T^{A})] / [U(TL + S_{T\text{-}1}^{B}(W\text{-}L) - X_{T\text{-}1}^{B} + (W\text{-}L) - x_T^{A}) - U(TL + S_{T\text{-}1}^{B}(W\text{-}L) - X_{T\text{-}1}^{B} - x_T^{A})] \quad \text{(A4)}$$

Now consider the $(T\text{-}1)^{th}$ contest. Here $Q_{T\text{-}1}^{A}$, Q_T^{A} and x_T^{A} are all random variables. Conditional on a choice $x_{T\text{-}1}^{A}$ we have

$$\mathrm{E}\, U_{T\text{-}2}^{A}(x_{T\text{-}1}^{A}) = \mathrm{E}\, U(TL + S_{T\text{-}2}^{A}(W\text{-}L) - X_{T\text{-}2}^{A} + (W\text{-}L)\, Q_{T\text{-}1}^{A} + (W\text{-}L)\, Q_T^{A} - x_{T\text{-}1}^{A} - x_T^{A}) =$$

$$U(TL + S_{T\text{-}2}^{A}(W\text{-}L) - X_{T\text{-}2}^{A} + (W\text{-}L) + (W\text{-}L) - x_{T\text{-}1}^{A} - x_T^{A})\, [\, f_{T\text{-}1}^{B}(x_{T\text{-}1}^{A} \mid S_{T\text{-}2}^{A}, X_{T\text{-}2}^{A})\, f_T^{B}(x_T^{A} \mid S_{T\text{-}2}^{A} + 1, X_{T\text{-}2}^{A} + x_{T\text{-}1}^{A})\,] +$$

$U(TL + S_{T\text{-}2}{}^{A}\,(W\text{-}L) - X_{T\text{-}2}{}^{A} + (W\text{-}L) + 0 - x_{T\text{-}1}{}^{A} - x_{T}{}^{A})\ [f_{T\text{-}1}{}^{B}(x_{T\text{-}1}{}^{A} \mid S_{T\text{-}2}{}^{A}, X_{T\text{-}2}{}^{A})(1 - f_{T}{}^{B}(x_{T}{}^{A} \mid S_{T\text{-}2}{}^{A} + 1, X_{T\text{-}2}{}^{A} + x_{T\text{-}1}{}^{A})) + (1 - f_{T\text{-}1}{}^{B}(x_{T\text{-}1}{}^{A} \mid S_{T\text{-}2}{}^{A}, X_{T\text{-}2}{}^{A})(\ f_{T}{}^{B}(x_{T}{}^{A} \mid S_{T\text{-}2}{}^{A}\ , X_{T\text{-}2}{}^{A} + x_{T\text{-}1}{}^{A}))] +$

$U(TL + S_{T\text{-}2}{}^{A}\,(W\text{-}L) - X_{T\text{-}2}{}^{A} + 0 + 0 - x_{T\text{-}1}{}^{A} - x_{T}{}^{A})[(1 - f_{T\text{-}1}{}^{B}(x_{T\text{-}1}{}^{A} \mid S_{T\text{-}2}{}^{A}, X_{T\text{-}2}{}^{A})\ (1 - f_{T}{}^{B}(x_{T}{}^{A} \mid S_{T\text{-}2}{}^{A}\ , X_{T\text{-}2}{}^{A} + x_{T\text{-}1}{}^{A})]$ (A5)

Rearranging terms gives:

$= f_{T\text{-}1}{}^{B}(x_{T\text{-}1}{}^{A} \mid S_{T\text{-}2}{}^{A}, X_{T\text{-}2}{}^{A})\ [U(TL + S_{T\text{-}2}{}^{A}\,(W\text{-}L) - X_{T\text{-}2}{}^{A} + (W\text{-}L) + (W\text{-}L) - x_{T\text{-}1}{}^{A} - x_{T}{}^{A})\ f_{T}{}^{B}(x_{T}{}^{A} \mid S_{T\text{-}2}{}^{A} + 1, X_{T\text{-}2}{}^{A} + x_{T\text{-}1}{}^{A}) + U(TL + S_{T\text{-}2}{}^{A}\,(W\text{-}L) - X_{T\text{-}2}{}^{A} + (W\text{-}L) + 0 - x_{T\text{-}1}{}^{A} - x_{T}{}^{A})\ (1 - f_{T}{}^{B}(x_{T}{}^{A} \mid S_{T\text{-}2}{}^{A} + 1, X_{T\text{-}2}{}^{A} + x_{T\text{-}1}{}^{A}))] +$

$(1 - f_{T\text{-}1}{}^{B}(x_{T\text{-}1}{}^{A} \mid S_{T\text{-}2}{}^{A}, X_{T\text{-}2}{}^{A})[\ U(TL + S_{T\text{-}2}{}^{A}\,(W\text{-}L) - X_{T\text{-}2}{}^{A} + (W\text{-}L) + 0 - x_{T\text{-}1}{}^{A} - x_{T}{}^{A})\ (\ f_{T}{}^{B}(x_{T}{}^{A} \mid S_{T\text{-}2}{}^{A}\ , X_{T\text{-}2}{}^{A} + x_{T\text{-}1}{}^{A})) + U(TL + S_{T\text{-}2}{}^{A}(W\text{-}L) - X_{T\text{-}2}{}^{A} + 0 + 0 - x_{T\text{-}1}{}^{A} - x_{T}{}^{A})\ (1 - f_{T}{}^{B}(x_{T}{}^{A} \mid S_{T\text{-}2}{}^{A}\ , X_{T\text{-}2}{}^{A} + x_{T\text{-}1}{}^{A})]$

$= f_{T\text{-}1}{}^{B}(x_{T\text{-}1}{}^{A} \mid S_{T\text{-}2}{}^{A}, X_{T\text{-}2}{}^{A})\ U(TL + S_{T\text{-}2}{}^{A}\,(W\text{-}L) - X_{T\text{-}2}{}^{A} + (W\text{-}L) + 0 - x_{T\text{-}1}{}^{A}) + (1 - f_{T\text{-}1}{}^{B}(x_{T\text{-}1}{}^{A} \mid S_{T\text{-}2}{}^{A}, X_{T\text{-}2}{}^{A})\ U(TL + S_{T\text{-}2}{}^{A}\,(W\text{-}L) - X_{T\text{-}2}{}^{A} + 0 + 0 - x_{T\text{-}1}{}^{A})$ (A6)

using the result (A2) for the T^{th} contest. This must be the same for all $x_{T\text{-}1}{}^{A}$ in [0, W-L]. When $x_{T\text{-}1}{}^{A} = 0$ and since $f_{T\text{-}1}{}^{B}(0 \mid S_{T\text{-}2}{}^{A}, X_{T\text{-}2}{}^{A}) = 0$, we have that

$$\text{E}\ U_{T\text{-}2}{}^{A} = U(TL + S_{T\text{-}2}{}^{A}\,(W\text{-}L) - X_{T\text{-}2}{}^{A} + 0 + 0) = U(TL + \text{S}_{\text{T-2}}{}^{A}\,(W\text{-}L) - X_{T\text{-}2}{}^{A}) \qquad \text{(A7)}$$

Thus

$f_{T\text{-}1}{}^{B}(x_{T\text{-}1}{}^{A} \mid S_{T\text{-}2}{}^{A}, X_{T\text{-}2}{}^{A}) = [U(TL + S_{T\text{-}2}{}^{A}(W\text{-}L) - X_{T\text{-}2}{}^{A}) - U(TL + S_{T\text{-}2}{}^{A}(W\text{-}L) - X_{T\text{-}2}{}^{A} - x_{T\text{-}1}{}^{A})]$ /

$[U(TL + S_{T\text{-}2}{}^{A}(W\text{-}L) - X_{T\text{-}2}{}^{A} + (W\text{-}L) - x_{T\text{-}1}{}^{A}) - U(TL + S_{T\text{-}2}{}^{A}(W\text{-}L) - X_{T\text{-}2}{}^{A} - x_{T\text{-}1}{}^{A})]$

so that $f_{T\text{-}1}{}^{B}(x_{T\text{-}1}{}^{B} \mid S_{T\text{-}2}{}^{A}, X_{T\text{-}2}{}^{A})$ is defined as

$$f_{T\text{-}1}{}^{B}(x_{T\text{-}1}{}^{B} \mid S_{T\text{-}2}{}^{A}, X_{T\text{-}2}{}^{A}) = [U(TL + S_{T\text{-}2}{}^{A}(W\text{-}L) - X_{T\text{-}2}{}^{A}) - U(TL + S_{T\text{-}2}{}^{A}(W\text{-}L) - X_{T\text{-}2}{}^{A} - x_{T\text{-}1}{}^{B})] / [U(TL + S_{T\text{-}2}{}^{A}(W\text{-}L) - X_{T\text{-}2}{}^{A} + (W\text{-}L) - x_{T\text{-}1}{}^{B}) - U(TL + S_{T\text{-}2}{}^{A}(W\text{-}L) - X_{T\text{-}2}{}^{A} - x_{T\text{-}1}{}^{B})] \quad \text{(A8)}$$

We now see that we can work backwards to find the general result. We know that

$E\,U_{T\text{-}2}{}^{A} = U(TL + S_{T\text{-}3}{}^{A}(W\text{-}L) + (W\text{-}L) - X_{T\text{-}3}{}^{A} - x_{t\text{-}2}{}^{A})\, f_{T\text{-}2}{}^{B}(x_{T\text{-}2}{}^{A} \mid S_{T\text{-}3}{}^{A}, X_{T\text{-}3}{}^{A}) + U(TL + S_{T\text{-}3}{}^{A}(W\text{-}\underline{L}) + 0 - X_{T\text{-}3}{}^{A} - x_{t\text{-}2}{}^{A})\,(1 - f_{T\text{-}2}{}^{B}(x_{T\text{-}2}{}^{A} \mid S_{T\text{-}3}{}^{A}, X_{T\text{-}3}{}^{A}))$

But this must have the same value for any $x_{t\text{-}2}{}^{A}$ in $[0, W\text{-}L]$, including for $x_{t\text{-}2}{}^{A} = 0$. In this case

$E\,U_{T\text{-}2}{}^{A} = U(TL + S_{T\text{-}3}{}^{A}(W\text{-}L) + (W\text{-}L) - X_{T\text{-}3}{}^{A} - x_{T\text{-}2}{}^{A})\, f_{T\text{-}2}{}^{B}(x_{T\text{-}2}{}^{A} \mid S_{T\text{-}3}{}^{A}, X_{T\text{-}3}{}^{A}) + U(TL + S_{T\text{-}3}{}^{A}(W\text{-}L) + 0 - X_{T\text{-}3}{}^{A} - x_{T\text{-}2}{}^{A})\,(1 - f_{T\text{-}2}{}^{B}(x_{T\text{-}2}{}^{A} \mid S_{T\text{-}3}{}^{A}, X_{T\text{-}3}{}^{A})) =$

$U(TL + S_{T\text{-}3}{}^{A}(W\text{-}L) - X_{T\text{-}3}{}^{A})$

And in general:

$$\mathrm{E}\, U_t^{\mathrm{A}} = U(TL + S_{t\text{-}1}^{\mathrm{A}}\,(W\text{-}L) + (W\text{-}L) - X_{t\text{-}1}^{\mathrm{A}} - x_t^{\mathrm{A}})\ f_t^{\mathrm{B}}(x_t^{\mathrm{A}} \mid S_{t\text{-}1}^{\mathrm{A}}, X_{t\text{-}1}^{\mathrm{A}}) + U(TL + S_{t\text{-}1}^{\mathrm{A}}\,(W\text{-}L) + 0 - X_{t\text{-}1}^{\mathrm{A}} - x_t^{\mathrm{A}})\ (1 - f_t^{\mathrm{B}}(x_t^{\mathrm{A}} \mid S_{t\text{-}1}^{\mathrm{A}}, X_{t\text{-}1}^{\mathrm{A}})) = U(TL + S_{t\text{-}1}^{\mathrm{A}}\,(W\text{-}L) - X_{t\text{-}1}^{\mathrm{A}}) \qquad \text{(A9)}$$

We then have

$$f_t^{\mathrm{B}}(x_t^{\mathrm{A}} \mid S_{t\text{-}1}^{\mathrm{A}}, X_{t\text{-}1}^{\mathrm{A}}) = [U(TL + S_{t\text{-}1}^{\mathrm{A}}\,(W\text{-}L) - X_{t\text{-}1}^{\mathrm{A}}) - U(TL + S_{t\text{-}1}^{\mathrm{A}}\,(W\text{-}L) + 0 - X_{t\text{-}1}^{\mathrm{A}} - x_t^{\mathrm{A}})]/[\,U(TL + S_{t\text{-}1}^{\mathrm{A}}\,(W\text{-}L) + (W\text{-}L) - X_{t\text{-}1}^{\mathrm{A}} - x_t^{\mathrm{A}}) - U(TL + S_{t\text{-}1}^{\mathrm{A}}\,(W\text{-}L) + 0 - X_{t\text{-}1}^{\mathrm{A}} - x_t^{\mathrm{A}})]$$

which implies the distribution functions (10A) and (10B).

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