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A note on bargaining over a finite number of feasible agreements

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Summary. In this note we show that the *uniqueness* of the subgame perfect equilibrium of Rubinstein's (1982) bargaining theory does not hold if the number of feasible agreements is *finite*. It will be shown that *any* Pareto-efficient agreement (belonging to the *finite* set of feasible agreements) can be supported as a subgame perfect equilibrium of the Rubinstein alternating-offers bargaining game, provided the length of a single bargaining period is sufficiently small.

1. Introduction

The bargaining problem refers to the following situation and question:

> "*Two individuals have before them several contractual agreements. Both have interests in reaching agreement but their interests are not entirely identical What "will be" the agreed contract, assuming that both parties behave rationally?*" (Rubinstein, 1982).

In his classic paper, Ariel Rubinstein presented an elegant model of a formulation of the bargaining problem. Basing his bargaining theory on "time preferences" he demonstrated that in many cases (for example, when the two bargainers have constant discount factors) the model has a *unique* equilibrium.

A feature underlying Rubinstein's formulation of the bargaining problem is that there is a *non-finite* number of feasible contractual agreements. This feature is retained by Ken Binmore in his generalization of Rubinstein's analysis (Binmore, 1987). In Rubinstein (1982) any partition of the unit size cake is a feasible agreement and in Binmore (1987) the Pareto frontier of the set of feasible agreements is connected.

In this note we shall comment on the bargaining problem when there are a *finite* number of feasible contractual agreements. We shall show that *any* Pareto-efficient agreement (belonging to the *finite* set of feasible agreements) can be supported as a subgame perfect equilibrium of the Rubinstein alternating-offers bargaining model, provided the length of a single bargaining period is sufficiently small[1].

The reason as to why Rubinstein's argument (and thus the "uniqueness" property) breaks down is as follows. A key ingredient of his argument is that for any length of a single bargaining period there will exist a pair of agreements, denote them by α and β, such that (i) β "today" is preferred (by a bargainer) to α "today" *and* (ii) α "today" is preferred (by the bargainer) to β "tomorrow". If the length of a single bargaining period is very small, then the agreement α and β will have to be very "close" (in terms of the preferences of the bargainer). If the set of agreements is a continuum (as in Rubinstein), then indeed there will exist such as "close" pair of agreements. But, if the set of agreements is finite, then there will not exist such a "close" pair of agreements.

2. The bargaining model

Denote the two bargainers by A and B. Let X denote the set of feasible agreements. The central assumption in this note is that X contains a *finite* number of elements. Thus, we write $X=\{x_1,x_2,\ldots,x_n\}$ where $x_i\{i=1,2,\ldots,n\}$ denotes a feasible agreement. Moreover, let d denote "disagreement".

The bargainers negotiate over X according to Rubinstein's infinite-horizon alternating-offers bargaining process. Each player, in turn suggests an agreement $x\in X$ and his opponent can either accept it or reject it. Acceptance of an agreement concludes the bargaining with the agreement $x\in X$. After rejection, the rejecting player then has to make a counter-proposal and so on. It is assumed that player A starts the process. The moves are made at points of time, 0, Δ, $2\Delta,\ldots,$ where Δ is the length of a single bargaining period.

Following Rubinstein we assume that the bargainers have preferences over the set $(XxT)U\{d\}$ where $T=[0,\infty)$ is the time space: (x,t) denotes "agreement $x\in X$ is reached at time $t\in T$". Let $\succsim_i$ $(i=A,B)$ denote player i's preference ordering over this set, which is assumed to satisfy the following assumptions.

A1. (Conflict of Interests): $(x_{k-1},0)\succ_A(x_k,0)$ and $(x_k,0)\succ_B(x_{k-1},0)$ for $k=2,3,\ldots,n$.
A2. (Mutually Beneficial Agreements): $(x_n,0)\succ_A d$ and $(x_1,0)\succ_B d$.
A3. (Monotonicity in Time): For any $x\in X$, $t,s\in T$ such that $t<s$, $(x,t)\succ_i(x,s)$ where $i=A,B$.
A4. (Stationarity): (i) For any $x,y\in X$, $t\in T$, $(x,t)\succ_i(y,t)$ if and only if $(x,0)\succ_i(y,0)$ where $i=A,B$. (ii) For any $x,y\in X$, $t,s\in T$, $\alpha>0$, if $(x,t)\underset{\precsim_i}{\succsim}(y,t+\alpha)$, then $(x,s)\underset{\precsim_i}{\succsim}(y,s+\alpha)$ where $i=A,B$.
A5. (Continuity): For any $k\in\{2,3,\ldots,n\}$ there exists $u_k^A>0$ and $u_k^B>0$ such that, $(x_{k-1},u_k^A)\sim_A(x_k,0)$ and $(x_k,u_k^B)\sim_B(x_{k-1},0)$.

This is a perfect information extensive form game. Moreover, we shall assume that there is a complete information.

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[1] Ariel Rubinstein brought my attention to a paper by Eric van Damme, Reinhard Selten and Eyal Winter (van Damme et al., 1990) who independently have obtained a similar result.

3. Equilibria

Before we state and prove the main result (Proposition 1) that deals with the equilibria of the game we state and prove a key result (Lemma 1) which is at the heart of the proof of Propositon 1.

Lemma 1. *Given A3–A5 there exists $\Delta^* > 0$ such that for any $k \in \{2, 3, \ldots, n\}$, $t \in T$ and $0 < \Delta < \Delta^*$,*

$$(x_{k-1}, (t+1)\Delta) \succ_A (x_k, t\Delta) \text{ and}$$

$$(x_k, (t+1) \succ_B (x_{k-1}, t\Delta).$$

Proof. From A5 we have, for any $k \in \{2, 3, \ldots, n\}$ there exists $u_k^A > 0$ such that $(x_{k-1}, u_k^A) \sim_A (x_k, 0)$. Therefore, from A4 (ii) we have, for any $k \in \{2, 3, \ldots, n\}$ there exists $u_k^A > 0$ such that for any $t \in T$ and $\Delta > 0$, $(x_{k-1}, t\Delta + u_k^A) \sim_A (x_k, t\Delta)$. Hence, from A3 we have, for any $k \in \{2, 3, \ldots, n\}$ there exists $u_k^A > 0$ such that for any $t \in T$, $\Delta > 0$ and $0 < \alpha_A < u_k^A$, $(x_{k-1}, t\Delta + \alpha_A) \succ_A (x_k, t\Delta)$. Put $\alpha_A = \Delta$. Thus, for any $k \in \{2, 3, \ldots, n\}$ there exists $u_k^A > 0$ such that for any $t \in T$ and $0 < \Delta < u_k^A$, $(x_{k-1}, (t+1)\Delta) \succ_A (x_k, t\Delta)$. Let $\Delta_A = \min\{u_k^A : k = 2, 3, \ldots, n\}$. Note that $\Delta_A > 0$. Therefore, for any $k \in \{2, 3, \ldots, n\}$, $t \in T$ and $0 < \Delta < \Delta_A$, $(x_{-1}, (t+1)\Delta) \succ_A (x_A, t\Delta)$.

Similarly, there exists $\Delta_B = \min\{u_k^B : k = 2, 3, \ldots, n\}$ where $\Delta_B > 0$ such that for any $k \in \{2, 3, \ldots, n\}$, $t \in T$ and $0 < \Delta < \Delta_B$, $(x_k, (t+1)\Delta) \succ_B (x_{k-1}, t\Delta)$.

Let $\Delta^* = \min\{\Delta_A, \Delta_B\}$. Q.E.D.

Proposition 1. *If $\succsim_A$ and $\succsim_B$ satisfy assumptions A1–A5, then for any $\Delta \in (0, \Delta^*)$ (where Δ is the length of a single bargaining period and Δ^* is as defined in Lemma 1) and for any $k \in \{1, 2, \ldots, n\}$ the agreement $x_k \in X$ can be supported as a subgame perfect equilibrium of the Rubinstein bargaining game.*

Proof. Consider the following (stationary) strategies. Player A offers x_k, and only accepts agreements belonging to the set $\{x_1, x_2, \ldots, x_k\}$. Player B offers x_k, and only accepts agreements belonging to the set $\{x_k, x_{k+1}, \ldots, x_n\}$. Given our hypothesis, these strategies are in subgame perfect equilibrium. In particular, since $\Delta \in (0, \Delta^*)$, if follows from Lemma 1 that each player strictly prefers to obtain the agreement x_k "tomorrow" to *any* worse agreement "today". Q.E.D.

References

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