Rules of Elasticities - Andrew Harkins (EC123)

EMEA (second edition) p. 242 (Section 7.7 'Why Economists Use Elasticities', Q10)

(1) Elasticity of a constant	$El_x(A) = 0$
(2) Product rule for elasticities	$El_x(f(x)\,g(x)) = El_x f(x) + El_x g(x)$
(3) Chain rule for elasticities	$El_x(f(g(x))) = El_u f(u)\,El_x u$ where $u = g(x)$
(4) Quotient rule for elasticities	$El_x\left(\frac{f}{g}\right) = El_x f(x) - El_x g(x)$
(5) Elasticity of a sum	$El_x(f+g) = \frac{f\,El_x f + g\,El_x g}{f+g}$
	$El_x(f-g) = \frac{f\,El_x f - g\,El_x g}{f-g}$

Clearly you can also apply rules (2)-(5) when f and g are not functions of x by using rule (1).

Example application of the product rule

Find $El_x\left(-10x^{-5}\right)$

Call $f = -10$ and $g = x^{-5}$. Using the product rule for elasticities $El_x\left(-10x^{-5}\right) = El_x(fg) = El_x f + El_x g$. Using rule (1) we get $El_x f + El_x g = 0 + El_x g = El_x x^{-5}$. Now directly computing $El_x x^{-5} = \frac{dx^{-5}}{dx}\frac{x}{x^{-5}}$ we get $El_x\left(-10x^{-5}\right) = -5$

Example application of the chain rule

Find $El_x\left(e^{2x^2}\right)$

Call $f(u) = e^u$ and $u = g(x) = 2x^2$. Using the chain rule for elasticities we get $El_x(f(g(x))) = El_u f(u)\,El_x u = El_u e^u El_x 2x^2$. Taking $El_u e^u$ first we compute this as $El_u e^u = \frac{de^u}{du}\frac{u}{e^u} = u = 2x^2$. Now looking at $El_x 2x^2$ and directly computing (or using the product rule) we get $El_x 2x^2 = \frac{d2x^2}{dx}\frac{x}{2x^2} = 2$. Putting these two together we arrive at $El_x\left(e^{2x^2}\right) = 4x^2$.

Question 5 from problem set #4

Q5: Find the elasticity with respect to x when $x^a y^b = Ae^{x/y^2}$ and a, b and A are strictly positive constants.

To solve this problem we take a similar approach to implicit differentiation. We will try to find the elasticity of y with respect to x on both sides of the equality and then rearrange to find an expression for $El_x y$. Notice here that y is an implicit function of x.

To find $El_x x^a y(x)^b$, first use the product rule to get:

$$El_x x^a y(x)^b = El_x x^a + El_x y(x)^b = a + El_x y(x)^b$$

Now we apply the chain rule to get

$$El_x y(x)^b = El_y y(x)^b El_x y(x) = bEl_x y(x)$$

For the right hand side of the equality we can see straight away (by (2) and (1)) that $El_x Ae^{x/y(x)^2} = El_x e^{x/y(x)^2}$. Use the chain rule for elasticities first to get:

$$El_x e^{x/y(x)^2} = El_u e^u El_x \frac{x}{y(x)^2}$$

From the previous example above we know that $El_u e^u = u = \frac{x}{y(x)^2}$. For $El_x \frac{x}{y(x)^2}$ we use the 'quotient rule for elasticities' to arrive at:

$$El_x \frac{x}{y(x)^2} = 1 - El_x y(x)^2 = 1 - 2El_x y(x)$$

Now, putting it all together (both sides) gives:

$$a + bEl_x y(x) = (\frac{x}{y(x)^2})\left(1 - 2El_x y(x)\right)$$

Rearranging we get:

$$El_x y(x) = \frac{\frac{x}{y(x)^2} - a}{2\frac{x}{y(x)^2} + b}$$

Multiplying numerator and denominator by y^2 we finally arrive at:

$$El_x y(x) = \frac{x - ay^2}{2x + by^2}$$

This is the same answer we got when using the 'natural log method' which I discussed in the seminar (i.e. rewrite $x^a y^b = Ae^{x/y^2}$ as $\hat{x} + \hat{y} = \hat{c} + e^{\hat{x} - 2\hat{y}}$ where $\hat{x} = ln(x)$, $\hat{y} = ln\left(y\right)$, and $\hat{c}$ is a constant, and then find $\frac{d\hat{y}}{d\hat{x}}$ via implicit differentiation).