Problem Set 6 - EC123

Q7 (a)

The first part of the question asks us to find the demand function of a hypothetical consumer with utility function $U = x^a + y$ and income m. We will use the Lagrange multiplier method to solve this particular problem. We are given the task of solving

$$\max_{x,y} x^a + y$$

$$\text{s.t. } px + y = m$$

Writing the Lagrangian we get

$$\mathcal{L} := x^a + y - \lambda\left(px + y - m\right)$$

Assuming an interior solution (as stated in the question) we use the first order conditions for maximisation to solve for x, y and λ. Starting with the FOCs

$$\frac{\partial \mathcal{L}}{\partial x} = ax^{a-1} - \lambda p = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 1 - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = px + y - m = 0$$

We can see that $\lambda = 1$ and so $ax^{a-1} = p$. This gives us $x = \left(\frac{p}{a}\right)^{\frac{1}{a-1}}$ and $y = m - p\left(\frac{p}{a}\right)^{\frac{1}{a-1}}$. It is possible to simplify the demand function for y as $m - p^{1+\frac{1}{a-1}}c = m - p^{\frac{a}{a-1}}c$ where we can define $c := a^{-\frac{1}{a-1}}$ to make the notation less messy.

Q7 (b)

Next we are asked to check the partial derivatives of these demand functions with respect to p and m. Computing for m is straightforward, giving $\frac{\partial x(p,m)}{\partial m} = 0$ and $\frac{\partial y(p,m)}{\partial m} = 1$. For partials with respect to p it is more complicated. As pointed out by Gor at the very end of the last seminar, the correct way to differentiate $x\left(p,m\right)$ is:

$$\frac{\partial x\left(p,m\right)}{\partial p} = \frac{1}{a-1}p^{\frac{1}{a-1}-1}c = \frac{c}{a-1}p^{\frac{2-a}{a-1}}$$

Here c is the constant I defined in part (a), which is always positive since $a \in (0,1)$, meaning that $\frac{c}{a-1} < 0$ since $a < 1$. Therefore, we see that increases in price p will lead to less demand for good x, as expected. Next for $\frac{\partial y(p,m)}{\partial p}$ we get

$$\frac{\partial y\left(p,m\right)}{\partial p} = \frac{a}{a-1}p^{\frac{a}{a-1}-1}c = -\frac{c \cdot a}{a-1}p^{-\frac{1}{a-1}}$$

which is positive since, again as expected.

Q7 (c)

We are now asked to calculate the elasticity of px with respect to p, i.e. to find $\mathrm{El}_p px$. The product rule of elasticities tells us that this is simply $1 + \mathrm{El}_p x = 1 + \frac{\partial x(p,m)}{\partial p}\frac{p}{x(p,m)}$ where we can find that $\mathrm{El}_p x$ is

$$\left(\frac{c}{a-1}p^{\frac{2-a}{a-1}}\right)\frac{p}{p^{\frac{1}{a-1}}c} = \left(\frac{1}{a-1}p^{\frac{2-a}{a-1}}\right)p^{1-\frac{1}{a-1}} = \left(\frac{1}{a-1}p^{\frac{2-a}{a-1}}\right)p^{\frac{a-2}{a-1}}$$

When we examine this further we see that $\mathrm{El}_p x$ simplifies to $\frac{1}{a-1}p^0 = \frac{1}{a-1}$, meaning that the final answer is $1 + \frac{1}{a-1} = \frac{a}{a-1}$.

Q7 (d)

Finally we must substitute in $a = \frac{1}{2}$ to $V = x\left(p,m\right)^a + y\left(p,m\right)$ and then differentiate with respect to p to prove that $\frac{\partial V}{\partial p} = -x\left(p,m\right)$. If we substitute in we get $x\left(p,m\right) = \left(2p\right)^{-2} = \frac{1}{4}p^{-2}$ and so $x\left(p,m\right)^{\frac{1}{2}} = \frac{1}{2}p^{-1}$, whilst $y\left(p,m\right) = m - p^{\frac{a}{a-1}}c$ becomes $m - p^{-1}\frac{1}{4}$. This gives us $V = \frac{1}{2}p^{-1} + m - \frac{1}{4}p^{-1} = m + \frac{1}{4}p^{-1}$. Differentiating yields

$$\frac{\partial V}{\partial p} = -\frac{1}{4}p^{-2} = -x\left(p,m\right)$$