Minimum Effort Games on Networks*

Andrew Harkins†

26th February 2016

Abstract

I study games played on networks where the benefit accrued across any pairwise relationship depends on the minimum effort made by either party. I show that the actions of agents in the Pareto dominant equilibrium of this game are completely characterised by the *k-core decomposition* of the network. In equilibrium, each agent plays effort equal to their highest k-core number or *coreness*.

The Pareto dominant equilibrium is shown to be 'knife edge' for a subset of networks and so vulnerable nodes in the network are identified. Using the potential function of the game I also analyse the long-run stability of equilibria. I introduce the *density decomposition* of a network and find that potential maximisers have a nested structure which is based on this decomposition.

Applications of this model include investments in human capital, user engagement in online social networks and technology adoption.

Keywords: Games on Networks, Threshold Games, Minimum Effort, Potential Function, k-core

JEL: D85, C72, C73

*I began this paper during my Ph.D. and I am grateful to Bhaskar Dutta and Daniel Sgroi for their advice and guidance. I thank Dan Bernhardt, Antonio Cabrales, Jonathan Cave, Peter Hammond, Timo Hiller and Dimitri Migrow for their comments and suggestions. I would also like to thank audiences at the 18th CTN workshop and the 2013 Warwick PhD Conference who gave their feedback on an early version of this paper titled 'Network Games with Perfect Complements'. Funding for this research from the ESRC and ANR (under contract ANR13-BSH1-0010) is gratefully acknowledged.

†CNRS and Université Paris 1, Centre d'Economie de la Sorbonne, 106-112 Boulevard de l'Hôpital, 75013 Paris, France. Contact: andrew.harkins@univ-paris1.fr

1 Introduction

Strategic complementarity plays an important role in almost every area of economics. Actions are strategic complements if an increased level of activity from one party raises the returns from that activity for others. For example, the more a firm's competitor invests in advertising, the more the firm itself needs to invest in order to compete; the more effort our friends or co-workers make to use a new piece of software, the more it pays for us to do the same; the more savers withdraw from a potentially failing bank, the more it benefits us to also withdraw; and so on.

In some of these situations, high levels of effort by one party can be an imperfect substitute for another party's lack of effort. In other situations, such as in the classic Stag Hunt game, the joint benefits will hinge on those who contribute least. If one party refuses to expend effort then it makes no difference what the other party does. For example, the benefit gained from socialising with friends or co-workers is limited by the socialising efforts of other individuals with whom you interact.

When actions are not substitutable and the returns from activity depend heavily on the concurrent activity of others, this can lead to coordination failures where parties become trapped in Pareto dominated equilibria (Cooper and John, 1988). If only a small number of individuals are involved, communication and joint action between parties can help to mitigate such coordination problems and attain Pareto dominant outcomes. Yet, this potential solution becomes less plausible when we shift focus from small numbers interactions towards interactions which take place in large and perhaps fragmented societies.

To analyse this issue further I construct a stylised model where individuals must invest in a continuous 'effort' variable which is privately costly but provides benefits if their neighbours also invest. To capture the non-substitutability of efforts, I assume that the benefit accrued across any pairwise relationship will depend on the minimum effort made by either party.[1] Furthermore, I allow for

[1]Previous authors have studied *group* minimum effort games (e.g. Bryant (1983)), often under the alternative name of 'weakest link' public goods games (Hirshleifer, 1983). My model is distinguished from previous work by focusing on pairwise interactions.

agents to vary in terms of their prominence by embedding them in a network. As was highlighted first by Schelling (1973) and later developed by Ellison (1993), the pattern of social interaction can have significant influence on both the levels and stability of equilibrium effort.

This model can be used to shed light on the role of network structure in technology adoption or the use of network goods. For example, situations where individuals spend time using a social networking product or invest effort in learning a new piece of software, with the hope of others using it. Another example of a possible application is to human capital investments made within firms. For firms in knowledge-intensive industries such as consulting, software development or R&D contracting, the principle input is the expertise of its employees. Investment in developing such skills is privately costly but may bring private benefits. Moreover, the effectiveness of knowledge sharing and collaboration between employees, which often involves highly specialised and technical information, is limited by the agent with the minimal level of expertise in any pairwise relationship. Empirical studies have shown that network structure (in particular cohesion) seems to play an important role in such settings (see Reagans and McEvily (2003)).[2]

As we shall later see, it is indeed network cohesion which plays a central role in determining equilibrium effort levels. An individual's effort in the Pareto dominant equilibrium will turn out to be completely characterised by their *k-core number* or *coreness*, which is a well-known measure of the extent to which an individual is a member of a cohesive core of the network. I will also analyse the stability properties of this equilibrium, characterising networks where it is 'knife edge', in addition to highlighting potentially vulnerable nodes in the network.

Previous authors such as Young (1998) and Jackson and Watts (2002) have focused on the effect of the network's structure on stochastically stable equilibria of similar discrete action 'Stag Hunt' games. In a close analogue to this analysis, I show that an individual's effort in the potential maximising equilibrium is always positive and also determined by the extent to which they inhabit a dense core of the network.

[2] See also Cremer et al. (2007) and Sobel (2012), who examine similar phenomena using a different approach.

Sections 3 and 4 contain these main results and Section 5 analyses the problem from a social planner's perspective. I will now discuss the relationship between this paper and the rest of the literature.

2 Related Literature

The games on networks literature has grown significantly in recent years and has been applied to study a diverse range of phenomena.[3] Many researchers in this literature have sought to establish relationships between equilibrium actions in simple games and both an individual's position in the network and the network structure as a whole. A foundational contribution is by Ballester et al. (2006) who examine a model with continuous action spaces and quadratic cost functions. Agents in their model have linear best responses and equilibrium efforts are shown to be related to their Bonacich centrality in the network. More recently Bramoullé et al. (2014) have examined such linear-quadratic games on networks in a framework which embeds both Ballester et al. (2006) and the earlier model of Bramoullé and Kranton (2007). As in the present paper, Bramoullé et al. (2014) make use of the potential function of the game to study the stability properties of equilibria. They highlight a relationship between the 'bipartiteness' of a network and equilibrium stability.[4]

A second strand of the literature on games on networks is associated with 'threshold games'. A typical setting is one similar to that described by Granovetter (1978) where a group of individuals face a collective action problem in the form of a binary choice (e.g. either to strike or not to strike) but prefer only to take a more risky action if at least some threshold percentage of the group do the same. Differences in thresholds can lead to cascading behaviours, where if one individual switches strategy this forces others to switch, leading to yet more switching until a new equilibrium is reached.

[3]The early work on games on networks has been surveyed by Jackson (2008), more recent surveys are presented by Jackson and Zenou (2014) and Bramoulle and Kranton (2015).

[4]Although I consider only fixed exogenous networks which are common knowledge, games where networks evolve over time (König et al., 2014) or are imperfectly known by agents (Galeotti et al., 2010) have also been studied.

Chwe (2000) examines how the communication structure of a society might enable individuals with heterogeneous thresholds to more easily coordinate on their preferred equilibrium in such a setting. His model shows that optimal networks can be formed from a series of interlocking cliques which allow all agents to take the more risky action by making all locally important thresholds common knowledge.

This is also closely related to the model of Morris (2000) who studies the role of network structure in a binary decision threshold game when the modeller is concerned about robustness of equilibria with respect to contagion. Morris shows that only sufficiently inward looking groups of nodes can be resilient to an invading cascade of switching. Another closely related contribution is by Young (1998) who is similarly concerned with the conditions on network structure which enable different regimes of play in different areas of the network.

Young (1998) however also highlights a negative result with regard to the ability of any network structure to prevent a risk-dominant equilibrium from prevailing as the unique stochastically stable state of play. This builds on the insights of the Ellison (1993) model, which has also been extended by Jackson and Watts (2002) to a setting where the network is endogenous. In their paper, Jackson and Watts (2002) find that stochastically stable equilibria may arise which are neither risk-dominant nor Pareto-dominant, if agents can select who they are linked to.

3 Model

A finite set of agents $N = \{1, \ldots, n\}$ inhabit a network given by the pair $(N, \mathbf{g})$ with $\mathbf{g}$ denoting a set of unordered pairs known as edges or links. The network is undirected and so a single link $ij \in \mathbf{g}$ implies that agent i is connected to agent j and vice versa. Where convenient I will simply refer to the network as $\mathbf{g}$ and may use $\mathbf{g} + ij$ to denote a network $\mathbf{g}'$ which is formed from $\mathbf{g}$ plus the addition of an undirected link between nodes i and j. The neighbourhood of an agent i in $\mathbf{g}$ will be denoted by $N_i(\mathbf{g}) = \{j \mid ij \in \mathbf{g}\}$ and each agent's degree will be denoted by $d_i(\mathbf{g}) = |N_i(\mathbf{g})|$.

Agents will interact with their neighbours by selecting a level of effort x_i from

$[0, \bar{x}]$, where $\bar{x} > \max_{i \in N} d_i(\mathbf{g})$.[5] This effort incurs a private cost which is assumed to be quadratic in x_i. In order to highlight the role of network structure in determining equilibrium effort, I assume that utility functions take the following simple form:

$$u_i(x_i, \mathbf{x}_{-i}) = \sum_{j \in N_i} \min\{x_i, x_j\} - \frac{1}{2}x_i^2$$

The simplicity of this utility function is mainly for ease of presentation and Appendix B contains an extension of the model to the case of weighted networks where agents also have heterogeneous private benefits and costs. Although the utility function is strictly concave and continuous in x_i, it is not continuously differentiable. In order to characterise best responses we use the left and right derivatives of u_i.

First consider the right derivative of u_i and let $\overline{d_i}(x_i, \mathbf{x}_{-i}) := \left|\left\{j \in N_i \mid x_j > x_i\right\}\right|$. The right derivative of u_i is given by

$$\frac{\partial_+ u_i}{\partial x_i} = \overline{d_i}(x_i, \mathbf{x}_{-i}) - x_i.$$

The gross marginal benefit from raising x_i increases linearly with the number of agents exerting strictly higher effort, yet the marginal cost of effort is x_i. This means that in any equilibrium all efforts satisfy

$$x_i \geq \overline{d_i}(x_i, \mathbf{x}_{-i}). \tag{1}$$

If $x_i < \overline{d_i}(\mathbf{x})$ then an agent could marginally raise their effort and increase their utility along $\overline{d_i}(\mathbf{x})$ links at a marginal cost of only x_i.

Consider now the left derivative of u_i and let $\underline{d_i}(\mathbf{x}) := \left|\left\{j \in N_i \mid x_j \geq x_i\right\}\right|$. The left derivative of u_i is

$$\frac{\partial_- u_i}{\partial x_i} = \underline{d_i}(x_i, \mathbf{x}_{-i}) - x_i.$$

[5] The upper bound to the effort level plays only a minor technical role and can be assumed to be arbitrarily large.

Therefore in any equilibrium all efforts satisfy

$$x_i \leq \underline{d_i}\left(x_i, \mathbf{x}_{-i}\right). \tag{2}$$

If $x_i > \underline{d_i}\left(\mathbf{x}\right)$ then decreasing x_i would lower utility linearly along $\underline{d_i}\left(\mathbf{x}\right)$ links but lower cost at rate x_i.

Conditions (1) and (2) are necessary and jointly sufficient for x_i to be a best response to $\mathbf{x}_{-i}$.[6] Since X_i is bounded and $u_i(x_i, x_{-i})$ is strictly concave and continuous in x_i, a unique best response satisfying (1) and (2) always exists:

Lemma 1. *For any agent $i \in N$, the unique best response to profile $\mathbf{x}_{-i}$ is given by*

$$BR_i\left(\mathbf{x}_{-i}\right) = \max\{x_i \mid x_i \leq \underline{d_i}\left(x_i, \mathbf{x}_{-i}\right)\} = \min\{x_i \mid x_i \geq \overline{d_i}\left(x_i, \mathbf{x}_{-i}\right)\}$$

Proof. All proofs are contained in the Appendix. □

The absolute number of neighbours playing a given action will therefore determine our optimal response. The pairwise minimum effort game is a 'threshold game' played on a network as in Morris (2000) and Young (1998).[7]

Moreover, for a given agent to sustain a high x_i in equilibrium we will require that they have a large number of neighbours with high x_j. This implies not only that d_i must be large, but also that d_j must be large for $j \in N_i$, and similarly for $j' \in N_j$, and so on. As we shall later see, dense and cohesive subgroups of the network will find it easier to sustain higher levels of effort in equilibrium.

3.1 Cohesive Subgroups

Notions of group cohesiveness in social networks have long been studied in the sociology literature and many such concepts are defined in standard texts such as Wasserman and Faust (1994). A variant which has been used in the economics

[6] This clearly holds if the best response is interior. If $BR_i\left(\mathbf{x}_{-i}\right) = 0$ then $x_j = 0$ for all $j \in N_i$ and (1) and (2) are both satisfied. Since $\bar{x} > \max_{i \in N} d_i\left(\mathbf{g}\right)$ due to the assumption on $\bar{x}$, $BR_i\left(\mathbf{x}_{-i}\right) = \bar{x}$ can never satisfy (2).

[7] The main technical differences between this paper and Morris (2000) or Young (1998) are firstly that effort is a continuous rather than binary variable, and secondly, that optimal efforts depend on the number of neighbours playing a weakly higher effort (rather than a proportion) since the effort cost is split across all neighbours.

literature is the notion of a p-cohesive subset, defined in Morris (2000). Formally, a subset of nodes S is said to be p-cohesive if every node in S has (at least) a proportion p of their neighbours in S. A related idea is also found in Young (1998) where a subset of nodes is called r-close-knit if for every $S' \subseteq S$ the proportion of links originating in S' and ending in S is at least r. Therefore, p-cohesiveness is a condition on the degrees of nodes, whereas r-close-knittedness is a condition on links within a subgroup and therefore a p-cohesive subgroup is $p/2$-close-knit.

This paper will employ a particularly useful concept originally defined by Seidman (1983) known as a *k-core*. Seidman (1983) considers subgraphs $\mathbf{g}_k \subseteq \mathbf{g}$ which can be induced by repeatedly pruning nodes of low degrees from the network in order uncover groups of densely connected individuals. The graph which is obtained by iteratively removing all nodes of degree less than k is known as a *core* of order k, or a *k-core*.[8] For any subgraph $\mathbf{g}_k \subseteq \mathbf{g}$ I will use N_k to denote the set of agents who have positive degree in that subgraph. A precise definition of a k-core of a graph $\mathbf{g}$ now follows:

Definition 1. A *k-core* of a graph $\mathbf{g}$ is a subgraph $\mathbf{g}_k \subseteq \mathbf{g}$ such that $d_i(\mathbf{g}_k) \geq k$ for each $i \in N_k$.

A k-core is therefore a subgraph of $\mathbf{g}$ where every agent with positive degree in that subgraph has at least degree k.[9] I will say that a group of nodes 'form' a k-core when the subgraph consisting of these nodes and links between them is a k-core. If an agent i is contained within a k-core then this implies that they have at least k neighbours of degree k or greater.[10]

Every connected graph trivially contains a 1-core, whilst the 2-core which can be formed using the least possible number of edges is the ring network. Also note that the definition implies that nodes belonging to k-cores of high orders are also members of some core of a lower order. If we focus on the union of cores of each

[8] Seidman (1983) in fact refers to the k-core as maximal subgraph which can be obtained by iteratively removing nodes of lower degree. I follow Wasserman and Faust (1994) and the more recent literature by referring to any core of order k as a k-core.

[9] This concept has been used in a recent working paper by (Gagnon and Goyal, 2015), although they focus on a single maximal core of a given order q and refer to this as *the q-core*.

[10] Applications of the concept outside of economics include the analysis of protein networks (Bader and Hogue, 2003) and the visualisation of large complex networks (Baur et al., 2004).

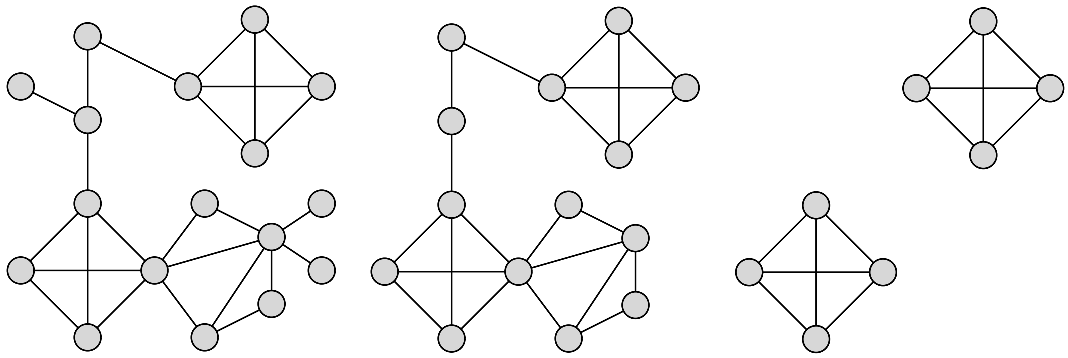

Figure 1: The k-cores of a network

order this permits a nested *k-core decomposition* of any given network. Figure 1 shows such a k-core decomposition of a graph for $1 \leq k \leq 3$.

Since cores of successive orders are nested within the previous core we can define a *coreness* value for each $i \in N$:

Definition 2. A node $i \in N$ has *coreness* $c_i(\mathbf{g}) = k$ if it is contained in a core of order k but not in a core of order k' for $k' > k$.

The coreness of a node can be interpreted as a coarse measure of its centrality. We can view coreness as a condition on a node's degree and the degree of other nodes in their neighbourhood. Nodes with high coreness may have important roles in the network since they have neighbours with high degrees (who in turn have neighbours with high degrees, etc). High coreness can often indicate that a given node is a member of a dense and cohesive subset of the network, since cliques of size n immediately form an $(n-1)$-core. The vector of coreness for all agents $i \in N$ will be denoted by $\mathbf{c}(\mathbf{g}) = (c_1(\mathbf{g}), \ldots, c_n(\mathbf{g}))$.[11]

As can be seen in the example in Figure 2, the coreness of individual nodes can depend on structural characteristics of the network which are relatively 'far away'. In this example, the addition of a single link between the remaining pair of nodes with degree 2 would raise the coreness of all nodes to 3. Adding links to a network cannot decrease coreness and it follows that $c_i(\mathbf{g}') \geq c_i(\mathbf{g})$ for any

[11]In what follows I refer to each agent's coreness in the network $\mathbf{g}$ as simply c_i. I will indicate by using $c_i(\mathbf{g}')$ when considering subgraphs of $\mathbf{g}$.

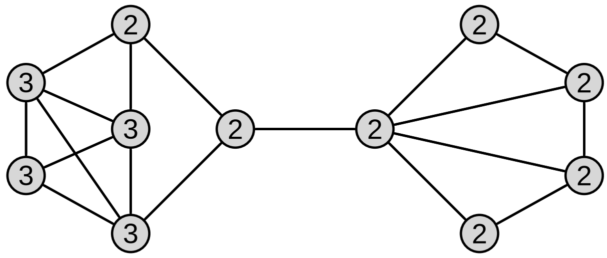

Figure 2: Coreness profile of a bridge network

$\mathbf{g}' \supseteq \mathbf{g}$. Moreover, the following lemma shows that the coreness of any given node can only decrease by 1 following link removal.

Lemma 2. *For all $l \in N$ and $\forall ij \in \mathbf{g}$, $c_l(\mathbf{g} - ij) \geq c_l(\mathbf{g}) - 1$*

Although the removal of one link cannot lower the coreness of any agent by more than 1, it can have a cascading effect which influences all nodes (e.g the transition from a ring to a line network). I will return to this stability issue in Section 4 but first I discuss the properties of equilibrium actions.

3.2 The Pareto Dominant Equilibrium

With the notion of coreness defined, I can now solve the model for action profiles $\mathbf{x} \in X = [0, \bar{x}]^n$ which constitute a Nash equilibrium of the game $\Gamma = \langle N, X, u \rangle$. The game will have multiple equilibria, so in this section I focus on the Pareto dominant Nash equilibrium. The game Γ is a supermodular game[12] so the results of Milgrom and Roberts (1990) imply that a greatest and least equilibrium must exist. Moreover, Milgrom and Roberts (1990) also show that if $\mathbf{x}$ and $\mathbf{x}'$ are equilibria of Γ where $\mathbf{x} \geqslant \mathbf{x}'$ then $\mathbf{x}$ Pareto dominates $\mathbf{x}'$. Since the set of equilibria of a supermodular game forms a complete lattice (Zhou, 1994), this implies that there exists a greatest equilibrium which Pareto dominates all others. With the existence of a Pareto dominant equilibrium established, we can now characterise this equilibrium in terms of the coreness of agents.

[12]Firstly, the strategy set $X = [0, \bar{x}]^n$ is a complete lattice under the usual partial order $\mathbf{x} \geqslant \mathbf{x}'$ if $x_k \geq x'_k$ for all $k = 1, \ldots, n$. By the definition of Milgrom and Roberts (1990), the game is supermodular since u_i has increasing differences in (x_i, x_{-i}), u_i is supermodular in x_i for fixed x_{-i}, and u_i is upper semi-continuous in X_i and order continuous in X_{-i} with a finite upper bound.

Theorem 1. *The Pareto dominant Nash equilibrium is* $\mathbf{x}^* = \mathbf{c}(\mathbf{g})$

Intuitively, a coreness of c_i for agent i guarantees that they have at least c_i neighbours, who have at least c_i neighbours, etc, who could feasibly play c_i in equilibrium. Furthermore, having coreness c_i implies that $x_i > c_i$ can never be played in equilibrium as there is an insufficient number of supporting nodes along paths from i. Supermodularity allows us to infer that the equilibrium where $\mathbf{x}^* = \mathbf{c}$ is Pareto dominant since it is maximal in terms of investment. We can also use the supermodularity property to verify that beginning with action profile $\bar{\mathbf{x}} = (\bar{x}, \ldots, \bar{x})$ and iterating the best responses of agents we arrive at $x_i = c_i$ for all i, implying that $\mathbf{x}^* = \mathbf{c}(g)$ is the maximal equilibrium (see Milgrom and Roberts (1990)).

3.3 Other Nash Equilibria

The extreme complementarity of actions in this model can result in multiple Nash equilibria. For example, the profile $(0, 0, \ldots, 0)$ is always an equilibrium in any network. In fact, the game will have infinitely many Nash equilibria for any non-empty network.[13]

This result is a consequence of the perfect complementarity assumption. The fact that each agent's effort cannot be used as an imperfect substitute for another agent's lack of effort leads to inertia for a large number of action profiles. Figure 3 illustrates three equilibria for the network presented earlier in Figure 1. The first panel shows the Pareto best equilibrium which corresponds to the coreness profile, whereas the final panel shows a Pareto inferior equilibrium. We can observe that there exist equilibria where those with lower coreness play strictly higher actions in equilibrium.

This multiplicity leads naturally on to the question of the relative stability of each equilibrium. For example, the equilibrium profile $(0, 0, \ldots, 0)$ requires only one node to deviate upwards to instigate a cascade of increases, whereas this is

[13] To show this, pick any equilibrium $\mathbf{x} \geqslant 0$ and consider the subset of agents playing the highest effort in that equilibrium. Reducing x_i by a sufficiently small ε for each agent in this subset must also be a Nash equilibrium as conditions (1) and (2) must still hold. Similarly, if $\mathbf{x} = 0$ then increasing the effort of all agents by a sufficiently small ε is also an equilibrium.

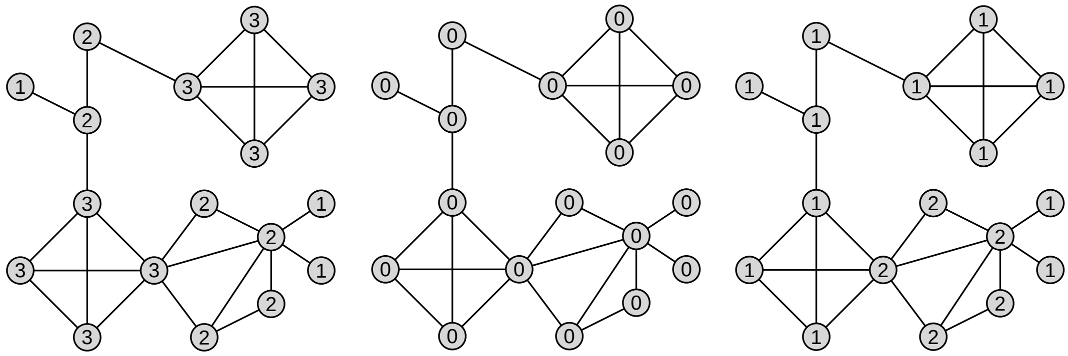

Figure 3: Some equilibria for the network in Figure 1

not the case for equilibrium profile $(1, 1, \ldots, 1)$. Moreover, the Pareto dominant equilibrium in Figure 3 is also unstable with respect to a downward shock to any of the nodes with $c_i = 3$.

This discussion highlights the fact that some equilibria may be more stable than others in the face of random shocks. This motivates a closer look at the stability of the Pareto dominant equilibrium with respect to random shocks to effort.

4 Stable Equilibria

The focus so far on the Pareto dominant equilibrium $\mathbf{x}^*$ can be justified when considering environments which allow some degree of pre-play communication or third party mediation. Since actions are complements, it is in the best interests of all agents to coordinate on the Pareto dominant equilibrium. However, without communication or mediation it may seem unlikely that individuals could tacitly coordinate on the Pareto dominant equilibrium, especially if n is large.

Furthermore, if tacit coordination is somehow achieved, then the question of the stability of $\mathbf{x}^*$ with respect to random shocks to effort is raised. As discussed previously, the failure of a single node can create a cascade of falling actions for all nodes in the network.

Consider the example in Figure 4. The coreness of all nodes in the network is $c_i = 2$ but even a temporary drop in the effort of any node would lead, via a sequence of best responses, to a new equilibrium where $x_i = 1$ for all i. A Pareto

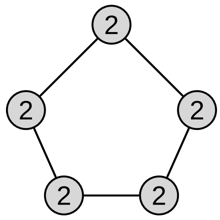

Figure 4: An unstable equilibrium

dominant equilibrium such as the one displayed in Figure 4 could not reasonably be considered stable in an environment where efforts may be subject to infrequent random shocks.

To study the stability properties of equilibria in this game I now examine two environments. First I focus on the properties of networks where the Pareto dominant equilibrium is stable with respect to small and isolated shocks to individual efforts. Following this, I examine the stability of equilibria in the presence of persistent random shocks. I order to do this, I will apply a common refinement technique for minimum effort games which uses the potential function to select equilibria which are most likely to be observed in the long run.

4.1 Stability of the Pareto Dominant Equilibrium

I now present a notion of equilibrium stability based on a small one period shock to the effort level of a single agent. Since this notion of stability is extremely weak, equilibria which do not satisfy this criterion should be considered to be very fragile.

A shock to equilibrium profile $\mathbf{x}$ is a profile $\hat{\mathbf{x}} \in X$ such that $\hat{x}_i = x_i + \varepsilon$ for exactly one agent i and $\hat{x}_j = x_j$ for all other agents j. Since I am focusing on small isolated shocks to actions I will assume that $|\varepsilon| \leq 1$. Following this shock, play will evolve over discrete time periods $t = \{0, 1, 2, \ldots\}$. Define the myopic best response dynamic as a sequence $\{\mathbf{x}^t\}$ in X such that $x_i^{t+1} = \text{argmax}_{x_i \in X_i} u_i\left(x_i, \mathbf{x}_{-i}^t\right)$ for each $i \in N$.

An equilibrium profile $\mathbf{x}^*$ will be considered *knife edge* if, for an arbitrarily small shock, efforts do not return to $\mathbf{x}$ via the sequence of myopic best responses. This notion of stability is similar to that used in Bramoullé and Kranton (2007), although slightly weaker as I restrict the shock to one individual node.

Focusing on the Pareto dominant equilibrium $\mathbf{x}^*$, we can ignore positive shocks as supermodularity will guarantee that best responses converge back to the largest equilibrium $\mathbf{x}^*$. However, an arbitrarily small negative shock has the potential to instigate a cascade of falling actions amongst neighbours (see Figure 4).

Recall that $\mathbf{g}_k$ denotes a subgraph of $\mathbf{g}$ such that $d_i(\mathbf{g}_k) \geq k$ for all $i \in N_k$. For a given $j \in N$ with coreness $c_j(\mathbf{g}) = k$, let $\mathcal{G}_j$ be the intersection of all subgraphs $\mathbf{g}_k \subseteq \mathbf{g}$ such that $c_j(\mathbf{g}_k) = k$. A useful definition and lemma now follow:

Definition 3. For any $i \in N$ and $j \in N_i$ where $c_i = c_j$ and $ij \in \mathcal{G}_j$ we say that i is *critical* for j, written $i \to j$.

Lemma 3. *If $i \to j$ then $c_j(\mathbf{g} - ij) = c_j(\mathbf{g}) - 1$*

It is then possible to construct a directed graph $\hat{\mathbf{g}}$ where $ij \in \hat{\mathbf{g}}$ if and only if $i \to j$. The directed graph $\hat{\mathbf{g}}$ identifies the possible transmission paths of small shocks which may propagate through the network. As this graph is directed, it becomes necessary to define in-degrees and out-degrees for each node. I therefore use $d_i^-(\hat{\mathbf{g}})$ to denote the in-degree of a node in $\hat{\mathbf{g}}$, and $d_i^+(\hat{\mathbf{g}})$ to denote their out-degree, with respective neighbourhoods $N_i^-(\hat{\mathbf{g}})$ and $N_i^+(\hat{\mathbf{g}})$.

We can also define a directed graph of *supporting* links denoted by $\check{\mathbf{g}}$, where $ij \in \check{\mathbf{g}}$ if and only $j \in N_i$ and $c_j \geq c_i$ but $ij \notin \mathcal{G}_j$. Graph $\check{\mathbf{g}}$ identifies neighbours of i who help to support x_i^* as an equilibrium action but are not affected by a shock $x_i - \varepsilon$. If agent i has a high out-degree in $\check{\mathbf{g}}$ and a low out-degree in $\hat{\mathbf{g}}$ then the effect of shock $x_i - \varepsilon$ on the local neighbourhood is minimal. I now formalise this intuition in the following proposition:

Proposition 1. *The equilibrium $\mathbf{x}^* = \mathbf{c}(\mathbf{g})$ is knife edge if and only if there exists $i \in N$ such that $d_i^+(\check{\mathbf{g}}) < c_i(\mathbf{g})$.*

Proposition 1 states that if there is a core whose members mutually support each other following a shock then this core is stable. It is a high out degree in $\hat{\mathbf{g}}$ which prevents a shock from converging back to $\mathbf{x}^*$, as i's lower effort simultaneously causes the efforts of many neighbours to fall, preventing i from reverting back to x_i^* in later periods.

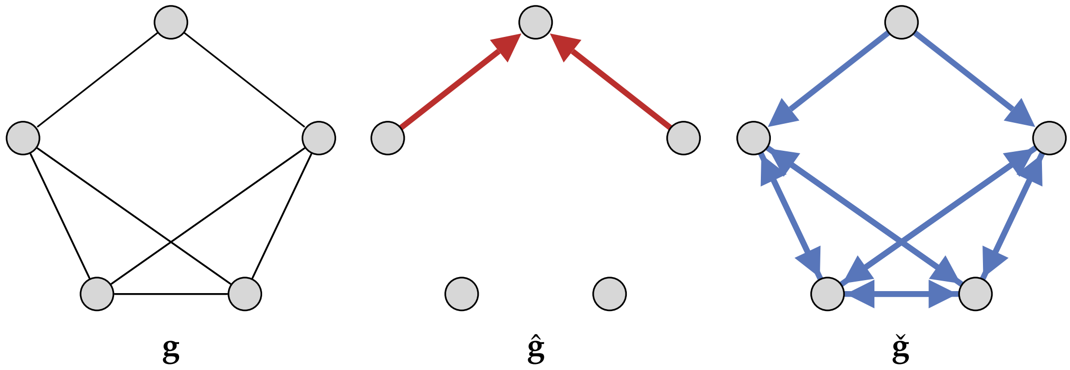

Figure 5: A network with a stable equilibrium

An alternative interpretation of Proposition 1 is that if we wish to improve the stability of equilibrium $\mathbf{x}^*$ by adding links then we should target agents where $d_i^+(\check{\mathbf{g}}) < c_i(\mathbf{g})$.[14] Links should be added between these nodes and nodes playing the highest effort in the network in order to minimise the effect of shocks.

Another implication is that subsidising or protecting critical nodes will minimise the effect of cascading failures when shocks are small. If agents potentially face larger shocks where $|\varepsilon| > 1$ then we must broaden the definition of node criticality so that a node i is considered critical for j if $c_i \geq c_j$ and $ij \in \mathcal{G}_j$. Unlike our original definition, there is always at least one such node in any network, since $d_j(\mathbf{g}) = c_j(\mathbf{g})$ for some $j \in N$, all $i \in N_j$ are critical for j. Since only critical nodes can instigate cascades, protecting them from shocks will ensure that the equilibrium cannot be influenced by small and isolated shocks to individual agents.

4.2 The Potential Maximising Equilibrium

The previous subsection assumed that shocks are rare and idiosyncratic, so the issue of persistent randomness in actions has yet to be addressed. I now discuss an equilibrium refinement which has been successful in the experimental economics literature for similar 'minimum effort' games. This refinement also takes into account some notion of persistent shocks to actions.

[14]Note that this may change the coreness profile of the network and hence the Pareto dominant equilibrium.

Experimental studies in Van Huyck et al. (1990), Goeree and Holt (2005), Chen and Chen (2011) and others[15] have identified the remarkable effectiveness of the potential function as a tool for equilibrium selection in 'minimum effort' coordination games. Despite the infinite number of equilibria, these studies demonstrate that the Nash equilibria which maximise the potential function of the game tend to be observed experimentally for a variety of parameters. Crawford (1991) puts forward an evolutionary explanation for these results, which is further strengthened by Anderson et al. (2001), who show that the distribution of strategies in the logit equilibrium maximises their stochastic potential function.

For games with discrete action sets Blume (1993) has shown that action profiles which are global maximisers of the potential function are stochastically stable equilibria under the logit best response dynamic. As suggested by Goeree and Holt (2005), one can view the potential maximising equilibria as being a close analogue of the stochastically stable equilibria in the case of continuous action sets. Beyond evolutionary arguments for the use of the potential function as a selection device, Carbonell-Nicolau and McLean (2014) have shown that unique potential maximisers are also trembling-hand perfect and strategically stable equilibria. I therefore single out the potential maximising equilibrium as being particularly robust and stable.

Monderer and Shapley (1996) define an exact potential function of a game as a function $\rho : X \to \mathbb{R}$ such that $\forall x_i, x_i' \in X_i$, $\forall x_{-i} \in X_{-i}$ and $\forall i \in N$

$$\rho\left(x_i, x_{-i}\right) - \rho\left(x_i', x_{-i}\right) = u_i\left(x_i, x_{-i}\right) - u_i\left(x_i', x_{-i}\right).$$

An exact potential function for this game is given by

$$\rho\left(\mathbf{x}\right) = \sum_{ij \in \mathbf{g}} \min\left\{x_i, x_j\right\} - \frac{1}{2}\sum_{i \in N} x_i^2 \tag{3}$$

By construction, any action profile which maximises ρ in each coordinate direction is also a Nash equilibrium and so the set of profiles which globally maximise ρ are a non-empty subset of these equilibria. The potential function in (3) inherits the properties of the utility functions u_i (e.g. it has increasing differences

[15] See Appendix F of Chen and Chen (2011) for details.

in x_i). The following lemma shows that it is also supermodular and strictly concave on X.

Lemma 4. *The potential function ρ is:*

(a) *Supermodular on X*

(b) *Strictly concave on X*

This lemma establishes that a maximiser of ρ exists and is unique. I now use the potential function as a tool for equilibrium refinement, as first suggested by Monderer and Shapley (1996) in their original article. I will later provide some further motivation for this choice by showing that the maximiser of ρ on X is an arbitrarily close approximation to the maximiser of ρ on a finely discretised version of X.

In order to find the potential maximising equilibrium I will take advantage of the hierarchical nature of equilibrium profiles. The optimal actions of those playing the highest x_i in any equilibrium cannot be influenced by the actions of those playing strictly lower efforts. This fact can be exploited to find the Nash equilibrium which maximises ρ by first partitioning the set of agents according to their actions in the potential maximising equilibrium $\tilde{\mathbf{x}}$. To do so I first must introduce some new notation and the concept of a density decomposition.

4.2.1 The Density Decomposition

Although there are various definitions of network cohesion, an overriding theme is that cohesive subgraphs have a large number of links between nodes in that subgraph relative to the rest of the network. As an absolute measure of a subgraph's cohesiveness, subgraph *density* is amongst one of the simplest. Given a subgraph $\bar{\mathbf{g}} \subseteq \mathbf{g}$ we define its *density* as $\frac{|\bar{\mathbf{g}}|}{|\bar{N}|}$ where $\bar{N}$ is the set of agents with positive degree in that subgraph. A subgraph's density is the ratio of the number of internal edges to the number of nodes and therefore is half the average internal degree.

In a similar manner to a *k-core* of $\mathbf{g}$, we can also let $\bar{\mathbf{g}}_k \subseteq \mathbf{g}$ denote the largest subgraph of $\mathbf{g}$ such that $\frac{|\bar{\mathbf{g}}_k|}{|\bar{N}_k|} \geq k$ for some $k \geq 0$.[16] Whereas a *k-core* orders sub-

[16]To require that $\bar{\mathbf{g}}_k$ must the 'largest' such subgraph means that $\bar{\mathbf{g}}_k$ is the union over all sub-

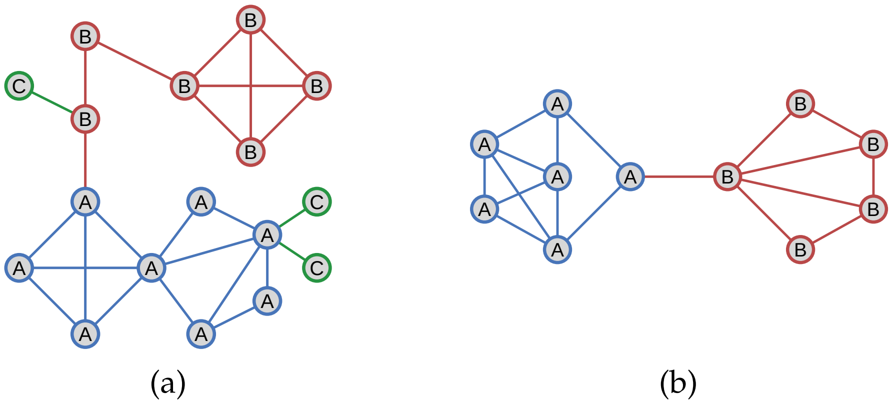

Figure 6: The density decompositions of two networks

graphs based on a restriction to their minimum degree, we now place a restriction on the average degree of nodes in that subgraph. Like the *k-core decomposition* from Section 3.1 we can also construct a *density decomposition* of $\mathbf{g}$, since all such $\bar{\mathbf{g}}_k$ are nested within maximal subgraphs of lower densities.[17]

To construct a corresponding node-level statistic we can let δ_i be the largest k such that $i \in \bar{N}_k$ for some $\bar{\mathbf{g}}_k \subseteq \mathbf{g}$. Now define a *density decomposition* of $\mathbf{g}$ as follows:

Definition 4. A *density decomposition* of a graph $\mathbf{g}$ is a partition $\mathcal{D} = \{D_1, \ldots, D_K\}$ of N such that $i \in D_k$ and $j \in D_k$ if and only if $\delta_i = \delta_j$.

Like the k-core decomposition, the *density decomposition* allows us to construct a nested hierarchy of the nodes based on their density values δ_i. However, unlike the k-core decomposition, it does not seem that there exists a corresponding 'pruning' algorithm which allows us to uncover this nested structure in linear time.[18]

graphs $\bar{\mathbf{g}}'_k \subseteq \mathbf{g}$ such that $\frac{|\bar{\mathbf{g}}'_k|}{|\bar{N}'_k|} \geq k$. Such a union always preserves the property $\frac{|\bar{\mathbf{g}}_k|}{|\bar{N}_k|} \geq k$.

[17] To see why $k' > k$ must imply that $\bar{\mathbf{g}}_{k'} \subseteq \bar{\mathbf{g}}_k$ we can assume to the contrary and observe that $\mathbf{h} = \bar{\mathbf{g}}_k \cup \bar{\mathbf{g}}_{k'}$ would be a subgraph with density which exceeds k, yet $\bar{\mathbf{g}}_k \subset \mathbf{h}$ which contradicts the assumption that $\bar{\mathbf{g}}_k$ is largest.

[18] In fact, finding the density decomposition of the network is closely related to a problem known in the computer science literature as the 'densest subgraph' problem. This was studied first by Goldberg (1984), who shows that the problem of finding the densest subgraph which can be induced using only k nodes can be solved by using a version of the celebrated max flow-min cut theorem.

I now use this notion to characterise the hierarchical nature of equilibrium actions at the potential maximising equilibrium $\tilde{\mathbf{x}}$.

4.2.2 The Potential Maximising Partition

Given a profile $\tilde{\mathbf{x}}$ let $\{\tilde{S}_1, \ldots, \tilde{S}_M\}$ be a partition of N according to equilibrium action such that $i, j \in \tilde{S}_m$ if and only if $\tilde{x}_i = \tilde{x}_j$. Index these subsets from 1 to M in increasing order of their equilibrium action. Let $\tilde{\mathbf{g}}_m$ denote the maximal subgraph of $\mathbf{g}$ such that $ij \in \tilde{\mathbf{g}}_m$ if and only if $i \in \tilde{S}_m$ and $j \in \tilde{S}_{m'}$ for $m' \geq m$.

Focusing on the subset of agents who are in $\tilde{S}_M$, their optimal decision cannot depend on the actions of agents in $N \setminus \tilde{S}_M$. The equilibrium actions of $\tilde{S}_M$ are only pivotal along links to other members of $\tilde{S}_M$, that is, along links $ij \in \tilde{\mathbf{g}}_M$. Therefore, we may optimise for members of $\tilde{S}_M$ whilst ignoring the actions of other subsets. Setting $\tilde{x}_i = \tilde{x}_j = \tilde{x}_M$ for all $i, j \in \tilde{S}_M$ the maximisation problem for agents in $\tilde{S}_M$ is

$$\max_{x_M} |\tilde{\mathbf{g}}_M| \, x_M - |\tilde{S}_M| \, \frac{1}{2} x_M^2$$

The solution to this maximisation problem is $\tilde{x}_M = \frac{|\tilde{\mathbf{g}}_M|}{|\tilde{S}_M|}$, which is the *density* of the subgraph $\tilde{\mathbf{g}}_M$. Since an agent in $\tilde{S}_{M-1}$ is pivotal along all links to others $\tilde{S}_{M-1}$ and those in $\tilde{S}_M$, the subgraph $\tilde{\mathbf{g}}_{M-1}$ includes links between agents in $\tilde{S}_{M-1}$ and from $\tilde{S}_{M-1}$ to $\tilde{S}_M$. In general for $\tilde{S}_m$ we have

$$\tilde{x}_m = \frac{|\tilde{\mathbf{g}}_m|}{|\tilde{S}_m|} \tag{4}$$

Condition (4) is a condition which any maximiser $\tilde{\mathbf{x}}$ must satisfy. By specifying a partition $\{\tilde{S}_1, \ldots, \tilde{S}_M\}$ of N we therefore also specify an equilibrium action profile via condition (4) and the implied subgraphs $\tilde{\mathbf{g}}_1, \ldots, \tilde{\mathbf{g}}_M$. Using the concept of a density decomposition I now characterise the Nash equilibrium profile which maximises the potential function.

Theorem 2. *The potential maximising partition of N is $\{\tilde{S}_1, \ldots, \tilde{S}_M\} = \mathcal{D}$ where the equilibrium action of each $i \in \tilde{S}_m$ is given by (4).*

Whilst the Pareto dominant equilibrium partitioned nodes into nested subgraphs based on minimum degree, the potential maximising equilibrium partitions nodes into nested subgraphs based on average degree.

In the potential maximising equilibrium, agents in the densest subgraph of $\mathbf{g}$ will play the highest action, followed by those remaining in the second densest subgraph, followed by those remaining in the third, and so on. Since costs are incurred at nodes but benefits are received along edges, agents in subgraphs which have a large number of edges spanned by a small number of nodes (i.e. high density) play higher equilibrium actions.

Two examples of a potential maximising partitions are shown in Figure 6. The densest subgraph of the network displayed in Figure 6 panel (a) is the subgraph formed by nodes in A. This subgraph has an average internal degree of 3.25, hence their equilibrium effort is $\tilde{x}_A = \frac{13}{8} = 1.625$. The subgraph formed by nodes in $A \cup B$ is the second most dense in $\mathbf{g}$ and so agents in B play $\tilde{x}_B = \frac{9}{6} = 1.5$. Finally, the network as a whole is the third most dense and so agents in C play $\tilde{x}_C = 1$. It is worth noting that agents with higher coreness do not necessarily play higher actions in the potential maximising equilibrium. In Figure 6 (a) there is a subset of agents in B with coreness 3 who play a lower action than the of agents in A with coreness 2 (see also Figure 3).

Returning to my justification for using the potential function as an equilibrium selection tool I conclude this section by showing that the maximiser of ρ on X is an arbitrarily close approximation to the maximiser of ρ on a discretised version of X. Let $X(z)$ denote a *discretisation* of X with parameter $z := \frac{1}{q}$ for some $q \in \mathbb{N}_+$. Let $X(z) = \prod_i X_i(z)$ where the set $X_i(z)$ is such that $x_i \in X_i(z)$ if and only if for some $p \in \mathbb{N}_0$ where $p \leq q$ we have that $x_i = \frac{p}{q}\bar{x}$. Taking $z \to 0$ allows us to approximate the continuous action game Γ by the discrete action game $\Gamma_z = \langle N, X(z), u \rangle$. The following lemma demonstrates that the potential maximiser of the continuous action game approximates that of the discrete game with arbitrary precision by taking z small enough.

Lemma 5. *For any discretised minimum effort game Γ_z we have that $\|\tilde{\mathbf{x}}(z) - \tilde{\mathbf{x}}\|_2 \to 0$ as $z \to 0$.*

Appealing to previously mentioned results on the stochastic stability of potential maximisers, I therefore view $\tilde{\mathbf{x}}$ as a close approximation to the stochastically stable outcome of a finely discretised minimum effort game on a network.[19]

5 Social Efficiency

I now consider the minimum effort game from a social planner's perspective. Focusing first on the case where a network designer can costlessly add edges between nodes, it is worth noting that each utility function $u_i\left(x_i, \mathbf{x}_{-i}\right)$ displays increasing differences in d_i.[20] Similarly, the potential function $\rho\left(\mathbf{x}, \mathbf{d}\right)$ also exhibits increasing differences in the vector of degrees $\mathbf{d}$.

An application of Theorem 2.8.1 from Topkis (1998) shows that if $d_i\left(\mathbf{g}\right) \geq d_i\left(\mathbf{g}'\right)$ for each i then $\mathbf{x}^*\left(\mathbf{g}\right) \geqslant \mathbf{x}^*\left(\mathbf{g}'\right)$ and $\tilde{\mathbf{x}}\left(\mathbf{g}\right) \geqslant \tilde{\mathbf{x}}\left(\mathbf{g}'\right)$. Although it is obvious that the coreness of nodes cannot decrease by adding more links to a network, the impact on the density decomposition is perhaps less clear. The application of Topkis' theorem therefore means that the complete network always permits the highest action profile in either the Pareto dominant or potential maximising equilibrium.

The next question one may ask is whether the Pareto dominant equilibrium is also socially efficient. Defining a utilitarian social welfare function $U\left(\mathbf{x}\right) = \sum_{i \in N} u_i\left(\mathbf{x}\right)$ we can examine the difference in welfare between the social optimum and Pareto dominant Nash equilibrium $\mathbf{x}^*$. As shown in Proposition 2 below, the Pareto dominant Nash equilibrium only maximises social welfare in one special case.

Proposition 2. *The Pareto dominant Nash equilibrium maximises social welfare if and only if the graph is regular.*

When the network is regular (i.e. $d_i\left(\mathbf{g}\right) = d_j\left(\mathbf{g}\right)$ for all i and j) then a social planner can implement the socially optimal level of effort in equilibrium without transfers. However, the equilibrium $\mathbf{x}^*$ in regular networks is unstable in the sense of Proposition 1. A trade-off therefore exists between the social efficiency of $\mathbf{x}^*$ and the stability of the equilibrium.

[19]The proof of Lemma 4 uses some special properties of the minimum effort game and therefore such approximations are not always valid.

[20]In other words, fixing the profile of others' actions at $\mathbf{x}_{-i} \in X_{-i}$, $u_i\left(x_i, \mathbf{x}_{-i}, d_i\right) - u_i\left(x_i', \mathbf{x}_{-i}, d_i\right) \geq u_i\left(x_i, \mathbf{x}_{-i}, d_i'\right) - u_i\left(x_i', \mathbf{x}_{-i}, d_i'\right)$ for $x_i > x_i'$ and $d_i > d_i'$

To compute the socially optimal effort profile for any network, it is possible to directly apply the ideas from Section 4.2, since $U(\mathbf{x})$ has a near identical structure to $\rho(\mathbf{x})$. It is straightforward to verify that the optimal solution will again partition agents according to the density decomposition $\mathcal{D}$ and that efforts will be such that $x_m = 2\tilde{x}_m$ for each $m \in \{1, \ldots, M\}$. Therefore, it is again dense subgraphs which permit the highest possible action by nodes in the network.

A social planner may be interested in bounds on the divergence between the Pareto dominant Nash equilibrium and socially optimal outcomes across different types of networks. This issue is examined using a concept known as the *price of stability*, which is defined as the ratio of the total utility surplus in the best Nash equilibrium to the total surplus at the social planner's optimum.[21]

As demonstrated in Proposition 2, the price of stability $PoS := U(x^*) / \max_{\mathbf{x} \in X} U(\mathbf{x})$ is only equal to 1 in the case of regular networks. However, it is possible to show PoS is also bounded below by $\frac{3}{4}$.

Proposition 3. *The price of stability lies in the interval* $(\frac{3}{4}, 1]$.

The value of the welfare function in the potential maximising equilibrium always places a lower bound on what can be achieved in $\mathbf{x}^*$. As shown in the proof of Proposition 3, this lower bound of $\frac{3}{4}$ is approached in the limit for a star network with a very large number of spokes. In contrast to regular networks, the equilibrium in the star network is both least efficient from the social planner's perspective and also knife edge, as the hub of the star is critical for all other agents. The line network is another example where the lower bound of $\frac{3}{4}$ is approached in the limit, yet the equilibrium in the line network can never be knife edge.

We can conclude therefore that while Pareto dominant equilibria in regular networks are socially efficient, they lie relatively 'far away' from the potential maximising equilibria and are therefore unlikely to be stable in the long run. On the other hand, while the Pareto dominant equilibria in the star and line networks are socially inefficient, they coincide with the potential maximising equilibria and are therefore likely to be more robust.

[21] See Nisan et al. (2007).

With reference to the results on equilibrium stability in Section 4, although adding links may increase the stability of $\mathbf{x}^*$ in the face of small random shocks, they also increase the *PoS* and so reduce the benefit from decentralising decisions to individual agents. Therefore, a trade off exists from the perspective of a network designer as redundant links bring stability but may be costly to maintain (both in terms of a link cost and the *PoS*).

6 Conclusion

Strategic complementarity and the need for coordination are central features of many economic decisions. Who we interact with in our social network and the way in which these networks interlink has an impact on these decisions. Previous research on games on networks has focused on the tractable 'linear-quadratic' case, yet very little is known about different (non-linear) functional forms. In this paper I have examined the case where the interaction between a pair of agents is given by the minimum effort between the two parties.

A major contribution has been to highlight the link between effort levels in two salient equilibria of the game and the network's cohesion and density. Both equilibria allow us to construct a hierarchical and nested decomposition of the nodes according to the extent of their presence in dense and cohesive subgraphs of the network.

In the Pareto dominant equilibrium of the model, agents play actions equal to their *coreness*, a well-known concept in the social networks literature. Coreness can also be regarded as a coarse measure of centrality, and so I provide a microfoundation for this measure by showing that it arises naturally from a threshold game played on a network.

I also look at equilibrium stability and show that the Pareto dominant equilibrium can be 'knife edge' in some networks. A small and temporary shock to the effort of one node can lead to a cascade of falling actions which prevents agents from returning to this equilibrium. Another key contribution is therefore my characterisation of networks which are robust to such shocks. I also identify nodes who can be targeted in order to prevent unravelling of the Pareto dominant equilibrium.

In the course of investigating equilibrium stability, I introduce a new concept known as the *density decomposition*. This decomposition characterises the potential maximising equilibrium, which I argue can be viewed as a close analogue of the stochastically stable outcome. Agents who are members of the densest subgraph (in terms of average degree) will play the highest effort in this regime.

Future work may wish to consider the role of a social planner in targeting nodes or links to subsidise. For example, a designer could pay a given node to increase their effort, raising the actions of others in equilibrium. For nodes who have coreness c_i but are first to be removed in the iterative pruning process used to uncover the $c_i + 1$ core, we can provide transfers to these nodes in order to prevent cascades of falling actions for neighbours j with $d_j > c_j = c_i$. Alternatively, a social planner could wish to identify particular links which, if added, would bring the greatest increase to the coreness of agents. Related questions have been examined in Bhawalkar et al. (2012) but remain an open area for study.

Appendices

A Proofs

Proof of Lemma 1. Fix a profile $\mathbf{x}_{-i}$ and define $\underline{BR}_i := \min \{x_i \mid x_i \geq \overline{d_i} (x_i, \mathbf{x}_{-i})\}$ and $\overline{BR}_i = \max \{x_i \mid x_i \leq \underline{d_i} (x_i, \mathbf{x}_{-i})\}$. By construction, it must be that $BR_i (\mathbf{x}_{-i})$ satisfies $\underline{BR}_i \leq BR_i$ and $BR_i \leq \overline{BR}_i$.

Since $BR_i = \underline{BR}_i$ satisfies (1), this is also true for $BR_i' > \underline{BR}_i$. Similarly, $BR_i'' < \overline{BR}_i$ must also satisfy (2). However if $\underline{BR}_i < \overline{BR}_i$ then at least two different profiles jointly satisfy (1) and (2). This contradicts the known fact that $BR_i (\mathbf{x}_{-i})$ is unique. So we conclude that $\underline{BR}_i = \overline{BR}_i = BR_i (\mathbf{x}_{-i})$. $\square$

Proof of Lemma 2. Take any agent $l \in N$ with $c_l (\mathbf{g}) = k$ and consider the subgraph $\mathbf{g}_k$ which forms the largest k-core in $\mathbf{g}$. Note firstly that the minimum degree of nodes in $\mathbf{g}_k$ is k and secondly that $l \in N_k$. By removing any single link from $\mathbf{g}_k$ the minimum degree of agents in $\mathbf{g}_k - ij$ decreases at most by 1. In the non-trivial case where $d_l (\mathbf{g}) \geq 2$ then agent l must have positive degree in $\mathbf{g}_k - ij$ and according to Definition 1 $\mathbf{g}_k - ij$ is at least a $(k-1)$-core. $\square$

Proof of Theorem 1. To show that $\mathbf{x}^* = \mathbf{c}(\mathbf{g})$ is an equilibrium, partition the agents into subsets $\{S_1, S_2, \ldots, S_K\}$ such that $c_i = k$ for all $i \in S_k$. Consider the set of agents S_K and note that if $x_i = c_i$ then condition (1) is satisfied for all $i \in S_K$, since no agents play higher efforts. The number of neighbours playing weakly higher effort is $c_i = x_i$ for $i \in S_K$ and so condition (2) also holds. Now consider S_{K-1} and note again that $x_{i'} \geq \overline{d_{i'}} (\mathbf{x})$ for all $i' \in S_{K-1}$, otherwise these agents would have K neighbours with coreness K. Furthermore $\underline{d_i} (x_i, \mathbf{x}_{-i}) \geq x_{i'}$ since $i' \in S_{K-1}$ are linked to at least $K-1$ neighbours with coreness $K-1$, so (2) again holds. This argument applies for all lower subsets and so the action profile $\mathbf{x}^* = \mathbf{c} (\mathbf{g})$ is a Nash equilibrium.

To show that this equilibrium is maximal assume that there exists another equilibrium vector of actions $\mathbf{x}'$ such that $\mathbf{x}' \geqslant \mathbf{x}^*$. Take any $i \in N$ playing

$x_i' > x_i^* = c_i$ in equilibrium. To be an equilibrium i must have at least $\lceil x_i' \rceil$ neighbours playing $x_j' \geq x_i'$. Moreover, these agents j must have at least $\lceil x_j' \rceil$ neighbours playing $x_k' \geq x_j'$. Continuing with this reasoning contradicts the assumption that the coreness of node i was $c_i < x_i'$ since we can now construct a subgraph containing i where each node has at least degree $\lceil x_i' \rceil$ within that subgraph. □

Proof of Lemma 3 . The fact that $i \rightarrow j \implies c_j(\mathbf{g} - ij) = c_j(\mathbf{g}) - 1$ follows immediately from Definition 3 and Lemma 2, since there is no $\mathbf{g}_k \subseteq \mathbf{g}$ which contains j but not i. □

Proof of Proposition 1. I first prove that if $d_i^+(\check{\mathbf{g}}) < c_i(\mathbf{g})$ for some $i \in N$ then $\mathbf{x}^*$ is *knife edge*. Pick any individual with $d_i^+(\check{\mathbf{g}}) < c_i(\mathbf{g})$ and lower their effort to $c_i(\mathbf{g}) - \varepsilon$ at $t = 0$. At period $t = 1$ i's effort returns to c_i but the efforts of $j \in N_i^+(\hat{\mathbf{g}})$ fall to $x_j < c_j$. Since there are only $d_i^+(\check{\mathbf{g}})$ unaffected neighbours of i at $t = 1$, i's action again falls at period $t = 2$, causing the actions of $j \in N_i^+(\hat{\mathbf{g}})$ to fall back to x_j in period $t = 3$. The pattern of periods 2 and 3 then cycles and so $\mathbf{x}^t$ does not return to $\mathbf{x}^*$.

To prove that if $d_i^+(\check{\mathbf{g}}) \geq c_i(\mathbf{g})$ for all $i \in N$ then $\mathbf{x}^*$ is not knife edge I will make use of two lemmas:

Lemma 6. *If $d_i^+(\check{\mathbf{g}}) \geq c_i(\mathbf{g})$ for all $i \in N$ then $\hat{\mathbf{g}}$ is acyclic.*

Proof. Assume to the contrary that a directed cycle $i \rightarrow j, \ldots, j' \rightarrow i$ in $\hat{\mathbf{g}}$ exists. Since $i \rightarrow j$ we know that $ij \in \hat{\mathbf{g}} \implies ij \notin \check{\mathbf{g}}$ and hence $d_i^+(\check{\mathbf{g}}) \geq c_i(\mathbf{g})$ implies that i is linked to at least $c_i + 1$ agents with coreness c_i or greater. However, $i \rightarrow j$ means that j has exactly $c_i = c_j$ neighbours of coreness c_i or greater, since $c_j(\mathbf{g} - ij) = c_j(\mathbf{g}) - 1$ by Lemma 3. Note that removing ij cannot indirectly lower $c_{j'}$ for $j' \neq j$ due to the fact that $d_i > c_i$. Since i has strictly more neighbours of coreness $c_i = c_j$ than j, we reach a contradiction by iterating along the cycle. □

Lemma 7. *If $i \to j$ and $i \to j'$ then $j' \notin N_j(\mathbf{g})$.*

Proof. Since j has c_i neighbours of coreness c_i and $c_i = c_j = c_{j'}$ this means that if $j' \in N_j$ then j' is one such agent of coreness c_i, hence $j' \to j$. This is also true for j meaning that $j \to j'$, which would create a cycle and therefore a contradiction. □

I now show that $d_i^+(\check{\mathbf{g}}) \geq c_i(\mathbf{g})$ for all $i \in N$ implies that $\mathbf{x}^*$ is not knife edge. Assume that a node i lowers action to $c_i - \varepsilon$ at $t = 0$. Since $d_i^+(\check{\mathbf{g}}) \geq c_i(\mathbf{g})$ and $\hat{\mathbf{g}}$ is acyclic, this permits i to revert back to playing c_i at $t = 1$. Agents $j \in N_i^+(\hat{\mathbf{g}})$ decrease action to $x_j < c_j$ at $t = 1$ but permanently return to c_j from $t = 2$. To see why, note first that j has exactly c_j neighbours of coreness c_j or greater. Now consider $k \in N_j$ such that $c_k = c_j$ and observe that $k \to j$. Yet, assuming $d_k^+(\check{\mathbf{g}}) \geq c_k(\mathbf{g})$ guarantees for any k in the neighbourhood of some $j \in N_i$ that k's effort cannot be affected even if all such $j \in N_i$ lower their efforts, since k has a supporting core which does not include any of the affected agents $j \in N_i$. Moreover, by Lemma 7 we know that k's effort cannot be directly affected by i's shock.

Therefore, agents $k \in N_j$ cannot be influenced by the shock and the cascade stops after 1 period. Since i reverts at $t = 1$ and all $j \in N_i^+(\hat{\mathbf{g}})$ revert at $t = 2$, efforts converge back to $\mathbf{x}^*$ at $t = 2$. □

Proof of Lemma 4. For (a) we need that $\rho(\mathbf{x} \vee \mathbf{x}') + \rho(\mathbf{x} \wedge \mathbf{x}') \geq \rho(\mathbf{x}) + \rho(\mathbf{x}')$ for any $\mathbf{x}$ and $\mathbf{x}'$. Applying (3) costs immediately cancel on both sides, resulting in

$$\textstyle\sum_{ij \in \mathbf{g}} \left(\min\left\{\max\{x_i, x_i'\}, \max\{x_j, x_j'\}\right\} + \min\left\{\min\{x_i, x_i'\}, \min\{x_j, x_j'\}\right\}\right) \geq \sum_{ij \in \mathbf{g}} \left(\min\{x_i, x_j\} + \min\{x_i', x_j'\}\right)$$

which clearly holds along each link.

Since $-\frac{1}{2}\sum_{i \in N} x_i^2$ is strictly concave in X, for part (b) of the lemma we must verify that $\sum_{ij\in\mathbf{g}} \min\left\{x_i, x_j\right\}$ is concave. We see that

$$\lambda \min\{x_i, x_j\} + (1-\lambda)\ \min\{x_i', x_j'\} \leq\ \min\{\lambda x_i + (1-\lambda)\, x_i', \lambda x_j + (1-\lambda)\, x_j'\}$$

holds with equality when $x_i \leq x_j$ and $x_i' \leq x_j'$. It is easily checked that when $x_i \leq x_j$ but $x_i' \geq x_j'$ the above also inequality holds (strictly if $x_i < x_j$ and $x_i' > x_j'$). The inequality therefore holds when summing over all links. Since ρ is then the sum of a concave and a strictly concave function we conclude that ρ is strictly concave. □

Proof of Theorem 2. Let $\{\tilde{S}_1, \ldots, \tilde{S}_K\}$ be the optimal partition of N where $\tilde{x}_i$ satisfies (4) for all $i \in N$. To see that $\{\tilde{S}_1, \ldots, \tilde{S}_K\} = \mathcal{D}$ note first that if $\tilde{S}_K \neq D_K$ then the subset of nodes D_K can change their action to δ_K to increase ρ. Since (4) holds at any potential maximum, $\tilde{x}_i \leq \delta_K$ for $i \in N$. Efforts must strictly increase for some $i \in D_K$, which must increase ρ due to supermodularity.

To show that $\tilde{S}_{K-1} = D_{K-1}$ we can apply the same logic. Taking the actions of $\tilde{S}_K = D_K$ as given, if $\tilde{S}_{K-1} \neq D_{K-1}$ then D_{K-1} can optimally increase efforts, which must increase ρ by the same argument as before. Noting that changes in x_i by nodes playing lower effort in equilibrium cannot influence incentives of those above, we can continue this reasoning downwards for all other subsets in the partition to complete the proof. □

Proof of Lemma 5. Any maximiser of ρ has a hierarchical structure where the group of agents exerting maximal effort $\tilde{x}_K$ do not depend on the effort levels of those below. As $\rho_K = |\tilde{\mathbf{g}}_K|\, \tilde{x}_K - \frac{|\tilde{S}_K|}{2}\tilde{x}_K^2$ is a strictly concave function, $\tilde{x}_{z,K}$ satisfies $|\tilde{x}_{z,K} - \tilde{x}_K| \leq z$. A similar argument holds for all other agents exerting optimal efforts $\tilde{x}_k$ for $k \in \{1, \ldots, K-1\}$. This implies that $\|\tilde{\mathbf{x}}\,(z) - \tilde{\mathbf{x}}\|_2 \leq \sqrt{n}z$ and the result immediately follows. □

***Proof of Proposition* 2.** To prove the 'if' direction we can note that for a *d-regular* graph $x_i = x_j$ for all $i, j \in N$ at the social optimum. This follows from the fact all actions are weak complements and all agents are identical. The problem then becomes

$$\max_{x} \quad 2\,|\mathbf{g}|\,x - \tfrac{|N|}{2}\,x^2$$

First order conditions imply that the socially optimal action is $x_i = 2\frac{|\mathbf{g}|}{|N|} = d = c_i$.

To prove the 'only if' direction I use a constrained optimisation formulation of the problem with a vector of dummy variables $\mathbf{y}$ such that $y_{ij} := \min\left\{x_i, x_j\right\}$

$$\begin{array}{cc} \max\limits_{\mathbf{x},\mathbf{y}} & 2\sum_{ij\in\mathbf{g}} y_{ij} - \frac{1}{2}\sum_{i\in N} x_i^2 \\ \text{subject to} & y_{ij} \leq x_i,\, y_{ij} \leq x_j \end{array}$$

At least one of the constraints must bind with equality for each $ij \in \mathbf{g}$. The Lagrangian for the reformulated problem is

$$\mathcal{L} = 2\sum_{ij\in\mathbf{g}} y_{ij} - \frac{1}{2}\sum_{i'\in N} (x_{i'})^2 - \sum_{ij\in\mathbf{g}} \lambda_{ij}\left(y_{ij} - x_i\right) - \sum_{ij\in\mathbf{g}} \mu_{ij}\left(y_{ij} - x_j\right)$$

Let λ_{ij} be the Lagrange multiplier for constraint $y_{ij} \leq x_i$ when $i < j$ and vice versa for μ_{ij}. The first order conditions with respect to y_{ij} and x_i are

$$\frac{\partial\mathcal{L}}{\partial y_{ij}} = 2 - \lambda_{ij} - \mu_{ij} = 0$$
$$\frac{\partial\mathcal{L}}{\partial x_{i'}} = \sum_{ij\in g: i<j} \lambda_{ij} + \sum_{ij\in g: i>j} \mu_{ij} - x_{i'} = 0$$

Rearranging and summing these over all edges and N respectively gives $2\,|\mathbf{g}| = \sum_{ij\in\mathbf{g}}\left(\lambda_{ij} + \mu_{ij}\right)$ and $\sum_{i'\in N} x_{i'} = \sum_{ij\in\mathbf{g}}\left(\lambda_{ij} + \mu_{ij}\right)$ and so $\sum_i x_i = 2\,|\mathbf{g}| = \sum_i d_i$ at any social optimum. If the network is not regular then $c_i < d_i$ for some $i \in N$ and so there is too little effort relative to the social optimum. □

***Proof of Proposition* 3.** The proof proceeds in three steps:

(1) I first establish that PoS is bounded from below by the ratio of the surplus generated in $\tilde{\mathbf{x}}$ to surplus generated at the social optimum. Using the ideas of Theorem 2 it is straightforward to see that if $\mathbf{x}$ is the action profile in the socially efficient case, then $\mathbf{x} = 2\tilde{\mathbf{x}}$. Since the action in the Pareto dominant equilibrium is weakly higher than $\tilde{\mathbf{x}}$ we know that $U(\tilde{\mathbf{x}}) \leq U(\mathbf{x}^*)$ and a lower bound on the price of stability is given by

$$PoS(\mathbf{g}) \geq \frac{U(\tilde{\mathbf{x}})}{U(\mathbf{x}_{opt})} = \frac{\sum_{ij\in\mathbf{g}} 2\min\{\tilde{x}_i, \tilde{x}_j\} - \sum_i \frac{1}{2}\tilde{x}_i^2}{\sum_{ij\in\mathbf{g}} 4\min\{\tilde{x}_i, \tilde{x}_j\} - \sum_i 2\tilde{x}_i^2} \equiv \underline{PoS}(\mathbf{g}) \tag{5}$$

(2) Using the approach in the proof of Proposition 2 we consider the maximisation problem

$$\begin{array}{cc} \max\limits_{\mathbf{x},\mathbf{y}} & \sum_{ij\in\mathbf{g}} y_{ij} - \frac{1}{2}\sum_{i\in N} x_i^2 \\ \text{subject to} & y_{ij} \leq x_i,\ y_{ij} \leq x_j \end{array}$$

This yields first order conditions

$$\lambda_{ij} + \mu_{ij} = 1 \qquad \forall ij \in \mathbf{g} \tag{6}$$

$$\sum_{ij\in\mathbf{g}:i<j} \lambda_{ij} + \sum_{ij\in\mathbf{g}:i>j} \mu_{ij} = \tilde{x}_i \qquad \forall i \in N \tag{7}$$

Equation 6 and the complementary slackness condition together imply that $y_{ij} = \lambda_{ij}\tilde{x}_i + \mu_{ij}\tilde{x}_j$. Summing this over all edges gives

$$\sum_{ij\in\mathbf{g}} y_{ij} = \sum_{i\in N} \tilde{x}_i \left(\sum_{ij\in\mathbf{g}:i<j} \lambda_{ij} + \sum_{ij\in\mathbf{g}:i>j} \mu_{ij} \right)$$

and using (7) we see that $\sum_{i\in N} \tilde{x}_i^2 = \sum_{ij\in\mathbf{g}} y_{ij}$.

We can also rearrange (5) to obtain

$$(\underline{PoS}(\mathbf{g}) - \frac{1}{4}) \sum_{i\in N} \tilde{x}_i^2 = (2\underline{PoS}(\mathbf{g}) - 1) \sum_{ij\in\mathbf{g}} y_{ij}$$

and so $PoS(\mathbf{g}) \geq \underline{PoS}(\mathbf{g}) = \frac{3}{4}$.

(3) Finally, I show that PoS in a large star network approaches $\frac{3}{4}$ as $n \to \infty$. Since the coreness of all agents in any star network is $c_i = 1$. As the star as a whole is the densest subgraph of the network, the potential maximising equilibrium action in a star network with 1 hub and M spokes is given by

$$\tilde{x}_i = \frac{M}{M+1}$$

So $\lim_{M\to\infty} PoS\left(\mathbf{g}_{star}\right)$ is therefore

$$\lim_{M\to\infty} \frac{2M\cdot 1 - \frac{1}{2}(M+1)\cdot 1^2}{4M\left(\frac{M}{M+1}\right) - 2(M+1)\left(\frac{M}{M+1}\right)^2} = \lim_{M\to\infty} \frac{M+1}{M} - \frac{1}{4}\left(\frac{M+1}{M}\right)^2 = \frac{3}{4}$$

□

B Private Benefits and Weighted Links

I now consider the general case where for each $i \in N$ we assume that utility functions are given by

$$u_i = a_{ii}x_i + \sum_{j\in N_i} a_{ij} \min\left\{x_i, x_j\right\} - \frac{\gamma_i}{2}x_i^2 \tag{8}$$

where $a_{ii} \geq 0$ is the marginal private benefit from increasing x_i and $\gamma_i \geq 0$ is the private cost parameter. The parameters a_{ij} are interpreted as the strength of the externality between agent i and j. We could in principle allow the weights a_{ij} to be negative, which would mean that $u_i = a_{ii}x_i - \sum_{j\in N_i} \left|a_{ij}\right| \min\left\{x_i, x_j\right\} - \frac{\gamma_i}{2}x_i^2$. We could then rewrite this utility function as

$$u_i = \left(a_{ii} + \sum_{j\in N} a_{ij}\right) x_i + \sum_{j\in N_i} a_{ij}\max\left\{0, x_j - x_i\right\} - \frac{\gamma_i}{2}x_i^2.$$

With a suitable condition on a_{ii} we could then interpret this as a model of conspicuous consumption on networks, as studied by Immorlica et al. (2015), who find a similar 'stratified' equilibrium which divides agents in to 'classes'. Since this is studied elsewhere I maintain the assumption that $a_{ij} \geq 0$ and that agent's efforts exhibit positive externalities.

In order to extend Theorem 2 I will assume that $a_{ij} = a_{ji}$, although Theorem 1 could easily be extended to the case of directed networks. We may now think of individuals as therefore being connected by a weighted but undirected network.

B.1 Generalised Coreness

To characterise equilibria in this broader case, the concept of the coreness of a node must be generalised to include properties other than their degree. Following Batagelj and Zaveršnik (2011) I will define a 'generalised coreness' for each node via the use of a node property function. For each subgraph $\mathbf{h} \subseteq \mathbf{g}$ the node property function will assign a value $\phi_i \in \mathbb{R}$ to each node, as a function of the link weights a_{ii}, a_{ij} and cost parameters γ_i. Letting $\mathcal{H}$ denote the set of possible subgraphs of $\mathbf{g}$, the node property function is a function $\phi : \mathbb{R}_+^{|\mathbf{g}|} \times \mathbb{R}_+^n \times \mathcal{H} \to \mathbb{R}^n$.

I now define a generalised core of order k as follows:

Definition 5. A *generalised core* of order k is a subgraph $\mathbf{h} \subseteq \mathbf{g}$ such that $\phi_i(\mathbf{a}_i, \gamma_i, \mathbf{h}) \geq k$ for each $i \in N(\mathbf{h})$.

The notion of degree coreness which was studied earlier in the paper can be considered a special case where we define the node property function to be $\phi_i = d_i(\mathbf{h})$. Analogously to the case of degree coreness, we can find *generalised cores* by repeatedly pruning nodes which have $\phi_i(\mathbf{a}, \mathbf{h}) < k$ from successive subgraphs of $\mathbf{g}$ (see Theorem 1 of Batagelj and Zaveršnik (2011)). Agents will therefore have a corresponding 'ϕ-coreness' denoted by c_i^{ϕ}, which is the maximal k such that agent i is in a generalised core of order k but not one of order k' for $k' > k$.

Definition 6. A node $i \in N$ has *ϕ-coreness* $c_i^{\phi}(\mathbf{g}) = k$ if it is contained in a *generalised core* of order k but not one of order k' for $k' > k$.

To characterise the maximal equilibrium in the case of heterogeneous costs and benefits the value assigned to each node by the node property function is given by

$$\phi_i(\mathbf{a}_i, \gamma_i, \mathbf{h}) = \frac{a_{ii} + \sum_{j \in N_i \cap N(\mathbf{h})} a_{ij}}{\gamma_i}$$

This specification of the node property function gives agent i's marginal benefit (relative to cost) from increasing effort when all other agents $j \in N(\mathbf{h})$ have strictly higher efforts.

Proposition 4. *The Pareto dominant Nash equilibrium profile in the generalised case is* $\mathbf{x}^* = \mathbf{c}^\phi$.

Proof. Since (8) is still concave in x_i given x_{-i} we need only verify that marginal changes in x_i cannot increase utility in order to show that $\mathbf{x}^* = \mathbf{c}^\phi$ is an equilibrium. Suppose that any node i increases their effort at marginal cost $\gamma_i c_i^\phi$. The gross marginal benefit is a_{ii} plus a_{ij} along every link to nodes with strictly higher effort (and therefore generalised coreness). This can only increase i's utility if this gross marginal benefit exceeds $\gamma_i c_i^\phi$, yet this is a contradiction as there must then exist a subgraph $h' \subseteq g$ where $\phi_i(\mathbf{a}, h') > c_i^\phi$.

Decreasing x_i below c_i^ϕ must also be suboptimal since we could increase u_i by marginally raising x_i. Letting h_i denote the subgraph spanned by all nodes $j \in N$ such that $c_j^\phi \geq c_i^\phi$ we see that if $x_i < c_i^\phi$ then our gross marginal benefit from increasing x_i is $a_{ii} + \sum_{j \in N_i \cap N(h_i)} a_{ij} = \gamma_i c_i^\phi > \gamma_i x_i$. To show that it is the maximal equilibrium of the game we can again repeat the argument of Theorem 1 by supposing it is not and then noting that this contradicts the definition of ϕ-coreness. □

Compared to the case in the main section of the paper, agents with higher a_{ii} parameters need lower weighted degrees to sustain a given level of effort. In the case where $a_{ii} = a$ for all $i \in N$ then weighted degree coreness will determine differences in Pareto dominant equilibrium efforts.

B.2 Potential Maximising Equilibrium

In order to replicate the other main result of the paper (Theorem 2) we must define a new potential function

$$\mathcal{P}(\mathbf{x}) = \sum_{ij \in g} a_{ij} \min\{x_i, x_j\} + \sum_{i \in N} (a_{ii} x_i - \frac{\gamma_i}{2} x_i^2)$$

We can then proceed to analyse the potential maximising equilibrium in this more general case by constructing a *weighted density decomposition* of the network in a similar vein to the density decomposition from Section 4.2. Where previously $\bar{g}_k \subseteq g$ denoted the largest subgraph such that $\frac{|\bar{g}_k|}{|\bar{N}_k|} \geq k$ for some $k \geq 0$, let $\bar{g}'_k$

denote the largest subgraph such that

$$\frac{\sum_{i\in\bar{N}'_k} a_{ii} + \sum_{ij\in\bar{g}'_k} a_{ij}}{\sum_{i\in\bar{N}'_k}\gamma_i} \geq k$$

As with the density decomposition in Section 4.2, these subgraphs are nested. To construct a corresponding node-level statistic we can let δ'_i be the largest k such that $i \in \bar{N}'_k$ for some $\bar{\mathbf{g}}'_k \subseteq \mathbf{g}$. We now have the following definition of a *weighted density decomposition*:

Definition 7. A *weighted density decomposition* of a graph $\mathbf{g}$ is a partition $\mathcal{D}' = \{D'_1, \ldots, D'_K\}$ of N such that $i \in D'_k$ and $j \in D'_k$ if and only if $\delta'{}_i = \delta'_j$

As in Section 4.2 the values of δ'_i will partition the nodes by their equilibrium efforts. In an identical manner to Section 4.2.2 we let $\{\tilde{S}'_1, \ldots, \tilde{S}'_M\}$ denote the potential maximising partition in the general case and let $\tilde{\mathbf{g}}'_m$ denote the maximal subgraph of $\mathbf{g}$ such that $ij \in \tilde{\mathbf{g}}'_m$ if and only if $i \in \tilde{S}'_m$ and $j \in \tilde{S}'_{m'}$ for $m' \geq m$. Given this partition, the corresponding optimal actions are

$$\tilde{x}_m = \frac{\sum_{i\in\tilde{S}'_m} a_{ii} + \sum_{ij\in\tilde{g}'_k} a_{ij}}{\sum_{i\in\tilde{S}'_m}\gamma_i} \tag{9}$$

Proposition 5. *The potential maximising partition of N is $\{\tilde{S}'_1, \ldots, \tilde{S}'_M\} = \mathcal{D}'$ where the equilibrium action of each $i \in \tilde{S}_m$ is given by (9).*

Proof. The proof is very similar to the proof of Theorem 2. We find the potential maximising partition as a result of optimising subset by subset. By the previous proof we know that D'_K must be the subset of agents exerting highest effort, otherwise they could jointly deviate to δ'_K to increase $\mathcal{P}$. Iterating this argument successive subsets of the partition in an identical manner to Theorem 2 we arrive at the result. □

If costs and private benefits were identical across agents such that $a_{ii} = a$ and $\gamma_i = \gamma$ for all $i \in N$ we see that the optimal partition of agents is based on average internal weighted degree. Propositions 4 and 5 therefore show that the main intuitions of the model extend to the case of agent heterogeneity and weighted networks.

References

Anderson, Simon P., Jacob K. Goeree and Charles A. Holt (2001), 'Minimum-Effort Coordination Games: Stochastic Potential and Logit Equilibrium', *Games and Economic Behavior* **34**(2), 177–199. [16]

Bader, GD and CWV Hogue (2003), 'An Automated Method for Finding Molecular Complexes in Large Protein Interaction Networks', *BMC Bioinformatics* **27**, 1–27. [8]

Ballester, Coralio, Antoni Calvó-Armengol and Yves Zenou (2006), 'Who's Who in Networks. Wanted : The Key Player', *Econometrica* **74**(5), 1403–1417. [4]

Batagelj, Vladimir and M. Zaveršnik (2011), 'Fast algorithms for determining (generalized) core groups in social networks', *Advances in Data Analysis and Classification* **5**(2), 129–145. [32]

Baur, Michael, Ulrik Brandes, Marco Gaertler and Dorothea Wagner (2004), Drawing the AS graph in 2.5 dimensions, *in* J.Pach, ed., 'Graph Drawing', number Lncs 3383, Springer, pp. 43–48. [8]

Bhawalkar, Kshipra, Jon Kleinberg, Kevin Lewi, Tim Roughgarden and Aneesh Sharma (2012), 'Preventing Unraveling in Social Networks: The Anchored k-core Problem', *39th International Colloquium, ICALP 2012* pp. 440–451. [24]

Blume, Larry (1993), 'The Statistical Mechanics of Strategic Interaction', *Games and Economic Behavior* **5**, 387–424. [16]

Bramoullé, Yann and Rachel Kranton (2007), 'Public Goods in Networks', *Journal of Economic Theory* **135**, 478–494. [4], [13]

Bramoulle, Yann and Rachel Kranton (2015), 'Games Played on Networks', *Oxford Handbook on the Economics of Networks* (Forthcoming). [4]

Bramoullé, Yann, Rachel Kranton and Martin D'Amours (2014), 'Strategic Interaction and Networks', *American Economic Review* **104**(3), 898–930. [4]

Bryant, John (1983), ‘A Simple Rational Expectations Keynes-Type Model’, *The Quarterly Journal of Economics* **98**(3), 525–528. [2]

Carbonell-Nicolau, Oriol and Richard P. McLean (2014), ‘Refinements of Nash equilibrium in potential games’, *Theoretical Economics* **9**(3), 555–582. [16]

Chen, Roy and Yan Chen (2011), ‘The Potential of Social Identity for Equilibrium Selection’, *The American Economic Review* **101**(October), 2562–2589. [16]

Chwe, Michael Suk-Young (2000), ‘Communication and Coordination in Social Networks’, *Review of Economic Studies* **67**, 1–16. [4]

Cooper, Russell and Andrew John (1988), ‘Coordinating coordination failures in Keynesian models’, *The Quarterly Journal of Economics* **103**(3), 441–463. [2]

Crawford, Vincent P (1991), ‘An "Evolutionary" Interpretation of Van Huyck, Battalio, and Beil’s Experimental Results on Coordination’, *Games and Economic Behavior* **3**(1), 25–59. [16]

Cremer, J, L Garicano and Andrea Prat (2007), ‘Language and the Theory of the Firm’, *The Quarterly Journal of Economics* (February), 373–407. [3]

Ellison, Glenn (1993), ‘Learning, Local Interaction and Coordination’, *Econometrica* **61**(5), 1047–1071. [3], [5]

Gagnon, Julien and Sanjeev Goyal (2015), Networks , Markets and Inequality. [8]

Galeotti, Andrea, Sanjeev Goyal, Matthew O. Jackson, Fernando Vega-Redondo and Leeat Yariv (2010), ‘Network Games’, *Review of Economic Studies* **77**(1), 218–244. [4]

Goeree, Jacob K. and Charles A. Holt (2005), ‘An Experimental Study of Costly Coordination’, *Games and Economic Behavior* **51**(2), 349–364. [16]

Goldberg, AV (1984), Finding a Maximum Density Subgraph. [18]

Granovetter, Mark (1978), ‘Threshold Models of Collective Behavior’, *The American Journal of Sociology* **83**(6), 1420–1443. [4]

Hirshleifer, Jack (1983), 'From weakest-link to best-shot: The voluntary provision of public goods', *Public Choice* **41**(3), 371–386. [2]

Immorlica, Nicole, Rachel Kranton, Mihai Manea and Greg Stoddard (2015), 'Social Status in Networks', *Working Paper* . [31]

Jackson, Matthew O. (2008), *Social and Economic Networks*, Princeton University Press. [4]

Jackson, Matthew O. and Alison Watts (2002), 'On the Formation of Interaction Networks in Social Coordination Games', *Games and Economic Behavior* **41**(2), 265–291. [3], [5]

Jackson, Matthew O. and Yves Zenou (2014), Games on Networks. [4]

König, Michael D., Claudio J. Tessone and Yves Zenou (2014), 'Nestedness in networks: A theoretical model and some applications', *Theoretical Economics* **9**(3), 695–752. [4]

Milgrom, Paul and John Roberts (1990), 'Rationalizability, Learning and Equilibrium in Games with Strategic Complementarities', *Econometrica* **58**(6), 1255–1277. [10], [11]

Monderer, Dov and Lloyd S Shapley (1996), 'Potential Games', *Games and Economic Behavior* **143**, 124–143. [16], [17]

Morris, Stephen (2000), 'Contagion', *Review of Economic Studies* **67**, 57–78. [5], [7], [8]

Nisan, N, T Roughgarden, E Tardos and V Vazirani (2007), *Algorithmic Game Theory*, Cambridge University Press. [22]

Reagans, Ray and Bill McEvily (2003), 'Network Structure and Knowledge Transfer: The Effects of Cohesion and Range', *Administrative Science Quarterly* **48**(2), 240–267. [3]

Schelling, Thomas C. (1973), 'Hockey Helmets, Concealed Weapons, and Daylight Saving: A Study of Binary Choices with Externalities', *The Journal of Conflict Resolution* **17**(3), 381–428. [3]

Seidman, Stephen B (1983), 'Network Structure and Minimum Degree', *Social Networks* **5**, 269–287. [8]

Sobel, Joel (2012), 'Complexity versus conflict in communication', *2012 46th Annual Conference on Information Sciences and Systems (CISS)* pp. 1–6. [3]

Topkis, Donald M. (1998), *Supermodularity and Complementarity*, Princeton University Press. [21]

Van Huyck, John B., Raymond C. Battalio and Richard O. Beil (1990), 'Tacit Coordination Games, Strategic Uncertainty, and Coordination Failure', *The American Economic Review* **80**(1), 234–248. [16]

Wasserman, Stanley and Katherine Faust (1994), *Social Network Analysis: Methods and Applications*, Cambridge University Press, Cambridge. [7], [8]

Young, H. Peyton (1998), *Individual Strategy and Social Structure*, Princeton University Press. [3], [5], [7], [8]

Zhou, L (1994), 'The set of Nash equilibria of a supermodular game is a complete lattice', *Games and Economic Behavior* . [10]