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Sequential bargaining and competition*

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Summary. This paper studies a sequential bargaining model of a decentralised market. A main objective is to explore the conditions under which the unique subgame perfect equilibrium outcome of the market game approximates the Walrasian outcome of the market. The three main messages that emerge from our results are as follows. First, contrary to conventional wisdom, frictionless markets need not be Walrasian. Second, the relative magnitudes of frictions can have a profound impact on the market outcome even in the limit as the absolute magnitudes of the frictions become negligible. And third, the relative magnitudes of certain types of frictions may have to be significantly large in order for markets to be Walrasian, reflecting that certain types of frictions are needed in the market in order to induce the Walrasian outcome.

1 Introduction

It is conventional wisdom that, under frictionless conditions, decentralised and centralised markets are Walrasian. Thus, under such conditions, the institutional structure of a market, which includes the particulars of the trading procedure, is supposed to have little impact on the market outcome. A main objective of this paper is to investigate this conventional wisdom. More generally, after describing a strategic bargaining model of a decentralised market, we explore the conditions under which the unique subgame perfect equilibrium outcome of the market game approximates the Walrasian outcome.

This paper is therefore a contribution to the growing literature on strategic bargaining models of markets, whose prime objective is to explore the scope and range of validity of the Walrasian equilibrium concept. For an excellent, and up-to-date, account of the various models that have been studied, and the results

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obtained, see Osborne and Rubinstein (1990). Although a few of the models differ from each other in some important respects, the central common idea underlying the models in this literature is, that agents of the opposite type meet in pairs, via a matching process, and then the matched pair engage in bilateral bargaining, according to some particular procedure, over the terms of trade.

A key feature in most of the models studied so far (see, for example, Gale, 1986 and 1987, and Rubinstein and Wolinsky, 1985 and 1990) is, that the bargaining procedure consists of the take-it-or-leave-it-offer game. Thus, if a matched pair do not reach agreement in their first bargaining session, then they are forced to abandon each other and rejoin the matching process. Indeed, they cannot choose to continue bargaining. The conclusions that are derived from such models tend to support the conventional wisdom, that frictionless markets are Walrasian. We note, that the results of this paper will reveal that the take-it-or-leave-it-offer procedure is very much responsible for such conclusions.

In this paper we construct a market game that does not incorporate the take-it-or-leave-it-offer procedure. Instead, the bargaining procedure is a Rubinstein-type infinite-horizon process, with the added feature that a matched agent can choose strategically, at certain exogenously specified points in the bargaining process, whether to opt out and rejoin the matching process or to continue bargaining. Indeed, for many self-evident and well-known reasons, such a trading procedure is far more appealing.

The conclusions we obtain from the analysis of the unique subgame perfect equilibrium of this market game are significantly different and quite revealing. First, contrary to conventional wisdom, it is shown that under frictionless conditions the market need not be Walrasian. Second, the relative magnitudes of the frictions have a profound impact on the market outcome even in the limit as the absolute magnitudes of the frictions approach zero. And third, the market is Walrasian if there exists some friction in the bargaining process but negligible friction in the matching process. The point is, that in a Walrasian market a matched pair of agents should be prevented from forming a bilateral bargaining relationship, but instead should be encouraged to move between such bargaining relationships. This last insight reveals that markets based on the take-it-or-leave-it-offer procedure will indeed be Walrasian (under frictionless conditions), since such a trading rule, by definition, prevents the formation of bilateral bargaining relationships.

The next two sections, respectively, describe the market game and characterise the unique subgame perfect equilibrium outcome. Then, section 4 contains the main results of, and the insights obtained from, this paper, on the conditions under which the equilibrium outcome approximates the Walrasian outcome.

2 The market game

The market considered in the model operates over the time interval $[0, +\infty]$. At time 0 the market opens with a finite number S of identical sellers and a finite number B of identical buyers, where $B > S \geqq 1$. It is assumed that after time 0 there is no entry into the market by either sellers or buyers. Each seller owns an indivisible object and her valuation of the object is normalised at zero, and each buyer's valuation of the object is normalised at one.

The market is modelled as a game in extensive form, which is built upon a random matching process, a Rubinstein-type sequential bargaining process and a procedure by means of which these two processes are interlaced. At any point in time a player will either have left the market after having executed a transaction, or be unmatched and thus, will be taking part in the random matching process, or be matched and thus, will be in a bargaining process. At time 0 all the players are unmatched.

The rate at which a match occurs is $nm\lambda$, given that n sellers and m buyers are taking part in the random matching process ($n = 0, 1, 2, \ldots, S$ and $m = 1, 2, 3, \ldots, B$), where $\lambda > 0$ is exogenously given. When a match occurs a seller and a buyer are paired randomly, in such a way that each of the n sellers and each of the m buyers are picked with equal probability.

When a seller and a buyer get matched, at some random time $t > 0$, they begin bargaining according to the following procedure. One of the players is selected randomly (with equal probability) to be the proposer, who immediately makes a price offer to the other player. The responder immediately will either accept the price offer (in which case the matched pair leave the market, at time t, having executed a transaction), or reject the offer and abandon the proposer (in which case, at time t, the pair rejoin the matching process), or reject the offer and continue bargaining according to this procedure $\Delta > 0$ units of time later, at time $t + \Delta$.

The payoffs to the players are as follows. If a seller (resp., a buyer) executes a transaction at time $t > 0$ and at the price $p \geqq 0$, then the (von Neumann–Morgenstern utility) payoff to the seller (resp., buyer) is pe^{-rt} (resp., $(1-p)e^{-rt}$), where $r > 0$ denotes the (common) rate of time preference. If a player stays in the market forever and thus, does not execute a transaction, then the (von Neumann–Morgenstern utility) payoff to the player is zero.

We make the following minimal assumptions concerning what a player knows about the history of the market when he has to make, or respond to, an offer. First, we assume that a player has perfect recall, in that he knows his personal history. Second, we assume that a player knows the number of sellers and the number of buyers taking part in the matching process, and he also knows the number of other matched pairs engaged in bargaining. We note, that the results of this paper are robust to any further information assumed to be known by a player.

Finally, we assume that the game is common knowledge amongst the players. The subgame perfect equilibrium concept (henceforth, SPE) will be employed to analyse the market game.

3 Market equilibrium

Proposition 1 below characterises the unique SPE path of the market game. The proof is by induction and is deferred to the Appendix, since the arguments involved are essentially based on the standard (but ingenious) arguments invented in Rubinstein (1982) and Shaked and Sutton (1984). An explanation of the result contained in this proposition is provided after we state the proposition.

Proposition 1. *For any $r > 0, \Delta > 0, \lambda > 0$ and $B > S \geqq 1$, the market game has a unique SPE path. In equilibrium, the ith matched seller and buyer (where $i = 1, 2, \ldots, S$) reach agreement immediately on the price x_i or y_i according to whether it is the buyer* *or the seller who is selected to be the proposer, with the values of* x_i *and* y_i *being independent of both the random time at which the ith match occurs and the particular buyer and seller who become the ith matched pair. The equilibrium prices* $(x_i, y_i)_{i=1}^S$ *are defined inductively by the following equations:*

$$x_i = \max\{V_i^s, R_i^s\} \tag{1}$$

$$1 - y_i = \max\{V_i^b, R_i^b\} \tag{2}$$

where, for $i = 1, 2, \dots, S$,

$$V_i^s = \left[\frac{\lambda(B-i+1)[(x_i+y_i)(1/2)+(S-i)V_{i+1}^s]}{r+(B-i+1)(S-i+1)\lambda}\right] \tag{3}$$

$$V_i^b = \left[\frac{\lambda(S-i+1)[(1-x_i+1-y_i)(1/2)+(B-i)V_{i+1}^b]}{r+(B-i+1)(S-i+1)\lambda}\right] \tag{4}$$

with $V_{S+1}^s = V_{S+1}^b = 0$ *and where, for* $i = 1, 2, \dots, S$,

$$R_i^s = e^{-r\Delta}\left[\sum_{q=i}^{S}[(x_q+y_q)(1/2)P_i^q]\right] \tag{5}$$

$$R_i^b = e^{-r\Delta}\left[\sum_{q=i}^{S}[(1-x_q+1-y_q)(1/2)P_i^q]\right] \tag{6}$$

with P_i^q *denoting the probability that* $q - i$ *matches occur in* Δ *units of time given that* $S - i$ *sellers and* $B - i$ *buyers are taking part in the matching process.*

An intuitive explanation for this proposition lies in the observation, that the essential strategic structure underlying the market game consists of Rubinstein's bilateral bargaining game with outside options, where a player can opt out only after he rejects his opponent's offer (cf. Osborne and Rubinstein, 1990, section 3.12.1). For each $i = 1, 2, \dots, S$, V_i^s (resp., V_i^b) is the equilibrium expected payoff to a seller (resp., a buyer) from being unmatched in the market with $i - 1$ sellers and $i - 1$ buyers having left the market and with a total of $S - i + 1$ sellers and $B - i + 1$ buyers taking part in the matching process. Indeed, V_i^s (resp., V_i^b) is the equilibrium value of the outside option to a seller (resp., buyer), who has just become part of the ith match. Furthermore, for each $i = 1, 2, \dots, S$, R_i^s (resp., R_i^b) is the equilibrium expected payoff to a seller (resp., buyer), who has just become part of the ith match, from rejecting the equilibrium price offer x_i (resp., y_i) and instead continuing to bargain with his matched opponent.

Equation 1 (resp., 2) is derived from the fact, that in equilibrium the matched buyer (resp., seller) will offer the lowest (resp., highest) price acceptable to the seller (resp., buyer). We now show how V_i^s and R_i^s are computed. We first compute V_i^s. If $S - i + 1$ sellers and $B - i + 1$ buyers are taking part in the matching process, then when a match occurs (at some random time t) the probability that any particular seller is the one who is matched is $1/(S - i + 1)$. Hence, in equilibrium, with probability $1/(S - i + 1)$ a seller obtains an expected payoff of $(x_i + y_i)/2$ and with probability $(S - i)/(S - i + 1)$ a seller obtains an expected payoff of V_{i+1}^s. Thus, the equilibrium expected payoff to a seller is

$$\underset{t \to +\infty}{\text{limit}} \int_0^t [(1/(S-i+1))((x_i+y_i)/2) + ((S-i)/(S-i+1))V_{i+1}^s] e^{-rt}(\lambda_1 e^{-\lambda_1 t})dt$$

where $\lambda_1 = (S-i+1)(B-i+1)\lambda$ is the rate at which a match occurs. It is easy to verify that this is equal to V_i^s as defined by equation 3.

R_i^s as defined by equation 5 is computed as follows. If the seller, who has just become part of the ith match, rejects the equilibrium price offer x_i and instead continues to bargain, then he has to wait for $\Delta > 0$ units of time before the second bargaining session can take place. During this time interval $q-i$ matches will occur (where $q = i, i+1, \ldots, S$) with some probability, denoted by P_i^q. Hence, with probability P_i^q, in equilibrium a further $q-i$ sellers and $q-i$ buyers will have left the market by the time the matched pair have their second bargaining session. Hence, in equilibrium, with probability P_i^q the matched seller and buyer will execute a transaction at the expected price of $(x_q + y_q)/2$.

4 Competition

This section addresses a main concern of this paper, namely, whether the unique SPE outcome converges, as the frictions in the market become arbitrarily small, to the Walrasian equilibrium outcome of the market. More generally, we investigate the conditions under which the SPE outcome approximates the Walrasian outcome. The unique Walrasian equilibrium price of the market is one, since $B > S$. Thus, in the Walrasian outcome a seller obtains all the surplus.

It is not surprising that the unique SPE prices $(x_i, y_i)_{i=1}^S$, defined in Proposition 1, need not be uniform and, moreover, need not be equal to one. After all, the market contains two different forms of frictions. First, there is a friction associated with the bargaining process, namely, that bargaining takes time (i.e., $\Delta > 0$) and that time is valuable (i.e., $r > 0$). Second, there is a friction associated with the matching process, namely, that matching takes time (since both B and λ are finite) and that time is valuable.

Although it is evident, that in the limit, as $r \to 0+$, both the bargaining and the matching related frictions would converge to zero, it turns out that the limiting SPE prices, as $r \to 0+$, need not be uniform, and this is because the market contains a finite number B of buyers. Hence, it would appear that a necessary condition for the SPE outcome to approximate the Walrasian outcome is, that the market contains an arbitrarily large number of buyers.

The next proposition characterises, for any finite $S \geqq 1$, the limiting SPE outcome of the market game, as $r \to 0+$, $B \to +\infty$, $\Delta \to 0+$ and $\lambda \to +\infty$. These conditions would appear to make the market as competitive as is possible. The formal (inductive) proof of this proposition is deferred to the Appendix.

Proposition 2. *Let α denote an arbitrary number in the closed interval $[0, +\infty]$. Furthermore, assume that in the limit, as $\Delta \to 0+$, $B \to +\infty$ and $\lambda \to +\infty$, the ratio $\Delta/(1/B\lambda) \to \alpha$. Then, for any $S \geqq 1$ in the limit, as $r \to 0+$, $\Delta \to 0+$, $B \to +\infty$, $\lambda \to +\infty$*

and $\Delta/(1/B\lambda)\to\alpha$, *for all* $i=1,2,\ldots,S$, $x_i\to z$, $y_i\to z$, $V_i^s\to z$ *and* $V_i^b\to 0$, *where*

$$z=\begin{cases}1/2 & \text{if}\quad 0\leqq\alpha\leqq 1\\ \alpha/(1+\alpha) & \text{if}\quad 1\leqq\alpha+\infty\\ 1 & \text{if}\quad \alpha=+\infty\end{cases}$$

Hence, from Proposition 2 we see, that under frictionless conditions the market is Walrasian if, in addition, the expected time $1/(B\lambda)$ taken by a seller to get matched is infinitely smaller than the time Δ between two consecutive offers within a bargaining game between a matched seller and buyer. Indeed, the market will be Walrasian provided there exists some friction in the bargaining process but not in the matching process. To phrase it more generally, a message contained in this proposition is, that Walrasian markets are characterised by the property, that in such markets it is costly to discuss the terms of trade but costless to switch between potential trading partners.

Before we provide an intuitive explanation, and an interpretation, of this proposition we emphasize the three general messages of fundamental importance that emerge from it. First, contrary to conventional wisdom, frictionless markets need not be Walrasian. Second, the relative magnitudes of frictions can have a profound impact on the market outcome even in the limit as the absolute magnitudes of the frictions become negligible. And third, the relative magnitudes of certain types of frictions may have to be significantly large in order for markets to be Walrasian, reflecting that certain types of frictions are needed in the market in order to induce the Walrasian outcome.

The result contained in Proposition 2 can be explained intuitively with reference to a basic insight derived from Rubinstein-type bilateral bargaining models, namely, that the share of the unit surplus obtained by a player depends on the relative magnitudes of Δ_s and Δ_b, where $\Delta_i>0$ $(i=s,b)$ denotes the amount of time that player i must wait after he rejects player j's offer ($j\neq i$ and $j=s,b$) and before he can make, or receive, another offer, and this is so even in the limit as $\Delta_s\to 0+$ and $\Delta_b\to 0+$. We now use this result to provide an intuitive explanation of Proposition 2. First, note that in the limiting SPE a buyer will never opt out and leave his matched seller, since his probability of being rematched with a seller is zero. Hence, in a bilateral bargaining game between a matched pair, $\Delta_b=\Delta$. Second, note that a seller will opt out, rather than continue bargaining with his matched buyer, if the expected time taken for a seller to get rematched with a buyer, which is (effectively) $1/(B\lambda)$, is smaller than Δ, and even in the limit, as $\Delta\to 0+$ and $1/(B\lambda)\to 0+$, the ratio $\Delta/(1/B\lambda)$ matters to that decision. Hence, $\Delta_s=\min\{\Delta,1/(B\lambda)\}$. The intuitive explanation for Proposition 2 is now evident.

An interpretation of Proposition 2 now follows, and it rests on interpreting the value of α as the number of bargaining-relationships that a seller can go through in a negligible amount of time, where a bargaining-relationship denotes an on-going bargaining game between a matched seller and buyer, and the expected time taken for a seller to move between two consecutive bargaining-relationships is $1/(B\lambda)$. Under frictionless conditions, the market is Walrasian if, in addition, it is the case that a seller can credibly threaten a buyer that he can go through an infinite number of bargaining-relationships in a negligible amount of time. Moreover, if this additional condition fails to hold, then the market is non-Walrasian. For example, if it takes the seller a non-negligible amount of time to go through an infinite number of bargaining-relationships, then, since time is valuable, a matched buyer can extract some positive surplus, thus generating a non-Walrasian outcome. The point is, that when a seller is matched with a buyer, he can extract full surplus if with negligible cost he can: move to another buyer, and then move yet again to another buyer, and again, ..., *ad infinitum*.

We conclude with two further observations. First, it should be noted, that the limiting SPE outcome, defined in Proposition 2, can be described by an asymmetric Nash bargaining solution, calculated with the payoff set $\{(u_s, u_b): u_s = p$ and $u_b = 1 - p$ where $0 \leqq p \leqq 1\}$, "disagreement point" (0, 0) and "bargaining powers" τ_s and τ_b to a seller and a buyer, respectively, where $\tau_s = \min\{\alpha, 1\}$ and $\tau_b = 1$. Indeed, under frictionless conditions, the market outcome approximates an asymmetric Nash bargaining solution and not the Walrasian outcome.

Second, we note, that if the market game were altered to contain the feature that a matched pair are forced to rejoin the matching process after some finite, and exogenously specified, number of bargaining sessions, then it can be established that, contrary to Proposition 2, frictionless markets are Walrasian. This sort of result has been established by a number of authors, including, Binmore and Herrero (1988), Gale (1986, 1987) and Rubinstein and Wolinsky (1985, 1990), all of whom model a market with the take-it-or-leave-it-offer bargaining procedure (see chapters 7 and 8 in Osborne and Rubinstein, 1990, for an account of these papers). The following intuition for this fundamental difference in results is quite instructive. In our market game players can choose to develop and maintain a particular bargaining-relationship, since they can choose, strategically, whether and when to opt out and break the bargaining-relationship. In contrast, in market games that contain the take-it-or-leave-it-offer bargaining procedure matched agents, by being forced to abandon each other after one bargaining session, cannot develop and maintain such bargaining-relationships. The results in this paper reveal, that frictionless markets are Walrasian if such bilateral bargaining-relationships are prevented from being formed. This is indeed the case in market games which contain the take-it-or-leave-it-offer procedure. Finally, it is interesting to note, that a market game with the take-it-or-leave-it-offer procedure can be interpreted as a special case, with $\Delta = +\infty$, of the market game analysed in this paper. Given this interpretation, it is evident from the results of this paper as to why such a frictionless market is Walrasian.

Appendix

This Appendix contains the proofs of Propositions 1 and 2.

Proof of Proposition 1

The proof involves the following inductive argument. First, the proposition is established for $S = 1$ and $B = m > 1$. Second, given the inductive hypothesis that the proposition is true for $S = n$ and $B = m + n - 1$ (where $1 \leqq n < +\infty$), it is deduced that the proposition must be true for $S = n + 1$ and $B = m + n$. It thus follows from the principle of mathematical induction, that for all $n \in \{1, 2, \ldots\}$ Proposition 1 is true for $S = n$ and $B = m + n - 1$. Hence, since m is an arbitrary natural number, Proposition 1 is indeed true for any B and S such that $B > S \geq 1$.

The arguments involved in establishing the proposition for $S = 1$ and $B = m$ are identical to those involved in establishing the inductive step (but with fewer computations). Hence, we proceed to establish the inductive step, and therefore we now analyse the SPEa of the market game with $S = n + 1$ and $B = m + n$. Given the underlying stationarity of the structure of the game we can define two sets A_s and A_b as follows. $A_s = \{y{:}y$ is a SPE payoff to a seller in a subgame starting with the seller's offer, given that the other n sellers and $m + n - 1$ buyers are unmatched in the market$\}$ and $A_b = \{x{:}x$ is a SPE payoff to a seller in a subgame starting with the buyer's offer, given that the other n sellers and $m + n - 1$ buyers are unmatched in the market$\}$. Let M_i (resp., m_i), where $i = b, s$, denote the supremum (resp., infimum) of the set A_i. Steps 1–4, stated below, will establish, that both $(x, y) = (M_b, M_s)$ and $(x, y) = (m_b, m_s)$ are solutions to the following pair of equations:

$$x = \max\{J_1(x, y), J_2(x, y)\} \tag{7}$$

$$1 - y = \max\{J_3(x, y), J_4(x, y)\} \tag{8}$$

where J_1 to J_4 are defined as follows.

$$J_1(x, y) = \left[\frac{\lambda(m+n)[(x+y)(1/2) + nV_1^s]}{r + (n+1)(m+n)\lambda}\right] \tag{9}$$

$$J_2(x, y) = e^{-r\Delta}\left[[(x+y)(1/2)P_1^1] + \sum_{q=2}^{n+1}[(x_{q-1} + y_{q-1})(1/2)P_1^q]\right] \tag{10}$$

$$J_3(x, y) = \left[\frac{\lambda(n+1)[(1 - x + 1 - y)(1/2) + (m+n-1)V_1^b]}{r + (n+1)(m+n)\lambda}\right] \tag{11}$$

$$J_4(x, y) = e^{-r\Delta}\left[[(1 - x + 1 - y)(1/2)P_1^1] + \sum_{q=2}^{n+1}[(1 - x_{q-1} + 1 - y_{q-1})(1/2)P_1^q]\right] \tag{12}$$

where $(x_{q-1}, y_{q-1})_{q=2}^{n+1}$ are the equilibrium prices defined in Proposition 1 with $S = n$ and $B = m + n - 1$, V_1^s and V_1^b are defined in Proposition 1 with $S = n$ and $B = m + n - 1$, and P_1^q (for $q = 1, 2, \ldots, n + 1$) denotes the probability that $q - 1$ matches occur in $\Delta > 0$ units of time given than n sellers and $m + n - 1$ buyers take part in the matching process.

Step 5, stated below, establishes that equations 7 and 8 have a unique solution. Therefore, $M_b = m_b$ and $M_s = m_s$. It then follows from standard arguments (cf. Osborne and Rubinstein, 1990, section 3.8) that, in equilibrium, in the market game with $S = n + 1$ and $B = m + n$ the first matched pair will reach agreement immediately on the price $M_b = m_b$ or $M_s = m_s$ according to whether it is the buyer or the seller who is selected to propose. Moreover, it is straightforward to verify, that $M_b = m_b = x_1$ and $M_s = m_s = y_1$ where x_1 and y_1 are as defined in Proposition 1 with $S = n + 1$ and $B = m + n$. Note, that after the first matched pair leave the market, then the market consists of n sellers and $m+n-1$ buyers, all of whom are unmatched. Thus, this establishes the inductive step, and hence completes the proof of Proposition 1.

Step 1. *If* $x \in A_b$, $y \in A_s$ *and* $w = \max\{J_1(x, y), J_2(x, y)\}$ *where* J_1 *and* J_2 *are defined by equations 9 and 10, then* $w \in A_b$.

Proof of Step 1. Let $x \in A_b$ and $y \in A_s$. Consider the following strategies in a subgame starting with a buyer's offer (and given that the other n sellers and $m+n-1$ buyers are unmatched in the market). The buyer offers price w, where w is as defined in Step 1, and the seller agrees to w and any price above it. Furthermore, if the seller rejects any offer and either opts out or continues bargaining, then all the players follow the SPE strategies that support a seller's payoff of x or y, according to whether in the next bargaining session (whenever it will take place and whomsoever are the matched pair) it is the buyer or the seller who is selected to propose. We shall now establish that these strategies constitute a SPE, and thus, it follows that $w \in A_b$.

We first show that the seller's behaviour of agreeing to w and any price above it is optimal. This is done by establishing that the seller's payoff from rejecting any offer and opting out is $J_1(x, y)$, and that the seller's payoff from rejecting any offer and continuing to bargain is $J_2(x, y)$.

We first compute the expected payoff to the seller from rejecting any offer and opting out. There would then be $n+1$ sellers and $m+n$ buyers taking part in the matching process. Hence, when a match occurs (at some random time t) the probability that any particular seller is the one who is matched is $1/(n+1)$. Hence, with probability $1/(n+1)$, the seller obtains an expected payoff of $(x+y)/2$ and with probability $n/(n+1)$ the seller obtains an expected payoff of V_1^s, which is defined by equation 3 in Proposition 1 with $S=n$ and $B=m+n-1$ (we are invoking the inductive hypothesis here). Thus, the expected payoff to the seller is equal to

$$\operatorname*{limit}_{t \to +\infty} \int_0^t [(1/(n+1))((x+y)/2) + (n/(n+1))V_1^s]e^{-rt}(\lambda_2 e^{-\lambda_2 t})dt$$

where $\lambda_2 = (n+1)(m+n)\lambda$ is the rate at which a match occurs. Upon solving this integral, it follows that this expression equals $J_1(x, y)$ as defined by equation 9.

We now compute the expected payoff to the seller from rejecting any offer and continuing to bargain. During the time interval $\Delta > 0$, $q-1$ matches will occur (where $q = 1, 2, \ldots, n+1$) with some probability, denoted by P_1^q. Hence, with probability P_1^1 the expected payoff to the seller is $(x+y)/2$. And, with probability P_1^q (where $q = 2, 3, \ldots, n+1$) the seller's expected payoff is $(x_{q-1} + y_{q-1})/2$, where $(x_{q-1}, y_{q-1})_{q=2}^{n+1}$ are the equilibrium prices defined in Proposition 1 with $S=n$ and $B=m+n-1$ (we are invoking the inductive hypothesis here). Hence, the seller's expected payoff is $J_2(x, y)$ as defined by equation 10.

We now argue that the buyer's behaviour of offering w is optimal. First of all suppose, that $J_1(x, y) \geqq J_2(x, y)$. If the buyer deviates and offers a price less than w, which is now equal to $J_1(x, y)$, then the seller will reject and opt out. By arguments similar to those given above, it can be shown that, then the buyers's payoff would equal $J_3(x, y)$ as defined by equation 11. Since $1 - w = 1 - J_1(x, y) \geqq J_3(x, y)$ such a deviation would not be profitable for the buyer. Now suppose, that $J_2(x, y) \geqq J_1(x, y)$. If the buyer deviates and offers a price less than w, which now equals $J_2(x, y)$, then the seller will reject and continue bargaining. Again, by arguments similar to those given above, it can be established that, then the buyer's payoff would equal $J_4(x, y)$ as defined by equation 12. Since $1 - w = 1 - J_2(x, y) \geqq J_4(x, y)$ such a deviation would not be profitable. QED

Step 2. *$M_b = \max\{J_1(M_b, M_s), J_2(M_b, M_s)\}$ where J_1 and J_2 are defined by equations 9 and 10.*

Proof of Step 2. From Step 1 it follows, that $M_b \geqq \max\{J_1(M_b, M_s), J_2(M_b, M_s)\}$. Suppose this inequality is strict. Then there exists a $w \in A_b$ such that $M_b \geqq w > \max\{J_1(M_b, M_s), J_2(M_b, M_s)\}$. We now obtain a contradiction. If the buyer deviates and instead offers the price of $\max\{J_1(M_b, M_s), J_2(M_b, M_s)\}$, then perfectness implies that the seller will accept. Since the buyer strictly prefers this, he will profit from such a defivation. QED

Step 3. *If $x \in A_b$, $y \in A_s$ and $1 - w = \max\{J_3(x, y), J_4(x, y)\}$ where J_3 and J_4 are defined by equations 11 and 12, then $w \in A_s$.*

Proof of Step 3. Identical arguments to those presented in the proof of Step 1. QED

Step 4. *$1 - M_s = \max\{J_3(M_b, M_s), J_4(M_b, M_s)\}$ where J_3 and J_4 are defined by equations 11 and 12.*

Proof of Step 4. From Step 3 it follows, that $M_s \geqq 1 - \max\{J_3(M_b, M_s), J_4(M_b, M_s)\}$. Suppose this inequality is strict. Then there exists a $w \in A_s$ such that $M_s \geqq w > 1 - \max\{J_3(M_b, M_s), J_4(M_b, M_s)\}$, i.e., $1 - M_s \leqq 1 - w < \max\{J_3(M_b, M_s), J_4(M_b, M_s)\}$. A contradiction is obtained, because if the buyer rejects any offer, then in equilibrium he gets at least an expected payoff of $\max\{J_3(M_b, M_s), J_4(M_b, M_s)\}$. QED

Arguments similar to those presented in the proofs of Steps 2 and 4 establish that $(x, y) = (m_b, m_s)$ is also a solution to equations 7 and 8.

Step 5. *Equations 7 and 8 have a unique solution.*

Proof of Step 5. The proof is based on the Banach contraction fixed point theorem. Rewrite equations 7 and 8 as: $(x, y) = H(x, y)$ where $H(\cdot) = (\max\{J_1(\cdot), J_2(\cdot)\}, 1 - \max\{J_3(\cdot), J_4(\cdot)\})$. H is a mapping from the complete metric space $[0, 1] \times [0, 1]$ into itself. If there exists a $0 < \mu < 1$ such that for any (x, y), $|\partial J_1(\cdot)/\partial x| + |\partial J_1(\cdot)/\partial y| \leqq \mu$, $|\partial J_2(\cdot)/\partial x| + |\partial J_2(\cdot)/\partial y| \leqq \mu$, $|\partial J_3(\cdot)/\partial x| + |\partial J_3(\cdot)/\partial y| \leqq \mu$ and $|\partial J_4(\cdot)/\partial x| + |\partial J_4(\cdot)/\partial y| \leqq \mu$, then H is a contraction mapping. By performing these simple computations using equations 9–12, it can be established that such a μ does indeed exist. QED

Proof of Proposition 2

The proof involves an inductive argument on the number of sellers. Step 1, stated below, establishes the proposition for $S = 1$. Then, given the inductive hypothesis that the proposition is true for $S = n$, Step 2 establishes that the proposition must therefore be true for $S = n + 1$. Proposition 2 then follows from the principle of mathematical induction.

Step 1. *Proposition 2 is true for $S = 1$.*

Proof of Step 1. Let $S = 1$. First solve explicitly for x_1, y_1, V_1^s and V_1^b using equations 1–6 defined in Proposition 1. Step 1 then follows immediately after taking the limits. For example, $x_1 = e^{-r\Delta}/2$ if $e^{-r\Delta} \geqq [B\lambda/(r + B\lambda)]$ and $e^{-r\Delta} \geqq [\lambda/(r + B\lambda)]$. In the limit, as $r \to 0+, \Delta \to 0+, \lambda \to +\infty, B \to +\infty$ and $\Delta/(1/B\lambda) \to \alpha$, this latter inequality is satisfied. Whether the former inequality will hold in this limit is made evident after the following manipulation. First, rewrite it as $re^{-r\Delta} \geqq B\lambda(1 - e^{-r\Delta})$. Second, substitute $r\Delta[1 + \phi(r, \Delta)]$ for $(1 - e^{-r\Delta})$, where $\phi(r, \Delta) = \sum_{n=1}^{\infty} [(-1)^n (r\Delta)^n/(n + 1)!]$. This substitution is derived by using the power series expansion of $e^{-r\Delta}$. Note, that $\phi(r, \Delta) \to 0+$ as $r \to 0+$ and $\Delta \to 0+$. Hence, the inequality becomes $e^{-r\Delta} \geqq [\Delta/(1/B\lambda)][1 + \phi(r, \Delta)]$, since $r > 0$. In the relevant limit this inequality will hold if $1 \geqq \alpha$. QED

Step 2. *If Proposition 2 is true for $S = n$, then it is true for $S = n + 1$.*

Proof of Step 2. Let $S = n + 1$. From the inductive hypothesis that Proposition 2 is true for $S = n$ we have in particular, that in the relevant limit, for $i = 2, 3, \ldots, n + 1$, $x_i \to z$, $y_i \to z$ and that $V_2^s \to z$ and $V_2^b \to 0$, where $(x_i, y_i)_{i=2}^{n+1}$ and (V_2^s, V_2^b) are defined in Proposition 1 with $S = n + 1$. In order to establish Step 2 we now have to show, that in the relevant limit, $x_1 \to z$, $y_1 \to z$, $V_1^s \to z$ and $V_1^b \to 0$ where x_1, y_1, V_1^s and V_1^b are defined in Proposition 1 with $S = n + 1$. This can be easily established, by using equations 1–6 (with $i = 1$ and $S = n + 1$) and the inductive hypothesis as stated above. QED

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