We started today talking about optimal strategies (so called dominant strategies).

We then learnt about Nash equilibrium (a method to make predictions about what is likely to happen in a game).

➢ Is it always optimal for me to play a Nash equilibrium strategy?

➢ Does every game have a Nash equilibrium?
Here there is no stable Nash equilibrium if:
1. Drivers can only pick either speed or don’t speed.
2. Police and only pick either monitor or don’t monitor.

The problem here is that both players want to be *unpredictable*. 
A Speeding Game

How often should the police monitor?

▷ Just enough to incentivise drivers not to speed.
▷ But not so much that we monitor unnecessarily.

How often do we monitor to make the drivers not want to speed all the time?

\[ p(0) + (1-p)0 \geq p(-50) + (1-p)20 \]

\[ 0 \geq 20 - 70p \]
\[ 70p \geq 20 \]
\[ p \geq \frac{2}{7} \]

Monitor no less than 2 days per week.

▷ Payoff for police: \( \left( \frac{2}{7} \right) 5 + \left( \frac{5}{7} \right) 35 = 26.43 \)
Other Examples of Monitoring Games in Game Theory

- **Employer-Employee monitoring**: Employers may monitor productivity (or effort) to prevent time wastage.
- **Environmental Agencies**: Government agencies may monitor factories to ensure compliance with pollution regulations.
- **Tax Authorities**: Random audits may be conducted to encourage accurate reporting.
- **Customs and Border Patrol**: Randomised searches may help to discourage smuggling of illegal items.
Rock, Paper, Scissors

Rules:

You will pick either ‘Rock’, ‘Paper’ or ‘Scissors’

‘Paper’ beats ‘Rock’
‘Rock’ beats ‘Scissors’
‘Scissors’ beats ‘Paper’
Suppose we each wager £1 to play this game. The winner therefore gains an extra £1 from winning and the loser will lose their £1.

<table>
<thead>
<tr>
<th></th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>0,0</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
<tr>
<td>Paper</td>
<td>1,-1</td>
<td>0,0</td>
<td>-1,1</td>
</tr>
<tr>
<td>Scissors</td>
<td>-1,1</td>
<td>1,-1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Side note: This is a special type of game called a ‘zero sum’ game.
Join at vevox.app

Or search Vevox in the app store

ID: 171-369-656

Pre-University Summer School

Join: vevox.app ID: 171-369-656
According to the now defunct ‘Rock-Paper-Scissors Society’, Rock is the most popular strategy over the long history of tournaments they had until 2010.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Frequency of Play (data from RPS tournaments)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>35.4%</td>
</tr>
<tr>
<td>Paper</td>
<td>29.6%</td>
</tr>
<tr>
<td>Scissors</td>
<td>35.0%</td>
</tr>
</tbody>
</table>

According to the now defunct ‘Rock-Paper-Scissors Society’, Rock is the most popular strategy over the long history of tournaments they had until 2010.

- If people play this way on average, then our best strategy is Rock.
- But if everyone else thinks this way, then our best strategy is Paper.
- And if everyone else thinks this way, then our best strategy is Scissors.
- …
Is there a strategy which guarantees that we will always win money?

- No. If one existed, then our opponent could copy it and we would always draw!

Is there a strategy which guarantees that we will never lose money?

A ‘mixed’ strategy for RPS

*Randomise between Rock, Paper and Scissors. Play each with 1/3 probability.*

**Expected Winnings:**
- If they play Rock = \((1/3) \cdot 0 + (1/3) \cdot 1 + (1/3) \cdot -1 = 0\)
- If they play Paper = \((1/3) \cdot -1 + (1/3) \cdot 0 + (1/3) \cdot 1 = 0\)
- If they play Scissors = \((1/3) \cdot 1 + (1/3) \cdot -1 + (1/3) \cdot 0 = 0\)

**Key feature:** We are unexploitable if the opponent’s choice is as difficult as possible. *Preventing them from easily picking a winning strategy will prevent us from losing!*

The Optimal Strategy?
A Penalty Shootout

- Kicker must shoot either Left or Right
- Goalkeeper must dive either Left or Right
- If the Kicker scores they get payoff 1, Goalkeeper gets payoff -1
- If the Goalkeeper saves they get payoff 1, Kicker gets payoff -1

Goalkeeper

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kicker</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Left</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
<tr>
<td>Right</td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
</tbody>
</table>

Side note: This is another example of a ‘zero sum’ game.
A Penalty Shootout

The kicker wants to be unpredictable.
They do not want to give the goalkeeper a clear choice of what to do.

What shooting strategy makes goalkeeper’s decision as difficult as possible?

What saving strategy makes the kicker’s decision as difficult as possible?

If they play like this then 50% of penalties are saved and 50% are scored.
(is this realistic?)

<table>
<thead>
<tr>
<th>Kicker</th>
<th>Goalkeeper</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Left</td>
<td>-1, 1</td>
</tr>
<tr>
<td>Right</td>
<td>1, -1</td>
</tr>
</tbody>
</table>
Now suppose the kicker has a strong right foot, so a shot to the left goes in 50% of the time even when the goalkeeper guesses correctly.

The payoffs of the game change to:

<table>
<thead>
<tr>
<th></th>
<th>Goalkeeper</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Kicker</td>
<td></td>
</tr>
<tr>
<td>Left</td>
<td>0, 0</td>
</tr>
<tr>
<td>Right</td>
<td>1, -1</td>
</tr>
</tbody>
</table>

Payoff for Kicker when (Left,Left) happens is:

\[(1/2)(1) + (1/2)(-1) = 0\]

Payoff for Goalkeeper when (Left,Left) happens is:

\[(1/2)(-1) + (1/2)(1) = 0\]
A Penalty Shootout (modified payoffs)

<table>
<thead>
<tr>
<th>Kicker</th>
<th>Goalkeeper</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Left</td>
<td>0, 0</td>
</tr>
<tr>
<td>Right</td>
<td>1, -1</td>
</tr>
</tbody>
</table>

▷ Is it still optimal for the kicker/goalkeeper to randomise 50-50?
▷ 50-50 is no longer a stable equilibrium, in fact, both players wish to adjust their strategy.

New equilibrium is actually:
▷ Shoot Left $2/3^{rd}$ of the time and Right $1/3^{rd}$ of the time.
▷ Dive Left $2/3^{rd}$ of the time and Right $1/3^{rd}$ of the time.

Now the kicker is guaranteed payoff of $1/3$  
(exercise: check this!)
Colonel Blotto is a game of *strategic mismatch*.

- 2 Players have T ‘troops’ each.
- There are N ‘fronts’ which must have a number of troops allocated.
- Whoever has more troops on a given front wins that front (payoff +1)
- Whoever has less troops on a given front loses that front (payoff -1)

**Applications:**
- ‘Troops’ = Advertising expenditure / ‘Fronts’ = Different product markets
- ‘Troops’ = R&D expenditure / ‘Fronts’ = Different product characteristics
- ‘Troops’ = Police vs. criminals / ‘Fronts’ = Areas of a city
- ‘Troops’ = Campaign spending / ‘Fronts’ = States in an election
Suppose $T=12$ and $N=3$.

<table>
<thead>
<tr>
<th>Player</th>
<th>Front 1</th>
<th>Front 2</th>
<th>Front 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>P2</td>
<td>5</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>P1'</td>
<td>6</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>P2'</td>
<td>8</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>P1''</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
</tr>
</tbody>
</table>

Any deterministic allocation of resources (troops) can be beaten by another!
Main insights of the Colonel Blotto game:

▷ There is no deterministic strategy which cannot be exploited.
▷ Playing an unpredictable strategy is beneficial.
▷ You do not always need all your troops to win.

If the game is not symmetric (e.g. one player has more troops) then it may be possible to guarantee a win.

▷ Weaker players can try to get around this by opening more ‘fronts’
To help make predictions in game theory we focus on stable outcomes.

A Nash equilibrium is an outcome where each player picks their best strategy, given the strategy of the opponent.

Sometimes these equilibrium strategies can involve randomisation.

In a zero-sum game (if I win, you lose) we are best off picking a strategy which makes our opponent’s decision as difficult as possible.