EFFICIENT PARTNERSHIP FORMATION IN NETWORKS

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ABSTRACT. We analyze the formation of partnerships in social networks. Players need favors at random times and ask their neighbors in the network to form exclusive long-term partnerships that guarantee reciprocal favor exchange. Refusing to provide a favor results in the automatic removal of the underlying link. Players agree to provide the first favor in a partnership only if they otherwise face the risk of eventual isolation. In equilibrium, players essential for realizing every maximum matching can avoid this risk and enjoy higher payoffs than inessential players. Although the search for partners is decentralized and reflects local partnership opportunities, the strength of essential players drives efficient partnership formation in every network. Equilibrium behavior is determined by the classification of nodes in the Gallai-Edmonds decomposition of the underlying network.

`JEL Classification Numbers: D85, C78.`

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1. INTRODUCTION

The idea that the power of an individual depends on his or her position in a certain social or economic network is well-established in a variety of contexts cutting across disciplines. For instance, social network analysis suggests that an individual's power cannot be explained by the individual's characteristics alone but must be combined with the structure of his or her relationships with others. Power arises from occupying advantageous positions in the relevant network and leveraging outside options. In particular, network exchange theory focuses on studying the relative bargaining power of individuals in bilateral exchanges with neighbors in social networks (see Willer (1999) for an overview). More recent research in economics develops game theoretical models aimed at understanding how an individual's position determines his or her bargaining power and selection of trading partners in markets

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with a network structure (Jackson (2008; Ch. 10), Easley and Kleinberg (2010; Ch. 11), and Manea (2016) survey this literature).

In this paper, we study the impact of the social and economic network structure on the relative strength of different positions in the network, the pattern of bilateral partnerships that emerge, and market efficiency. In our model, players form exclusive partnerships to exchange favors with one another. Favors could be small—e.g., advice on a particular issue, a small loan, and help on a school project—or large—e.g., sharing one's life with another person and forming a long-term business relationship. Every player needs favors at random points in time and can receive them from neighbors in the network. All players benefit equally from receiving favors and incur the same cost for providing them. The benefit exceeds the cost, so it is socially desirable to exchange favors. If a player agrees to provide a favor to another, the two players form an exclusive partnership which requires that they leave the network together and do each other favors whenever needs arise in the future. A player who needs a favor and has not yet formed a partnership approaches his remaining neighbors in the network in random order to request the favor. Each neighbor asked for the favor decides whether to provide it and enter the partnership or refuse to do so and irreversibly lose his link with the player requesting it.

Our modeling of the search for partners is admittedly specific but captures important features of applications such as marriage and joint business ventures. We implicitly assume that the payoffs received by the favor seeker and favor giver are asymmetric. In the context of courtship for example, the intensity of preferences of the two partners is often asymmetric, and one partner is more eager than the other to form the partnership. In our model, players are symmetric, but the favor seeker is always more eager to form the partnership than the favor giver. This introduces an incentive for one of the two parties to delay the formation of the partnership. However, this incentive is modulated by the evolution of the network and the decline in partnership opportunities over time. Turning down favor requests results in the loss of potential future partners both because links leading to rejections are deleted from the network and because other neighbors form exclusive partnerships and leave the network. Hence, players face a trade-off between the cost of accepting a partner and the risk of not finding one in the future. Returning to the example of dating, players prefer not to commit to a relationship until they start worrying that they will grow old alone. We find that precisely this feature of the search process leads to efficient matching outcomes.

Since all partnerships create the same value in our game, efficiency for high discount factors requires that a maximum number of partnerships be forged in the network. In the language of graph theory, the emerging partnerships should form a maximum matching, i.e., a subset of disjoint links that has the greatest cardinality. Some links are inefficient because they are not part of any maximum matching; efficiency dictates that favor requests are declined when such links are activated. Other links are indispensable for efficiency since they belong to every maximum matching; to achieve an efficient market outcome, such links should result in partnerships when activated. Keeping track of which partnerships are efficient in our dynamic game is challenging because the structure of maximum matchings and the efficiency of links evolves as partnerships form and favor requests are turned down.

Given the global considerations involved in identifying maximum matchings in a general network and the local decentralized nature of the search for partners as favor needs arise in our game, one might not expect a strong link between equilibrium outcomes and efficiency in the game. Nevertheless, our main result proves that when players are sufficiently patient, the game admits a unique subgame perfect equilibrium and that the equilibrium pattern of partnerships is efficient for every network structure. This result is surprising in the context of existing research on matching and trade in networks. Indeed, in a survey of the literature on bilateral trade in networks, Manea (2016) argues that the disconnect between global efficiency and local incentives explains why decentralized trade often generates inefficient market outcomes. Abreu and Manea (2012a, b) and Elliott and Nava (2016) reach this conclusion for Markov perfect equilibria in two natural models of bargaining in networks.[1] By contrast, the seminal work in this area of Kranton and Minehart (2001), Corominas-Bosch (2004), and Polanski (2007) showed that centralized matching is conducive to efficient trade.

Somewhat paradoxically, the absence of prices or direct transfers in our model drives the efficiency result and the divergence from the conclusions of Abreu and Manea (2012a, b) and Elliott and Nava (2016). Players can experience only three types of outcomes in our model: remain single; commit to a partnership by granting a favor; and enter a partnership by way of receiving a favor. A player has an incentive to provide the first favor in a partnership only if refusing to do so puts him at risk of eventually becoming single. We find that a key structural property of nodes determines whether a player ever faces the risk of becoming single in equilibrium. Specifically, a node is said to be essential if it belongs to all maximum matchings of the network and under-demanded otherwise. A further partition of essential nodes into over-demanded nodes—neighbors of under-demanded nodes—and perfectly matched nodes—the remaining ones—is central to the Gallai-Edmonds decomposition, which characterizes the structure of maximum matchings.[2] These concepts also play a prominent role in the equilibrium analyses of Corominas-Bosch (2004), Polanski (2007), and Abreu and Manea (2012a). We prove that in the equilibrium of our game, essential players always find partners, while under-demanded players remain single with positive probability. Hence, essential players obtain higher payoffs than under-demanded players.

[1]However, Abreu and Manea (2012b) construct a complex system of punishments and rewards with a non-Markovian structure that implements an efficient subgame perfect equilibrium.

[2]The intuitive economic terminology for the Gallai-Edmonds partition of nodes has been introduced by Bogomolnaia and Moulin (2004) and Roth et al. (2005).

The conclusion that under-demanded players are relatively weaker than essential ones is common across the bargaining models discussed above.[3] However, our model highlights a new channel leading to this conclusion. In previous models, the weakness of under-demanded players is caused by their vulnerability to isolation following some sequences of efficient trades. Thus, the fact that under-demanded nodes are left out by some maximum matchings, which is their defining property, is directly involved in the argument. In the present model, the analysis relies on a latent property of under-demanded nodes: if an under-demanded node is removed from the network, all its neighbors become essential in the remaining network. Backward induction then implies that when an under-demanded player needs a favor, his neighbors have incentives to turn him down in sequence and ultimately reach desirable essential positions. Hence, every under-demanded player who requests a favor remains single.

Our characterization of the subgame perfect equilibrium relies on the intuition that under-demanded players are weak, and players commit to partnerships via doing favors in order to avoid occupying under-demanded positions in the network formed by surviving links. We prove that when an over-demanded player needs a favor, the first under-demanded neighbor he approaches has to provide it. When a perfectly matched player requests a favor, the last neighbor in the order with whom he shares an efficient link—another perfectly matched player—agrees to provide the favor. Refusing to enter this last possible efficient partnership and removing the corresponding link would result in both players switching from essential to under-demanded positions, with the player who requested the favor ending up single and the other player preserving his under-demanded status in the ensuing network.

Our proof additionally shows that a player does not have an incentive to grant a favor to a neighbor if refusing to do so and losing the link with the neighbor leaves him in an essential position. However, under-demanded players turn down favor requests from other under-demanded neighbors and remain temporarily under-demanded anticipating that no player will agree to provide the favor and they will become essential after the chain of rejections and link removals. Therefore, our favor exchange game reveals a deep connection between decentralized incentives for efficient partnership formation and the Gallai-Edmonds structure.

We investigate the robustness of our theoretical conclusions with respect to several modeling assumptions. One important ingredient for our analysis is the assumption that links generating rejections are permanently severed.[4] We show that this assumption is crucial

[3]Polanski (2016) emphasizes this point in a range of bargaining environments, including the cooperative game of Kleinberg and Tardos (2008) and the stationary market of Manea (2011) in addition to the models already mentioned.

[4]This assumption appears in other models of favor exchange. For instance, Jackson et al. (2012) motivate the removal of links following rejections on "behavioral (e.g., emotional) or pro-social grounds." Gere and MacDonald (2010) discuss psychological evidence indicating that rejected or ostracized individuals often reciprocate with antisocial behavior. In our setting, players may also passively lose links and effectively disappear from the network if they do not find a partner when needed.

for our conclusions—in a version of the model in which links are never removed from the network, multiple equilibria may exist and equilibria are not necessarily efficient. However, our equilibrium characterization continues to apply if the game is perturbed so that links generating rejections are maintained with small probability.

We find that our results are not sensitive to other modeling assumptions. In particular, the efficiency result does not change if we assume that the order in which a player asks neighbors for favors is selected strategically or specified exogenously instead of being generated randomly as in the benchmark model. We extend our results to a setting with reduced-form payoffs in which, in the spirit of standard matching and bargaining models, all gains from a partnership are realized at its creation (or equivalently each player requires a single favor at a random time), and the player initiating a partnership enjoys a first-mover advantage. We also discuss extensions of the model with payoff heterogeneity and continuous time.

Bloch et al. (2018) test the predictions of the model in a laboratory experiment. They find that a large fraction of subjects play according to the subgame perfect equilibrium, but a subject's ability to select the equilibrium action depends on the complexity of the network as well as on his or her position in the network. Deviations from equilibrium behavior primarily involve subjects agreeing to grant favors when equilibrium play prescribes declining the request.

Besides the literature on bilateral trade in networks discussed above, our model contributes to research on favor exchange. Möbius and Rozenblat (2016) survey existing research in the latter area. Many models in this literature—in particular, Bramoullé and Kranton (2007), Bloch et al. (2008), Karlan et al. (2009), Jackson et al. (2012), and Ambrus et al. (2014)—share the basic structure of our model: players request favors or transfers at different points in time and cooperation is enforced through reciprocation in the future. The model of Jackson et al. (2012) is closest to ours. However, in that model favor needs are link-specific and pairs of players meet too infrequently to sustain bilateral exchange in isolation. Jackson et al. show that clustered social quilts support cooperation via the social threat of losing links with multiple neighbors following deviations from cooperative behavior.

The rest of the paper is organized as follows. Section 2 introduces the partnership formation game, and Section 3 illustrates its equilibria for two networks. In Section 4, we formalize the relationship between efficient partnerships and maximum matchings and review the Gallai-Edmonds decomposition. Section 5 presents the main result, which establishes the uniqueness and the efficiency of the equilibrium and shows that equilibrium decisions are closely tied to the Gallai-Edmonds decomposition. In Section 6, we analyze alternative versions of the model. Section 7 provides concluding remarks.

2. Model

We study a partnership formation game played by a finite set $N = \{1, 2, \ldots, n\}$ of players who constitute the nodes in an undirected network G. Since the network of potential partnerships evolves over time and the collection of existing partnerships forms a matching, it is useful to provide general definitions for networks and matchings. An *undirected network* G linking the set of nodes N is a subset of $N \times N$ such that $(i, i) \notin G$ and $(i, j) \in G \Leftrightarrow (j, i) \in G$ for all $i, j \in N$. The condition $(i, j) \in G$ is interpreted as the existence of a *link* between nodes i and j in the network G; in this case, we say that i is *linked* to j, or that i is a *neighbor* of j in G. We use the shorthand ij for the pair (i, j) and identify the links ij and ji. A node is *isolated* in G if it has no neighbors in G. For any network G, let $G \setminus ij, kh, \ldots$ denote the network obtained by removing links $ij, kh, \ldots$ from G, and $G \setminus i, j, \ldots$ denote the network in which all links of nodes $i, j, \ldots$ in G are removed (but nodes $i, j, \ldots$ remain, isolated, in the network). A *matching* is a network in which every node has at most one link. A *matching of the network* G is a matching that is a subset of G. We say that a matching *covers* a node if the node has one link in the matching (and that a matching covers a set of nodes if it covers every node in that set).

The partnership formation game proceeds in discrete time at dates $t = 0, 1, \ldots$. At every date t, there is a set of partnerships that have already formed represented by a matching M_t and a prevailing network of potential future partnerships G_t. At date t, one player i randomly selected—each with probability $1/n$—from the set N needs a favor. Partnerships are assumed to be permanent and guarantee reciprocal favor exchange, so if player i has a partner j under M_t, then j automatically provides the favor to i. Otherwise, player i randomly chooses one of his neighbors j_0 in the network $G_{t0} := G_t$ and asks him for the favor. Player j_0 decides whether to provide the favor or not. If player j_0 declines to do the favor for i, then the link ij_0 is permanently removed from the network, and player i continues searching for a partner in the network $G_{t1} := G_{t0} \setminus ij_0$. In general, after k rejections, player i randomly chooses one of his neighbors j_k in the remaining network $G_{tk} := G_{t(k-1)} \setminus ij_{k-1}$ to ask for the favor. If player j_k agrees to provide the favor to player i at date t, player i receives a payoff $v > 0$ and player j_k incurs a cost $c \in (0, v)$. In this case, i and j_k form a long-term partnership, so that the set of ongoing partnerships becomes $M_{t+1} = M_t \cup ij_k$, and the game proceeds to date $t + 1$ on the network $G_{t+1} = G_t \setminus i, j_k$. If none of i's neighbors in G_t agrees to provide the favor to i, then i remains isolated and the game continues to period $t + 1$ on the network $G_{t+1} = G_t \setminus i$. All players discount future payoffs by a factor of δ per period.

We assume that the game has perfect information and use the solution concept of subgame perfect equilibrium. We allow for mixed strategies but will show that as players become patient, the subgame perfect equilibrium is unique and involves only pure strategies.

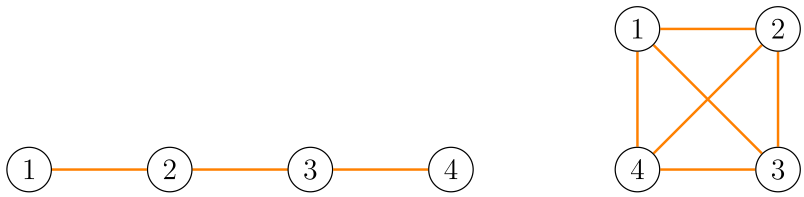

FIGURE 1. The line and complete networks with four nodes

Let V denote the expected discounted payoff obtained by a player who is matched with a partner with whom he reciprocates favors,

$$V = \frac{v - c}{n(1 - \delta)}.$$

Since providing the first favor in a partnership costs c and leads to a continuation payoff of δV, a necessary condition for a player to rationally agree to provide the first favor in equilibrium is that $\delta V \geq c$, which is equivalent to

$$\delta \geq \underline{\delta} := \frac{n}{r + n}$$

where

$$r := \frac{v - c}{c}$$

represents the *return to favors*. Hence, if $\delta < \underline{\delta}$, then all favor requests are turned down and every player receives zero payoff in equilibrium.

As customary, the *normalized* payoff of a player is defined by his expected payoff in the game multiplied by $1 - \delta$. Thus, the normalized payoff accruing to a player who is matched with a partner with whom he reciprocates favors is $(v - c)/n$.

Two types of networks that will be useful for illustrations. The *line* network with n players consists of the links $(1, 2), \ldots, (n-1, n)$. A network is *complete* if it links every pair of nodes.

3. EXAMPLES

In this section, we analyze the partnership formation game in the two examples shown in Figure 1: the line and the complete networks with four players. Assume that $\delta \geq \underline{\delta}$, so that at least one partnership forms in any equilibrium in either network.

Consider first the four-player line network. Suppose that player 1 needs a favor in the first period of the game. In this case, only player 2 can provide the favor to 1. If 2 agrees to provide the favor, he obtains an expected payoff of $-c + \delta V$. If 2 turns 1 down, then the link $(1, 2)$ is removed from the network. In the remaining network, if player 2 or 4 requests the next favor, then player 3 has no incentive to provide it. Indeed, declining such a request leaves player 3 in a network with a single link, which generates an expected continuation payoff of δV for player 3, while accepting such a request results in the lower expected payoff of $-c + \delta V$. If instead player 3 requires the first favor in the remaining network, then he approaches each of players 2 and 4 with probability 1/2, and under the assumption that $\delta \geq \underline{\delta}$, either player accepts the request because he would otherwise remain isolated.

It follows that the expected continuation payoff of player 2 following the rejection of 1's favor request is δW^L, where W^L solves the equation

$$W^L = \frac{1}{4}\left(\delta W^L + \frac{1}{2}(-c + \delta V) + \delta V\right).$$

In this equation, the term δW^L represents the continuation payoff of player 2 in the event that player 1 requires the second-period favor as well. Player 2 receives payoff 0 if he needs the second-period favor (as player 3 refuses to provide it) and payoff δV if 4 needs the favor (and 3 turns him down). The expected payoff of player 2 in the event that player 3 needs a favor in the second period is $(-c+\delta V)/2$, reflecting the fact that 3 asks 2 for the favor with probability 1/2. The solution to the equation is

$$W^L = \frac{3\delta V - c}{2(4-\delta)}.$$

Player 2 then has an incentive to grant the favor in the first period to player 1 only if $-c + \delta V \geq \delta W^L$, which is equivalent to

$$\delta V(8 - 5\delta) \geq c(8 - 3\delta).$$

Using the formula $V = (v - c)/(4(1 - \delta))$, the inequality above can be rewritten as

$$r = \frac{v - c}{c} \geq \frac{4(1-\delta)(8-3\delta)}{\delta(8-5\delta)}.$$

For $\delta \in [0, 1)$, this inequality is equivalent to

$$\delta \geq \delta^{*L} := \frac{2(11 + 2r - \sqrt{25 + 4r + 4r^2})}{12 + 5r}.$$

For instance, for $r = 1$, which means that $v = 2c$, we have that $\delta^{*L} \approx 0.854$.

When player 2 needs the first favor, player 3 does not have an incentive to provide it because he can count on always receiving favors from 4. However, for $\delta \geq \underline{\delta}$, player 1 has an incentive to do the favor to 2 because refusing to do so would leave him isolated. Symmetric arguments apply to the situations in which players 3 and 4 require the first favor.

Therefore, the structure of equilibria in this network is as follows:

- For $\delta < \underline{\delta}$, no favors are ever granted in equilibrium.
- For $\delta \in (\underline{\delta}, \delta^{*L})$, if player 1 (or 4) needs the first favor, then player 2 (3) turns him down; in the remaining network, if player 2 or 4 (1 or 3) needs the next favor, player 3 (2) turns him down, while if player 3 (2) needs the next favor, the first neighbor he approaches agrees to provide it. If player 2 (3) needs the first favor instead, then player 1 (4) provides it.

- For $\delta \in (\delta^{*L}, 1)$, every player who needs a favor receives it and the partnerships $(1, 2)$ and $(3, 4)$ form (players 2 and 3 reject each other's favor requests on the equilibrium path).

For any δ, welfare maximization requires that the partnerships $(1, 2)$ and $(3, 4)$ form, so the equilibrium is efficient only for $\delta > \delta^{*L}$.

Consider next the complete network with four players. If player 1 needs the first favor and is rejected by all his neighbors, then the other three players are left in a complete network. In this case, an argument similar to the one above shows that when one of the remaining three players requires a favor, the other two turn him down. Thus, if all players refuse to do player 1 the favor, each player $i \in \{2, 3, 4\}$ enjoys a continuation payoff δW^C, where

$$W^C = \frac{1}{4}(\delta W^C + 2\delta V).$$

This payoff equation is analogous to the one defining W^L. In particular, the term $2\delta V$ captures the events in which one of two players different from 1 and i needs the next favor and remains single, effectively leaving i in a bilateral partnership with the fourth player starting in the third period. Solving the equation, we obtain

$$W^C = \frac{2\delta V}{4 - \delta}.$$

The last neighbor approached by player 1 has an incentive to provide the first-period favor to 1 only if $-c + \delta V \geq \delta W^C$, which is equivalent to

$$r \geq \frac{4(1 - \delta)(4 - \delta)}{\delta(4 - 3\delta)}.$$

The last inequality reduces to

$$\delta \geq \delta^{*C} := \frac{2(5 + r - \sqrt{9 - 2r + r^2})}{4 + 3r}.$$

For $r = 1$, we obtain $\delta^{*C} \approx 0.906$.

To summarize, the structure of equilibria in the complete network is as follows:

- For $\delta < \underline{\delta}$, no favors are granted in equilibrium.
- For $\delta \in (\underline{\delta}, \delta^{*C})$, the player who needs the first favor is refused by all other players. The next player requiring a favor is also turned down by his remaining neighbors. The third player who needs a favor receives it from his only remaining neighbor, and a single partnership forms.
- For $\delta \in (\delta^{*C}, 1)$, the first player who needs a favor receives it from the last player he approaches, and two partnerships form.

As in the line network, for any δ, welfare maximization requires that every player who needs a favor receives it, so the equilibrium is efficient only for $\delta > \delta^{*C}$.

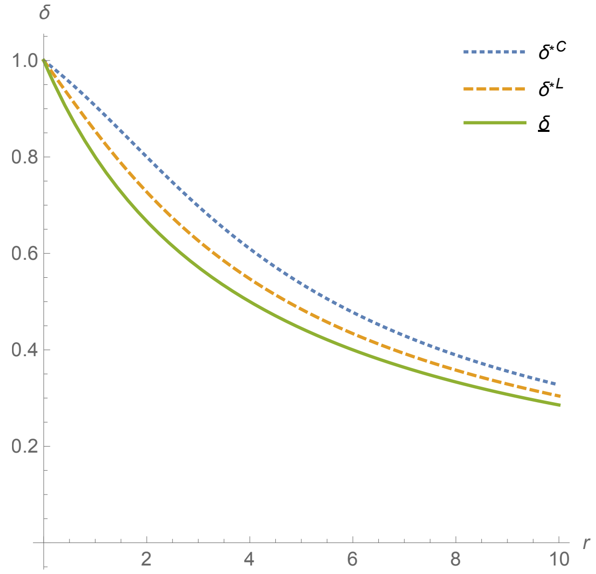

FIGURE 2. Thresholds $\underline{\delta}, \delta^{*L}$, and δ^{*C} as a function of r

Figure 2 depicts the thresholds $\underline{\delta}, \delta^{*L}$, and δ^{*C} as a function of the return to favors r. Note that $\underline{\delta} < \delta^{*L} < \delta^{*C}$ for all values of $r > 0$. The inequality $\delta^{*L} < \delta^{*C}$ reflects the fact that player 2's continuation payoff in the event that player 1 requires the first favor and his neighbors turn him down is smaller in the line network than in the complete network, $W^L < W^C$. Hence, it is easier to provide an incentive for player 2 to form an efficient partnership with player 1 in the line than in the complete network. The two examples demonstrate that adding links to a network does not always facilitate the efficient formation of partnerships. Enlarging the set of links increases the number of potential matchings but may also increase the continuation values of players after links are severed, making it more difficult to sustain efficient partnerships.

We will prove that the conclusions regarding the uniqueness and efficiency of equilibria for high δ in the examples of this section extend to all networks. We will also show that the equilibrium decisions to form partnerships for high δ are directly determined by the classification of the corresponding pairs of nodes in the Gallai-Edmonds decomposition, which we introduce next.

4. Efficient Partnerships and Maximum Matchings

The previous section reveals a close relationship between efficient partnership formation and maximum matchings. We say that a matching M is a *maximum matching* of G if there exists no matching of G that contains a greater number of links than M. For any network G, let $\mu(G)$ denote the *size of the maximum matching* of G, i.e., the number of links in a maximum matching of G.

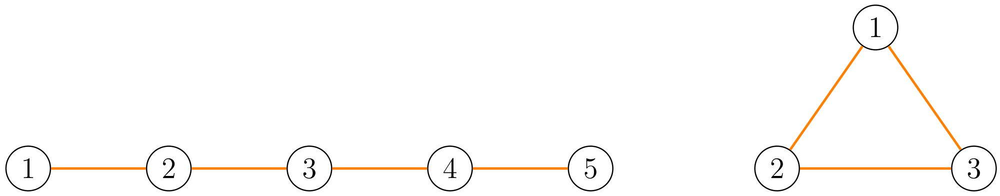

FIGURE 3. The line with five nodes and the complete network with three nodes

Any strategy profile σ along with the random moves by nature—the list of players needing favors and the sequence of neighbors they approach at every date—induce a probability distribution over outcomes at every date. We view $(M_t)_{t\geq 0}$ and $(G_t)_{t\geq 0}$ as random variables in this space. Clearly, we have $M_t \subseteq M_{t+1}$ and $G_{t+1} \subseteq G_t$ for all $t \geq 0$. Let $\bar{M}$ and $\bar{G}$ denote the limits as $t \to \infty$ of the variables $(M_t)_{t\geq 0}$ and $(G_t)_{t\geq 0}$ defined by $\bar{M} = \cup_{t\geq 0} M_t$ and $\bar{G} = \cap_{t\geq 0} G_t$. We call the random variable $\bar{M}$ the *long-run matching* induced by σ. This motivates the following definition.

Definition 1. *A strategy profile is* long-run efficient *if the long-run matching it induces is a maximum matching of G with probability 1.*

In the Appendix, we confirm the intuition that a strategy profile maximizes the limit of the sum of normalized expected payoffs of all players as $\delta \to 1$ only if it is long-run efficient.

The welfare analysis of equilibria in our partnership formation game thus naturally leads us to examine the structure of maximum matchings. Gallai (1964) and Edmonds (1965) developed a characterization of maximum matchings that not only proves useful in analyzing welfare properties of equilibria but captures the structure of incentives in our game in a precise way. Gallai and Edmonds' result relies on the following partition of the set of nodes in a network G. A node is *under-demanded* in G if it is not covered by some maximum matching of G. A node is *over-demanded* in G if it is not under-demanded but has an under-demanded neighbor in G. A node is *perfectly matched* in G if it is neither under- nor over-demanded in G. For example, in both the line and the complete networks with four players from Figure 1, all nodes are perfectly matched. In the line with five nodes shown in the left panel of Figure 3, nodes 1, 3, and 5 are under-demanded, while nodes 2 and 4 are over-demanded. In the complete network with three nodes from the right panel of Figure 3, all nodes are under-demanded.

Theorem GE (Gallai-Edmonds Decomposition [19]). *Every maximum matching of a network links each perfectly matched node to another perfectly matched node and each over-demanded node to an under-demanded node.*[5]

[5]The original theorem provides a comprehensive description of the structure of maximum matchings. We only state the part of the result needed for our analysis here.

The following result, whose proof appears in the Appendix, describes the evolution of the Gallai-Edmonds partition as partnerships form or as links are removed following declined favor requests. We say that i is an *efficient partner* of j in G or that ij is an *efficient link* if the link ij belongs to a maximum matching of G and that a node is *essential* in G if it is either over-demanded or perfectly matched in G.

Lemma 1. *For every network G and any link $ij \in G$, the following statements hold:*

(1) If i is under-demanded in G, then i is under-demanded in the network $G \setminus ij$, and j is essential in the network $G \setminus i$.

(2) Suppose that i is perfectly matched in G and that j is not the only efficient partner of i in G. Then, the sets of perfectly matched, over-demanded, and under-demanded nodes coincide in G and $G \setminus ij$. Moreover, the set of efficient partners of i in $G \setminus ij$ consists of all efficient partners of i in G except for j (in case ij is an efficient link).

(3) If i is perfectly matched in G, and j is i's only efficient partner in G, then both i and j are under-demanded in the network $G \setminus ij$, and j is under-demanded in $G \setminus i$.

(4) If i is over-demanded and j is under-demanded in G, then ij is an efficient link in G, and i is an essential node in $G \setminus ij$.

(5) If both i and j are perfectly matched, and ij is an efficient link in G, then the sets of under-demanded nodes different from i and j in $G \setminus i, j$ and G coincide.

5. Equilibrium Partnerships and Efficiency

We provide a complete characterization of subgame perfect equilibria in the partnership formation game for high δ that relies on the classification of nodes in the Gallai-Edmonds decomposition in the prevailing network. From this characterization, we infer that equilibria are long-run efficient and that all essential players find partners when players are patient. The characterization also implies that for $\delta \to 1$, all essential players receive limit normalized payoffs of $(v-c)/n$, while under-demanded players obtain limit normalized payoffs of at most $(n-1)/n \times (v-c)/n$. Therefore, as players become patient, essential players fare better than under-demanded players.

Theorem 1. *Fix a network G with n players. The following statements hold for sufficiently high δ.*

- ***Equilibrium Uniqueness:*** *The partnership formation game played on the network G has a unique subgame perfect equilibrium, which is in pure strategies.*
- ***Outcomes:*** *In the equilibrium, each essential player in G receives favors any time he needs them and enjoys a limit normalized expected payoff of $(v-c)/n$, while each under-demanded player in G remains single with probability at least $1/n$ and obtains a limit normalized expected payoff of at most $(n-1)/n \times (v-c)/n$.*
- ***Behavior and Partnerships:*** *When player i requests a favor in the initial network G,*

- *if i is under-demanded in G, all neighbors deny i's request and i remains single;*
- *if i is perfectly matched in G, all neighbors turn i down until there is only one remaining neighbor among i's efficient partners in G; this neighbor agrees to provide the favor to i;*
- *if i is over-demanded in G, all neighbors turn i down until i approaches his first under-demanded neighbor in G, who grants the favor.*

- ***Efficiency:*** *In equilibrium, $\mu(G)$ partnerships form with probability 1. The equilibrium is long-run efficient.*

As discussed in the introduction, existing research (e.g., Abreu and Manea (2012a, b) and Elliott and Nava (2016)) reveals a tension between decentralized trade in networks and efficient market outcomes. A combination of new modeling assumptions delivers efficiency in our decentralized setting: direct transfers are not possible within partnerships, so every player can experience only a discrete set of outcomes; a player who needs a favor gets the opportunity to propose partnerships sequentially to all his neighbors; and links leading to rejections are irreversibly removed from the network. Corominas-Bosch (2004), Polanski (2007), and Abreu and Manea (2012a) also reach the conclusion that under-demanded players are weaker than essential ones. Their results rely on the vulnerability of under-demanded players to isolation as neighbors form partnerships and exit the network. This vulnerability stems from the fact that, by definition, under-demanded players are excluded by some maximum matchings. Our analysis points to a conceptually distinct quality of under-demanded players in the structure of maximum matchings: removing an under-demanded player from the network makes all his neighbors essential. For this reason, when an under-demanded player requests a favor, his neighbors anticipate that his other potential partners will refuse the request in order to reach attractive essential positions. Hence, under-demanded players are marginalized in the original network via immediate link deletions triggered by rejections rather than being exposed to the standard gradual decline in partnership opportunities.

We present the proof of Theorem 1 in the Appendix. To develop some intuition for this result, note that every player i can experience three types of outcomes in the partnership formation game in the network G: (1) remaining single; (2) entering a partnership by way of providing a favor to a neighbor who requires one; (3) initiating a partnership via having the first favor he needs granted. The expected payoffs of player i when these situations arise are given by 0, $-c+\delta V$, and $v+\delta V$, respectively. For $\delta > \underline{\delta}$, we have that $0 < -c+\delta V < v+\delta V$. In scenarios (2) and (3), player i always receives the benefit v when he needs a favor and has to pay the cost c any time his partner requires a favor. However, scenario (3) saves player i some early costs c of providing favors before he needs one, so i does not have an incentive to accept a partnership of type (2) unless there is some risk that refusing to enter such a partnership exposes him to some risk of facing scenario (1). Therefore, every player prefers scenario (3) most and would like to delay accepting a partnership of type (2) for as long as this does not make him vulnerable to remaining single as in scenario (1).

The proof shows by induction on the number of links in network G that in equilibrium, essential players always form partnerships and end up in scenario (2) or (3), while under-demanded players reach scenario (1) with probability at least $1/n$ for δ close to 1. It is then optimal for a player to provide a favor when asked only if he becomes under-demanded in the network in which his link with the player needing the favor is severed. Lemma 1.1 implies that if an under-demanded player i needs a favor and all his neighbors turn him down, then i's neighbors become essential in the remaining network. The induction hypothesis and backward induction then imply that every neighbor of i is guaranteed an outcome classified as scenario (2) or (3) above and thus does not have an incentive to do i the favor. Hence, in any subgame, every player who is under-demanded in the remaining network faces scenario (1) in the event he needs the next favor, which happens with probability $1/n$.

When a perfectly matched player i requires a favor, we argue that his last efficient partner j whom he asks should provide the favor to i. Otherwise, the link ij is removed from the network, and the first part of Lemma 1.3 shows that both i and j are under-demanded in the resulting network $G \setminus ij$. The induction hypothesis for $G \setminus ij$ implies that no neighbor whom i approaches after j grants him the favor. Then, the second part of Lemma 1.3 shows that j becomes under-demanded in the ensuing network $G \setminus i$. By Lemma 1.2, every player whom i asks for the favor before reaching his last efficient partner is essential and maintains his role in the Gallai-Edmonds decomposition following his refusal to do i the favor. Hence, these players do not have incentives to partner with i.

When an over-demanded player i requires a favor, Theorem GE implies that no essential neighbor changes status in the Gallai-Edmonds decomposition by refusing to provide the favor and losing the link with i. Then, no such neighbor has an incentive to do i the favor. However, if the over-demanded player i asks an under-demanded neighbor j for the favor, player j has an incentive to do it. This requires a more delicate analysis of the evolution of the positions of i and j in the Gallai-Edmonds decomposition in the subgame in which the link ij is removed and i approaches other neighbors with the request. Lemma 1.4 shows that i remains essential in $G \setminus ij$, and the induction hypothesis implies that i will eventually reach a neighbor k who is willing to partner with him. However, it is possible that the partnership between i and his specific neighbor k improves j's position from being under-demanded in G (as well as $G \setminus ij$ according to Lemma 1.1) to becoming essential in $G \setminus i, k$.

This situation is illustrated in the network from Figure 4. Suppose that in this network, the over-demanded player i asks his under-demanded neighbor j for a favor, and j turns him down. If i requests the favor from h next, then h accepts to provide the favor anticipating that he would otherwise remain under-demanded. Following the formation of the partnership (i, h), player j becomes perfectly matched in the remaining network and eventually partners with g. Player j's continuation payoff in this event is δV, which is greater than the expected

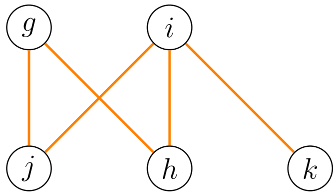

FIGURE 4. A key step in the proof of Theorem 1

payoff $\delta V - c$ derived from providing the favor to i. However, if i asks player k instead of h for the favor after j's rejection, k agrees to provide the favor, in which case j is left under-demanded and exposed to a probability $1/2$ of remaining single, which for high δ is significantly less desirable than the expected payoff $\delta V - c$ guaranteed by the partnership with i. This is where the assumption that player i asks his neighbors for the favor in random order is crucial for the argument—after being rejected by j, player i is equally likely to ask the favor from h and k; while j is slightly better off not providing the favor to i in case i partners with h, he is considerably worse off in case i partners with k. Then, for high δ, player j prefers to provide the favor to i if asked first. The proof shows that a player acting like k—willing to form a partnership with i in equilibrium that leaves j under-demanded—always exists in a general network in which i is over-demanded and j is under-demanded. Lemma 1.4 is used to conclude that the set of partnerships that emerge in equilibrium form a maximum matching.

In Section 6.1, we discuss how the conclusions of Theorem 1 adjust if we alternatively assume that the order in which a player asks neighbors for favors is selected strategically or specified exogenously. Only the prediction of exactly which under-demanded player provides the favor to an over-demanded player change in these alternative specifications of the model.

6. Alternative Models

In this section, we test the robustness of our predictions with respect to several modeling variations. We show that the assumption that players who need favors ask neighbors in random order is not essential for our main findings. We also develop a version of the model with reduced-form payoffs in which all benefits from a partnership accrue at the time of its creation and there is some advantage for the player initiating the partnership. The results extend to such settings. We then discuss extensions of the model with payoff asymmetries and continuous time. Finally, we show that inefficient equilibria emerge if we assume that links leading to rejections are not removed from the network, but efficiency is preserved in a perturbation of the game whereby links generating rejections are maintained with small probability.

6.1. **Requesting Favors in Strategic or Exogenous Order.** We first comment on the implications of alternative modeling assumptions regarding the order in which players ask neighbors for favors. The example from Figure 4 shows that the fact that the player who needs a favor approaches neighbors in random order without revealing the order at the beginning of the period and instead picking neighbors sequentially is essential for the partnership outcomes described by Theorem 1. Indeed, if player i in the network from Figure 4 chose the order in which he approaches neighbors for the favor randomly and announced that it would be (j, h, k), then player j would anticipate that i will form a partnership with h and would optimally decide to turn down i's request knowing that he will always have the option to partner with g at a later stage. The same conclusion would carry over to a specification of the model in which players who need favors approach neighbors in an exogenous deterministic order.

Furthermore, the equilibrium uniqueness established by Theorem 1 does not extend to a version of the model in which unmatched players who need favors choose the order in which they ask neighbors strategically. To fix ideas, assume that the player who needs a favor does not broadcast the order at the beginning of the period but rather decides whom to approach next following every rejection. Consider, for instance, the line network with three players. In the model with endogenous orders, for every $p \in [0, 1]$, there exists an equilibrium in which when player 2 needs the first favor, he approaches player 1 first with probability p. The limit normalized payoffs of player 1 range from $(v - c)/9$ to $2(v - c)/9$ between the extreme cases $p = 0$ and $p = 1$.

While the characterization of partnerships formed by over-demanded players from Theorem 1 does not extend to models with alternative assumptions regarding the order in which players needing favors approach neighbors, we show that in any of these model specifications, every over-demanded player who needs a favor receives it from some under-demanded neighbor. All remaining properties of the structure of equilibria uncovered by Theorem 1 extend to the alternative model specifications. The proof is provided in the Appendix.

Theorem 2. *Subgame perfect equilibria of the version of the partnership formation game with strategic or exogenous orderings for sufficiently high δ satisfy all the properties outlined in Theorem 1 with the following exceptions. The equilibrium is not unique in the case with endogenous orderings. When an over-demanded player requests a favor, some under-demanded neighbor provides the favor (possibly after rejections by other under-demanded neighbors).*

Given the binary nature of possible equilibrium outcomes for a player requiring a favor—either finding a long-term partner or becoming isolated with probability 1—in any subgame and every equilibrium for all game specifications, Theorems 1 and 2 imply that players who need favors are indifferent among all orderings in which they can ask neighbors. In particular, the equilibrium of the game with random orderings and the equilibrium of the game with any exogenous orderings constitute equilibria for the game with strategic choice of orderings.

6.2. **Reduced-Form One-Time Payoffs.** Our model assumes that players who agree to form partnerships exchange favors and collect payoffs at arbitrarily many dates. Nevertheless, Theorems 1 and 2 extend to a setting in which, similarly to the models of bargaining in networks [1, 2, 8, 11, 17, 18, 20, 23, 24], each player consumes all benefits of a partnership immediately and receives a payoff only at the time the partnership is created. An alternative interpretation of this modeling assumption is that each player requires a single favor at a random time. If the player is unable to find a partner willing to provide the favor at that time, he vanishes from the network along with all his links. This interpretation of the model opens the door to applications with a "ticking clock" such as fertility in mating, deadlines for collaborative projects, and emergencies that require immediate support from a friend.

In this specification of the model, the need for a favor represents an opportunity for a player to propose a partnership. When player i proposes a partnership to neighbor j, and j accepts the proposal, players i and j receive one-time payoffs v_1 and v_2, respectively, and exit the game permanently. To match the premise of the benchmark model that each player prefers receiving the first favor in a partnership, we assume that the proposer enjoys a first-mover advantage, i.e., $v_1 > v_2 > 0$. Similarly, to capture the idea that entering any partnership is more desirable than running the risk of remaining single, we require that $v_2 > (n-1)/n\ v_1$. We maintain the assumptions of a common discount factor δ and of a constant arrival rate of $1/n$ per period of opportunities for proposing partnerships for players who have not yet entered partnerships.

The statements regarding equilibrium uniqueness or existence, the structure of equilibrium partnerships, and efficiency of equilibrium outcomes from Theorems 1 and 2 carry over to this setting. For Theorem 1, we need to impose the stronger hypothesis that $v_2 > (1 - 1/((n-1)n))v_1$. This condition is needed for the more detailed characterization of equilibrium partnerships from Theorem 1 establishing that an over-demanded proposer reaches an agreement with the first under-demanded player he encounters and is involved in checking incentives for the step of the proof illustrated in Figure 4. Intuitively, the conditions $v_1 > v_2 > (n-1)/n\ v_1$ and $v_1 > v_2 > (1-1/((n-1)n))v_1$ require that the risk of not finding a partner outweighs the first-mover advantage in a partnership, so the first-mover advantage should not be too large.

The proofs of the results for this setting rely on exactly the same structural properties of nodes invoked by the analogous steps in the proofs of Theorems 1 and 2 and minor modifications in payoff bound computations. In particular, the optimality of accepting a proposal whose rejection would leave a player vulnerable to isolation is driven by the assumptions that $v_2 > (1 - 1/((n-1)n))v_1$ and $v_2 > (n-1)/n\ v_1$, respectively. Similarly, the inequality $(v_1 + (n-1)v_2)/n > v_2$ implies that a player has an incentive to reject a proposal and forgo the second-mover payoff v_2 if doing so does not put him at risk of not finding a partner and makes him eligible for the first-mover payoff v_1 in the event he has the opportunity to propose a partnership in the remaining network in the next period.

6.3. **Player Heterogeneity and Continuous Time.** The conclusions of Theorems 1 and 2 extend to a setting in which time is continuous and players require favors at random times that have independent and identical Poisson distributions. The structure of equilibria does not change and maximum matchings always emerge in equilibrium if the benefit of receiving favors and the cost of providing favors are player specific.[6] It should be noted that the connection between efficiency and maximum matchings breaks down for this generalization. Indeed, if the benefit from receiving favors and the cost of providing favors for player i are v_i and c_i (with $v_i > c_i$), respectively, then long-run efficiency requires that the long-run matching $\bar{M}$ describing the structure of partnerships satisfies

$$\bar{M} \in \arg \max_{\text{matchings } M \text{ of } G} \sum_{i \text{ covered by } M} (v_i - c_i).$$

There is no general relationship between maximizers of the expression above and maximum matchings. Nevertheless, for small levels of variation in $v_i - c_i$ over $i \in N$, the value achieved by maximum matchings is close to the optimal solution, so we can conclude that equilibria of our partnership formation game are approximately efficient.

6.4. **No Link Removal.** Our results rely critically on the assumption that refusing to provide a favor over a link results in its removal from the network. Let us now consider a version of the model in which if a player refuses to provide a favor to a neighbor, the underlying link is not removed from the network and can be used for exchanging favors in the future (but cannot be reactivated to request the same favor in the current period). As in the benchmark model, the player who needs a favor asks neighbors in random order and partnerships are permanent once formed. The main conclusions of Theorem 1—uniqueness of the subgame perfect equilibrium and its asymptotic efficiency—do not extend to this model.

To illustrate equilibrium multiplicity, consider a network with two players, 1 and 2, linked with each other. For high discount factors δ, we can identify the following subgame perfect equilibria. In one equilibrium, player 1 always refuses to provide the first favor, while player 2 always agrees to provide the first favor. Under these strategies, it is optimal for player 1 to refuse to provide favors as long as a partnership is not in place because player 2 will grant favors to player 1 whenever player 1 requires them. The optimality of player 2's strategy can be checked using the single-deviation principle. The described strategies require that if player 1 needs the first favor at date t, player 2 should provide it. This behavior generates an expected payoff of $-c + \delta V > 0$ for player 2. If player 2 deviates from the date t action by turning player 1 down and then play confirms to the prescribed strategies, player 2 will enter a partnership the next time player 1 requires a favor. If this happens $t' > 0$ periods later, player 2's conditional expected payoff is $\delta^{t'}(-c + \delta V)$, which is smaller than $-c + \delta V$

[6]However, the analysis becomes intractable if benefits and costs are link specific. The homogeneity assumptions embedded in our model allow us to neatly separate network effects from other player asymmetries.

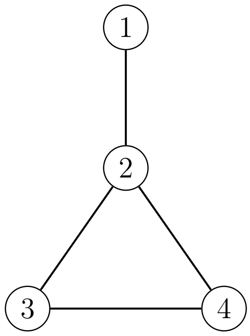

Figure 5. Long-run inefficient equilibria without link removals

for all t'. This clearly constitutes the best equilibrium outcome for player 1 and the worst equilibrium outcome for player 2.

To obtain another stationary equilibrium, we can simply switch the behavior of players 1 and 2 in the strategies constructed above. Non-stationary equilibria can be built relying on these two equilibria. For instance, another equilibrium prescribes that the player who asks for the first favor receives it, and a partnership forms in the first period. This equilibrium relies on the threat that if a player turns down the first favor request, then play reverts to the equilibrium in which the opponent never provides the first favor. Note that this equilibrium generates expected payoffs of V for each player and is welfare optimal.[7]

Every subgame perfect equilibrium in the two-player network is long-run efficient. To see this, note that for δ close to 1, if an equilibrium involves only a small probability of agreement after a certain date, then either player would have an incentive to deviate and provide the favor at that date.

We next present an example in which long-run inefficient subgame perfect equilibria exist. Consider the 4-player network from Figure 5, which is also used to illustrate inefficient equilibria in Abreu and Manea (2012a). In the Appendix, we verify that the following strategies constitute a subgame prefect equilibrium for high δ. Before any partnership forms, players 1, 3, and 4 always provide favors to any neighbor who asks for one regardless of the set of players who previously refused to do the same favor, while player 2 never agrees to provide a favor. In a subgame in which partnership $(1, 2)$ or $(3, 4)$ formed, play between the remaining pair of (linked) players proceeds according to certain equilibrium strategies for the two-player setting discussed above.[8] In subgames in which the partnership $(3, 4)$ formed, the

[7]In another equilibrium, the first-period favor request is turned down, and then play proceeds according to the welfare optimal equilibrium. These strategies constitute an equilibrium because providing the first favor generates an expected payoff of $-c + \delta V$, while refusing to do so results in a continuation payoff of $\delta V > -c+\delta V$. There exists also a symmetric stationary equilibrium in mixed strategies, in which conditional on not having entered the partnership, both players agree to provide the next favor with equal probability.

[8]Note that the payoffs induced by these strategies are not the same as in the two-player setting because in these subgames each player requires favors with probability $1/4$ rather than $1/2$, but the verification of equilibrium incentives for high δ is analogous.

strategies specify that the best equilibrium for player 2 is played in the relationship between players 1 and 2. In subgames in which player 2 requires a favor and is first refused by both players 3 and 4 (in either order) and then ends up partnering with player 1, the strategies prescribe that the worst equilibrium for the second player who turned 2 down is played in the relationship between players 3 and 4. In other subgames with two linked players remaining, equilibrium play can be specified arbitrarily.

Under these strategies, each of the inefficient partnerships $(2, 3)$ and $(2, 4)$ forms with probability $1/4 \times 1/3 = 1/12$ in the first period of any subgame in which no partnership has formed (in the event player 2 requests a favor from player 3 or 4, respectively), with probability $1/4 \times 1/12$ in the second period (in the event player 1 requested the first-period favor and got turned down, then 2 requested the second-period favor), and so on.

Long-run efficient equilibria also exist in this example. Such an equilibrium can be constructed as follows. Player 2 never agrees to provide the first favor to a neighbor in any subgame. Player 1 provides a favor when asked by player 2 in every subgame. As long as no partnership has formed, there are two regimes depending on whether 3 and 4 ever refused a favor request from each other. Play starts in the "cooperative" regime, in which players 3 and 4 agree to provide favors to each other whenever one asks the other. If the pair $(3, 4)$ ever deviates from this norm, play switches to a "non-cooperative" regime, in which players follow the inefficient equilibrium strategies constructed above. In subgames in which the partnership $(1, 2)$ has formed, players 3 and 4 play according to the welfare optimal equilibrium strategies for the two-player game whereby they exchange favors on the equilibrium path in every period.

Given these strategies, it is not optimal for players 3 and 4 to ever provide favors to 2 as long as they have not deviated from the norm of the cooperative regime because they know that they can always count on each other to exchange favors when needed. By refusing to partner with 2, each of the players 3 and 4 receive the benefit v of favors when needed, but postpone paying the cost c. Players 3 and 4 have incentives to provide favors to each other in the cooperative regime because failing to do so results in player 2 also refusing the favor request (if asked second in the order by either 3 or 4), and the ensuing inefficient continuation play in the non-cooperative regime leaves each of them isolated with probability greater than 1/12. Incentives in other interactions involving players 1 and 2 can be checked as in the case of the inefficient equilibrium.[9]

[9]There also exists a mixed-strategy efficient equilibrium that generates identical payoffs for all players similar to the symmetric stationary equilibrium for the two-player setting described in footnote 7. In this equilibrium, only the efficient partnerships $(1, 2)$ and $(3, 4)$ form and each of the four players mixes with the same probability between providing and not providing the first favor across these two links. Every player is indifferent between doing a favor when asked and waiting for the game to proceed to the next period with the same structure of partnerships in place.

6.5. **Small Probability of Link Preservation.** The weakness of each under-demanded player i driving our main result stems from the incentives of i's neighbors to decline his favor request in order to achieve essential positions following the removal of all of i's links. When rejections do not always trigger link removals, i's neighbors are not guaranteed to land essential positions following their rejections of i. If links are preserved with significant probability following rejections, then some of i's neighbors would prefer to grant the favor and enter a partnership with i instead of running the risk of remaining under-demanded and eventually being left single. Hence, equilibrium dynamics and bargaining power depend crucially on the probability with which rejection-generating links are preserved. However, we will argue that our main result is robust with respect to small probabilities of link preservation following rejections.

Consider a perturbation of the partnership formation game in which rejections do not necessarily lead to link removals. Instead, assume that a player whose favor request has been declined by all his neighbors maintains each of his links independently with some small probability $\epsilon \geq 0$.[10] We refer to this version of the partnership formation game as the *ϵ-perturbed game*.

The proof of Theorem 1 shows that incentives in the subgame perfect equilibrium of the unperturbed game are strict in every subgame for sufficiently high discount factors. A standard continuity argument implies that the subgame perfect equilibrium of the unperturbed game (described by $\epsilon = 0$) also constitutes a subgame perfect equilibrium of the ϵ-perturbed game for ϵ close to 0. The proof of Theorem 1 can be straighforwardly adjusted to establish that this is the only equilibrium of the ϵ-perturbed game.

The conclusion that a maximum matching emerges in equilibrium in the long-run extends to a version of the perturbed game in which links leading to rejections are immediately removed with high probability and players subsequently asked for favors observe which prior rejections generated link removals. However, short-run dynamics are sensitive to the realizations of perturbed paths under this model specification. It is no longer the case that perfectly matched players find a partner as soon as they need a favor. Indeed, when a perfectly matched player i who has multiple efficient partners needs a favor and some of these partners turn him down without losing their links with i, then all neighbors subsequently approached by i must turn him down since i is not their unique efficient partner in the remaining network. Nevertheless, with probability 1, repeated rejections eventually leave i with a single efficient partner, who should agree to form a partnership with i in equilibrium. The prospects of under-demanded players who require favors are also affected by perturbations. When an under-demanded player i is rejected by a neighbor but maintains the link with that neighbor, another neighbor may risk remaining under-demanded if he turns i down

[10]The argument holds for any specification whereby non-empty subsets of rejection-generating links are maintained with small (history-dependent) probabilities.

and loses his link with i. This neighbor should then partner with i in equilibrium, and the ensuing partnership is efficient (since all links of under-demanded players are efficient).

7. Conclusion

This paper studies the formation of bilateral partnerships that guarantee reciprocal exchange of favors in social and economic networks. We find that the structure of equilibrium partnerships, the strengths of players, and market efficiency are driven by the configuration of nodes that are essential for achieving all maximum matchings. In particular, essential players always find partners, while inessential players remain single with positive probability in equilibrium. This implies that essential players obtain higher equilibrium payoffs than inessential players. Even though the search for partners is decentralized and incentives for entering partnerships depend on local network conditions, we prove that the possibility that each inessential player might be unable to find a partner implies that partnerships form efficiently in every network. This result is striking in the context of existing research, which has found that local incentives for forming partnerships are not usually aligned with global welfare maximization in markets with decentralized matching.

More generally, we show exactly how each player's equilibrium decisions are determined by his (evolving) position in the Gallai-Edmonds decomposition. Prior research on trade in networks has established similar but less detailed connections between equilibrium outcomes and the Gallai-Edmonds decomposition (mainly in markets with centralized matching). However, there is a conceptual novelty in the mechanism underlying our result. In our setting, the weakness of inessential players is inflicted by neighbors actively marginalizing them via severing links with them when they request favors in the original network, while in previous models, it is indirectly precipitated by the possibility of remaining isolated in the network as neighbors forge agreements with other players.

The contribution of this research lies at the intersection of game theory and graph theory. Our analysis delivers a precise relationship between the classic Gallai-Edmonds structure of maximum matchings and incentives driving the efficient formation of partnerships in a natural favor exchange game with decentralized matching.

Appendix: Proofs

Long-run efficiency and total welfare. To understand the connection between long-run efficiency and total welfare in our model, fix a strategy profile σ. Let $\mathbf{P}$ denote the probability measure over outcomes induced by σ. Let T denote the lowest t such that $M_t = \bar{M}$ and $G_t = \bar{G}$. For $t \geq T$, we have that $M_t = \bar{M}$ and only players who are matched under $\bar{M}$ are granted favors at date t. Hence, players collectively receive the net benefit of $v - c$ from favor exchange at date $t \geq T$ with probability $2\mu(\bar{M})/n$. Starting at any date $t < T$, there is a sequence of n or fewer draws by nature of unmatched players asking for favors for the first time and either entering partnerships or becoming isolated, which leads to $M_{t+n} = \bar{M}$

and $G_{t+n} = \bar{G}$. The probability of such a sequence being drawn by nature conditional on (G_t, M_t) is at least $1/n^n$. Therefore, $\mathbf{P}(T > t+n | T > t) \leq 1 - 1/n^n$, so

$$\mathbf{P}(T > t+n) = \mathbf{P}(T > t)\mathbf{P}(T > t+n|T > t) \leq \mathbf{P}(T > t)(1 - 1/n^n).$$

It follows that $\mathbf{P}(T > kn) \leq (1 - 1/n^n)^k$ for all $k \geq 0$. Given this exponential bound on the tail of the distribution of T and the fact that the total expected payoffs of all players under σ average to $2\mu(\bar{M})/n(v-c)$ at dates $t \geq T$, the limit of the total normalized expected payoffs of all players under σ as $\delta \to 1$ does not exceed $2\mu(G)/n(v-c)$ and achieves the maximum of $2\mu(G)/n(v-c)$ only if $\bar{M}$ is a maximum matching of G with probability 1. We conclude that σ maximizes the limit of the sum of normalized expected payoffs of all players as $\delta \to 1$ only if it is long-run efficient. $\square$

Proof of Lemma 1. Fix the network G and the link $ij \in G$.

(1) Suppose that i is under-demanded in G, and let M be a maximum matching of G that does not cover i. Since $\mu(G \setminus ij) \leq \mu(G)$ and M is a matching of $G \setminus ij$, it must be that $\mu(G \setminus ij) = \mu(G)$ and M is also a maximum matching of $G \setminus ij$. As M does not cover i, it follows that i is under-demanded in $G \setminus ij$. Similarly, $\mu(G \setminus i) \leq \mu(G)$ combined with the fact that M constitutes a matching for $G \setminus i$ implies that $\mu(G \setminus i) = \mu(G)$. If j is not essential in $G \setminus i$, then there exists a maximum matching M' of $G \setminus i$ that does not cover j. Adding the link ij to M' generates a matching of G with $\mu(G \setminus i) + 1 = \mu(G) + 1$ links, contradicting the definition of $\mu(G)$.

(2) Suppose that i is perfectly matched in G and that j is not the only efficient partner of i. If j is over-demanded in G, then Theorem GE implies that ij is not an efficient link in G, so G and $G \setminus ij$ have the same set of maximum matchings and the same Gallai-Edmonds decomposition. Moreover, the set of efficient partners of i in G does not contain j and is identical to the set of efficient partners of i in $G \setminus ij$.

Suppose next that j is perfectly matched in G. Since j is not the only efficient partner of i, we have that $\mu(G \setminus ij) = \mu(G)$ and every maximum matching of $G \setminus ij$ is a maximum matching of G. In particular, every player who is under-demanded in $G \setminus ij$ is also under-demanded in G, and all efficient partners of i in $G \setminus ij$ are his efficient partners in G as well. Consider now a player k who is under-demanded in G. Then, there exists a maximum matching M of G that does not cover k. Since j is not the only efficient partner of i, there exists a maximum matching M' of G that does not contain the link ij. By Theorem GE, both M and M' link perfectly matched players with one another. Construct a third matching M'' that consists of the links of M among under- and over-demanded players in G and the links of M' among perfectly matched players in G. Then, M'' is a maximum matching of $G \setminus ij$ which does not cover k. Thus, k is under-demanded in $G \setminus ij$. The arguments above show that the sets of under-demanded players in G and $G \setminus ij$ coincide. Since neither i nor j is under-demanded in G, the sets of neighbors in the two networks of the common set of under-demanded players in G and $G \setminus ij$ are also identical. This implies that the sets of over-demanded players in G and $G \setminus ij$ are the same. It follows that the sets of perfectly matched players in G and $G \setminus ij$ also coincide. To prove that every efficient partner $k \neq j$ of i in G is also an efficient partner of i in $G \setminus ij$, it is sufficient to note that every maximum matching of G that contains the link ik continues to be a maximum matching of $G \setminus ij$.

(3) Suppose that i is perfectly matched in G and that j is i's only efficient partner in G. We know that $\mu(G) - 1 \leq \mu(G \setminus ij) \leq \mu(G)$. If $\mu(G \setminus ij) = \mu(G)$, then there exists a maximum matching of G that does not contain the link ij, contradicting the assumption that i is perfectly matched in G and j is i's only efficient partner in G. Hence, $\mu(G \setminus ij) = \mu(G) - 1$. Let M be a maximum matching of G. As i is perfectly matched in G and j is i's only efficient partner in G, the link ij is necessarily contained in M. Removing ij from M produces a matching in $G \setminus ij$ with $\mu(G) - 1 = \mu(G \setminus ij)$ links. This constitutes a maximum matching for $G \setminus ij$, which does not cover i or j. It follows that both i and j are under-demanded in the network $G \setminus ij$.

Since i is perfectly matched in G, we have that $\mu(G \setminus i) = \mu(G) - 1$. Then, removing the link ij from any maximum matching of G generates a maximum matching for $G \setminus i$ that does not cover j. It follows that j is under-demanded in $G \setminus i$.

(4) Suppose that i is over-demanded and j is under-demanded in G. Then, there exists a maximum matching M of G that does not cover node j and hence excludes the link ij. As i is over-demanded in G, M must cover i. If we replace i's link under M with ij, we obtain another maximum matching of G that contains the link ij. Therefore, ij is an efficient link in G. Since $\mu(G \setminus ij) \leq \mu(G)$ and M is a matching of $G \setminus ij$, it must be that $\mu(G \setminus ij) = \mu(G)$, so any maximum matching of $G \setminus ij$ is also a maximum matching of G. As any maximum matching of G covers i, it must be that every maximum matching of $G \setminus ij$ also covers i, so i is essential in $G \setminus ij$.

(5) Suppose that both i and j are perfectly matched and ij is an efficient link in G. Then, we have that $\mu(G \setminus i, j) = \mu(G) - 1$ and every maximum matching of $G \setminus i, j$ can be completed to a maximum matching of G by adding the link ij. Hence, every under-demanded player in $G \setminus i, j$ different from i and j is under-demanded in G. Conversely, reasoning similar to the proof of part (2) shows that any maximum matching of G that does not cover a given node can be transformed into a maximum matching of $G \setminus i, j$ with the same property by rewiring the links among perfectly matched players in G using a maximum matching of G that contains the link ij. It follows that every under-demanded node in G is under-demanded in $G \setminus i, j$. $\square$

Proof of Theorem 1. Fix a set of n nodes and consider a network G linking them. The result follows from claims (I1)-(I7) below concerning *subgame perfect equilibrium* behavior and outcomes in the network G for sufficiently high δ.

(I1) When player i asks player j for a favor in G, player j refuses to provide the favor if j is essential in $G \setminus ij$.

(I2) When a player i who is under-demanded in G first asks a neighbor for a favor, the neighbor turns him down. In equilibrium, no neighbor of i grants the favor and i remains single.

(I3) When a player i who is perfectly matched in G first asks a neighbor j for a favor, j does the favor for i if and only if j is the only efficient partner of i in G.[11] In equilibrium, the last efficient partner of i in G whom i approaches grants the favor to i.

(I4) When an over-demanded player i in G asks his first neighbor for a favor, the neighbor agrees to provide the favor if and only if he is under-demanded in G. In equilibrium, the first under-demanded player j in G whom i approaches provides the favor to i; the resulting partnership between i and j is efficient in G.

(I5) In a subgame in which an essential player in G happens to need a favor at the beginning of a period (before asking any neighbor), each under-demanded player in G remains single with probability at least $1/((n-1)n)$ and receives a limit normalized expected payoff of at most $(1 - 1/((n-1)n)) \times (v - c)/n$.

(I6) In every subgame perfect equilibrium for the network G, each essential player in G receives favors any time he needs them and enjoys a limit normalized expected payoff of $(v - c)/n$, while each under-demanded player in G is left single with probability at least $1/n$ and obtains a limit normalized expected payoff of at most $(n - 1)/n \times (v - c)/n$.

(I7) There exists a unique subgame perfect equilibrium for the network G. In equilibrium, exactly $\mu(G)$ partnerships form with probability 1. The equilibrium is long-run efficient.

We prove claims (I1)-(I7) simultaneously and in this sequence by induction on the number of links in G. The induction base case for a network G with a single link can be verified without difficulty. We need to establish the claims for a network G assuming they are true for any network with fewer links. The proof of the inductive step relies on the existence and uniqueness of the subgame perfect equilibrium for subnetworks of G different from G, which follows from the induction hypothesis (I7). For brevity, we do not explicitly state this fact at every instance it is needed.

Let $V = v/((1-\delta)n)$ denote the expected value of favors provided to a player in a partnership and $\bar{C} = (1 + \delta/((1-\delta)n)c$ the maximum expected cost a player pays upon committing to a partnership via providing a favor in the current period.

To prove the inductive step for claim (I1), suppose that player i first asks player j for a favor in G and that j is essential in $G \setminus ij$. If j decides to grant the favor, then his expected

[11] If i is perfectly matched in G, the condition that j is the only efficient partner of i in G is equivalent to the condition that i is the only efficient partner of j in G.

payoff is $\delta V - \bar{C}$. If j refuses to provide the favor, then i continues to ask for favors in $G \setminus ij$. By the induction hypothesis (I6) applied to network $G \setminus ij$, player j is guaranteed to receive favors whenever he needs them in the future. Hence, j enjoys the same favor benefits, with an expected discounted value of δV, regardless of whether he agrees to partner with i. However, the cost $\bar{C}$ of partnering with i is greater than the expected cost of entering another partnership at any later date, which is at most $\delta\bar{C}$, so j has a strict incentive to refuse i's request in any subgame perfect equilibrium for G.

To establish claim (I2), suppose that an under-demanded player i in G first asks his neighbor j for a favor. Player j obtains an expected payoff of $\delta V - \bar{C}$ if he grants the favor to i. If j refuses to provide the favor, then Lemma 1.1 implies that player i continues to be under-demanded in the resulting network $G \setminus ij$. By the induction hypothesis for the second part of (I2) applied to the network $G \setminus ij$, all of i's other neighbors refuse to partner with him after j turns him down. Then, player j is left in $G \setminus i$ following his rejection of i's request. Since j is linked to the under-demanded player i in G, Lemma 1.1 implies that j is essential in $G \setminus i$. By the induction hypothesis (I6) for the network $G \setminus i$, player j always finds a partner when he needs a favor. Player j's payoff following his rejection of i's request is thus at least $\delta V - \delta\bar{C}$, which is greater than $\delta V - \bar{C}$. Hence, j has a strict incentive to turn i down as asserted. In a subgame perfect equilibrium, player j must turn i down, and then as argued above, all other neighbors of i in G should also turn him down in sequence leaving him ultimately isolated.

To prove the first part of claim (I3), suppose that a player i who is perfectly matched in G first asks his neighbor j for a favor. Since there are no links between under-demanded and perfectly matched nodes in G, is must be that j is an essential player in G. If j is not the only efficient partner of i in G, then Lemma 1.2 implies that j continues to be essential in $G \setminus ij$. Then, (I1) implies that j should refuse i's request in any subgame perfect equilibrium.

Suppose instead that j is the only efficient partner of i in G. Then, by Lemma 1.3, both i and j are under-demanded in $G \setminus ij$. Assume that j refuses i's request. Since i is under-demanded in $G \setminus ij$, the induction hypothesis (I2) for network $G \setminus ij$ implies that no neighbor whom i approaches after j grants the favor to i. Hence, player j ends up in the network $G \setminus i$ following his refusal to partner with i. By Lemma 1.3, player j is under-demanded in $G \setminus i$. Then, the induction hypothesis (I6) for network $G \setminus i$ implies that player j's limit normalized expected payoff following the rejection of i's request does not exceed $(n-1)/n \times (v-c)/n$. Since j can attain a limit normalized expected payoff of $(v-c)/n$ by partnering with i, player j has a strict incentive to accept i's request for high δ.

To prove the inductive step for the second part of claim (I3), note that repeated use of Lemma 1.2 implies that the sets of perfectly matched, over-demanded, and under-demanded nodes do not change as we remove i's links following rejections from neighbors until we reach his last efficient partner in G, while his set of efficient partners in the remaining network consists of his efficient partners in G except for those neighbors who have already rejected him. When i approaches his last efficient partner j in G, it is the case that j is also his only efficient partner in the remaining network. Then, the induction hypothesis (I3) applied for the subnetworks of G resulting from the sequence of rejections (along with the arguments for the first part of claim (I3) above) implies that j must grant the favor to i in any subgame perfect equilibrium.

To demonstrate the first part of claim (I4), suppose that an over-demanded player i in G asks his first neighbor j for a favor. If j is essential in G, then Theorem GE implies that no maximum matching of G contains the link ij, and thus j is essential in $G \setminus ij$. Then, (I1) implies that j should refuse i's request in any subgame perfect equilibrium.

Suppose instead that j is under-demanded in G. Then, Lemma 1.1 shows that j continues to be under-demanded in $G \setminus ij$, while Lemma 1.4 implies that i remains essential in $G \setminus ij$. If j refuses to grant the favor to i, then he finds himself under-demanded in the network $G \setminus ij$ in a subgame where the essential player i needs a favor. By the induction hypothesis (I5) for the network $G \setminus ij$, player j obtains a limit normalized expected payoff of at most $(1 - 1/((n-1)n)) \times (v-c)/n$ in this subgame, which is smaller than the limit normalized expected payoff of $(v-c)/n$ guaranteed to him upon committing to a partnership with i. Hence, player j has a strict incentive to grant the favor to i for sufficiently high δ.

The second part of claim (I4) follows from observing that the removal of any set of links between i and essential players in G does not affect the set of maximum matchings or the Gallai-Edmonds partition because none of these links belongs to a maximum matching in G according to Theorem GE. We can then apply the induction hypothesis (I4) to all subnetworks of G resulting from i's request being denied by his essential neighbors in G. By Lemma 1.4, the eventual partnership that i forms with the first under-demanded neighbor in G whom he asks for a favor is efficient in the remaining network as well as in G.

To prove the inductive step for claim (I5), suppose that an essential player i in G needs a favor at the beginning of a period and fix an under-demanded player j. Consider first the case in which i is perfectly matched in G. In this case, (I3) implies that one of i's efficient partners k in G agrees to grant the favor and forms a partnership with i. By Lemma 1.5, player j remains under-demanded in $G \setminus i, k$. With probability $1/n$, player j needs a favor in the network $G \setminus i, k$ in the next period. Since j is under-demanded in $G \setminus i, k$, the induction hypothesis (I2) for network $G \setminus i, k$ implies that in this event all neighbors turn j down and j remains single.

Consider next the case in which i is over-demanded in G. Since j is under-demanded in G, there exists a maximum matching M of G that does not cover j. Player i has to be matched under M because he is over-demanded in G. Let k be player i's partner under M. By Theorem GE, player k is under-demanded in G. With probability at least $1/(n-1)$, player k is the first neighbor whom i asks for the favor. In this event, (I4) implies that k grants the favor to i. Note that the matching M without the link ik constitutes a maximum matching of $G \setminus i, k$ that does not cover j. Hence, player j is under-demanded in $G \setminus i, k$.

By the same argument as the one used in the first case, j remains single with a conditional probability of at least $1/n$ following the agreement between i and k.

In either case, we have shown that conditional on player i needing a favor in network G, player j remains single with probability at least $1/((n-1)n)$. Since the payoff from being single is 0 and the limit normalized expected payoff from forming a partnership is $(v-c)/n$, player j's limit normalized expected payoff in the subgame cannot exceed $(1-1/((n-1)n)) \times (v-c)/n$.

We now establish claim (I6). Consider an essential player i in G. If i needs a favor in the first period of the game, then claims (I3) and (I4) imply that one of i's neighbors will agree to provide the favor to him. If another player asks i for a favor in the first period, then i forms a partnership with that player in the situation described by (I3) and receives favors as needed thereafter. Otherwise, (I2), (I3), and (I4) imply that in the first period, either an under-demanded player in G (different from the essential player i) needs a favor and is left single or a pair of players different from i form an efficient partnership. Since every maximum matching for the remaining network is a maximum matching of G in the former case and can be completed to form a maximum matching of G by adding the link connecting the partners in the latter case, i continues to be an essential player in the second period network. Then, the induction hypothesis (I6) for the remaining network implies that i always receives favors when he needs them in the subgame starting in the second period. We have shown that i receives favors at any instance he needs them.

If i is an under-demanded player in G, then (I2) implies that i remains single in the event that he needs a favor in the first period of the game. This event has probability $1/n$.

The statements of claim (I6) regarding payoffs follow from the fact every player receives 0 payoff when left single and a limit normalized expected payoff of $(v-c)/n$ upon entering a partnership.

We finally prove claim (I7). Suppose that player i needs a favor and first asks neighbor j in the network G. By the induction hypothesis (I7), there is a unique subgame perfect equilibrium for the network $G \setminus ij$ arising in the event that j turns i's request down. By (I2), (I3), and (I4), the optimal response of player j to i's request is uniquely determined given the equilibrium play in $G \setminus ij$. It follows that there exists at most one subgame perfect equilibrium in G. Using the single-deviation principle, one can easily prove that the strategies described by (I2), (I3), and (I4) indeed constitute a subgame perfect equilibrium for G.

Claims (I2), (I3), and (I4) also show that the partnerships formed along any equilibrium path constitute a maximum matching with probability 1. Indeed, (I2) proves that when an under-demanded player i in G requires the first favor, no neighbor does i the favor and i remains single in the network $G \setminus i$. Since i is under-demanded in G, there exists a maximum matching M of G that does not cover i. Then, M is also a maximum matching of $G \setminus i$, so $\mu(G \setminus i) = \mu(G)$. The inductive hypothesis (I7) implies that $\mu(G \setminus i) = \mu(G)$ partnerships emerge in the subgame played on the resulting network $G \setminus i$. Similarly, (I3) shows that when a perfectly matched player i in G requires the first favor, the last efficient partner j of i in G approached by i grants the favor to i. As the link ij is efficient in G, we have that $\mu(G \setminus i, j) = \mu(G) - 1$. The inductive hypothesis (I7) applied to $G \setminus i, j$ implies that $\mu(G \setminus i, j) = \mu(G) - 1$ partnerships form in the subgame played in the network $G \setminus i, j$ after i partners with j in G. Hence, a total of $\mu(G)$ partnerships emerge in this case as well. An analogous argument deals with the case in which an over-demanded player in G needs the first favor and, according to (I4), forms an efficient partnership with the first under-demanded player in G he approaches. Since every player needs favors at some points in time with probability 1, we conclude that exactly $\mu(G)$ partnerships form with probability 1, and hence the equilibrium is long-run efficient. □

Proof of Theorem 2. Fix a set of n nodes and consider a network G linking them. The result follows from claims (J1)-(J7) below—which reflect appropriate modifications of claims (I1)-(I7) from the proof of Theorem 1—concerning *subgame perfect equilibrium* behavior and outcomes in either specification of the game on the network G for sufficiently high δ.

- (J1) When player i asks player j for a favor in G, if j is essential in $G \setminus ij$, then j refuses to provide the favor to i.
- (J2) When a player i who is under-demanded in G first asks a neighbor j for a favor, j turns him down. In equilibrium, no neighbor of i grants the favor, and i remains single.
- (J3) If player i needs a favor in network G and neighbor j agrees to provide it after some sequence of rejections by other neighbors, then the link ij is efficient in G.
- (J4) When a player i who is perfectly matched in G first asks a neighbor j for a favor, j does the favor for i if and only if j is the only efficient partner of i in G. In equilibrium, the last efficient partner of i in G whom i approaches grants the favor to i.
- (J5) When an essential player needs a favor in G, he forms a partnership with probability 1 in equilibrium.
- (J6) In every subgame perfect equilibrium for the network G, each essential player in G receives favors any time he needs them and enjoys a limit normalized expected payoff of $(v - c)/n$, while each under-demanded player in G is left single with probability at least $1/n$ and obtains a limit normalized expected payoff of at most $(n - 1)/n \times (v - c)/n$.
- (J7) In equilibrium, exactly $\mu(G)$ partnerships form. The equilibrium is long-run efficient.

As for Theorem 1, we prove claims (J1)-(J7) by induction on the number of links in G. The main departure from the approach of Theorem 1 is that in the alternative specifications of the game, the identity of the neighbor who agrees to provide the favor to an over-demanded player depends on the order in which the over-demanded player plans to request the favor from neighbors subsequently. The conclusion (I4) that an over-demanded player receives favors from the first under-demanded player he asks is not true in this setting, as the discussion from Section 6.1 demonstrates. That conclusion is replaced by the weaker claims (J3) and (J5), which in light of Theorem GE jointly imply that any over-demanded player who requests a favor receives it from some under-demanded neighbor.

The induction base case for a network G with a single link can be verified immedaitely. We set out to prove the claims for a network G assuming they hold for any network with fewer links.

The proof of the inductive step for claim (J1) relies on hypothesis (J6) applied to network $G \setminus ij$ and parallels the arguments establishing that (I1) is a consequence of the induction hypothesis (I6) in the proof of Theorem 1. Similarly, the inductive step for claim (J2) follows from applying the induction hypotheses (J2) and (J6) and arguments analogous to those proving that (I2) follows from (I2) and (I6) for networks with fewer links.

We prove (J3) by contradiction. Suppose that player i receives a favor from player j possibly after a sequence of rejections in network G and that the link ij is not efficient in G. Then, both players i and j must be essential in G. For instance, if i is under-demanded in G, then there exists a maximum matching M of G that does not cover i. If M does not cover j either, then the matching $M \cup ij$ of G has greater cardinality than M, a contradiction. If M does cover j, then replacing j's link in M with ij creates another maximum matching of G. This matching contains the link ij, which contradicts the assumption that ij is not an efficient link in G.

Suppose that i's favor request is turned down by neighbors in the order $j_0, j_1, \ldots, j_k$ before j agrees to provide the favor. Note that $\mu(G) - 1 \leq \mu(G \setminus ij_0, \ldots, ij_k) \leq \mu(G)$.

If $\{j_0, j_1, \ldots, j_k\}$ includes all efficient partners of i in G, then it cannot be that $\mu(G \setminus ij_0, \ldots, ij_k) = \mu(G)$. Indeed, if that was the case, then any maximum matching of $G \setminus ij_0, \ldots, ij_k$ should be a maximum matching of G. Since i is essential in G, it must also be essential in $G \setminus ij_0, \ldots, ij_k$. Then, there exists a maximum matching of both $G \setminus ij_0, \ldots, ij_k$ and G that contains a link ih with $h \notin \{j_0, j_1, \ldots, j_k\}$, which means that h is an efficient partner of i in G, a contradiction. This proves that if $\{j_0, j_1, \ldots, j_k\}$ contains all efficient partners of i in G, then $\mu(G \setminus ij_0, \ldots, ij_k) = \mu(G) - 1$. We can then derive a maximum matching of $G \setminus ij_0, \ldots, ij_k$ by removing i's link in any maximum matching of G. Therefore, i is under-demanded in the network $G \setminus ij_0, \ldots, ij_k$. Claim (J2) for network $G \setminus ij_0, \ldots, ij_k$ then shows that player j should not agree to provide the favor to j, contradicting our original assumption.

We are left to consider the case in which $\{j_0, j_1, \ldots, j_k\}$ does not contain all efficient partners of i in G. Since j is assumed to not be an efficient partner of i in G, node i has a neighbor $h \notin \{j_0, j_1, \ldots, j_k, j\}$ such that ih is an efficient link in G. Hence, ih belongs to a maximum matching M of G. Since M is also a matching in the network $G \setminus ij_0, \ldots, ij_k, ij$, it must be that $\mu(G \setminus ij_0, \ldots, ij_k, ij) = \mu(G)$. This implies that every maximum matching of $G \setminus ij_0, \ldots, ij_k, ij$ is a maximum matching of G. Since j is essential in G, it is covered by every maximum matching of G and consequently also by every maximum matching of $G \setminus ij_0, \ldots, ij_k, ij$. Therefore, j is also essential in $G \setminus ij_0, \ldots, ij_k, ij$. Claim (J1) applied to network $G \setminus ij_0, \ldots, ij_k$ then proves that it is not optimal for j to provide the favor to i, again contradicting the original assumption.

The proof of the inductive step (J4) relies on the induction hypotheses (J2) for network $G \setminus ij$ and (J6) for network $G \setminus i$ in the same fashion (I3) follows from the induction hypotheses (I2) for $G \setminus ij$ and (I6) for $G \setminus i$.

To demonstrate claim (J5), we proceed by contradiction. Suppose that in a subgame perfect equilibrium, the essential player i is rejected by all neighbors with positive probability when he needs a favor in G and asks them in the order $j_0, j_1, \ldots, j_{\bar{k}}$. Let j_k be the last node in this sequence with the property that ij_k is an efficient link in G. Then, there exists a maximum matching of G that contains the link ij_k. This matching also constitutes a maximum matching for the network $G \setminus ij_0, \ldots, ij_{k-1}$. It follows that $\mu(G) = \mu(G \setminus ij_0, \ldots, ij_{k-1})$ and every maximum matching of $G \setminus ij_0, \ldots, ij_{k-1}$ is a maximum matching of G. In particular, i should be an essential node in $G \setminus ij_0, \ldots, ij_{k-1}$ as it is essential in G. Moreover, if i had an efficient partner in $G \setminus ij_0, \ldots, ij_{k-1}$ other than j_k, then that node would also be an efficient partner of i in G. Since j_k is the last node in the sequence $j_0, j_1, \ldots, j_{\bar{k}}$ that is an efficient partner of i in G, it must be that that j_k is the only efficient partner of i in $G \setminus ij_0, \ldots, ij_{k-1}$. As i is essential and j_k is the only efficient partner of i in $G \setminus ij_0, \ldots, ij_{k-1}$, the link ij_k must belong to all maximum matchings of $G \setminus ij_0, \ldots, ij_{k-1}$. It follows that node j_k is also essential in $G \setminus ij_0, \ldots, ij_{k-1}$. As both i and j_k are essential and the link ij_k is efficient in $G \setminus ij_0, \ldots, ij_{k-1}$, Theorem GE implies that i and j_k are perfectly matched in $G \setminus ij_0, \ldots, ij_{k-1}$. Therefore, i is perfectly matched and j_k is his only efficient partner in $G \setminus ij_0, \ldots, ij_{k-1}$. Claim (J4) applied to $G \setminus ij_0, \ldots, ij_{k-1}$ implies that if i requests the favor from j_k after being turned down by $j_1, \ldots, j_{k-1}$, then j_k should provide the favor to i with probability 1, a contradiction with our initial assumption.

The proofs of claims (J6) and (J7) use similar arguments to those establishing (I6) and (I7) in the proof of Theorem 1. Steps that rely on the detailed identification of who partners with over-demanded players in those arguments are replaced by claims (J2), (J3) and (J5), which show that no partnerships form when under-demanded players need favors and that efficient partnerships form whenever essential players request favors at the beginning of a period in network G.

The existence of a subgame perfect equilibrium can be established constructively based on the characterization of equilibrium behavior and outcomes revealed by claims (J1)-(J7). Equilibrium uniqueness for the game with exogenous orderings follows from a backward induction argument proving uniqueness of optimal responses in subgames in which over-demanded players require favors. $\square$

Proof for Section 6.4. We first check that under the constructed strategies, there are no profitable one-shot deviations for players 1, 3, and 4 when player 2 asks for a favor. After 2 has been rejected by two of his three neighbors, the last player $i \in \{1, 3, 4\}$ whom player 2 asks for the favor has an incentive to do it. Player i anticipates that refusing to do so leads with probability greater than 1/12 to the formation of each of the partnerships $(2, 3)$ and $(2, 4)$, at least one of which leaves i isolated. For high δ, player i is better off providing the requested favor and partnering with player 2. Similarly, the first player i approached by player 2 has an incentive to provide the favor because refusing to do so leads player 2 to request the favor with probability at least 1/2 from either player 3 or 4 next, which results in the formation of the partnership $(2, 3)$ or $(2, 4)$ and the isolation of player i.

The second player i whom player 2 asks for the favor has an incentive to provide it if the last player left to ask is either 3 or 4 since under the prescribed strategies the last player would agree to do the favor and inefficiently partner with player 2, leaving player i isolated. If player 1 is last in the order player 2 approaches neighbors for the favor, then player 2 must have asked neighbors in either order $(3, 4, 1)$ or $(4, 3, 1)$. In the former case, following player 3's refusal to do the favor to player 2, player 4 is asked next, and if he also refuses, then player 1 provides the favor to 2. Under the prescribed strategies, players 3 and 4 play the worst equilibrium for player 4 in the subgame ensuing after the formation of partnership $(1, 2)$. In this continuation game, player 3 never provides the first favor to player 4, and player 4 starts receiving favors only after entering a partnership with player 3 via providing a favor. Then, player 4 is better off agreeing to do the favor and forming a partnership with player 2 when asked. Indeed, from the perspective of player 4, players 2 and 3 are interchangeable partners, and joining the partnership with 2 earlier is preferred to waiting for 3 to require a favor in the subgame in which 1 partners with 2 (we have shown that early agreements are optimal for a player if the opponent never provides the first favor). An analogous argument applies for the ordering $(4, 3, 1)$.

Finally, we check incentives for subgames in which players other than 2 require a favor in the original network. If player 1 asks player 2 for a favor, it is optimal for player 2 to decline because he is guaranteed to find a partner whenever he needs a favor (even if the partnership $(3, 4)$ forms in the meanwhile). We next show that player 2 does not have an incentive to provide the first favor to either 3 or 4. If asked first in the order by either player 3 or 4, player 2 anticipates that the partnership $(3, 4)$ will form and then play between 1 and 2 proceeds according to the best equilibrium for 2, under which player 2 always receives favors when needed. If asked second in the order by either player 3 or 4, it is optimal for player 2 to refuse because under the prescribed strategies player 2 receives favors in any circumstance he needs them in the future. Player 3 has an incentive to provide the first favor to 4 and vice versa because refusing to do so leads to no one providing the favor (player 2 declines regardless of his position in the order), in which case the game proceeds to the next period and each of players 3 and 4 remains isolated with probability greater than 1/12. $\square$

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