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Bargaining in a non-stationary environment ☆

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Abstract

We study an alternating offers bargaining model in which the set of possible utility pairs evolves through time in a non-stationary, but smooth manner. In general, there exists a multiplicity of subgame perfect equilibria. However, we show that in the limit as the time interval between two consecutive offers becomes arbitrarily small, there exists a unique subgame perfect equilibrium. Furthermore, we derive a powerful characterization of the unique (limiting) subgame perfect equilibrium payoffs. We then explore the circumstances under which Nash's bargaining solution implements this bargaining equilibrium. Finally, we extend our results to the case when the players have time-varying inside options.

JEL classification: C7; D4

Keywords: Alternating offers; Non-stationary environment; Subgame perfect equilibrium

1. Introduction

This paper studies Rubinstein's alternating offers bargaining game [15] in a non-stationary setting. Binmore [1, Section 6] considers this case and shows through an example that for any positive time interval, Δ, between two consecutive offers, a continuum of subgame perfect equilibria (SPE) is possible. We adopt the same assumptions, but also assume that the Pareto frontier evolves smoothly through time. Although multiple SPE are possible for any $\Delta > 0$, we show that as $\Delta \to 0$, the set of SPE converges to a unique (limiting) SPE. Moreover the limiting SPE is described by a simple differential equation, whose solution has a clear geometric

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interpretation and which is tractable for dynamic applications. We also identify conditions under which the Nash bargaining solution (with appropriate threat-points) implements this solution.

There is a large matching literature where equilibrium is determined by bargaining. Typically that literature adopts the static Nash bargaining approach (see for example [14]). Binmore [1] and Binmore et al. [2] show this can be reasonable in steady-state situations where payoffs do not change over time. However, more recently the matching literature has considered non-steady-state equilibria (see for example [5,7,10,17]) and bargaining situations where agents have time-varying payoffs (for example, a worker's unemployment benefit entitlement may be about to expire, or a worker's job skills may decline while unemployed, e.g. [4,8,14]). In such environments, the strategic bargaining approach determines the equilibrium terms of trade in a way which is consistent with how payoffs are expected to evolve over time. In contrast, determining the terms of trade using the static Nash bargaining approach may be restrictive. For example, Coles and Wright [5] demonstrate that the two bargaining approaches can support qualitatively different matching equilibria; in the context of a monetary economy they show that equilibria with strategic bargaining can exhibit trading cycles, while the application of the static Nash bargaining approach rules out such equilibria.

Coles and Wright [5] consider the non-stationary bargaining problem studied here. This paper supplements their results in three ways. First, Coles and Wright [5] do not establish that their 'differential equation' describes the limiting equilibrium to the bargaining game as $\Delta \to 0$. They simply assume it to be the case. This is not obvious given Binmore's continuum example. Second, Coles and Wright [5] provide a uniqueness argument which applies only to a special case; that agents use Markov strategies, that payoffs are additively separable (i.e., are of the form $u_i(x) + v_i(t)$) and that the $v_i(t)$ converge to some limit as $t \to \infty$.[1] We do not impose these restrictions, especially as the latter would require that the underlying market equilibrium converges to a steady state (which is formally inconsistent with their limit cycle example, and to extended models which allow for say endogenous growth and/or technology shocks). Third, our paper provides a nice geometric interpretation for the limiting equilibrium which shows how the strategic bargaining approach and the Nash bargaining approach are properly related.

Other related work includes Merlo and Wilson [9] and Cripps [6]. Those papers assume that two agents negotiate over some pie (x_1, x_2) satisfying $x_1 + x_2 \leqslant M_t$, where agreement at time t implies that agent i's payoff is $e^{-rt}u_i(x_i)$. If M_t evolves deterministically, then that preference structure is a special case of those considered in this paper—it describes a one-shot bargaining game where given agreement, the traders then exit the market for good. In essence that bargaining structure describes an optimal tree-felling problem where M_t evolves according to an (exogenous,

[1] In the context of their monetary economy, $u_i(x)$ is the instantaneous payoff to agent i from obtaining x units of a "pie", while $v_i(t)$ is the expected payoff to an unmatched agent i at time t in a market equilibrium.

stationary) Markov process. The frameworks are different and, in general, one is not a subset of the other.

The rest of this paper is organized as follows. Section 2 describes the model and Section 3 considers its SPE. Section 4 considers the relationship between the unique (limiting) SPE and the Nash bargaining approach. Section 5 provides an illustrative example in which agents have time-varying discount rates. Section 6 extends the analysis to time-varying inside options.

2. The model

Two players, A and B, bargain according to an alternating-offers procedure, where the set $\Omega(t)$ of possible utility pairs available at time t is a non-empty subset of $\Re^2$. Bargaining begins at time s, where the players negotiate according to the following procedure. At time $s + n\Delta$ (where $n \in \mathbb{N} \equiv \{0, 1, 2, \dots\}$, and $\Delta > 0$), player i makes an offer to player j ($j \neq i$), where $i = \mathrm{A}$ if n is even (i.e., $n = 0, 2, 4, 6, \dots$) and $i = \mathrm{B}$ if n is odd (i.e., $n = 1, 3, 5, \dots$). An offer at time $s + n\Delta$ is a utility pair $(u_\mathrm{A}, u_\mathrm{B})$ from the set $\Omega(s + n\Delta)$. Player j then decides whether to accept or reject the proposed offer. If she accepts the offer, then the bargaining game ends. Otherwise, Δ time units later, at time $s + (n+1)\Delta$, player j makes a counteroffer to player i.

The payoffs are as follows. If the players reach agreement at time $s + n\Delta$ (where $n \in \mathbb{N}$) on $(u_\mathrm{A}, u_\mathrm{B}) \in \Omega(s + n\Delta)$, then player i's ($i = \mathrm{A}, \mathrm{B}$) payoff is u_i. On the other hand, if the players perpetually disagree (i.e., each player always rejects any offer made to her), then each player obtains a payoff of zero.

Let $\Omega^\mathrm{P}(t)$ denote the Pareto frontier at time t—that is, the set of Pareto efficient utility pairs available at time t. We assume that $\Omega^\mathrm{P}(t)$ is a connected set. Furthermore, there exists $\bar{u}^t_\mathrm{A} > 0$ and $\bar{u}^t_\mathrm{B} > 0$ such that $(0, \bar{u}^t_\mathrm{B}) \in \Omega^\mathrm{P}(t)$ and $(\bar{u}^t_\mathrm{A}, 0) \in \Omega^\mathrm{P}(t)$. For convenience, we describe this frontier by a function ϕ where $u_\mathrm{B} = \phi(u_\mathrm{A}, t)$ if and only if $(u_\mathrm{A}, u_\mathrm{B}) \in \Omega^\mathrm{P}(t)$, where the assumptions above imply ϕ is strictly decreasing in u_A for all $u_\mathrm{A} \in [0, \bar{u}^t_\mathrm{A}]$. The following two assumptions are standard in the literature:

Assumption 1 (Concave Pareto frontiers). For each $t \geqslant 0$, $\phi(., t)$ is concave in u_A on the interval $[0, \bar{u}^t_\mathrm{A}]$.

Assumption 2 (Shrinking and vanishing Pareto frontiers). (i) For any $t \geqslant 0$ and $u_\mathrm{A} \in [0, \bar{u}^t_\mathrm{A}]$, $\phi(u_\mathrm{A}, t') > \phi(u_\mathrm{A}, t)$ for all $t' < t$, and (ii) for any $\varepsilon > 0$ there exists a $T > 0$ such that $\bar{u}^t_\mathrm{A} < \varepsilon$ and $\bar{u}^t_\mathrm{B} < \varepsilon$ for all $t > T$.

Our third assumption replaces the (standard) stationarity assumption—we only require that the Pareto frontier evolves smoothly over time:

Assumption 3 (Smoothly evolving Pareto frontiers). ϕ is continuously differentiable in t and u_A.

As Assumption 1 implies that (for any t) ϕ is differentiable in u_A *almost everywhere*, the main role of Assumption 3 is that it ensures that the time derivative exists—the Pareto frontier evolves smoothly over time. This plays no role when $\Delta > 0$, but is important in the limit as $\Delta \to 0$.

3. Characterizing equilibria

Using Assumptions 1–3, our objective here is to characterize the set of equilibria in the limit as $\Delta \to 0$.[2] In Section 3.1, we first characterize the set of Markov SPE, and establish their existence. Then, in Section 3.2, we define our *candidate limiting equilibrium*. Section 3.3 establishes our convergence results: Theorem 1 establishes that in the limit as $\Delta \to 0$, any Markov SPE converges to the candidate limiting equilibrium, and then Theorem 2 establishes that in this limit any non-Markov SPE also converges to this limiting equilibrium.

Before restricting attention to Markov SPE, note that in any SPE of any subgame beginning at any time t, player i's ($i = \mathrm{A}, \mathrm{B}$) equilibrium payoff lies between zero and $\bar{u}_i^t$.

3.1. Markov equilibria: characterization and existence

For any $\Delta > 0$ (fixed), we first consider the set of all Markov SPE. Let Γ_i ($i = \mathrm{A}, \mathrm{B}$) denote the set of times at which player i makes an offer, i.e.

$$\Gamma_\mathrm{A} = \{s, s + 2\Delta, s + 4\Delta, \ldots\} \quad \text{and} \quad \Gamma_\mathrm{B} = \{s + \Delta, s + 3\Delta, s + 5\Delta, \ldots\},$$

and define $\Gamma = \Gamma_\mathrm{A} \cup \Gamma_\mathrm{B}$.

Now consider an arbitrary Markov SPE. For each $t \in \Gamma$, let $v(t) = (v_\mathrm{A}(t), v_\mathrm{B}(t))$ (where $v(t) \in \Omega(t)$) denote the equilibrium offer made at time t. It is straightforward to show that Assumption 2(i) (shrinking pie) and the restriction to Markov strategies imply there is no delay to trade in equilibrium, i.e. for any $t \in \Gamma$, the equilibrium offer $v(t)$ is accepted. This implies that at any time $t \in \Gamma_i$ ($i = \mathrm{A}, \mathrm{B}$), in equilibrium player j ($j \neq i$) accepts an offer $(u_\mathrm{A}, u_\mathrm{B}) \in \Omega(t)$ if and only if $u_j \geqslant v_j(t + \Delta)$. It thus follows that the equilibrium offer $v(t)$ at time $t \in \Gamma_i$ satisfies two standard properties, which are formally stated below in (1) and (2). Eq. (1) states that in equilibrium player j is indifferent between accepting and rejecting player i's equilibrium offer $v(t)$ made at time $t \in \Gamma_i$, and Eq. (2) states that the equilibrium offer $v(t)$ lies on the Pareto frontier:

$$v_j(t) = v_j(t + \Delta) \quad \text{for } t \in \Gamma_i \ (j \neq i), \tag{1}$$

$$v_\mathrm{B}(t) = \phi(v_\mathrm{A}(t), t). \tag{2}$$

[2] See for example, Binmore [1, Section 8] and Muthoo [11, Section 3.2.4] who argue that the limiting SPE is the most interesting case.

For $t \in \Gamma_A$, these equations imply that the sequence $\langle v_A(t) \rangle_{t \in \Gamma_A}$ must satisfy the following recursive equation:

$$\phi(v_A(t), t) = \phi(v_A(t + 2\Delta), t + \Delta). \tag{3}$$

Furthermore, as was noted above, it must also satisfy

$$v_A(t) \in [0, \bar{u}^t_A] \quad \text{for all } t \in \Gamma_A. \tag{4}$$

This argument implies the following proposition.

Proposition 1 (Characterization of Markov SPE). *Fix $\Delta > 0$. Given any sequence $\langle v_A(t) \rangle_{t \in \Gamma_A}$ satisfying* (3) *and* (4), *there corresponds a unique Markov SPE, with the following pair of strategies*:

- *At time $t \in \Gamma_A$ player* A *offers $(v_A(t), \phi(v_A(t), t))$, and at times $t \in \Gamma_B$ she accepts an offer $u \in \Omega(t)$ if and only if $u_A \geqslant v_A(t + \Delta)$.*
- *At times $t \in \Gamma_B$ player* B *offers $(v_A(t + \Delta), \phi(v_A(t + \Delta), t))$, and at times $t \in \Gamma_A$ she accepts an offer $u \in \Omega(t)$ if and only if $u_B \geqslant \phi(v_A(t + 2\Delta), t + \Delta)$.*

There exists no other Markov SPE.

This proposition implies that a Markov SPE exists if and only if a sequence $\langle v_A(t) \rangle_{t \in \Gamma_A}$ satisfying (3) and (4) exists. By slightly amending the arguments used in [1], it is straightforward to establish that such a sequence always exists, and, hence, that a Markov SPE exists.[3]

Proposition 2 (Existence of Markov SPE). *For any $\Delta > 0$ there exists a Markov SPE.*

As Binmore [1] demonstrates, given any $\Delta > 0$, multiple solutions to (3) and (4) may exist. However, we now focus on the set of Markov SPE in the limit as $\Delta \to 0$.

3.2. A candidate limiting equilibrium

Using Assumption 3, a first-order Taylor expansion of Eq. (3) implies

$$\begin{aligned}\phi(v_A(t + 2\Delta), t + \Delta) = \phi(v_A(t), t) &+ [v_A(t + 2\Delta) - v_A(t)]\phi_u(v_A(t), t) \\ &+ \Delta\phi_t(v_A(t), t) + R,\end{aligned} \tag{5}$$

where ϕ_u and ϕ_t denote the first-order derivatives of ϕ w.r.t. u_A and t, respectively, and R is the remainder term. Using (3) to substitute for $\phi(v_A(t + 2\Delta), t + \Delta)$ in (5), rearranging and dividing by 2Δ, it follows that

$$\frac{v_A(t + 2\Delta) - v_A(t)}{2\Delta} = -\frac{1}{2}\frac{\phi_t(v_A(t), t)}{\phi_u(v_A(t), t)} - \frac{R}{2\Delta}. \tag{6}$$

[3] A proof is available upon request.

This equation motivates our candidate for the limiting equilibrium. If we could argue that the ratio of the remainder term to Δ disappears in the limit as $\Delta \to 0$, we might interpret (6) as a differential equation describing how player A's equilibrium payoff changes over time in the limiting equilibrium. We define such a solution as our candidate limiting equilibrium.

Definition 1 (CLE). A *candidate limiting equilibrium* (CLE) is a pair of functions $(v^*_{\mathrm{A}}(.), v^*_{\mathrm{B}}(.))$ such that

$$\text{for all } s \geqslant 0, \quad v^*_{\mathrm{B}}(s) = \phi(v^*_{\mathrm{A}}(s), s), \tag{7}$$

where $v^*_{\mathrm{A}}(.)$ is a solution to the differential equation

$$\frac{dv_{\mathrm{A}}}{ds} = -\frac{1}{2}\frac{\phi_t(v_{\mathrm{A}}, s)}{\phi_u(v_{\mathrm{A}}, s)} \tag{8}$$

$$\text{subject to} \quad v_{\mathrm{A}}(s) \in (0, \bar{u}^s_{\mathrm{A}}) \quad \text{for all } s \geqslant 0. \tag{9}$$

Notice that the CLE describes a path $(v^*_{\mathrm{A}}(s), v^*_{\mathrm{B}}(s))$ for all s, while in the previous section s was fixed, but arbitrary. Also note the boundary condition (9) reflects two facts—that a SPE requires $v_{\mathrm{A}}(s) \in [0, \bar{u}^s_{\mathrm{A}}]$ for all s, and that Assumption 2(i) (shrinking pie) implies that a proposer can always obtain some surplus (so that $v_i(s) = 0$ cannot be an equilibrium outcome). We now establish that such a CLE exists and is unique.

Lemma 1 (Uniqueness). *If a CLE exists, then it is unique.*

Proof. By contradiction—suppose at least two CLE exist. Let x^*_1, x^*_2 denote two different solutions to the differential equation (8) satisfying (9), and let $y^*_i(s) = \phi(x^*_i(s), s)$ (which is well defined as x^*_i satisfies (9)). Note that Assumption 2(ii) (vanishing Pareto frontiers) implies $\bar{u}^s_{\mathrm{A}} \to 0$ as $s \to \infty$, and so $\lim_{s\to\infty} (x^*_i(s), y^*_i(s)) = (0, 0)$; i.e. both paths converge asymptotically to $(0, 0)$.

For each s, define the distance between the two trajectories as

$$\Psi(s) = -[x^*_1(s) - x^*_2(s)][y^*_1(s) - y^*_2(s)].$$

Note that $x^*_1(s) \neq x^*_2(s)$ implies $\Psi(s) > 0$. Differentiating Ψ with respect to s, using (8) and rearranging gives

$$\begin{aligned}\Psi'(s) = &\frac{1}{2}\frac{[x^*_1(s) - x^*_2(s)]\phi_t(x^*_1(s), s)}{\phi_u(x^*_1(s), s)}\left[\frac{y^*_1(s) - y^*_2(s)}{x^*_1(s) - x^*_2(s)} - \phi_u(x^*_1(s), s)\right] \\ &- \frac{1}{2}\frac{[x^*_1(s) - x^*_2(s)]\phi_t(x^*_2(s), s)}{\phi_u(x^*_2(s), s)}\left[\frac{y^*_1(s) - y^*_2(s)}{x^*_1(s) - x^*_2(s)} - \phi_u(x^*_2(s), s)\right].\end{aligned}$$

Concavity of ϕ (Assumption 1) and ϕ decreasing in t (Assumption 2) imply $\Psi'(s) \geqslant 0$. Hence, since $x^*_1(s') \neq x^*_2(s')$ for some $s' \geqslant 0$, it follows that $\lim_{s\to\infty} \Psi(s) \geqslant -[x^*_1(s') - x^*_2(s')][y^*_1(s') - y^*_2(s')] > 0$, which contradicts $\lim_{s\to\infty} (x^*_i(s), y^*_i(s)) = (0, 0)$. □

The proof reflects a simple geometric property of the CLE. Differentiating (7) with respect to s and using (8) to substitute out ϕ_t implies

$$\frac{dv^*_{\mathrm{B}}(s)/ds}{dv^*_{\mathrm{A}}(s)/ds} = -\phi_u(v^*_{\mathrm{A}}(s), s). \tag{10}$$

The right-hand side of (10) is the marginal rate of utility substitution along the Pareto frontier at the equilibrium outcome. The left-hand side is the ratio of the agents' rate of utility loss by delay at the equilibrium outcome. Strategic bargaining implies these two margins are the same. Geometrically, it says that the slope of the CLE $(v^*_{\mathrm{A}}(s), v^*_{\mathrm{B}}(s))$ at time s *equals* the absolute value of the slope of the Pareto frontier $\Omega^{\mathrm{P}}(s)$ at that point. As drawn in Fig. 1 (which is in the appendix), concavity of Ω^{P} implies that any two trajectories satisfying (10) will tend to diverge (where a larger value of v_{A} implies a steeper trajectory).

An interesting feature of the uniqueness proof is that it uses a Liapunov-type function whose structure is closely related to that of the Nash-product (which, recall, is a key object in the definition of the Nash bargaining solution). Indeed, the proof of the Convergence Theorem stated below relies on the same construction. However, before doing that, we next establish existence of a CLE.

Lemma 2 (Existence). *A CLE exists.*

Proof. In the appendix.

Having established the existence and uniqueness of the CLE, we now establish our main convergence results.

3.3. *The unique limiting subgame perfect equilibrium*

To emphasize the dependence of the set of Markov SPE on Δ, it is helpful to define the following sets. For each $\Delta > 0$, let $\mathscr{F}(\Delta)$ denote the set of all sequences $\langle v_{\mathrm{A}}(t) \rangle_{t \in \Gamma_{\mathrm{A}}}$ which satisfy (3) and (4).[4] Moreover, for each $\Delta > 0$, let $\mathscr{G}(\Delta)$ denote the set of all Markov SPE *payoffs* to player A. Formally,

$$\mathscr{G}(\Delta) = \{u_{\mathrm{A}}:\ \text{there exists a sequence} \langle v_{\mathrm{A}}(t) \rangle_{t \in \Gamma_{\mathrm{A}}} \in \mathscr{F}(\Delta)\ \text{s.t.}\ v_{\mathrm{A}}(s) = u_{\mathrm{A}}\}.$$

Of course as Δ changes, the set $\mathscr{G}(\Delta)$ changes. Theorem 1 below establishes that $\mathscr{G}(\Delta)$ converges to a single point, namely $v^*_{\mathrm{A}}(s)$, as $\Delta \to 0$.

Theorem 1 (The Convergence theorem). *Fix an arbitrary s. For any $\varepsilon > 0$ there exists $\bar{\Delta}$ such that for all $\Delta < \bar{\Delta}$*

$$\max_{u_{\mathrm{A}} \in \mathscr{G}(\Delta)} |u_{\mathrm{A}} - v^*_{\mathrm{A}}(s)| < \varepsilon.$$

[4] Where Proposition 1 implies $\mathscr{F}(\Delta)$ essentially defines the set of Markov SPE (given Δ), and Proposition 2 establishes this set is non-empty.

Proof. In the appendix.

Theorem 1 implies that the Hausdorff distance between the set $\mathscr{G}(\Delta)$ and $\{v_A^*(s)\}$ converges to zero as $\Delta \to 0$. The proof uses the distance measure $\Psi(t)$ defined above but with $x_1^*(t) = v_A(t)$ where $\langle v_A(t) \rangle_{t \in \Gamma_A} \in \mathscr{F}(\Delta)$ describes an equilibrium sequence of payoffs given $\Delta > 0$, and $x_2^*(t) \equiv v_A^*(t)$. The intuition is that if the distance between $v_A^*(s)$ and $v_A(s)$ does not become small as $\Delta \to 0$, then for Δ small enough, the resulting trajectories $\langle v_A(t) \rangle_{t \in \Gamma_A}$ and $\langle v_A^*(t) \rangle_{t \in \Gamma_A}$ necessarily diverge as $t \to \infty$ (which contradicts their respective boundary conditions—that they both asymptote to zero).

Having established convergence for Markov SPE, we now establish it for any non-Markov SPE.

Theorem 2 (Unique limiting SPE). *In the limit as $\Delta \to 0$, any SPE converges to the CLE.*

Proof. In the appendix.

4. The relationship with Nash's bargaining solution

As is well known, the unique SPE of Rubinstein's [15] bargaining model can be described by the Nash [12] bargaining solution of an appropriately defined bargaining problem (cf., for example, [11,13]). We now extend this result to non-stationary environments.

Theorem 2 has established that in the limit as $\Delta \to 0$, the non-stationary bargaining game possesses a unique SPE. In this limiting SPE, agreement is always struck immediately where if the players begin negotiations at time s, then player A's equilibrium payoff is $v_A^*(s)$, where $v_A^*(.)$ satisfies the differential equation (8) subject to (9), and player B's equilibrium payoff is $v_B^*(s) = \phi(v_A^*(s), s)$.

In contrast, the Nash bargaining solution (NBS) is

$$(v_A^N(s), v_B^N(s)) = \arg \max_{(u_A, u_B) \in \Omega(s)} (u_A - d_A(s))(u_B - d_B(s)),$$

where $(d_A(s), d_B(s))$ is the as yet unspecified *disagreement point*. If the disagreement point $(d_A(s), d_B(s)) = (0, 0)$ then the NBS $(v_A^N(s), v_B^N(s))$ satisfies

$$v_B = \phi(v_A, s), \tag{11}$$

$$\frac{v_B}{v_A} = -\phi_u(v_A, s). \tag{12}$$

Note the NBS picks a point on the Pareto frontier where the absolute value of the slope of the frontier at that point equals the slope of the line joining the disagreement point $(0, 0)$ and the NBS. The following lemma establishes conditions under which the NBS and the limiting SPE payoff pair coincide for all s.

Lemma 3. *The NBS* $(v_A^N(s), v_B^N(s))$ *with disagreement point* $(0,0)$ *is identical to the limiting SPE payoff pair* $(v_A^*(s), v_B^*(s))$ *for all s if and only if* $\phi_u(v_A^*(s), s)$ *is constant for all s.*

Proof. In the appendix.

If $\phi_u(v_A^*(s), s)$ is constant for all s, (10) implies that the locus $(v_A^*(s), v_B^*(s))$ describes a straight line which, by (9), must pass through the origin. This corresponds to the Nash bargaining solution—a ray out of the origin with slope equal to the absolute value of the Pareto frontier.

The condition which guarantees that the strategic bargaining solution describes a ray out of the origin is that the Pareto frontier shrinks homothetically, which requires a Pareto frontier of the form

$$\gamma(u_A, u_B) = \alpha(t), \tag{13}$$

and γ is a homogenous function. Note that homotheticity requires that the time component affects the players equally over time.

Proposition 3 (Nash equivalence under homotheticity). *If the Pareto frontier shrinks homothetically, then the NBS with disagreement point* (0,0) *and the unique limiting SPE payoff pair coincide for all s.*

Proof. In the appendix.

In fact, assuming both agents are risk neutral and have a common discount rate guarantees homotheticity. However, we defer discussion of this example to Section 6 where we also consider time-varying inside options.

5. A worked example

Suppose players A and B bargain over the partition of a unit size cake, where negotiations begin at time $s = 0$. Player i's payoff from obtaining $x_i \in [0, 1]$ units of the cake at time $t \geqslant 0$ is $u_i = x_i \delta_i(t)$, where

$$\delta_i(t) = \exp\left[-\int_0^t r_i(z)\, dz\right]$$

and $r_i(z) > 0$ denotes i's instantaneous rate of time preference at time z. As $x_A + x_B = 1$, this implies that the Pareto frontier is defined by the implicit function

$$\frac{u_A}{\delta_A(t)} + \frac{u_B}{\delta_B(t)} = 1.$$

Assuming r_i continuous and bounded away from zero, Assumptions 1–3 are satisfied and hence Theorems 1 and 2 apply. Notice that unless $r_A = r_B$ almost everywhere, the Pareto frontier does not shrink homothetically and so a Nash bargaining solution cannot be applied. Instead, we have to solve directly for the CLE. As the Pareto frontier is described by $\phi = \delta_B(t)[1 - u_A/\delta_A(t)]$, the CLE implies $v_A^*(s)$ satisfies

$$\frac{dv_A}{ds} = -\frac{1}{2}[v_A[r_A(s) - r_B(s)] + r_B(s)\delta_A(s)].$$

Given the boundary condition (9), it is straightforward to verify that

$$v_A^*(s) = \delta_A(s)\left[\frac{1}{2} + \frac{1}{4}\int_s^\infty \left[\frac{\delta_A(t)\delta_B(t)}{\delta_A(s)\delta_B(s)}\right]^{1/2}[r_B(t) - r_A(t)]\,dt\right]$$

is the (unique, limiting) equilibrium outcome. Putting $s = 0$ implies player A obtains share

$$v_A^*(0) = \frac{1}{2} + \frac{1}{4}\int_0^\infty [\delta_A(t)\delta_B(t)]^{1/2}[r_B(t) - r_A(t)]\,dt$$

which is a discounted weighting of the difference between the players' discount rates in the entire future. The more impatient player B is, the higher the payoff to player A. If they have equal discount rates then the bargaining game is perfectly symmetric and they split the pie evenly. This solution can also be written as[5]

$$v_A^*(0) = 1 - \int_0^\infty \left[\frac{1}{2}r_A(t)\right][e^{-\int_0^t \frac{1}{2}[r_A(s)+r_B(s)]\,ds}]\,dt,$$

which says that the share of the pie that player A "loses" depends on the weighted average of his/her future discount rate. If A becomes arbitrarily patient ($r_A \to 0$) while $r_B > 0$, then A obtains the whole pie.

6. An extension to time-varying inside options

The previous sections assumed that the pie evolves over time in a non-stationary way. But a different class of problems arise if the agents' inside options are time varying. For example, when bargaining with a striking union, the firm might sell out of its inventory of finished goods where such sales reduce the cost of the strike to the firm; see, for example, [3]. A different example is an unemployed worker who is bargaining with a firm for a job and who receives duration dependent unemployment insurance (UI) payments. The purpose of this section is to extend the previous results for time-varying inside options and so demonstrate the robustness of this approach.

Two players, A and B, bargain according to the alternating-offers procedure as previously described. An offer at time t is a utility pair (u_A, u_B) from the set $\Omega(t)$, where $u_B = \phi(u_A, t)$ describes the Pareto frontier. If the offer is rejected then over the intervening period, player i obtains flow payoff $f_i(t)\Delta \geqslant 0$ (which is measured in period zero utils, i.e., it is discounted back to time zero).[6] Define

[5] As $\int_0^\infty z(t)\exp[-\int_0^t z(s)\,ds]\,dt = 1$ (when $z > 0$ is bounded away from zero).

[6] For example, if player i obtains UI payments $b(t)$, then his/her flow payoff during disagreement might be described as $f_i = e^{-rt}u_i(b(t))$.

$d_i(t) = \int_t^{\infty} f_i(z)\,dz \geqslant 0$, which is player i's discounted payoff at time t should they perpetually disagree.

Assumption 1′ (Concave Pareto frontiers). For each $t \geqslant 0$, $\phi(.,t)$ is concave in u_{A} on the interval $[d_{\mathrm{A}}(t), \bar{u}^t_{\mathrm{A}}]$.

Assumption 2′ (Positive, shrinking and vanishing Pareto frontiers). (i) For any $t \geqslant 0$, $d_{\mathrm{B}}(t) < \phi(d_{\mathrm{A}}(t), t)$, (ii) for any $t \geqslant 0$ and $u_{\mathrm{A}} \in [d_{\mathrm{A}}(t), \bar{u}^t_{\mathrm{A}}]$, $\phi_t(u_{\mathrm{A}}, t) + f_{\mathrm{B}}(t) - \phi_u(u_{\mathrm{A}}, t) f_{\mathrm{A}}(t) < 0$, and (iii) for any $\varepsilon > 0$ there exists a $T > 0$ such that $\bar{u}^t_{\mathrm{A}} < \varepsilon$ and $\bar{u}^t_{\mathrm{B}} < \varepsilon$ for all $t > T$.

Assumption 3′ (Smoothly evolving Pareto frontiers). ϕ is continuously differentiable, and $f_{\mathrm{A}}, f_{\mathrm{B}}$ are continuous.

Condition (i) in Assumption 2′ implies that there is always some partition both players would prefer to perpetual disagreement—a gain to trade always exists.[7] This implies $0 \leqslant d_i(t) < \bar{u}^t_i$ for all t and $i = \mathrm{A}, \mathrm{B}$. Condition (ii) is the appropriate shrinking pie condition. To see why, suppose rather than agree some (Pareto efficient) partition $(u_{\mathrm{A}}, u_{\mathrm{B}})$ at time t, the agreement is deferred to $t + dt$. Player A is no worse off as long as the partition $(u'_{\mathrm{A}}, u'_{\mathrm{B}})$ at time $t + dt$ satisfies $f_{\mathrm{A}}(t)\,dt + u'_{\mathrm{A}} \geqslant u_{\mathrm{A}}$. As player B's maximal payoff is $\phi(u'_{\mathrm{A}}, t + dt) + f_{\mathrm{B}}(t)\,dt$, then the stated condition (ii) guarantees delay makes player B strictly worse off.

Again consider an arbitrary Markov SPE where $v(t) = (v_{\mathrm{A}}(t), v_{\mathrm{B}}(t))$ denotes the equilibrium offer made at time $t \in \Gamma$. As before shrinking pie and Markov strategies imply there is no delay in equilibrium. Hence the equilibrium offer $v(t)$ at time $t \in \Gamma_i$ satisfies

$$v_j(t) = f_j(t)\Delta + v_j(t + \Delta) \quad \text{for } t \in \Gamma_i \ (j \neq i),$$

$$v_{\mathrm{B}}(t) = \phi(v_{\mathrm{A}}(t), t),$$

where the first condition says the proposer extracts maximal rents from the responder, and the second says the offer is Pareto efficient. For any $t \in \Gamma_{\mathrm{A}}$, these equations imply the difference equation

$$\phi(v_{\mathrm{A}}(t), t) = f_{\mathrm{B}}(t)\Delta + \phi(f_{\mathrm{A}}(t + \Delta)\Delta + v_{\mathrm{A}}(t + 2\Delta), t + \Delta).$$

As before, our main interest is characterising the limiting equilibria as $\Delta \to 0$. A first order Taylor expansion implies

$$0 = f_{\mathrm{B}}(t)\Delta + [f_{\mathrm{A}}(t + \Delta)\Delta + v_{\mathrm{A}}(t + 2\Delta) - v_{\mathrm{A}}(t)]\phi_u + \Delta\phi_t + R.$$

Rearranging and taking the limit $\Delta \to 0$ suggests that a candidate limiting equilibrium (CLE) is a pair of functions $(v_{\mathrm{A}}(.), v_{\mathrm{B}}(.))$ such that for all $s \geqslant 0$, $v_{\mathrm{B}}(s) = \phi(v_{\mathrm{A}}(s), s)$,

[7] This assumption is convenient rather than critical. If it does not hold, then shrinking pie implies a (unique) T where $d_{\mathrm{B}}(T) = \phi(d_{\mathrm{A}}(T), T)$. A gain to trade then exists for $t < T$, but not for $t > T$. As equilibrium implies no trade for $t \geqslant T$, we would then use backward induction from $t = T$ with boundary condition $v_i(T) = d_i(T)$.

where $v_A(.)$ is a solution to the differential equation

$$\frac{dv_A}{ds} = -\frac{1}{2}\frac{[f_B(s) + f_A(s)\phi_u(v_A, s) + \phi_t(v_A, s)]}{\phi_u(v_A, s)},$$
$$\text{subject to} \quad v_A(s) \in [d_A(s), \phi^{-1}(d_B(s), s)] \quad \text{for all } s \geqslant 0. \tag{14}$$

There are several points worth noticing. First Assumption 2′(ii) (shrinking pie) and (14) imply $dv_A/ds + f_A(s) < 0$; along the CLE, delay always makes player A worse off. Also, using $dv_B/ds = \phi_u(v_A, s)\, dv_A/ds + \phi_t(v_A, s)$, it follows that $dv_B/ds + f_B(s) < 0$. Delay makes both players strictly worse off.

Given the corresponding solution for dv_B/ds, (14) can be rewritten as

$$\frac{dv_B/ds + f_B(s)}{dv_A/ds + f_A(s)} = -\phi_u(v_A, s),$$

which implies the geometric interpretation obtained previously. $dv_B/ds + f_B(s)$ is the (rate of) utility gain to player B through delay (which is negative). Strategic bargaining implies the ratio of the agents' rates of utility loss by delay at the equilibrium outcome equals the marginal rate of utility substitution along the Pareto frontier.

Establishing existence of a CLE is straightforward. The key is to note that the previous expression can also be written as

$$\frac{\frac{d}{ds}(v_B - d_B)}{\frac{d}{ds}(v_A - d_A)} = -\phi_u(v_A, s).$$

At each point in time, strategic bargaining shares the increase in surplus by reaching agreement today rather than deferring another instant, where the ratio depends on the slope of the Pareto frontier. By defining "surplus" variables $\tilde{x} \equiv v_A - d_A, \tilde{y} \equiv v_B - d_B$, the proof of Lemma 2 can be applied to establish existence of a solution where $\tilde{x}, \tilde{y} > 0$ for all s (as required).[8]

To establish uniqueness, suppose there exist (at least) two solutions to (14) which we denote $x_1(s), x_2(s)$. Further, let $y_i(s) = \phi(x_i(s), s)$ and define

$$\Psi(s) = -[x_1 - x_2][y_1 - y_2],$$

where $x_i = x_i(s), y_i = y_i(s)$. Note that $\Psi(s) > 0$ if $x_1 \neq x_2$ and vanishing pie requires $\Psi(s) \to 0$ as $s \to \infty$. But

$$\Psi'(s) = -[x'_1 - x'_2][y_1 - y_2] - [x_1 - x_2][y'_1 - y'_2],$$

and as the CLE implies $y'_i(s) + f_B = -\phi_u(x_i, s)[x'_i(s) + f_A]$, we can substitute out the y'_i and rearrange to get

$$\Psi'(s) = [x'_1 + f_A][y_2 - [y_1 + (x_2 - x_1)\phi_u(x_1, s)]] + [x'_2 + f_A][y_1 - [y_2 + (x_1 - x_2)\phi_u(x_2, s)]].$$

[8] In particular, define $\tilde{\phi}(x, s) \equiv \phi(x + d_A(s), s) - d_B(s)$. Then Pareto efficiency implies $\tilde{y} = \tilde{\phi}(\tilde{x}, s)$, and the CLE implies $\frac{d\tilde{y}/ds}{d\tilde{x}/ds} = -\tilde{\phi}_x(\tilde{x}, s)$. Further, given ϕ, d_i satisfy Assumptions 1′–3′, direct inspection shows that $\tilde{\phi}$ satisfies Assumptions 1–3. Hence the proof of Lemma 2 implies a path exists where $\tilde{x}, \tilde{y} > 0$ for all s.

As an equilibrium solution implies $x'(s) + f_A < 0$ (see above), then concavity of ϕ with respect to u implies $\Psi'(s) \geqslant 0$ which is the required contradiction. In the same way, we can adapt the limiting argument demonstrated in the proof of Theorem 1 and so establish the corresponding Convergence Theorem.

6.1. An important special case

There is one simple special case—that both players are risk neutral and have common discount rate $r > 0$. Together these assumptions imply the Pareto frontier is of the form

$$u_B + \alpha u_A = \gamma(t),$$

where α is a positive constant and γ is a positive, decreasing function. As $\phi_u \equiv -\alpha$, the CLE above implies

$$\frac{dv_A}{ds} = \frac{1}{2\alpha}[f_B(s) - \alpha f_A(s) + \gamma'(s)].$$

A vanishing frontier requires $\gamma(s) \to 0$ as $s \to \infty$ and integration then gives

$$v_A^*(s) = d_A(s) + \frac{1}{2\alpha}[\gamma(s) - \alpha d_A(s) - d_B(s)],$$

where as previously defined, $d_i(s)$ is player i's expected discounted payoff from perpetual disagreement. Hence we have established the following claim:

Claim. *When both players are risk neutral, have a common discount rate $r > 0$, and payoffs satisfy Assumptions $1'$–$3'$, then the limiting strategic bargaining equilibrium implies*

$$(v_A^*(s), v_B^*(s)) = \arg \max_{u_A, u_B \in \Omega(s)} [u_A - d_A(s)][u_B - d_B(s)],$$

i.e. it corresponds to the Nash bargaining solution with threatpoints $d_A(s)$ and $d_B(s)$ being the players' expected payoffs (as of time s) from perpetual disagreement.

7. Conclusion

This paper has extended the Rubinstein bargaining model to a non-stationary environment. Although in general multiple equilibria are possible for $\Delta > 0$, it has been established that with an appropriate continuity assumption, equilibrium is always unique in the limit as $\Delta \to 0$. It has also been shown for a very special but commonly used case—where all agents are risk neutral and have the same discount rate—that the bargaining equilibrium corresponds to the Nash bargaining solution with threatpoints being the agents' expected payoffs from perpetual disagreement. If the pie does not shrink homothetically then there is no simple relationship between the two approaches. Nevertheless, as Coles and Wright [5] demonstrate, the strategic bargaining approach may still be tractable in dynamic applications and identifies dynamically consistent equilibria.

Appendix

Proof of Lemma 2. To establish existence of a CLE, consider the initial value problem:

$$\frac{dx}{ds} = -\frac{1}{2}\frac{\phi_t(x,s)}{\phi_u(x,s)} \tag{A.1}$$

$$\text{subject to} \quad x(0) = x_0 \in (0, \bar{u}_A^0). \tag{A.2}$$

Conditional on x_0, let $\hat{x}(s; x_0)$ denote the solution to this initial value problem and define $\hat{y}(s; x_0) = \phi(\hat{x}, s)$. This defines a trajectory $\{\hat{x}(s; x_0), \hat{y}(s; x_0)\}$ in the (x, y) plane where $(\hat{x}(s; .), \hat{y}(s; .)) \in \Omega^P(s)$ (see Fig. 1).

Given Assumption 3 (ϕ is continuously differentiable) and that $\phi_u(x, s) < 0$ for all $x \in [0, \bar{u}_A^s]$ and for all $s \geqslant 0$, the *Fundamental Theorem of Differential Equations* (cf., for example, [16, Section 24.4]) implies that a solution exists to this initial value problem while $\hat{x}(s; x_0) \in (0, \bar{u}_A^s)$.

Note that while $\hat{x}(s; x_0) \in (0, \bar{u}_A^s)$, (A.1) and the definition of $\hat{y}$ imply $[d\hat{y}/ds]/[d\hat{x}/ds] = -\phi_u > 0$, which implies that any trajectory $\{\hat{x}(s; x_0), \hat{y}(s; x_0)\}$ has strictly positive slope in the (x, y) plane. Hence as depicted in Fig. 1, any trajectory $\{\hat{x}, \hat{y}\}$ is either (i) always strictly in the positive quadrant, or (ii) meets the x-axis in finite time, or (iii) meets the y-axis in finite time. Also note that the proof of Lemma 1 implies that trajectories cannot cross.

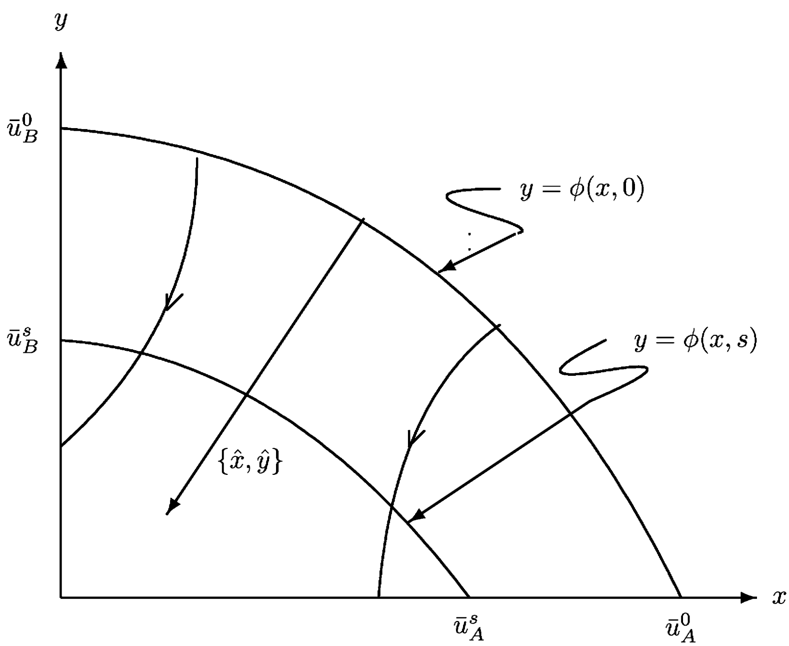

Fig. 1.

Define $\Upsilon_x = \{x_0 \in [0, \bar{u}_A^0] :$ there exists an $S < \infty$ such that $\hat{x}(S; x_0) > 0$ and $\hat{y}(S; x_0) = 0\}$, i.e. Υ_x is the set of initial values whose trajectory $\{\hat{x}(s; x_0), \hat{y}(s; x_0)\}$ reaches the x-axis in finite time. Similarly define $\Upsilon_y = \{x_0 \in [0, \bar{u}_A^0]$: there exists an $S < \infty$ such that $\hat{x}(S; x_0) = 0$ and $\hat{y}(S; x_0) > 0\}$, and $\Upsilon^* = \{x_0 \in [0, \bar{u}_A^0]$: for all $s \geqslant 0$, $\hat{x}(s; x_0) > 0$ and $\hat{y}(s; x_0) > 0\}$. Claim A.1 now shows that Υ^* is non-empty. As such a trajectory satisfies boundary condition (9), Claim A.1 establishes the lemma.

Claim A.1. Υ^* *is non-empty.*

Proof. By contradiction—suppose Υ^* is empty and so Υ_x and Υ_y form a complete partition of $[0, \bar{u}_A^0]$. Since $0 \in \Upsilon_y$ and $\bar{u}_A^0 \in \Upsilon_x$, these two sets are non-empty. Furthermore, since trajectories do not cross, the respective supports of Υ_x and Υ_y are connected. Hence, since Υ_x and Υ_y partition the interval $[0, \bar{u}_A^0]$, one of these two sets is closed. Suppose, without loss of generality, that Υ_x is closed—that is, there exists $x^c \in (0, \bar{u}_A^0)$ such that $\Upsilon_x = [x^c, \bar{u}_A^0]$. Hence, there exists a corresponding $S < \infty$ such that $\hat{x}(S; x^c) = \bar{u}_A^S > 0$ and $\hat{y}(S; x^c) = 0$.

The contradiction is now obtained by backward induction. Consider the alternative trajectory $\{\tilde{x}(s), \tilde{y}(s)\}$ where (i) $\tilde{x}(s)$ satisfies the boundary condition $\tilde{x}(S+1) = \bar{u}_A^{S+1} > 0$, (ii) $\tilde{x}(s)$ satisfies the differential equation (A.1) for $s \leqslant S+1$, and (iii) $\tilde{y}(s) = \phi(\tilde{x}(s), s)$. This trajectory is obtained by iterating the differential equation (A.1) backwards through time, starting at $s = S+1$ with $\tilde{x}(S+1) = \bar{u}_A^{S+1}$.

As trajectories cannot cross then, as drawn in Fig. 2, backward induction implies a trajectory $\{\hat{x}(s; x_0'), \hat{y}(s; x_0')\} = \{\tilde{x}(s), \tilde{y}(s)\}$ exists where $x_0' \in \Upsilon_x$ and $x_0' < x^c$, which is the required contradiction. □

Proof of Theorem 1. Fix an arbitrary sequence $\langle \Delta_n \rangle$ such that $\Delta_n > 0$ (for all $n \in \mathbb{N}$) and $\Delta_n \to 0$ as $n \to \infty$. This defines a sequence $\langle \mathscr{F}_n \rangle$ where $\mathscr{F}_n \equiv \mathscr{F}(\Delta_n)$. Now define a sequence $\langle x_n \rangle$ where for each $n \in \mathbb{N}$, x_n is an arbitrary element of $\mathscr{F}_n$. That is, for each $n \in \mathbb{N}$, x_n is an arbitrary sequence $\langle x_n(t) \rangle_{t \in \Gamma_A^n}$ that satisfies

$$\phi(x_n(t), t) = \phi(x_n(t + 2\Delta_n), t + \Delta_n) \tag{A.3}$$

and $x_n(t) \in [0, \bar{u}_A^t]$ for all $t \in \Gamma_A^n$, where $\Gamma_A^n \equiv \Gamma_A(\Delta_n) = \{s, s + 2\Delta_n, s + 4\Delta_n, \ldots\}$. We have to show that the sequence $\langle x_n(s) \rangle$ converges to $v_A^*(s)$.

For each $n \in \mathbb{N}$ and $t \in \Gamma_A^n$ define

$$\Psi(n, t) = -[v_A^*(t) - x_n(t)][v_B^*(t) - y_n(t)], \tag{A.4}$$

where (v_A^*, v_B^*) is the unique CLE and $y_n(t) = \phi(x_n(t), t)$. One might interpret $\Psi(n, t)$ as a measure of the distance between the CLE $(v_A^*(t), v_B^*(t))$ and the SPE payoff pair $(x_n(t), y_n(t))$. In particular, $\Psi(n, t) = 0$ if and only if $x_n(t) = v_A^*(t)$, and $\Psi(n, t) > 0$ for $x_n(t) \neq v_A^*(t)$. Most importantly, by establishing that $\Psi(n, s) \to 0$ as $n \to \infty$ it follows that $x_n(s) \to v_A^*(s)$ in this limit. Hence we establish the theorem by proving that for any $\epsilon > 0$ there always exists an N such that $\Psi(n, s) < \epsilon$ for all $n > N$.

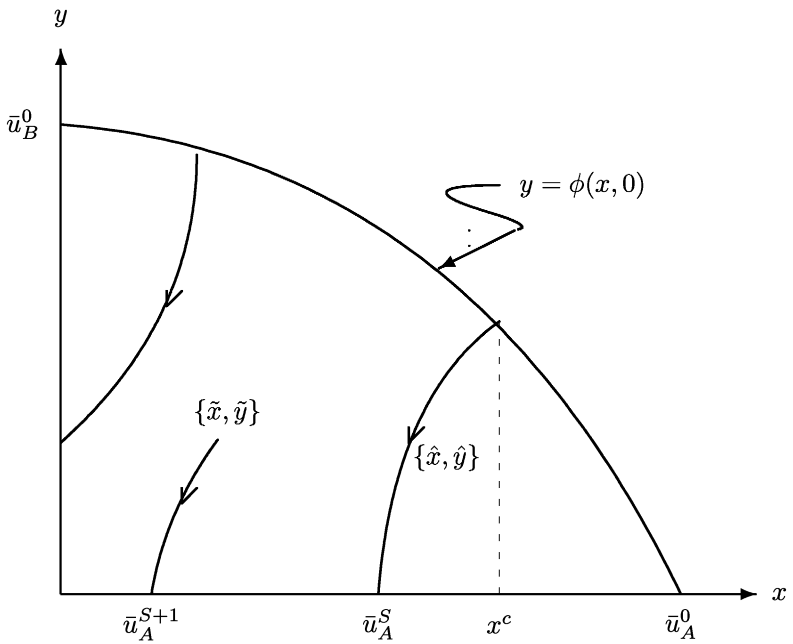

Fig. 2.

Fix an arbitrary $\epsilon > 0$. If $\bar{u}_A^s \bar{u}_B^s \leqslant \epsilon/2$, then $\Psi(n, s) \leqslant \bar{u}_A^s \bar{u}_B^s \leqslant \epsilon/2$ (for all $n \in \mathbb{N}$), and we are done. Now suppose that $\epsilon < 2\bar{u}_A^s \bar{u}_B^s$. Define T such that $\bar{u}_A^T \bar{u}_B^T = \epsilon/2$. Assumptions 2 and 3 imply T exists, is unique and is strictly greater than s. Also $\Psi(n, T) \leqslant \bar{u}_A^T \bar{u}_B^T = \epsilon/2$ for all $n \in \mathbb{N}$. Furthermore, define for each $n \in \mathbb{N}$,

$$M_n = \min\{m \in \mathbb{N}\colon\ m \geqslant (T - s)/2\Delta_n\} \quad \text{and} \quad T_n = s + 2M_n\Delta_n.$$

Notice that $T_n \in \Gamma_A^n$ for all $n \in \mathbb{N}$. Further $T_n \geqslant T$, and Assumption 2 implies that $\Psi(n, T_n) \leqslant \bar{u}_A^{T_n} \bar{u}_B^{T_n} \leqslant \epsilon/2$.

Now for any $n \in \mathbb{N}$,

$$\Psi(n, s) = \Psi(n, T_n) - \sum_{i=0}^{M_n - 1} [\Psi(n, s + 2(i+1)\Delta_n) - \Psi(n, s + 2i\Delta_n)].$$

Claim A.2—which is stated below—implies

$$\Psi(n, s) = \Psi(n, T_n) - \sum_{i=0}^{M_n - 1} [F(n, s + 2i\Delta_n)\Delta_n + o(\Delta_n)],$$

where $o(\Delta_n)$ denotes a remainder term that is of order smaller than Δ_n (i.e. $o(\Delta_n)/\Delta_n$ converges to zero as $n \to \infty$). As Claim A.2 also implies $F(n, t) \geqslant 0$ for all $t \in \Gamma_A^n$, this now implies

$$\Psi(n, s) \leqslant \Psi(n, T_n) - \sum_{i=0}^{M_n - 1} o(\Delta_n).$$

But $M_n = O(1/\Delta_n)$ and so it follows that $\sum_{i=0}^{M_n-1} o(\Delta_n)$ converges to zero as $n \to \infty$. Hence there exists an N such that for any $n > N$, $|\sum_{i=0}^{M_n-1} o(\Delta_n)| < \epsilon/2$. As $\Psi(n, T_n) \leqslant \epsilon/2$, this implies $\Psi(n, s) < \epsilon$ for all $n > N$ (as required). □

Claim A.2. *For any $n \in \mathbb{N}$ and $t \in \Gamma_{\mathrm{A}}^n$:*

$$\Psi(n, t + 2\Delta_n) - \Psi(n, t) = F(n, t)\Delta_n + o(\Delta_n),$$

where $F(n, t)$ is defined by

$$F(n,t) = \frac{[v_{\mathrm{A}}^*(t) - x_n(t)]\phi_t(v_{\mathrm{A}}^*(t), t)}{\phi_u(v_{\mathrm{A}}^*(t), t)}\left[\frac{v_{\mathrm{B}}^*(t) - y_n(t)}{v_{\mathrm{A}}^*(t) - x_n(t)} - \phi_u(v_{\mathrm{A}}^*(t), t)\right] - \frac{[v_{\mathrm{A}}^*(t) - x_n(t)]\phi_t(x_n(t), t)}{\phi_u(x_n(t), t)}\left[\frac{v_{\mathrm{B}}^*(t) - y_n(t)}{v_{\mathrm{A}}^*(t) - x_n(t)} - \phi_u(x_n(t), t)\right].$$

Furthermore, for any $n \in \mathbb{N}$ and $t \in \Gamma_{\mathrm{A}}^n$: $F(n, t) \geqslant 0$.

Proof. As $v_{\mathrm{A}}^* : [0, \infty) \to \Re$ satisfies the differential equation in (8), then for any $n \in \mathbb{N}$ and $t \in \Gamma_{\mathrm{A}}^n$,

$$v_{\mathrm{A}}^*(t + 2\Delta_n) - v_{\mathrm{A}}^*(t) = -\frac{\phi_t(v_{\mathrm{A}}^*(t), t)}{\phi_u(v_{\mathrm{A}}^*(t), t)}\Delta_n + o(\Delta_n). \tag{A.5}$$

Further, Assumption 3 (differentiability) implies that we can consider a first-order Taylor expansion of $\phi(v_{\mathrm{A}}^*(t + 2\Delta_n), t + 2\Delta_n)$ around $\phi(v_{\mathrm{A}}^*(t), t)$, and (A.5) then implies that for any $n \in \mathbb{N}$ and $t \in \Gamma_{\mathrm{A}}^n$:

$$\phi(v_{\mathrm{A}}^*(t + 2\Delta_n), t + 2\Delta_n) = \phi(v_{\mathrm{A}}^*(t), t) + [v_{\mathrm{A}}^*(t + 2\Delta_n) - v_{\mathrm{A}}^*(t)]\phi_u(v_{\mathrm{A}}^*(t), t) + 2\Delta_n\phi_t(v_{\mathrm{A}}^*(t), t) + o(\Delta_n). \tag{A.6}$$

Recalling that $x_n(t)$ satisfies (A.3), Assumption 3 (differentiability) implies that for any $n \in \mathbb{N}$ and $t \in \Gamma_{\mathrm{A}}^n$:

$$x_n(t + 2\Delta_n) - x_n(t) = -\frac{\phi_t(x_n(t), t)}{\phi_u(x_n(t), t)}\Delta_n + o(\Delta_n). \tag{A.7}$$

Now consider a first order Taylor expansion of $\phi(x_n(t + 2\Delta_n), t + 2\Delta_n)$ around $\phi(x_n(t), t)$. Eq. (A.7) then implies that for any $n \in \mathbb{N}$ and $t \in \Gamma_{\mathrm{A}}^n$:

$$\phi(x_n(t + 2\Delta_n), t + 2\Delta_n) = \phi(x_n(t), t) + [x_n(t + 2\Delta_n) - x_n(t)]\phi_u(x_n(t), t) + 2\Delta_n\phi_t(x_n(t), t) + o(\Delta_n). \tag{A.8}$$

Given the definition of Ψ in (A.4), and using (A.5)–(A.8) to substitute out terms dated at time $t + 2\Delta_n$, straightforward (but messy) algebra establishes the equations stated in the claim. $F(n, t) \geqslant 0$ follows from the concavity of ϕ, and from $\phi_t < 0$ and $\phi_u < 0$. □

Proof of Theorem 2. To prove this theorem we first establish that both the sequence of *maximum* SPE payoffs to player A and the sequence of *minimum* SPE payoffs to player A satisfy the same recursive equation (namely, Eq. (3)) which describes the sequence of Markov SPE payoffs (to player A). The convergence argument given in the proof of Theorem 1 then implies that in the limit as $\Delta \to 0$, all non-Markov SPE payoffs converge to the CLE.

Fix $\Delta > 0$. For each $i = \mathrm{A}, \mathrm{B}$ and $t \in \Gamma_i$, let $G_i(t)$ denote the set of SPE payoffs to player i in any subgame beginning at time t. Formally, $G_i(t) = \{g_i$: there exists an SPE in any subgame beginning at time t (when player i makes an offer) that gives player i a payoff of $g_i\}$. Since $G_i(t)$ is bounded, we denote its supremum and infimum by $M_i(t)$ and $m_i(t)$, respectively.

It follows from Claim A.3 below that both the sequence $\langle M_\mathrm{A}(t) \rangle_{t \in \Gamma_\mathrm{A}}$ and the sequence $\langle m_\mathrm{A}(t) \rangle_{t \in \Gamma_\mathrm{A}}$ are elements of the set $\mathscr{F}(\Delta)$. Theorem 1 implies that in the limit, as $\Delta \to 0$, the set $\mathscr{F}(\Delta)$ converges to a unique element. Hence, it follows (by appealing to Claim A.3) that in the limit, as $\Delta \to 0$, the set of SPE payoffs to the players in any subgame are *uniquely* defined: in the limit as $\Delta \to 0$, any SPE in any subgame gives player A a payoff of $v^*_\mathrm{A}(s)$ and player B a payoff of $v^*_\mathrm{B}(s)$. This implies that in any limiting (as $\Delta \to 0$) SPE, each player's offer (in any subgame when she has to make an offer) is accepted by her opponent. Hence, it immediately follows that in the limit as $\Delta \to 0$, any SPE converges to the CLE. □

Claim A.3. *Fix* $\Delta > 0$. $\forall t \in \Gamma_\mathrm{A}$, $M_\mathrm{A}(t) = \phi^{-1}(m_\mathrm{B}(t+\Delta), t)$ *and* $m_\mathrm{A}(t) = \phi^{-1}(M_\mathrm{B}(t+\Delta), t)$, *and* $\forall t \in \Gamma_\mathrm{B}$, $M_\mathrm{B}(t) = \phi(m_\mathrm{A}(t+\Delta), t)$ *and* $m_\mathrm{B}(t) = \phi(M_\mathrm{A}(t+\Delta), t)$.

Proof. The proof—which is available upon request—follows from a straightforward adaptation of standard arguments (which are, for example, presented in [13, Chapter 3; 11, Chapter 3]). □

Proof of Lemma 3. We first establish *sufficiency*. If $\phi_u(v^*_\mathrm{A}(s), s)$ is constant for all s, then (10) implies that the locus $\{(v^*_\mathrm{A}(s), v^*_\mathrm{B}(s))\colon s \geqslant 0\}$ is a straight line, being a ray through the origin with slope equal to the absolute value of the slope of the frontier $\Omega^\mathrm{P}(s)$ at $(v^*_\mathrm{A}(s), v^*_\mathrm{B}(s))$. Hence, for all s the NBS and the limiting SPE payoff pair are identical. We now establish *necessity*. If $v^\mathrm{N}_\mathrm{A}(s) = v^*_\mathrm{A}(s)$ for all s, then (10) and (12) imply

$$\frac{dv^*_\mathrm{B}(s)/ds}{dv^*_\mathrm{A}(s)/ds} = \frac{v^\mathrm{N}_\mathrm{B}(s)}{v^\mathrm{N}_\mathrm{A}(s)} \quad \text{for all } s. \tag{A.9}$$

Suppose, to the contrary, that there exists $s'' > s'$ such that $\phi_u(v^*_\mathrm{A}(s''), s'') \neq \phi_u(v^*_\mathrm{A}(s'), s')$. Then (10) and (A.9) imply

$$\frac{v^\mathrm{N}_\mathrm{B}(s'')}{v^\mathrm{N}_\mathrm{A}(s'')} \neq \frac{v^\mathrm{N}_\mathrm{B}(s')}{v^\mathrm{N}_\mathrm{A}(s')}.$$

But this implies that there exists $s \in (s', s'')$ such that

$$\frac{dv_{\rm B}^*(s)/ds}{dv_{\rm A}^*(s)/ds} \neq \frac{v_{\rm B}^{\rm N}(s)}{v_{\rm A}^{\rm N}(s)},$$

which contradicts (A.9). □

Proof of Proposition 3. Given (13), the Nash bargaining solution satisfies

$$\gamma(v_{\rm A}^{\rm N}(t), v_{\rm B}^{\rm N}(t)) = \alpha(t), \tag{A.10}$$

$$\frac{v_{\rm B}^{\rm N}(t)}{v_{\rm A}^{\rm N}(t)} = \frac{\gamma_{\rm A}(v_{\rm A}^{\rm N}(t), v_{\rm B}^{\rm N}(t))}{\gamma_{\rm B}(v_{\rm A}^{\rm N}(t), v_{\rm B}^{\rm N}(t))}, \tag{A.11}$$

where $\gamma_i \equiv \partial\gamma/\partial u_i$. Homogeneity of γ and (A.11) implies $v_{\rm B}^{\rm N}(t) = \lambda v_{\rm A}^{\rm N}(t)$, where λ is defined by

$$\lambda = \frac{\gamma_{\rm A}(1, \lambda)}{\gamma_{\rm B}(1, \lambda)}.$$

Assumptions 1–3 guarantee a solution exists and is unique.[9] Given that solution, $v_{\rm A}^{\rm N}(t)$ is then uniquely determined by

$$\gamma(v_{\rm A}^{\rm N}(t), \lambda v_{\rm A}^{\rm N}(t)) = \alpha(t).$$

Direct inspection shows that this solution also satisfies (10) and therefore satisfies the FBE.

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[9] Pick any point on the Pareto frontier. λ is the slope of the line from the origin to this point, while the right-hand side is the (absolute) slope of the Pareto frontier at this point. The right-hand side is positive, decreasing in λ and is continuous.

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