Selecting a Winner with External Referees*

Francis Bloch†, Bhaskar Dutta‡ and Marcin Dziubiński§

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Abstract

We consider a problem of mechanism design without money, where a planner selects a winner among a set of agents with binary types and receives outside signals (like the report of external referees). We show that there is a gap between the optimal Dominant Strategy Incentive Compatible (DSIC) mechanism and the optimal Bayesian Incentive Compatible (BIC) mechanism. In the optimal BIC mechanism, the planner can leverage the outside signal to elicit information about agents' types. BIC mechanisms are lexicographic mechanisms, where the planner first shortlists agents who receive high reports from the referees and then uses agents' reports to break ties among agents in the shortlist. We compare the "self-evaluation" mechanism with a "peer evaluation" mechanism where agents evaluate other agents, and show that for the same signal precision, the self- evaluation mechanism outperforms the peer evaluation mechanism. We show that optimal Ex Post Incentive Compatible (EPIC) mechanisms give the planner an intermediate value between the optimal DSIC and BIC mechanisms

Keywords: Peer selection, Mechanism design without money, Bayesian incentive compatibility, Dominant strategy incentive compatibility, Ex post incentive compatibility

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†Université Paris 1 and Paris School of Economics, 48 Boulevard Jourdan, 75014 Paris, France. francis.bloch@univ-paris1.fr

‡University of Warwick and Ashoka University, CV4 7AL, Coventry, UK b.dutta@warwick.ac.uk

§Institute of Informatics, University of Warsaw, Banacha 2, 02-097, Warsaw, Poland m.dziubinski@mimuw.edu.pl

1 Introduction

We consider a selection problem without money, where a planner awards a single, indivisible, prize to one of n competing agents. The prize can be a coveted position in an organization, a project or research proposal to be funded, or the recognition of a particular achievement. Each individual can either be *suitable* or *unsuitable* for the award. The planner wants to select a suitable candidate but does not observe the suitability of the candidates.

In the absence of transfers, the planner is unable to elicit the suitability of candidates. Irrespective of their types, all agents have an incentive to report that they are suitable, in order to be considered for the award. In this paper, we show that *if the planner is able to observe an outside signal on the candidate's types, for example a report by an external referee, she can leverage this information to elicit agents' suitability for the award.*

More precisely, we suppose that both candidates and external referees receive a signal on agents' suitability, with precision levels denoted by p and r, where $1 > p \geq r > 1/2$, reflecting the fact that agents have a better (although imperfect) perspective on their suitability than an external referee. We analyze *direct selection mechanisms*, termed "self-evaluation" mechanisms, where the planner asks candidates and referees to report their signals, and chooses a probability of assigning the award to any candidate as a function of the $2n$-tuple of reports. A dominant strategy incentive compatible (DSIC) mechanism must ignore the reports of the agents. The optimal DSIC mechanism will utilize the reports of the external referees but at an efficiency loss, which will be particularly pronounced when the precision of the referee's signal, r, is low. We then weaken the incentive compatibility requirement and focus attention on Bayesian incentive compatible (BIC) mechanisms.

Our first result shows that there is a gap between optimal DSIC and BIC mechanisms and that *an optimal BIC mechanism gives a higher expected payoff to the planner than the optimal DSIC mechanisms.* Correlation between the signals of the agents and the external referees can be exploited by the planner to improve upon any mechanism which only relies on the referee reports. The intuition is best illustrated when there are only two contestants. In the optimal DSIC mechanism, when the two candidates receive the same report from the external referee, each is chosen with equal probability. The planner can do better when agents face BIC constraints. She can discriminate among the agents when they submit different reports, by increasing the probability of selecting an agent who reports high type when the two referee reports are high and decreasing that probability when the two referee reports are low. Given that the signals of the agents and referees are positively correlated, the referee is more likely to report that the agent is suitable when the agent receives a high signal, so that the planner can strictly increase the expected probability of selecting a suitable agent, while keeping the BIC constraint satisfied.

This intuition can be used to *characterize the optimal BIC mechanism*, both with two contestants and in the general model with an arbitrary number of candidates. The optimal mechanisms are lexicographic. The planner first follows the reports of the external referees, and shortlists agents who receive a high report from the referees. The planner then selects, with equal probability, one of the agents in the short list who reports a high signal. When all agents receive low reports from the referees, the planner selects with high probability an agent who reports a low signal.

We also consider a second class of mechanisms, the "peer evaluation" mechanisms, where candidates are asked to report not their own type, but the suitability of another candidate. There are several settings where close analogues of peer evaluation mechanisms are used. In computer science conferences, program committee members who review and rank papers also typically submit papers to the conference. The National Science Foundation has asked applicants to review and rank other applicants for the same funding program.[1] These practices have given rise to a recent literature on DSIC peer evaluation mechanisms. Formally, the self-evaluation and peer evaluation mechanisms may be interpreted as "duals" of each other, but give very different predictions. First, we show that, in

[1]See the article by Jeffrey Marvis in *Science* on July 17, 2014, "Want a grant? First review someone else's proposal!", https://www.science.org/content/article/want-grant-first-review-someone-elses-proposal.

contrast to the self-evaluation mechanism, the optimal BIC and DSIC peer evaluation mechanisms coincide when there are two contestants. When the number of individuals exceeds two, the optimal BIC peer evaluation mechanism performs better than the DSIC mechanism, but still yields a lower value than the optimal BIC self-evaluation mechanism for the same signal precision. The superiority of the self-evaluation mechanism stems from the fact that the planner can more easily construct compensating probabilities when the binding incentive constraint affects agents reporting that they have a low type than when it affects agents reporting that their competitors have a high type.

Finally, we consider Ex Post Incentive Compatibility (EPIC) mechanisms, where agents have an incentive to tell the truth, for any truthful announcement of the other agents, and taking expectations over the reports of the referees. When the number of agents is equal to two, the optimal EPIC mechanisms give the same value as the optimal DSIC mechanism. In general, EPIC mechanisms always generate a lower value than optimal BIC mechanisms, and when the number of agents is greater than two, the optimal EPIC mechanism gives a value which is intermediate between the optimal DSIC and BIC mechanisms.

The proofs of the main characterization theorem is given in the Appendix. In Online Appendix 1, we provide the proofs of the other results in the paper. In Online Appendix 2, we run several robustness checks, showing that the main results of the paper remain true if the underlying information structure is non-symmetric, if we consider wasteful and non-anonymous mechanisms, and if the number of signals is greater than two.

2 Related Literature

Our paper belongs to the growing literature on mechanism design without monetary transfers. To the best of our knowledge, the paper by Kattwinkel and Knoepfle [?] is the first to point out how (positive) correlation between an outside signal and the agent's report can be used to provide incentives in a setting without monetary transfers. Kattwinkel and Knoepfle [?] consider a principal-agent model, where the principal, for instance, has to allocate an indivisible project to the agent. The principal observes a signal on the cost of the project while the agent reports a value. Costs and values are positively correlated. Kattwinkel and Knoepfle [?] show that the principal can exploit the correlation in costs and values to design a direct mechanism that induces truth-telling and improves upon a constant mechanism.

A recent paper by Pereyra and Silva [?] exploits the same intuition to show that imperfect signals can improve the allocation mechanism in an assignment model with a continuum of agents. Inspired by the problem of assigning students to colleges, Pereyra and Silva [?] consider a setting where colleges receive an imperfect signal correlated with the students' abilities. They characterize the optimal allocation mechanism which involves students sorting by selecting among two tracks with different admission thresholds.

Our paper is complementary to these papers, as we focus on a problem of selection among *multiple but finite number of agents*. The structure of the optimal mechanisms is very different from the structure of the optimal mechanisms in these papers. We find that any optimal mechanism is "lexicographic": it first follows the recommendation of the referees and uses signals of the agents to break ties. We also compare self-evaluation and peer evaluation mechanisms and show that the former outperforms the latter. Such results cannot be obtained in the setups studied either in Kattwinkel and Knoepfle [?] or Pereyra and Silva [?].

Our paper is of course also related to the older literature on allocation without transfers but with costly verification. This literature was initiated by Ben Porath, Dekel and Lipman [?] who derive a very simple optimal mechanism, the "favored agent mechanism", where the good is allocated to one given agent, except if another agent reports a value above a threshold. In that case, the agent's report is checked for accuracy. Mylonanov and Zapechelnuyk [?] assume that reports can only be verified ex post, after the object has been allocated and punishments are limited. When the number of agents is large, the optimal mechanism is, like in our case, a *shortlisting method*. Each agent reports whether her type is above a threshold, and a winner is chosen randomly among the shortlisted candidates and checked. Finally, in a recent paper, Li [?] assumes ex ante costly verification and limited punishment and shows that the optimal mechanism is a version of a shortlisting method.

Other papers on mechanism design without transfers focus on different specific settings. For instance, Börger and Postl [?] study BIC mechanisms in a voting problem with three alternatives, while Gerkshov et al. [?] characterize optimal DSIC rules as sequential voting rules in a general model of voting over social states. Miralles [?] and Halafir and Miralles [?] focus on mechanism design issues in matching problems. Antic and Steverson [?] analyze a problem where a principal selects among a set of agents, and show how complementarities in preferences can be used to circumvent the absence of transfers.

Our paper also connects to the large literature in economics and computer science on incentive compatibility and peer selection. This literature has focused on dominant strategy mechanisms. Holzman and Moulin [?] provide an axiomatic analysis of "external" voting rules when individuals nominate a single individual for office. Tamura and Ohseto [?], Berga and Gjorjiev [?], Mackenzie [?] and [?], are other contributions that explore various facets of the axiomatic analysis of dominant strategy in peer selection problems. Alon et al. [?] consider the problem of designing a dominant strategy mechanism to select k individuals from a group of peers. They show that no deterministic efficient dominant mechanism exists, and then go on to construct approximately efficient, stochastic, dominant strategy mechanisms. Fischer and Klimm [?], Bousquet, Norin and Vetta [?] improve on the partition algorithm proposed in Alon et al. [?]. Kurokawa et al. [?] and Aziz et al. [?] and [?] propose other algorithms to improve on the partition algorithm. Bjelde, Fischer and Klimm [?] and Babichenko, Dean and Tenneholtz [?] consider other aspects of the peer selection problem.

3 The model

We consider a set of agents, $N = \{0, \dots, n-1\}$. Each agent $i \in N$ is characterized by a type $\Theta_i \in \{0, 1\}$, which can be either *high* ($\Theta_i = 1$) (a "suitable candidate") or *low* ($\Theta_i = 0$) (an "unsuitable candidate") with equal probability, $\mathbf{Pr}(\Theta_i = 0) = \mathbf{Pr}(\Theta_i = 1) = 1/2$. Types are distributed independently.

Each agent i receives a signal, $X_i \in \{0, 1\}$, about her type. For any agent

$i \in N$, let $p = \mathbf{Pr}(X_i = 1 \mid \Theta_i = 1) = \mathbf{Pr}(X_i = 0 \mid \Theta_i = 0)$ be the precision of the signal. We assume that $p \in (1/2, 1)$.

The planner also has access to other signals, called *referee reports*, $(Y_i)_{i \in N}$ where $Y_i \in \{0, 1\}$ is a signal about the type of agent i. Conditional on the type of agent i, the signal received by the agent, x_i, and the referee report y_i are independently distributed.

For any agent $i \in N$, let $r = \mathbf{Pr}(Y_i = 1 \mid \Theta_i = 1) = \mathbf{Pr}(Y_i = 0 \mid \Theta_i = 0)$ be the precision of the referee report. We assume that $r \in (1/2, p]$, so that the referee reports are less accurate than the signals that agents receive about their own type.[2] Notice that, in the benchmark model, the information structure we consider is completely symmetric. In Online Appendix 2, we show that our results extend to non-symmetric information structures.

Formally, a direct mechanism without transfers is a mapping, $\pi : \{0, 1\}^N \times \{0, 1\}^N \to \Delta(N)$,[3] that assigns a probability of selecting agent i, $\pi_i(\boldsymbol{x}, \boldsymbol{y})$, given the vectors of reports $\boldsymbol{x} = (x_i)_{i \in N}$ and $\boldsymbol{y} = (y_i)_{i \in N}$. [4] Throughout the paper we use $\mathcal{S} = \Delta(N)^{\{0,1\}^N \times \{0,1\}^N}$ to denote the set of all direct mechanisms without transfers.[5]

We assume that an agent receives a payoff of 1 when she is selected and a payoff of 0 otherwise. The planner receives a value of 1 if she selects a suitable candidate and a value of 0 if she selects an unsuitable candidate, so that the expected value of the planner is given by[6]

$$\Pi = \mathbf{E}\left(\sum_{i \in N} \pi_i(\boldsymbol{X}, \boldsymbol{Y}) [\Theta_i = 1] \right).$$

We are interested in mechanisms under which the agents have incentives to report their signals truthfully and, within such sets of mechanisms, in the mechanisms

[2]This Assumption is not needed to prove that the optimal BIC mechanism dominates the optimal DSIC mechanism, but will prove convenient to characterize the class of optimal BIC mechanisms.

[3]Given a nonempty set S we use $\Delta(S)$ to denote the set of probability distributions on S.

[4]Notwithstanding the absence of transfers, the mechanism design problem we consider is a classical problem, and the revelation principle applies. There is thus no loss of generality in focussing on direct mechanisms.

[5]We denote by the lower case x_i (respectively y_i, θ_i) a realization of the signal X_i (respectively Y_i, Θ_i).

[6]Given logical proposition φ, $[\varphi]$ is the Iverson bracket taking value 1 when φ is true and value 0 otherwise.

that maximize the planner's expected value. We consider two concepts of incentive compatibility, Dominant Strategy and Bayesian Incentive Compatibility.

Definition 1 (Dominant Strategy Incentive Compatibility)**.** *A mechanism* $\pi \in \mathcal{S}$ *is* dominant strategy incentive compatible *(DSIC) if and only if, for any agent* $i \in N$*, any tuples of signals* $\boldsymbol{x} \in \{0,1\}^N$ *and* $\boldsymbol{y} \in \{0,1\}^N$*, and any report* $x_i' \in \{0,1\}$ *of agent* i*,*[7]

$$\pi_i(\boldsymbol{x}, \boldsymbol{y}) \geq \pi_i(x_i', \boldsymbol{x}_{-i}, \boldsymbol{y}). \tag{1}$$

Dominant Strategy Incentive Compatibility is a strong notion of incentive compatibility. Under mild conditions (that are automatically satisfied in the two-person case), a DSIC mechanism must ignore the reports of the agents and depend only upon the referee reports.

We next define the weaker Bayesian Incentive Compatibility property.

Definition 2 (Bayesian Incentive Compatibility)**.** *A mechanism* $\pi \in \mathcal{S}$ *is* Bayesian incentive compatible *(BIC) if and only if, for any agent* $i \in N$*, any signal* $x_i \in \{0,1\}$ *and any report* $x_i' \in \{0,1\}$ *of agent* i*,*

$$\mathbf{E}(\pi_i(x_i, \boldsymbol{X}_{-i}, \boldsymbol{Y}) | X_i = x_i) \geq \mathbf{E}(\pi_i(x_i', \boldsymbol{X}_{-i}, \boldsymbol{Y}) | X_i = x_i) \,. \tag{2}$$

A mechanism is Bayesian Incentive Compatible if agents have no incentive to misreport their signals when the other agents report truthfully. Clearly, any DSIC mechanism also satisfies BIC, but the converse need not be true.

We restrict attention to anonymous mechanisms under which the probability of selecting an agent only depends on the reports received on the agents and not on their identities.

Definition 3 (Anonymity)**.** *A mechanism* $\pi : \{0,1\}^N \times \{0,1\}^N \to \Delta(N)$ *is* anonymous *if and only if, for any agent* $i \in N$*, any vector of reports* $\boldsymbol{x} \in \{0,1\}^N$ *of the agents, any vector of public signals* $\boldsymbol{y} \in \{0,1\}^N$*, and any permutation* $\sigma : N \to N$*,*

$$\pi_i(\boldsymbol{x}, \boldsymbol{y}) = \pi_{\sigma(i)}\big((x_{\sigma(i)})_{i \in N}, (y_{\sigma(i)})_{i \in N}\big) \,. \tag{3}$$

[7]Throughout the paper, given a vector $\boldsymbol{x}$ and indices i and j, we denote by $\boldsymbol{x}_{-i}$ the vector obtained from $\boldsymbol{x}$ by omitting the value at index i and $\boldsymbol{x}_{-i,j}$ the vector obtained from $\boldsymbol{x}$ by omitting the values at indices i and j.

Our main goal in this paper is to characterize mechanisms under the anonymity and incentive compatibility constraints that maximize the expected value of the social planner.

Definition 4 (Optimality)**.** *A anonymous DSIC (BIC) mechanism* $\pi \in \mathcal{S}$ *is* optimal *if there is no other anonymous DSIC (resp BIC) mechanism* $\pi' \in \mathcal{S}$ *such that*

$$\mathbf{E}\left(\sum_{i \in N} \pi'_i(\boldsymbol{X}, \boldsymbol{Y})[\Theta_i = 1]\right) > \mathbf{E}\left(\sum_{i \in N} \pi_i(\boldsymbol{X}, \boldsymbol{Y})[\Theta_i = 1]\right). \tag{4}$$

4 Selecting a winner among two agents

We first analyze the planner's problem when there are only two candidates. The social planner has access to four reports: (x_0x_1, y_0y_1) where x_0 and x_1 denote the reports of the two agents and y_0 and y_1 the reports of the external referees. A direct mechanism is thus determined by the probabilities $\pi_0(x_0x_1, y_0y_1)$ and $\pi_1(x_0x_1, y_0y_1)$, for each of the 16 possible reports $(x_0x_1, y_0y_1) \in \{0, 1\}^4$. By anonymity, $\pi_0(x_0x_1, y_0y_1) = \pi_1(x_1x_0, y_1y_0)$ so we can focus attention on the 16 probabilities of selecting agent 0, and drop the subscript, letting $\pi(x_0x_1, y_0y_1) = \pi_0(x_0x_1, y_0y_1)$.

With these notations in hand, we write the expected value of the planner as

$$\begin{aligned}\Pi = \frac{1}{2}&((1-p)(1-r)Q(\pi(00,00) + \pi(01,01)) + (1-p)(1-r)(1-Q)(\pi(00,01) + \pi(01,00)) \\ &+ p(1-r)Q(\pi(10,00) + \pi(11,01)) + p(1-r)(1-Q)(\pi(10,01) + \pi(11,00)) \\ &+ (1-p)rQ(\pi(00,10) + \pi(01,11)) + (1-p)r(1-Q)(\pi(00,11) + \pi(01,10)) \\ &+ prQ(\pi(10,10) + \pi(11,11)) + pr(1-Q)(\pi(10,11) + \pi(11,10)))\end{aligned}$$

where $Q = pr + (1-r)(1-p)$ denotes the expected probability that the two reports (of the agent and the referee) coincide.

It is straightforward to observe that, in a DSIC mechanism, the planner can only condition the probability on the report of the referees. She selects with equal probability the two projects when they receive the same report, and always selects the project with the high report when the projects receive different referee reports.

The BIC constraint for the low-signal agent is given by

$$\begin{aligned}
&Q^2(\pi(00,00) - \pi(10,00) + \pi(01,01) - \pi(11,01)) \\
&+Q(1-Q)(\pi(00,01) - \pi(10,01) + \pi(01,00) - \pi(11,00)) \\
&+(1-Q)Q(\pi(00,10) - \pi(10,10) + \pi(01,11) - \pi(11,11)) \\
&+(1-Q)^2(\pi(00,11) - \pi(10,11) + \pi(01,10) - \pi(11,10)) \geq 0.
\end{aligned}$$

We first show that the planner can exploit the correlation between the agent's signals and the referee reports to improve upon the optimal DSIC mechanism. The planner's value would increase if she could use the agents' reports, and select, with higher probability, an agent who reports a high signal. However, this runs against the incentives of the low-signal agent, who has an incentive to report a high signal. In order to mitigate this effect, the planner can increase the probability of selecting an agent announcing high signal for some reports of the referees, and decrease the probability for other reports. Because the reports of the referees are correlated with the agents' signals, a low-signal agent is less likely to receive a high report from the referee. Hence, by increasing the probability of selecting an agent reporting a high signal when both referee reports are high, and decreasing this probability when both referee reports are low, the planner can increase her value while keeping the incentive constraint of the two types of agents satisfied.

We now proceed to characterize the optimal BIC mechanism. Given that the planner does not suffer any utility loss from selecting an unsuitable candidate, there is no waste in the optimal mechanism: for any vector of reports, one of the two agents is always selected. We then compute

$$\pi(x_1x_0, y_1y_0) = 1 - \pi(x_0x_1, y_0y_1)$$

and the mechanism is fully determined by the following six probabilities:

$$\pi_1 = \pi(10,10), \pi_2 = \pi(10,00), \pi_3 = \pi(00,10), \pi_4 = \pi(10,01); \pi_5 = \pi(10,11), \pi_6 = \pi(11,10).$$

We consider the *relaxed* problem where the social planner only faces the BIC constraint for the low-signal agent. The social planner solves a linear programming problem given by

$$\begin{aligned} \max_{\pi_i} \quad & \tfrac{1}{2} + (1-p)(1-p(2r-1)) + \sum a_i \pi_i \\ s.to \quad & \tfrac{3}{2} - Q - \sum b_i \pi_i \geq 0 \\ & 0 \leq \pi_i \leq 1 \end{aligned}$$

where

$$a_1 = \left(p^2r^2 - (1-p)^2(1-r)^2\right), a_2 = a_5 = \left(p^2r(1-r) - (1-p)^2r(1-r)\right),$$
$$a_3 = a_6 = \left(r^2p(1-p) - (1-r)^2p(1-p)\right), a_4 = \left(p^2(1-r)^2 - (1-p)^2r^2\right)$$
$$b_1 = b_2 = Q, b_3 = 0, b_4 = b_5 = 1-Q, b_6 = 1-2Q.$$

We consider the *relaxed* problem where the planner only faces the BIC constraint of the low type but subsequently check that the solution satisfies the BIC constraint for the high type) which is given by

$$\frac{3}{2} - Q - b_1\pi_1 - b_2\pi_2 - b_3\pi_3 - b_4\pi_4 - b_5\pi_5 - b_6\pi_6 \geq 0. \tag{5}$$

where

$$b_1 = b_2 = Q, b_3 = 0, b_4 = b_5 = 1-Q, b_6 = 1-2Q.$$

This is a linear relaxation of the knapsack problem and the solution to this problem is well known. (See, for example Theorem 2.2.1 in [**?**].) As $b_3 = 0$ and $b_6 < 0$, the items 3 and 6 are always included. For the other items, we compute the efficiency indices $\gamma_i = -a_i/b_i$ as

$$\gamma_1 = p + r - 1 > \gamma_5 = \frac{r(1-r)(2p-1)}{1-Q} > \gamma_2 = \frac{r(1-r)(2p-1)}{Q} > \gamma_4 = p - r.$$

It is easy to check that items 1 and 5 are always included, item 4 never included, and item 2, the split item, is included with probability $\pi_2 = 1 - 1/(2Q)$.

The optimal BIC mechanism is thus a *lexicographic mechanism* where the planner first considers the reports of the referees. If the reports of the referees differ, the planner always selects the agent who receives the high report, $\pi(00,10) = \pi(10,10) = \pi(01,10) = \pi(11,10) = 1$. If the two agents receive the same report from the referees and report the same signal, they are selected with equal probability, $\pi(00,00) = \pi(00,11) = \pi(11,00) = \pi(11,11) = 1/2$. It is when the two agents receive the same referee report but report different signals that the optimal BIC mechanism differs from the optimal DSIC mechanism. Instead of selecting each agent with equal probability, the social planner selects the agent reporting the high signal with probability one when the referees send high reports, $\pi(10,11) = 1$, but with a probability smaller than 1/2 when the referees send low reports, $\pi(10,00) = 1 - 1/(2Q)$.[8]

5 Selecting a winner among n agents

We now extend the characterization of the optimal mechanism to an arbitrary number of agents and show that the optimal BIC mechanism again has a lexicographic structure, where the planner only uses the agents' reports to break ties among candidates who receive the same reports from external referees. In this extension, different régimes can arise, and new notations are needed. We will distinguish between situations where the precision of the signal of the referee is high, $r \geq r^*$, or low, $r < r^*$, where the threshold value r^* is defined as follows. Define $\varphi_n : [0,1) \to \mathbb{R}$ to be the function

$$\varphi_n(x) = \frac{(2-x)^n}{1-x} - x(1-x)^{n-2}.$$

We show in the Appendix that, for any $n \geq 2$, there exists a unique $x \in [1/2, 1)$ such that $\varphi_n(x) = 2^n$. Then[9]

$$r^* = \frac{\varphi_n^{(-1)}(2^n) + p - 1}{2p - 1}.$$

Since we consider anonymous mechanisms, the probability of selecting candidate $i \in N$ only depends on the scores received by i, (x_i, y_i) and the number of other agents receiving different combinations of scores: $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Given any natural number m let

$$S_4(m) = \left\{(z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}) \in \mathbb{N}^4 : z_{0,0} + z_{0,1} + z_{1,0} + z_{1,1} = m\right\}$$

[8]When $r = 1/2$, $Q = 1/2$. In this case, the mechanism reduces to the optimal DSIC mechanism.

[9]We observe that the threshold value $r^*(p,n)$ is strictly decreasing in p and it is equal to $\varphi_2^{(-1)}(2^n)$ when $p = 1$. It is also equal to 1/2, for all $p \in [1/2, 1]$, when $n = 2$. Hence in the case of $n = 2$ it is always "sufficiently high".

denote the set of all four-partitions of m into the sum of four natural numbers. Any $\boldsymbol{z} = (z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}) \in S_4(m)$ contains a record of numbers of different scores received among m agents. In some situations we will be interested only in the numbers of agents receiving different self-scores among the agents receiving low scores from the referee. Given any natural number m let

$$S_2(m) = \left\{(z_{0,0}, z_{1,0}) \in \mathbb{N}^2 : z_{0,0} + z_{1,0} = m\right\}$$

denote the set of all bipartitions of m.

We are now ready to state the theorem providing a full characterization of optimal anonymous BIC mechanisms.

Theorem 1. *A mechanism $\pi \in \mathcal{S}$ is an optimal anonymous BIC mechanism if and only if*

- *It selects one of the agents from the set M of agents receiving maximal reports from the referee.*
- *If all agents in M receive the same self-reports then one of them is selected with probability $1/|M|$.*
- *If some agents in M receive low self-reports (set M_0) and some agents in M receive high self-reports (set M_1) then*
  - *(1) if $r < r^*$ then*
    - *(a) if agents in M received high report from the referee then an agent in M_0 is selected with probability $\alpha_{\boldsymbol{z}}/|M_0|$ or an agent in M_1 is selected with probability $(1 - \alpha_{\boldsymbol{z}})/|M_1|$,*
    - *(b) if agents in M received low report from the referee then one of the agents in M_0 is selected with probability $1/|M_0|$,*
  - *(2) if $r \geq r^*$ then*
    - *(a) if agents in M received high report from the referee then one of the agents in M_1 is selected with probability $1/|M_1|$,*
    - *(b) if agents in M received low report from the referee then an agent in M_0 is selected with probability $\beta_{\boldsymbol{z}}/|M_0|$ or an agent in M_1 is selected with probability $(1 - \beta_{\boldsymbol{z}})/|M_1|$,*

where $(\alpha_{\boldsymbol{z}})_{\boldsymbol{z}\in S_4(n)} \in [0,1]^{S_4(n)}$ *satisfies*

$$\sum_{\substack{\boldsymbol{z}\in S_4(n)\\ z_{0,1}>0\\ z_{1,1}>0}} \alpha_{\boldsymbol{z}} \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} \frac{Q^{z_{0,0}+z_{1,1}}(1-Q)^{z_{0,1}+z_{1,0}}}{2^n} = \frac{(1-Q)2^n + Q(1-Q)^{n-1} - (2-Q)^n}{2^n}. \quad (6)$$

and $(\beta_{\boldsymbol{z}})_{\boldsymbol{z}\in S_2(n)} \in [0,1]^{S_2(n)}$ *satisfies*

$$\sum_{\substack{\boldsymbol{z}\in S_2(n)\\ z_{0,0}>0\\ z_{1,0}>0}} \beta_{\boldsymbol{z}} \binom{n}{z_{0,0}, z_{1,0}} \frac{Q^{z_{0,0}}(1-Q)^{z_{1,0}}}{2^n} = \frac{(1-Q)^2 2^n + Q\,(1-Q^n) - (1-Q)(2-Q)^n}{Q2^n}, \quad (7)$$

As in the case of two agents, optimal BIC mechanisms are *lexicographic*. They first follow the reports of the referees to establish a shortlist of agents with maximal reports and only uses the reports of the agents to break ties in the shortlist. If all agents in the shortlist report the same signal, they are selected with equal probability. Otherwise, the planner adjusts the probability of selecting an agent in the shortlist according to agents' reports and the numbers of agents receiving different combinations of reports, $\boldsymbol{z} \in S_4(n)$. This is done differently depending on whether precision of referee's signal is high or low and depending on whether the score of the shortlisted agents received from the referee is high or low.

If the precision of the referee's signal is low and the score of the shortlisted agents is low, the ties are broken in favor of the shortlisted agents receiving low self-reports. In that case, the mechanism goes "against" self-reports and chooses to inefficiently select an agent with low self-report. If the precision of the signal is low and the score of the shortlisted agents is high , then the mechanism selects one of the shortlisted agents with high self-report with a probability greater than $\frac{1}{2}$, thereby improving upon the optimal DSIC mechanism. The probability of going against self-reports of the agents, denoted $\alpha_{\boldsymbol{z}}$ is not uniquely determined but the *sum of probabilities* $\sum_{\boldsymbol{z}\in S_4(n)} \in [0,1]^{S_4(n)}(\alpha_{\boldsymbol{z}})$ must satisfy Equation (6), which is determined by the BIC constraint for the low type agent.

If the precision of the referee's signal is high and the score of the shortlisted agents is high, the ties are broken in favor of the shortlisted agents receiving high self-reports, improving upon the optimal DSIC mechanism. If the precision of the signal is high and the score of the shortlisted agents is low, then the mechanism selects one of the shortlisted agents with low self-report with a probability greater than $\frac{1}{2}$, introducing an inefficiency by going against self-reports. The probability of going against self-reports of the agents, $\beta_{\boldsymbol{z}}$, is again not uniquely determined and only the sum $\sum_{\boldsymbol{z}\in S_2(n)} \in [0,1]^{S_2(n)}(\beta_{\boldsymbol{z}})$ is pinned down by Equation (7) which is determined by the BIC constraint for the low type.

The complete proof of Theorem 1 is given in the Appendix. It amounts to computing the solution of a linear relaxation of a knapsack problem with a large number of variables and multiple constraints. We first show that, for any set of reports $\boldsymbol{z}$ such that there exist agents submitting high reports who receive different scores from the referees (the case $(11, 10)$ when $n = 2$), the agents receiving low referee reports are never selected. We then prove that a transfer of probability from the agent receiving a low report to the agent receiving a high report results both in an increase in value (because high-type agents are more likely to generate high reports from referees) and a relaxation of the BIC constraints. This result is contained in Lemma 1 in the Appendix.

This first observation, together with the probability constraints, allows us to rewrite the objective function of the social planner and the BIC constraints only in terms of the probabilities of selecting agents submitting low reports. When at least one other agent submits a high report, the coefficient of these probabilities in the objective function of the social planner are negative. Hence, if the low-type BIC constraint were slack, the planner could reduce these probabilities and increase value. This argument shows that the BIC constraint for the low-type agent must be satisfied with equality (Lemma 2).

Next, we consider a relaxed optimization problem, where the only binding constraint is the BIC constraint of the low type. In the relaxed problem, we first show that, when there exist agents submitting low reports who receive different scores from the referees, (the case $(00, 10)$ when $n = 2$) the agents receiving low referee reports are never selected. This result appears as Lemma 3.

The next step in the argument considers the effect of probability transfers and provides an implicit ranking of the probabilities, showing which ones are selected or are critical at the optimum. Lemma 4 summarizes this discussion.

The logic behind the computation of the optimal probabilities $\alpha_{\boldsymbol{z}}$ and $\beta_{\boldsymbol{z}}$ is similar to the logic in the two agent case. As a low-type agent is more likely to generate a low report from the referee, the social planner can increase the value of a BIC mechanism by increasing the probability of selecting an agent reporting a high type when referee reports are high while simultaneously reducing the probability of selecting an agent reporting a high type when referee reports are low. With more than two agents, the event that all referee reports are low becomes less likely than the event that some referee reports are high, creating an imbalance which may prevent the social planner from compensating probabilities.

As in the two-agent problem, we find that the social planner first follows the reports of the referees and uses agents' reports to break ties when they receive the same reports from the referees. As opposed to the two-agent case, an optimal mechanism is not unique. Through Equations (6) and (7), an optimal mechanism pins down the weighted sums of probabilities $(\alpha_{\boldsymbol{z}})_{\boldsymbol{z}\in S_4(n)} \in [0,1]^{S_4(n)}$ and $(\beta_{\boldsymbol{z}})_{\boldsymbol{z}\in S_2(n)} \in [0,1]^{S_2(n)}$ of selecting agents with low reports when some agents receive high referee reports. In particular, for all $\boldsymbol{z} \in S_4(n)$, $\alpha_{\boldsymbol{z}}$ could be set to the same value, $\alpha^{\mathrm{c}}(Q)$, where

$$\begin{aligned}\alpha^{\mathrm{c}}(Q) &= \frac{(1-Q)2^n + Q(1-Q)^{n-1} - (2-Q)^n}{\sum_{\substack{\boldsymbol{z}\in S_4(n)\\ z_{0,1}>0\\ z_{1,1}>0}} \binom{n}{z_{0,0},z_{0,1},z_{1,0},z_{1,1}} Q^{z_{0,0}+z_{1,1}}(1-Q)^{z_{0,1}+z_{1,0}}}\\ &= \frac{(1-Q)2^n + Q(1-Q)^{n-1} - (2-Q)^n}{2^n - (1+Q)^n - (2-Q)^n + 1}\end{aligned}$$

and, for all $\boldsymbol{z} \in S_2(n)$, $\beta_{\boldsymbol{z}}$ could be set to $\beta^{\mathrm{c}}(Q)$, where

$$\begin{aligned}\beta^{\mathrm{c}}(Q) &= \frac{(1-Q)^2 2^n + Q\left(1-Q^n\right) - (1-Q)(2-Q)^n}{Q\sum_{\substack{\boldsymbol{z}\in S_2(n)\\ z_{0,0}>0\\ z_{1,0}>0}} \binom{n}{z_{0,0},z_{1,0}} Q^{z_{0,0}}(1-Q)^{z_{1,0}}}\\ &= \frac{(1-Q)^2 2^n + Q\left(1-Q^n\right) - (1-Q)(2-Q)^n}{Q(1-Q^n-(1-Q)^n)}\end{aligned}$$

Figure 1 shows, for $n=3$, the probabilities of selecting an agent with low report, $\alpha^{\mathrm{c}}(Q)$ and $\beta^{\mathrm{c}}(Q)$, as a function of Q. This figure displays the two possible regimes, as a function of the precision Q. If Q is low, even when the planner always selects a low report agent when all agents receive low report from the referee, he cannot select a high report agent with probability 1 when the shortlisted agents receive high report from the referee and different self-reports. As Q increases, the probability of selecting a low self-report agent from among shortlisted agents with high report from the referee, $\alpha^c(Q)$, decreases until a point where a low self-report agent is chosen with probability 0. From that point on, as Q increases, the planner is able to decrease the probability of selecting a low self-report agent when all shortlisted agents receive low report from the referee, while keeping the probability of selecting a low self-report agent from among the shorlisted agents with high self-reports equal to 0.

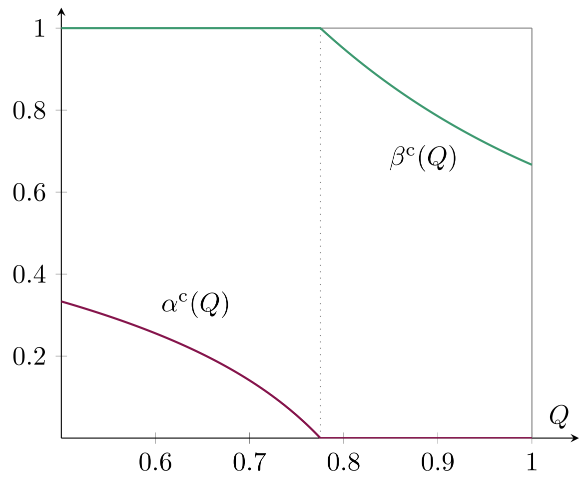

Figure 1: $\alpha^c(Q)$, and $\beta^c(Q)$ for $n = 3$.

We conclude the analysis of the general model by comparing the value of the social planner's objective function at the optimal BIC and DSIC mechanisms. Under mild and well-known conditions, a DSIC mechanism must ignore the reports of the agents.[10] The optimal DSIC mechanism thus only uses the reports of the referees and selects one of the agents with maximal report uniformly. The proposition below provides the values of the social planner's objective function at the optimal BIC and DSIC mechanisms.

Proposition 1. *The probability of selecting a high quality project under an optimal*

[10] If we exclude "bossy" mechanisms (Satterthwaite and Sonnenschein [**?**]) where agents can by their reports change the assignment probability of other agents without affecting their own assignment probability, the DSIC mechanisms must be independent of the report of the agents.

anonymous BIC mechanism is given by

$$\Pi^{\mathsf{B}}(p,r) = r - \frac{2r-1}{2^n}\left(1 - \left(\frac{r(1-r)(2p-1)^2}{Q(1-Q)}\right)\right.$$
$$\left.\min\left(1-(1-Q)^{n-1}, \frac{2^n(1-Q)-(2-Q)^n+Q}{Q}\right)\right)$$

and probability of selecting a high quality project under an optimal anonymous DSIC mechanism is given by

$$\Pi^{\mathsf{D}}(p,r) = r - \frac{2r-1}{2^n}.$$

Except for the two limit cases where the reports of the referees are uninformative or perfectly informative ($r = \frac{1}{2}$ and $r = 1$), the value of an optimal BIC mechanism is strictly larger than the value of the optimal DSIC mechanism. Figure 2 illustrates these values when $n = 3$ and agents have nearly perfect signals about their types, close to $p = 1$, as a function of r. The kink in the value of the BIC mechanism corresponds to the change in regime at $r = r^*$, where the sum of probabilities of selecting a low-type agent when all agents receive low reports from the referees reaches one.

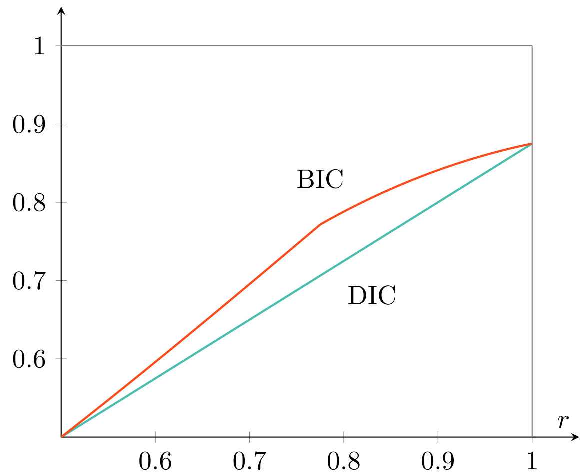

Figure 2: Payoffs $\Pi^{\mathsf{B}}(p,r)$ and $\Pi^{\mathsf{D}}(p,r)$ for $n = 3$ and and the limit case of $p = 1$.

We close this section with a figure which shows how the difference in optimal values for the BIC and DSIC mechanisms changes when n varies.

Figure 3 shows that the maximal difference between the value of an optimal BIC and DSIC mechanism initially increases from $n = 2$ to $n = 3$, and then

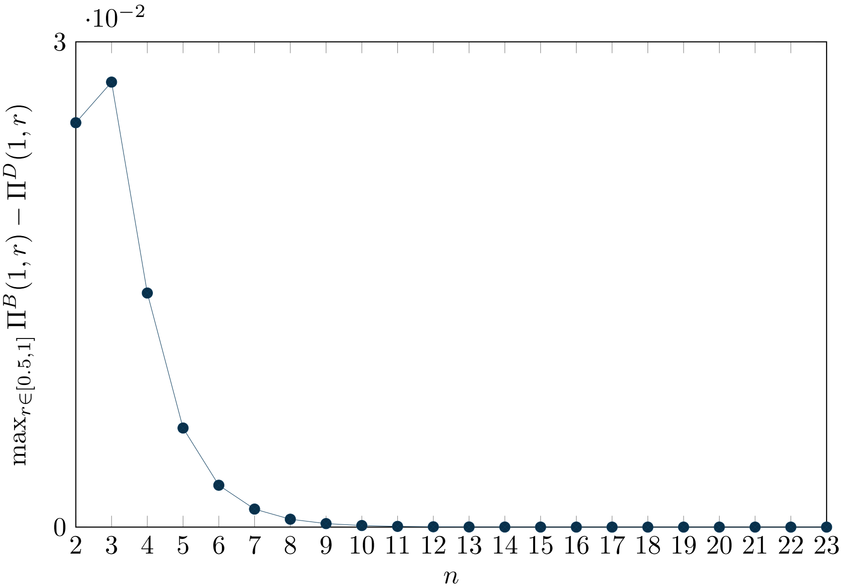

Figure 3: Difference in payoffs $\Pi^{\mathsf{B}}(p,r) - \Pi^{\mathsf{D}}(p,r)$ as a function of n.

quickly decreases, to become negligible for $n \geq 10$. The convergence between the value of an optimal BIC and the optimal DSIC mechanism when n becomes large is easily interpreted. An optimal BIC mechanism differs from the optimal DSIC mechanism by setting low probabilities to agents with high reports when all the agents receive low reports from the referee and high reports when the shortlisted agents receive high reports from the referee and different self-reports. But as n increases, the probability that all agents receive low reports from the referees becomes smaller and smaller. Therefore, as n grows large, the difference in value between the optimal BIC and DIC mechanisms vanishes.

6 Peer evaluation

In this section, we analyze a different class of mechanisms, which are often used in practice to select papers in conferences or applicants to research funds. Agents are asked to evaluate the agents with whom they compete for the allocation of a prize. In the model we assume that agents, who are not external, serve as referees and evaluate other agents, whom they compete with. In addition, a social planner has access to reports of external, less informed, referees.

We suppose that agents are ordered arbitrarily and each agent i is asked to evaluate his immediate successor in the order, agent $i \oplus 1 = (i+1) \mod n$.[11] In these *peer evaluation mechanisms*, each agent $i \in N$ receives two signals: a signal on her own quality, $V_i \in \{0,1\}$, and a signal on the quality of agent $i \oplus 1$, $X_{i\oplus 1} \in \{0,1\}$. We denote by $p \in (\frac{1}{2}, 1)$ the precision of the signal on own quality and by $q \in (\frac{1}{2}, 1)$ the precision of the signal on the quality of the other agent, $\mathbf{Pr}(V_i = 1 \mid \Theta_i = 1) = \mathbf{Pr}(V_i = 0 \mid \Theta_i = 0) = p$ and $\mathbf{Pr}(X_{i\oplus 1} = 1 \mid \Theta_{i\oplus 1} = 1) = \mathbf{Pr}(X_{i\oplus 1} = 0 \mid \Theta_{i\oplus 1} = 0) = q$. We suppose that each agent receives a more precise signal on her quality than on the quality of her competitor so that $q \leq p$.

While, in principle, the social planner could ask each agent to report her two signals, we suppose, as is customary in peer selection mechanisms, that agents only report their evaluation of the quality of their competitor. In addition, as in the benchmark, self-evaluation model studied in previous sections, the planner has access to a report by an referee, $Y_i \in \{0,1\}$, of precision $r = \mathbf{Pr}(Y_i = 1 \mid \Theta_i = 1) = \mathbf{Pr}(Y_i = 0 \mid \Theta_i = 0)$ with $r \in (1/2, q)$. That is, we suppose that the signal of the referee is less informative than the signal of the peer evaluator.

As in the benchmark model, a mechanism associates to every vector of reports $(\boldsymbol{x}, \boldsymbol{y})$ a probability distribution $\pi(\boldsymbol{x}, \boldsymbol{y})$ in the simplex $\Delta(N)$. The objective function of the social planner is given by

$$\Pi = \mathbf{E}\left(\sum_{i \in N} \pi_i(\boldsymbol{X}, \boldsymbol{Y}) | \theta_i = 1\right),$$

and is hence identical to the objective function in the self-evaluation model, once we replace the precision of the signal on own quality, p, with the precision on the signal on the quality of the competitor, q. The DSIC constraints are given by

$$\pi_i(\boldsymbol{x}, \boldsymbol{y}) \geq \pi_i(x'_{i\oplus 1}, \boldsymbol{x}_{-i\oplus 1}, \boldsymbol{y}),$$

for any $(2n-1)$-tuple of signals $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\{0,1\}^N$, any $i \in N$, and any report $x'_{i\oplus 1}$. The BIC constraints are different in the peer evaluation mechanism and in the benchmark mechanism. Because agents receive two signals $(V_i, X_{i\oplus 1})$, even

[11] As long as the mechanism satisfies anonymity and does not distinguish agents according to their position in the directed evaluation graph, any evaluation graph would give rise to the same optimal anonymous mechanism. We can therefore consider as well a simple evaluation graph where agents are organized in a directed cycle of order n.

though the mechanism only uses signal $X_{i\oplus 1}$, each agent faces *four* BIC constraints, one for each possible realization of the signals $(V_i, X_{i\oplus 1})$. The BIC constraints are thus given by

$$\mathbf{E}\left(\pi_i(x_{i\oplus 1}, \boldsymbol{X}_{-i\oplus 1}, \boldsymbol{Y}) \mid (V_i, X_{i\oplus 1}) = (v_i, x_{i\oplus 1})\right) \geq \mathbf{E}\left(\pi_i(x'_{i\oplus 1}, \boldsymbol{X}_{-i\oplus 1}, \boldsymbol{Y}) \mid (V_i, X_{i\oplus 1}) = (v_i, x_{i\oplus 1})\right),$$

for all signals $x_i \in \{0,1\}$ and reports $x'_i \in \{0,1\}$. Our first result shows that, when there are only two agents, an optimal BIC peer evaluation mechanism does not perform better than the optimal DSIC mechanism (the proof is provided in the on-line appendix).

Proposition 2. *Suppose that $n = 2$. Any optimal BIC peer evaluation mechanism results in the same value for the social planner as the optimal DSIC mechanism.*

Proposition 2 stands in sharp contrast to the analysis of the benchmark model. In the peer evaluation mechanism, the planner cannot exploit correlations between the agent's types and the report of the referee to improve on the optimal DSIC mechanism. To understand this result, recall that, in the benchmark model, the planner is able to improve upon the optimal DSIC mechanism by increasing the probability of selecting an agent who reports a high type, when both reports of the referee are high and lower the probability of selecting an agent who reports a high type when both reports of the referee are low. This result was obtained because the only binding constraint in the baseline model is the BIC constraint for the low type agent. In the peer-evaluation mechanism, there are four BIC constraints (every type of agent (high or low) must prefer to report the true type of the other agent) and one cannot assume that a single BIC constraint is binding as in the self-evaluation case. We claim that the adjustment of probabilities becomes impossible. To see this, consider a relaxed problem where the only binding BIC constraints are the two constraints faced by the low-type agent. An agent who observes that his competitor's type is low will assign a higher probability to the fact that the referee receives a low signal. Similarly, an agent who observes that his competitor's type is high will assign a higher probability to the fact that the referee receives a high signal. But this implies that any adjustment which increases the probability of selecting an agent who reports that her competitor is low (or high) will result in one of the two BIC constraints to be violated. In order to satisfy simultaneously both BIC constraints, the planner must assign the same probability of selecting the agent who reports a low (or high) type of his competitor when the referee reports that both types are high and both types are low. The optimal BIC mechanism is now identical to the optimal DSIC mechanism.

Proposition 2 does not extend to more than two players. When $n \geq 3$, numerical examples show that the optimal BIC mechanism gives a strictly higher value to the planner than the optimal DSIC mechanism. However, as we argue below, the optimal BIC mechanism under peer evaluation always results in a lower value than the optimal BIC mechanism of the benchmark model, where agents are asked to report on their own types.

As a full characterization of the optimal anonymous mechanism under peer evaluation for $n \geq 3$ is difficult, we consider instead a simpler mechanism where agents do not receive any signal about their own type, V_i, but only a signal on the quality of their competitor, X_i. In this artificial scenario, the planner only faces two BIC nstraints, which correspond to the average of the BIC constraints of the peer-evaluation mechanism, where the average is taken over the two possible types V_i of the agents. Clearly, this new problem is a relaxation of the original problem: any mechanism satisfying the four BIC constraints in the original problem also satisfies the two BIC constraints in the new problem, but the converse may not be true.

The characterization of the optimal anonymous BIC mechanism in the relaxed problem follows exactly the same steps as the characterization of the optimal anonymous BIC mechanism in the benchmark model, and is given in the Online Appendix. The optimal value of the mechanism is computed as a function of q and $Q = qr + (1-q)(1-r)$, giving an upper bound on the value of the peer evaluation mechanism.

Proposition 3. *An upper bound on the value of the optimal BIC peer evaluation mechanism is given by:*

$$\overline{\Pi}^{PB}(q,r) = r - \frac{2r-1}{2^n}\left(1 - \frac{r(1-r)(2q-1)^2(1-Q^{n-1})}{Q(1-Q)}\right).$$

If $r \in (1/2, 1)$ then $\Pi^{\mathsf{B}}(p,r) \geq \overline{\Pi}^{PB}(p,r)$ with equality only in the case $n = 2$.

Proposition 3 has an important implication. It shows that even when the two signals have the same precision, $q = p$, the value to the planner is higher in the scenario where agents are asked to self-evaluate, strictly higher when $n \geq 3$. The intuition underlying this comparison can be grasped by noting the differences in the incentives to misreport in the two scenarios. The binding constraints are different in the optimal self-evaluation and peer-evaluation mechanisms. In the self-evaluation mechanism, the low-type agent must be given incentives to report her true type. In the peer-evaluation mechanism, an agent observing a high type for her competitor must be given an incentive to report truthfully. Because low types are more likely to generate low referee reports, and high types are more likely to generate high referee reports, this implies that the compensation probabilities chosen by the planner will be diametrically opposed in the two mechanisms. In the self-evaluation mechanism, the planner chooses an agent reporting a high type with low probability when all agents receive low referee reports, and an agent reporting a high type with high probability when some agents receive high referee reports. By contrast, in the peer evaluation mechanism the planner chooses an agent who receives a high report from a competitor with high probability when all agents receive low referee reports and an agent who receives a high report from a competitor with low probability when some agents receive high referee reports. There is an imbalance between the expected probability of the two events and: the first event is much less likely than the second one (except for the case $n = 2$), and this difference increases with the number of agents. This implies that the situations where the compensation probabilities increase the value of the objective function in the self-evaluation mechanism are more likely to arise than the situations where the compensation probabilities increase the value of the objective function in the peer evaluation mechanism. In addition, note that, as agents who receive low referee reports are never shortlisted, when p is sufficiently large, changes in the probability of selecting an agent when all agents receive low report from the referee have a small effect on the incentive constraint in the self-evaluation mechanism. By contrast, in the peer evaluation mechanism, agents ignore the report they will receive, and hence their incentive constraint is affected by changes in $Z_\alpha(n)$. This second effect limits further the planner's ability to exploit compensations in probabilities to increase the value of the objective function in the peer evaluation mechanism.

We conclude this section with two diagrams illustrating the upper bound on the optimal payoffs in the scenario with peer evaluation and optimal payoffs in the scenario with self-evaluation.

Figure 4 illustrates the upper bound and the optimal payoffs under the two scenarios for three agents and perfect signals about own quality, $p = 1$. Figure 5 illustrates the difference between the optimal payoff and the upper bound for $n = 3$ agents, $r \in (1/2, 1)$, and $p \in (r, 1)$. These figures show that the difference in value between the two mechanisms is highest when agents have perfect information and the precision of the referee reports is intermediate.

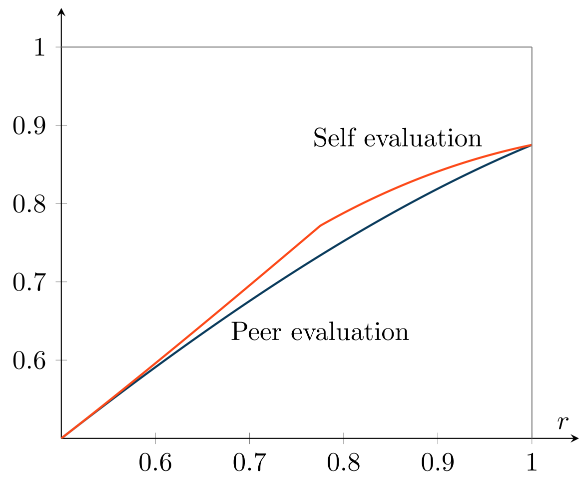

Figure 4: Payoffs $\Pi^{\mathsf{B}}(p, r)$ and $\overline{\Pi}^{PB}(p, r)$ for $n = 3$ and $p = 1$.

7 Ex Post Incentive Compatibility

In the preceding analysis, we have considered two incentive compatibility notions: Dominant Strategy Incentive Compatibility, where the players have an incentive to report the truth *for all reports of the other players and of the external referees* and Bayesian Incentive Compatibility , where the players have an incentive to report the truth, assuming that the other players also tell the truth and *taking expectations over the reports of the other players and external referees.* In

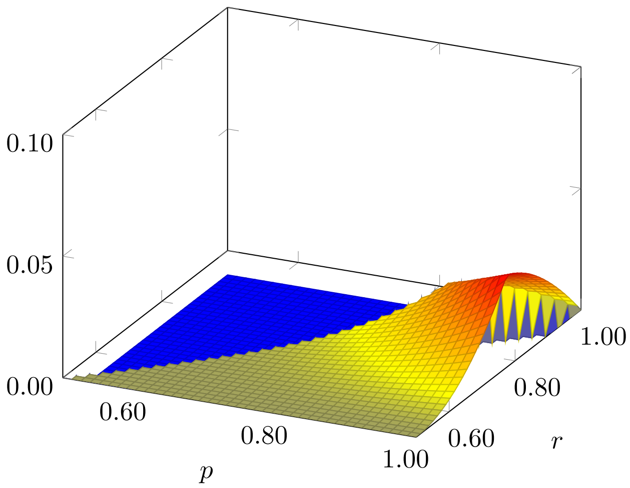

Figure 5: Difference in payoffs $\Pi^{\mathsf{B}}(p,r) - \overline{\Pi}^{PB}(p,r)$ when $n = 3$.

our setting, where external referees are not strategic players, there is meaningful third, intermediate, incentive compatibility constraint where the players have an incentive to report the truth *for all reports of the other players but taking expectations over the reports of the external referees.* In this Section, we explore this intermediate Incentive Compatibility condition for the self-evaluation mechanism. We will assume that players evaluate the probability of the signals of the external referees conditional on the fact that the other agents report the truth, so that the notion is analogous to Ex Post Incentive Compatibility.

Definition 5 (Ex Post Incentive Compatibility). *A mechanism $\pi : \{0,1\}^N \times \{0,1\}^N \to \Delta(N)$ is ex post incentive compatible (EPIC) if and only if, for any agent $i \in N$, any tuple of agents signals $\boldsymbol{x} \in \{0,1\}^N$ and any report x'_i of agent i*

$$\mathbf{E}(\pi_i(\boldsymbol{x}, \boldsymbol{Y} \mid \boldsymbol{X} = \boldsymbol{x})) \geq \mathbf{E}(\pi_i(x'_i, \boldsymbol{x}_{-i}, \boldsymbol{Y} \mid \boldsymbol{X} = \boldsymbol{x})).$$

Proposition 4. *Suppose that $n = 2$. Any optimal EPIC results in the same value for the social planner as the optimal DSIC mechanism.*

Proposition 4 shows that the gap between the optimal BIC and DSIC mechanisms is due to the fact that the planner can exploit the correlation between the referees' signals and the agents' announcements rather than the fact that the planner can take expectations over the signals of the referees. With two players, the optimal EPIC mechanism does not give a higher value to the planner than the optimal DSIC mechanism. In general, the characterization of the optimal EPIC mechanism is very complex and we have been unable to obtain a precise characterization of the optimal EPIC mechanism. However, we can show that the optimal BIC mechanism does not satisfy the EPIC constraints so that the optimal EPIC mechanism always gives the planner a smaller value than the optimal BIC mechanism:

Proposition 5. *For any $n \geq 2$, any $r \in (1/2, 1)$ and any $p \in (r, 1)$, the payoff from an optimal anonymous EPIC mechanism is strictly lower than the payoff from an optimal anonymous BIC mechanism.*

When the number of agents exceeds two, we also know that the optimal EPIC mechanism gives the planner a higher value than the optimal DSIC mechanism, as illustrated by the following example with three agents.

Example 1 (EPIC mechanism for $n = 3$ agents better than optimal DSIC mechanism)**.** *Consider a lexicographic mechanism such that*

$$\begin{aligned}
\varrho(0,1,(1,0,0,1)) &= \frac{1}{3} - \frac{1}{6}\min\left(1, \frac{2Q-1}{(1-Q)^2}\right) \\
\varrho(0,1,(0,1,0,1)) &= \frac{1}{3} \\
\varrho(0,0,(1,0,1,0)) &= \frac{1}{3} + \frac{1}{6}\min\left(1, \frac{2(1-Q)^2}{Q^2}\right) \\
\varrho(0,1,(0,0,1,1)) &= \frac{1}{3} - \frac{1}{6}\min\left(1, \frac{2Q-1}{(1-Q)^2}\right) \\
\varrho(0,1,(0,0,0,2)) &= \frac{1}{3} \\
\varrho(0,0,(0,0,2,0)) &= \frac{2}{3} + \frac{1}{3}\min\left(1, \frac{2Q-1}{(1-Q)^2}\right)
\end{aligned}$$

and the rest of the mechanism is determined by the lexicographic property and by the probability constraints.

The mechanism satisfies the EPIC constraints and yields payoff

$$\Pi^{\mathsf{EP}}(p,r) = r - \frac{2r-1}{8}\left(1 - \frac{r(1-r)(2p-1)^2}{Q}\min\left(2, \frac{Q^2}{(1-Q)^2}\right)\right)$$

which, for any $r \in (1/2, 1)$ and $p \in (1/2, 1)$, is strictly higher than the payoff from the constant mechanism, ϱ^{c}. Figure 6 illustrates these two payoffs (as a function of r) for the case when agents have nearly perfect signals about their types.

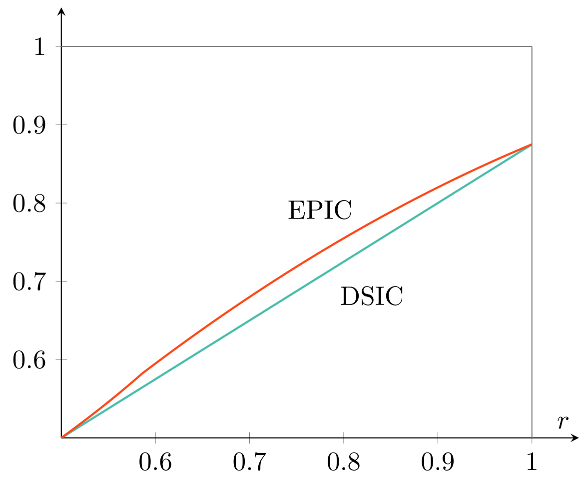

Figure 6: Payoffs from the EPIC mechanism in Example 1 and $\Pi^{\mathsf{D}}(p, r)$ for $n = 3$ and and the limit case of $p = 1$.

8 Conclusions

We consider a problem of mechanism design without money, where a planner selects a winner among a set of agents with binary types. In addition to the reports of the agents, the planner has access to a binary signal on the agent's types (the reports of external referees). We characterize the optimal Bayesian Incentive (BIC) mechanisms as a lexicographic mechanisms, where the planner first shortlists agents who receive high reports from the referees and then uses agents' reports to break ties among agents in the shortlist. We show that the planner can exploit the correlation between the reports of referees and the agents' types to improve upon the optimal Dominant Strategy Incentive Compatible (DSIC) mechanism and that the gain in value is highest when the precision of the signal of the referee is intermediate. We compare "self-evaluation" mechanisms with "peer evaluation" mechanism where agents obtain signals and are asked to report on the types of other agents, and show that for the same signal precision, the self- evaluation mechanism always results in a higher value than the peer evaluation mechanism. Finally, we explore Ex Post Incentive Compatible (EPIC) self-evaluation mechanisms and show that they give the planner an intermediate value, between the optimal DSIC and BIC mechanisms.

Our analysis can be used to guide the design of selection procedures for conference papers, prizes or positions. It suggests that self-evaluation reports are useful, but that they should be used to break ties after agents are shortlisted using the reports of referees. It also suggests that one should sometimes go against the report of the agents - and select with high probability an agent who sends a low report – in order to provide the correct incentives for truth-telling. Finally, if the cost of obtaining external signals is increasing in their precision, the analysis indicates that one only needs to recruit referees with intermediate signal precision, as this will be sufficient to discipline agents and result in the highest efficiency gain with respect to the DSIC mechanism.

We are aware of a number of limitations in our study, which should be tackled in future research. First, our mechanism relies on a strong commitment power of the planner. Once the planner receives the reports, she clearly has an incentive to ignore the reports of the referees and follow the agents' reports instead - and hence must be able to commit to the mechanism. Second, in order to keep the model with an arbitrary number of agents tractable, we focus attention on a binary model. Allowing for more categories, or for a continuum of types, is clearly and important item in our future research agenda. Finally, we consider a very simple peer evaluation mechanism, where each agent reports on another agent, and does not report on herself. In future work, we plan on exploring more complex peer evaluation mechanisms, where agents could receive multiple reports, both from themselves and from other agents.

A Appendix

The Appendix contains the proof of Theorem 1 . The proofs of other results in the paper are provided in Online Appendix 1. An anonymous mechanism π is fully described by a mapping $\varrho : \{0,1\}^2 \times S_4(n-1) \to [0,1]$ such that $\pi_i(\boldsymbol{x}, \boldsymbol{y}) = \varrho(x_i, y_i, \boldsymbol{z})$ where, for any $(b, b') \in \{0,1\}^2$, $z_{b,b'} = |\{j \in N \setminus \{i\} : (x_j, y_j) = (b, b')\}|$.

The objective function of the planner can then be written as:

$$\Pi = \frac{n}{2} \sum_{(x_0,y_0)\in\{0,1\}^2} \sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\zeta^1(x_0, y_0)\varrho(x_0, y_0, \boldsymbol{z}), \tag{8}$$

where, given $\boldsymbol{z} \in S_4(n-1)$,

$$U(\boldsymbol{z}) = \frac{1}{2^{n-1}} \binom{n-1}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} Q^{z_{1,1}+z_{0,0}} (1-Q)^{z_{0,1}+z_{1,0}}, \tag{9}$$

is the probability that, for each $(a,b) \in \{0,1\}^2$, $z_{a,b}$ out of $n-1$ agents receive reports ab and

$$\zeta^a(x_0, y_0) = ([x_0 = a]p + [x_0 \neq a](1-p))([y_0 = a]r + [y_0 \neq a](1-r)), \tag{10}$$

is the probability that an agent receives signal x_0 and a referee report y_0 when the true type of the agent is a.

The probability constraints stemming from the no-waste condition take the form:[12]

$$(z_{x_0,y_0}+1)\varrho(x_0,y_0,\boldsymbol{z}) + \sum_{(b,b')\in\{0,1\}^2\setminus(x_0,y_0)} z_{b,b'}\,\varrho(b,b',\boldsymbol{z}_{-(b,b'),(x_0,y_0)}, z_{b,b'}-1, z_{x_0,y_0}+1) = 1. \tag{11}$$

We now generalize the BIC constraints and compute the optimal anonymous BIC mechanism. The two BIC constraints take the form

$$\sum_{y_0\in\{0,1\}} \sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\zeta(0,y_0)\left(\varrho(0,y_0,\boldsymbol{z}) - \varrho(1,y_0,\boldsymbol{z})\right) \geq 0. \tag{12}$$

$$\sum_{y_0\in\{0,1\}} \sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\zeta(1,y_0)\left(\varrho(1,y_0,\boldsymbol{z}) - \varrho(0,y_0,\boldsymbol{z})\right) \geq 0. \tag{13}$$

where

$$\zeta(x_0,y_0) = \zeta^0(x_0,y_0) + \zeta^1(x_0,y_0) = [x_0 = y_0]Q + [x_0 \neq y_0](1-Q) \tag{14}$$

is twice the probability that an agent receives signal x_0 and referee report y_0.

The proof of Theorem 1 follows from a number of auxiliary lemmas.

Lemma 1. *For any optimal mechanism $(\varrho(x_0,y_0,\boldsymbol{z}))_{(x_0,y_0)\in\{0,1\}^2,\boldsymbol{z}\in S_4(n-1)}$, and any $\boldsymbol{z}\in S_4(n-1)$, if $z_{1,1}>0$ then $\varrho(1,0,\boldsymbol{z})=0$.*

Proof. Let $(\varrho(x_0,y_0,\boldsymbol{z}))_{(x_0,y_0)\in\{0,1\}^2,\boldsymbol{z}\in S_4(n-1)}$ be an optimal mechanism. Suppose that there exists $\boldsymbol{z}\in S_4(n-1)$ with $z_{1,1}>0$ such that $\varrho(1,0,\boldsymbol{z})>0$.

Consider a modification to the mechanism where $\varrho(1,0,\boldsymbol{z})$ is decreased by $\varepsilon/(z_{1,0}+1)$ and $\varrho(1,1,z_{0,0},z_{0,1},z_{1,0}+1,z_{1,1}-1)$ is increased by $\varepsilon/z_{1,1}$, where $\varepsilon\in(0,(z_{1,0}+1)\varrho(1,0,\boldsymbol{z}))$. This modification maintains the probability constraint (11).

Moreover, for any $A\in\mathbb{R}$ and $B\in\mathbb{R}$, the change in the sum

$$U(\boldsymbol{z})A\varrho(1,0,\boldsymbol{z}) + U(\boldsymbol{z}_{-(1,0),(1,1)}, z_{1,0}+1, z_{1,1}-1)B\varrho(1,1,\boldsymbol{z}_{-(1,0),(1,1)}, z_{1,0}+1, z_{1,1}-1)$$

resulting from this modification is equal to

$$-U(\boldsymbol{z})\frac{\varepsilon}{z_{1,0}+1}A + U(\boldsymbol{z}_{-(1,0),(1,1)}, z_{1,0}+1, z_{1,1}-1)\frac{\varepsilon}{z_{1,1}}B = \frac{1}{n2^{n-1}}\binom{n}{z_{0,0},z_{0,1},z_{1,0}+1,z_{1,1}}Q^{z_{0,0}+z_{1,1}}(1-Q)^{z_{0,1}+z_{1,0}+1}\varepsilon\left(\frac{B}{Q}-\frac{A}{1-Q}\right).$$

Thus the sign of the change is the same as the sign of $B(1-Q)-AQ$.

Consider the change to the value of the objective function resulting from the change to the mechanism. We have $A=\zeta^1(1,0)=p(1-r)$, $B=\zeta^1(1,1)=pr$, and

$$B(1-Q)-AQ = p(2r-1)(1-p) > 0.$$

[12] We adopt the convention that $\varrho(x_0,y_0,\boldsymbol{z})$ can have negative argument when it is multiplied by zero.

Thus the change results in an increase in the value of the objective function.

Consider now the BIC constraints and the change to the value of the LHS of a constraint resulting from the change to the mechanism. Suppose that $x_0 = 0$. Then $A = -\zeta(0,0) = -Q$, $B = -\zeta(1,1) = Q - 1$, and

$$B(1-Q) - AQ = -(1-Q)^2 + Q^2 = 2Q - 1 > 0,$$

as $Q > 1/2$. Hence the LHS of the constraint increases and the constraint remains satisfied. Suppose that $x_0 = 1$. Then $A = \zeta(1,0) = 1 - Q$, $B = \zeta(1,1) = Q$, and

$$B(1-Q) - AQ = Q(1-Q) - Q(1-Q) = 0.$$

Again, the LHS of the constraint is unchanged and the constraint remains satisfied.

□

By Lemma 1, either $\varrho(1,0,\boldsymbol{z}) = 0$ or $z_{1,1} = 0$. Thus, using the probability constraints (11), we obtain:

$$\varrho(1,1,\boldsymbol{z}) = \frac{1 - z_{0,0}\varrho(0,0,\boldsymbol{z}_{-(0,0),(1,1)}, z_{0,0}-1, z_{1,1}+1) - z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,1),(1,1)}, z_{0,1}-1, z_{1,1}+1)}{z_{1,1}+1} \tag{15}$$

and, in the case of $z_{1,1} = 0$,

$$\varrho(1,0,\boldsymbol{z}) = \frac{1 - z_{0,0}\varrho(0,0,\boldsymbol{z}_{-(0,0),(1,0)}, z_{0,0}-1, z_{1,0}+1) - z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,1),(1,0)}, z_{0,1}-1, z_{1,0}+1)}{z_{1,0}+1}. \tag{16}$$

Using these identities, we rewrite the objective function as

$$\begin{aligned}
\frac{n}{2}\Bigg(& \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\,(1-p-r)\,\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\left(\frac{r(1-r)(1-2p)}{Q}\right)\varrho(0,1,\boldsymbol{z}) \\
&+ \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}>0, z_{1,1}=0}} U(\boldsymbol{z})\left(\frac{r(1-r)(1-2p)}{1-Q}\right)\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}>0, z_{1,1}=0}} U(\boldsymbol{z})\,(r-p)\,\varrho(0,1,\boldsymbol{z}) \\
&+ \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,0}>0}} \frac{(1-p)(Q+r-1)}{Q} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})\Bigg) + C,
\end{aligned} \tag{17}$$

where the constant C is given by

$$C = \frac{1}{2^n}\left(\frac{2^n - (2-Q)^n}{Q}pr + \frac{(2-Q)^n - 1}{1-Q}p(1-r) - \frac{(1-r)(1-(1-Q)^n) + rQ(1-Q)^{n-1}}{Q}\right). \tag{18}$$

Similarly we rewrite the BIC constraint in the case of $x_0 = 0$ as

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} \left(\frac{1-Q}{Q}\right) U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} \left(\frac{Q}{1-Q}\right) U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) \geq \\
&\frac{1}{n2^{n-1}}\left(\frac{2^n(1-Q)^2 + (2Q-1)(2-Q)^n - Q}{Q(1-Q)}\right),
\end{aligned} \tag{19}$$

and in the case of $x_0 = 1$ as

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=z_{1,0}=0\\ z_{0,0}>0}} \left(\frac{2Q-1}{Q}\right) U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) \leq \frac{1}{n2^{n-1}} \left(\frac{2^n Q - 1 - (2Q-1)(1-Q)^{n-1}}{Q}\right). \tag{20}$$

Thus Lemma 1 allows us to rewrite the optimization problem as $\max_{\varrho(0,0,\boldsymbol{z}),\varrho(0,1,\boldsymbol{z})}$ (17) subject to BIC constraints (19) and (20), the probability constraints (11), and the constraints that all the variables $\varrho(\cdot,\cdot,\cdot)$ are non-negative. Using the fact that the variables $\varrho(1,0,\cdot)$ and $\varrho(1,1,\cdot)$ do not appear in the objective function and in the BIC constraints we can show that at an optimum the BIC constraint for $x_0 = 0$ is satisfied with equality. To show that we need to show, in particular, that we are able to adjust the values of variables $\varrho(1,0,\cdot)$ and $\varrho(1,1,\cdot)$ so that the probability constraints are still satisfied when the values of variables $\varrho(0,0,\cdot)$ and $\varrho(0,1,\cdot)$ are modified.

Lemma 2. *At any optimal mechanism, the BIC constraint for $x_0 = 0$ is satisfied with equality.*

Proof. We show first that at any optimal mechanism either the BIC constraint for $x_0 = 0$ is satisfied with equality or, for all $\boldsymbol{z} \in S_4(n-1)$ with $z_{1,0} + z_{1,1} > 0$, $\varrho(0,0,\boldsymbol{z}) = \varrho(0,1,\boldsymbol{z}) = 0$. This follows from the fact that the coefficients at $\varrho(0,0,\boldsymbol{z})$ and $\varrho(0,1,\boldsymbol{z})$ in the objective function (17) in the case $z_{1,0}+z_{1,1} > 0$ are negative. Hence one can increase the value of the objective function while maintaining the probability and BIC constraints, either by reducing the value of $\varrho(0,0,\boldsymbol{z})$ or $\varrho(0,1,\boldsymbol{z})$ by a sufficiently small amount and simultaneously increasing the value of $\varrho(1,1,\boldsymbol{z}_{-(0,0),(1,1)}, z_{0,0}+1, z_{0,1}, z_{1,0}, z_{1,1}-1)$ or the value of $\varrho(1,0,\boldsymbol{z}_{-(0,0),(1,0)}, z_{0,0}+1, z_{0,1}, z_{1,0}-1, z_{1,1})$ (in the case of decreasing $\varrho(0,0,\boldsymbol{z})$) or increasing the value of $\varrho(1,1,\boldsymbol{z}_{-(0,1),(1,1)}, z_{0,0}, z_{0,1}+1, z_{1,0}, z_{1,1}-1)$ or the value of $\varrho(1,0,\boldsymbol{z}_{-(0,1),(1,0)}, z_{0,0}, z_{0,1}+1, z_{1,0}-1, z_{1,1})$ (in the case of decreasing $\varrho(0,1,\boldsymbol{z})$). This contradicts the fact that the mechanism is optimal and proves the first step.

Second, we show that for some $\boldsymbol{z} \in S_4(n-1)$ with $z_{1,0}+z_{1,1} > 0$ either $\varrho(0,0,\boldsymbol{z}) > 0$ or $\varrho(0,1,\boldsymbol{z}) > 0$. Assume that this is not the case. Then the LHS of the BIC constraint for $x_0 = 0$ is equal to 0. On the other hand the RHS is equal to

$$\frac{1}{n2^{n-1}} \left(\frac{2^n(1-Q)^2 + (2Q-1)(2-Q)^n - Q}{Q(1-Q)}\right) >$$
$$\frac{1}{n2^{n-1}} \left(\frac{2(1-Q)^2 + (2Q-1)(2-Q) - Q}{Q(1-Q)}\right) = 0,$$

as $n \geq 2$, $Q < 1$, and $2 - Q > 1$. This shows that the BIC constraint is not satisfied, a contradiction which completes the proof of the Lemma. □

We will now characterize the solutions of a relaxed optimization problem, dropping the BIC constraint for $x_0 = 1$. We will then show that all these solutions satisfy the BIC constraint for $x_0 = 1$ and, therefore, are also solutions of the original problem.

Lemma 3. *Consider a relaxed optimization problem, without the BIC constraint for $x_0 = 1$. For any optimal mechanism $(\varrho(x_0, y_0, \boldsymbol{z}))_{(x_0,y_0)\in\{0,1\}^2, \boldsymbol{z}\in S_4(n-1)}$, and any $\boldsymbol{z} \in S_4(n-1)$, if $z_{0,1} > 0$ then $\varrho(0, 0, \boldsymbol{z}) = 0$.*

Proof. Let $(\varrho(x_0, y_0, \boldsymbol{z}))_{(x_0,y_0)\in\{0,1\}^2, \boldsymbol{z}\in S_4(n-1)}$ be an optimal mechanism of the relaxed optimization problem. Assume, to the contrary, that there exists $\boldsymbol{z} \in S_4(n-1)$ with $z_{0,1} > 0$ such that $\varrho(0, 0, \boldsymbol{z}) > 0$.

Consider a modification to the mechanism where $\varrho(0, 0, \boldsymbol{z})$ is decreased by $\varepsilon/(z_{0,0} + 1)$ and $\varrho(0, 1, z_{0,0} + 1, z_{0,1} - 1, z_{1,0}, z_{1,1})$ is increased by $\varepsilon/z_{0,1}$, where $\varepsilon \in (0, (z_{0,0} + 1)\varrho(0, 0, \boldsymbol{z}))$. This modification maintains the probability constraint (11). Moreover, for any $A \in \mathbb{R}$ and $B \in \mathbb{R}$, the change in the sum

$$U(\boldsymbol{z})A\varrho(0, 0, \boldsymbol{z}) + U(\boldsymbol{z}_{-(0,0),(0,1)}, z_{0,0} + 1, z_{0,1} - 1)B\varrho(0, 1, \boldsymbol{z}_{-(0,0),(0,1)}, z_{0,0} + 1, z_{0,1} - 1)$$

resulting from this modification is equal to

$$\begin{aligned}
&- U(\boldsymbol{z})\frac{\varepsilon}{z_{0,0} + 1}A + U(\boldsymbol{z}_{-(0,0),(0,1)}, z_{0,0} + 1, z_{0,1} - 1)\frac{\varepsilon}{z_{0,1}}B = \\
&\frac{1}{n2^{n-1}}\binom{n}{z_{0,0} + 1, z_{0,1}, z_{1,0}, z_{1,1}}Q^{z_{0,0}+z_{1,1}+1}(1 - Q)^{z_{0,1}+z_{1,0}}\varepsilon\left(\frac{B}{1 - Q} - \frac{A}{Q}\right).
\end{aligned}$$

Thus the sign of the change is the same as the sign of $BQ - A(1 - Q)$.

Consider the change to the value of the objective function resulting from the change to the mechanism. If $z_{1,1} > 0$ then $A = 1 - p - r$, $B = r(1 - r)(1 - 2p)/Q$ and

$$BQ - A(1 - Q) = p(1 - p)(2r - 1) > 0.$$

If $z_{1,1} = 0$ and $z_{1,0} > 0$ then $A = r(1 - r)(1 - 2p)/(1 - Q)$, $B = r - p$ and

$$BQ - A(1 - Q) = p(1 - p)(2r - 1) \geq 0.$$

In both cases the change results in an increase in the value of the objective function.

Consider now the BIC constraint for $x_0 = 0$ given by Equation (19) If $z_{1,1} > 0$ then $A = 1$, $B = (1 - Q)/Q$, and $BQ - A(1 - Q) = 0$. If $z_{1,1} = 0$ and $z_{1,0} > 0$ then $A = Q/(1 - Q)$, $B = 1$, and $BQ - A(1 - Q) = 0$. If $z_{1,1} = z_{1,0} = 0$ then $A = B = 0$ and, again, $BQ - A(1 - Q) = 0$. Hence the LHS of the constraint remains unchanged and the constraint remains satisfied. ☐

Using Lemmas 2 and 3, the BIC constraint for $x_0 = 0$ can be rewritten as follows:

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1) \\ \text{s.t. } z_{0,1}=0, z_{1,1}>0}} U(\boldsymbol{z})\varrho(0, 0, \boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1) \\ \text{s.t. } z_{1,1}>0}} \left(\frac{1 - Q}{Q}\right) U(\boldsymbol{z})\varrho(0, 1, \boldsymbol{z}) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1) \\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{1,0}>0}} \left(\frac{Q}{1 - Q}\right) U(\boldsymbol{z})\varrho(0, 0, \boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1) \\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0, 1, \boldsymbol{z}) = \\
&\frac{1}{n2^{n-1}}\left(\frac{2^n(1 - Q)^2 + (2Q - 1)(2 - Q)^n - Q}{Q(1 - Q)}\right).
\end{aligned} \tag{21}$$

By Lemma 3 and the probability constraints, in the case of any $\boldsymbol{z} \in S_4(n-1)$ such that $z_{1,0} = z_{1,1} = 0$ and $z_{0,0} > 0$, $\varrho(0, 1, \boldsymbol{z}) = 1/(z_{0,1} + 1)$. This, together with Lemma 3 and (17) allows us to rewrite the objective as:

$$\frac{n}{2}\Bigg(\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{1,1}>0}} U(\boldsymbol{z})\,(1-p-r)\,\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z}) \left(\frac{r(1-r)(1-2p)}{Q}\right) \varrho(0,1,\boldsymbol{z})$$
$$+ \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z}) \left(\frac{r(1-r)(1-2p)}{1-Q}\right) \varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}>0, z_{1,1}=0}} U(\boldsymbol{z})\,(r-p)\,\varrho(0,1,\boldsymbol{z})\Bigg)$$
$$+ C', \tag{22}$$

where the constant C' is equal to

$$C' = C + \frac{n}{2} \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,0}>0}} \frac{(1-p)(Q+r-1)}{Q} U(\boldsymbol{z}) \frac{1}{z_{0,1}+1},$$

and the constant C is given by (18). Since

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=z_{1,0}=0\\ z_{0,0}>0}} U(\boldsymbol{z}) \frac{1}{z_{0,1}+1} = \frac{1}{n2^{n-1}} \left(\frac{1-Q^n-(1-Q)^n}{1-Q}\right),$$

C' is equal to

$$C' = C + \frac{1}{2^n}\left(\frac{(1-p)(Q+r-1)\,(1-Q^n-(1-Q)^n)}{Q(1-Q)}\right). \tag{23}$$

Before characterizing the solutions of the relaxed optimization problem, we prove a simple mathematical fact about the function

$$\varphi_n(x) = \frac{(2-x)^n}{1-x} - x(1-x)^{n-2}.$$

Fact 1. *For any $n \geq 2$ there exists a unique $x_n^* \in [1/2, 1)$ such that $\varphi_n(x_n^*) = 2^n$. In addition, for all $x < x_n^*$, $\varphi_n(x) < 2^n$, and for all $x > x_n^*$, $\varphi_n(x) > 2^n$. Moreover, $x_2^* = 1/2$ and for any $n \geq 3$, $x_n^* \in (2/3, 1)$.*

Proof. Let $\psi_n(x) = \varphi_n(x) - 2^n$. Notice that

$$\frac{\partial \psi_n(x)}{\partial x} = \frac{(2-x)^{n-1}(x(n-1)-(n-2)) + (1-x)^{n-1}(x(n-1)-1)}{(1-x)^2}.$$

Let $N(x) = (2-x)^{n-1}(x(n-1)-(n-2)) + (1-x)^{n-1}(x(n-1)-1)$. Since

$$\frac{\partial N(x)}{\partial x} = n(2-x)^{n-2}(1-x) + (1-x)^{n-2}(2-nx) = (2-x)^{n-3},$$

and, for $x \in [0,1)$, $\frac{\partial N(x)}{\partial x} > 0$ if $n = 2$ and, in the case of $n \geq 3$,

$$\begin{aligned}\frac{\partial N(x)}{\partial x} &= (2-x)^{n-3} n(2-x)(1-x) + (1-x)^{n-3}(1-x)(2-nx)\\ &> (1-x)^{n-3}(n(2-x)(1-x) + (1-x)(2-nx)) = 2(1-x)^{n-3}(1-x)(n(1-x)+1) > 0,\end{aligned}$$

so $N(x)$ is strictly increasing for any natural $n \geq 2$. In addition, if $n \geq 2$, $N(0) = -2^{n-1}(n-2) - 1 < 0$ and $N(1) = 1$. Hence, on the interval $[0, 1)$, $N(x)$ is first negative and then positive. Therefore $\partial\psi_n(x)/\partial x$ is first negative and then positive on $[0, 1)$. It follows that $\psi_n(x)$ is first decreasing and then increasing on $[0, 1)$. Since $\psi_n(0) = 0$, $\lim_{x\uparrow 1} \psi_n(x) = +\infty$, and $\psi_n(x)$ is first decreasing and then increasing on $[0, 1)$ and continuous everywhere, there exists a unique $x_n^* \in (0, 1)$ such that $\psi_n(x) = 0$. In addition, for all $x < x_n^*$, $\psi(x) < 0$, and for all $x > x_n^*$, $\psi(x) > 0$. This establishes the first part of the Fact. Notice that

$$\psi_2(1/2) = \frac{2 \cdot 9 - 16 - 2}{2^n} = 0,$$

and

$$\psi_n(2/3) = \frac{4^n - 2 - 2^n 3^{n-1}}{3^{n-1}} = \frac{2^n \left(2^n - 3^{n-1}\right)}{3^{n-1}} < 0,$$

for $n \geq 3$, because in this case

$$\sqrt[n]{3} \leq \sqrt[3]{3} < \frac{3}{2}.$$

Hence it follows that if $n \geq 3$ then $x_n^* \in [2/3, 1)$. □

Lemma 4. *Consider the relaxed optimization problem, without the BIC constraint for $x_0 = 1$. Mechanism $(\varrho(x_0, y_0, \boldsymbol{z}))_{(x_0,y_0)\in\{0,1\}^2, \boldsymbol{z}\in S_4(n-1)}$ is optimal if and only if for any $\boldsymbol{z} \in S_4(n-1)$:*

1. *If $z_{0,1} > 0$ then $\varrho(0, 0, \boldsymbol{z}) = 0$ and if $z_{1,1} > 0$ then $\varrho(1, 0, \boldsymbol{z}) = 0$,*
2. *If $z_{0,1} = 0$ then*
   - (a) *if $z_{1,1} > 0$ then $\varrho(0, 0, \boldsymbol{z}) = 0$,*
   - (b) *if $z_{0,0} > 0$ then $\varrho(1, 1, \boldsymbol{z}) = \frac{1}{z_{1,1}+1}$.*
3. *If $z_{1,1} = 0$ then*
   - (a) *if $z_{1,0} > 0$ then $\varrho(0, 1, \boldsymbol{z}) = \frac{1}{z_{0,1}+1}$,*
   - (b) *if $z_{0,1} > 0$ then $\varrho(1, 0, \boldsymbol{z}) = 0$.*
4. *If $Q < \varphi_n^{(-1)}(2^n)$ then*
   - (a) *if $z_{1,1} = z_{0,1} = 0$ and $z_{1,0} > 0$ then $\varrho(0, 0, \boldsymbol{z}) = \frac{1}{z_{0,0}+1}$,*
   - (b) *if $z_{1,1} = z_{0,1} = 0$ and $z_{0,0} > 0$ then $\varrho(1, 0, \boldsymbol{z}) = 0$,*
   - (c) $\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}>0}} U(\boldsymbol{z})\varrho(0, 1, \boldsymbol{z}) = \frac{1}{n2^{n-1}} \left(\frac{2^n(1-Q)+Q(1-Q)^{n-1}-(2-Q)^n}{1-Q}\right)$.
5. *If $Q \geq \varphi_n^{(-1)}(2^n)$ then*
   - (a) *if $z_{0,1} > 0$ then $\varrho(1, 1, \boldsymbol{z}) = \frac{1}{z_{1,1}+1}$,*
   - (b) *if $z_{1,1} > 0$ then $\varrho(0, 1, \boldsymbol{z}) = 0$.*
   - (c) $\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,1}=0,\ z_{1,0}>0}} U(\boldsymbol{z})\varrho(0, 0, \boldsymbol{z}) = \frac{1}{n2^{n-1}} \left(\frac{2^n(1-Q)^2+Q(1-Q^n)-(2-Q)^n(1-Q)}{Q^2}\right)$.

Proof. We first show that any optimal mechanism under relaxed optimization problem satisfies the characterization given in the lemma.

Point 1 follows directly from Lemmas 1 and 3.

Throughout the proof we will use the following sets of possible reports on the other agents:

$$Z_1 = \{\boldsymbol{z} \in S_4(n-1) : z_{0,1} = 0 \text{ and } z_{1,1} > 0\} \qquad Z_2 = \{\boldsymbol{z} \in S_4(n-1) : z_{1,1} > 0\}$$
$$Z_3 = \{\boldsymbol{z} \in S_4(n-1) : z_{0,1} = z_{1,1} = 0,\ z_{1,0} > 0\} \quad Z_4 = \{\boldsymbol{z} \in S_4(n-1) : z_{1,1} = 0 \text{ and } z_{1,0} > 0\}.$$

Set Z_1 consists of reports on $n-1$ agents where at least one agent received two high reports and there is no agent who reported low on himself and received high report from the referee. Set Z_2 consists of reports on $n-1$ agents where at least one agent received two high reports. Set Z_3 consists of reports on $n-1$ agents where no agent received a high report from the referee and at least one agent reported a high type and received low report from the referee. Lastly, set Z_4 consists of reports on $n-1$ agents where no agent reported a high type and received a high report from the referee and there exists an agent who reported a high type and received a high report from the referee

Let

$$D = \frac{1}{n2^{n-1}}\left(\frac{2^n(1-Q)^2 + (2Q-1)(2-Q)^n - Q}{Q(1-Q)}\right).$$

By (21), the BIC constraint for $x_0 = 0$ is

$$\sum_{\boldsymbol{z}\in Z_1} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_2}\left(\frac{1-Q}{Q}\right)U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_3}\left(\frac{Q}{1-Q}\right)U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_4} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) = D. \quad (24)$$

By (22) the objective function can be rewritten as

$$\frac{n}{2}\Bigg(\sum_{\boldsymbol{z}\in Z_1} U(\boldsymbol{z})\,(1-p-r)\,\varrho(0,0,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_2} U(\boldsymbol{z})\left(\frac{r(1-r)(1-2p)}{Q}\right)\varrho(0,1,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_3} U(\boldsymbol{z})\left(\frac{r(1-r)(1-2p)}{1-Q}\right)\varrho(0,0,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_4} U(\boldsymbol{z})\,(r-p)\,\varrho(0,1,\boldsymbol{z})\Bigg) + C'. \quad (25)$$

Since

$$(p+r-1)(1-Q) - (2p-1)(1-r)r = (2p-1)(1-r)r - (p-r)Q = p(1-p)(2r-1) > 0$$

(for $1/2 < r \le p < 1$ and $Q > 1-Q$) so

$$1-p-r < \frac{r(1-r)(1-2p)}{1-Q} < \frac{r(1-r)(1-2p)}{Q} < r-p. \quad (26)$$

For the left to right implication, we will show that if there exists $\boldsymbol{z}$ for which one of the points given in the lemma is not satisfied for mechanism ϱ then the mechanism can be adjusted so that all the constraints remain satisfied and the value of the objective function increases, which contradicts optimality. By Lemmas 1 and 3, either $\varrho(1,0,\boldsymbol{z}) =$

0 or $z_{1,1} = 0$ and either $\varrho(0,0,\boldsymbol{z}) = 0$ or $z_{0,1} = 0$. Therefore, depending on x_0, y_0, and $\boldsymbol{z}$, the probability constraints (11) can be rewritten as follows:

$$
\begin{aligned}
(z_{0,0}+1)\varrho(0,0,\boldsymbol{z}) + z_{1,1}\varrho(1,1,\boldsymbol{z}_{-(0,0),-(1,1)}, z_{0,0}+1, z_{1,1}-1) &= 1, \text{ if } z_{0,1}=0 \text{ and } z_{1,1}>0\\
(z_{0,0}+1)\varrho(0,0,\boldsymbol{z}) + z_{1,0}\varrho(1,0,\boldsymbol{z}_{-(0,0),-(1,0)}, z_{0,0}+1, z_{1,0}-1) &= 1, \text{ if } z_{0,1}=0,\ z_{1,1}=0, \text{ and } z_{1,0}>0\\
(z_{0,0}+1)\varrho(0,0,\boldsymbol{z}) &= 1, \text{ if } z_{0,1}=0,\ z_{1,1}=0, \text{ and } z_{1,0}=0\\
(z_{0,1}+1)\varrho(0,1,\boldsymbol{z}) + z_{1,1}\varrho(1,1,\boldsymbol{z}_{-(0,1),-(1,1)}, z_{0,1}+1, z_{1,1}-1) &= 1, \text{ if } z_{1,1}>0\\
(z_{0,1}+1)\varrho(0,1,\boldsymbol{z}) + z_{1,0}\varrho(1,0,\boldsymbol{z}_{-(0,1),-(1,0)}, z_{0,1}+1, z_{1,0}-1) &= 1, \text{ if } z_{1,1}=0 \text{ and } z_{1,0}>0\\
(z_{0,1}+1)\varrho(0,1,\boldsymbol{z}) &= 1, \text{ if } z_{1,1}=0 \text{ and } z_{1,0}=0\\
(z_{1,0}+1)\varrho(1,0,\boldsymbol{z}) + z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,1),-(1,0)}, z_{0,1}-1, z_{1,0}+1) &= 1, \text{ if } z_{1,1}=0 \text{ and } z_{0,1}>0\\
(z_{1,0}+1)\varrho(1,0,\boldsymbol{z}) + z_{0,0}\varrho(0,0,\boldsymbol{z}_{-(0,0),-(1,0)}, z_{0,0}-1, z_{1,0}+1) &= 1, \text{ if } z_{1,1}=0,\ z_{0,1}=0, \text{ and } z_{0,0}>0\\
(z_{1,0}+1)\varrho(1,0,\boldsymbol{z}) &= 1, \text{ if } z_{1,1}=0,\ z_{0,1}=0, \text{ and } z_{0,0}=0\\
(z_{1,1}+1)\varrho(1,1,\boldsymbol{z}) + z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,1),-(1,1)}, z_{0,1}-1, z_{1,1}+1) &= 1, \text{ if } z_{0,1}>0\\
(z_{1,1}+1)\varrho(1,1,\boldsymbol{z}) + z_{0,0}\varrho(0,0,\boldsymbol{z}_{-(0,0),-(1,1)}, z_{0,0}-1, z_{1,1}+1) &= 1, \text{ if } z_{0,1}=0 \text{ and } z_{0,0}>0\\
(z_{1,1}+1)\varrho(1,1,\boldsymbol{z}) &= 1, \text{ if } z_{0,1}=0 \text{ and } z_{0,0}=0.
\end{aligned}
\tag{27}
$$

We will refer to the following sets of variables of the optimization problem:

$$
\begin{aligned}
V_1 &= \{\varrho(0,0,\boldsymbol{z}) : \boldsymbol{z} \in Z_1\}, & \bar{V}_1 &= \{\varrho(1,1,\boldsymbol{z}) : \boldsymbol{z} \in S_4(n-1),\ z_{0,1}=0 \text{ and } z_{0,0}>0\}\\
V_2 &= \{\varrho(0,1,\boldsymbol{z}) : \boldsymbol{z} \in Z_2\}, & \bar{V}_2 &= \{\varrho(1,1,\boldsymbol{z}) : \boldsymbol{z} \in S_4(n-1),\ z_{0,1}>0\}\\
V_3 &= \{\varrho(0,0,\boldsymbol{z}) : \boldsymbol{z} \in Z_3\}, & \bar{V}_3 &= \{\varrho(1,0,\boldsymbol{z}) : \boldsymbol{z} \in S_4(n-1),\ z_{0,1}=z_{1,1}=0,\ z_{0,0}>0\}\\
V_4 &= \{\varrho(0,1,\boldsymbol{z}) : \boldsymbol{z} \in Z_4\}, & \bar{V}_4 &= \{\varrho(1,0,\boldsymbol{z}) : \boldsymbol{z} \in S_4(n-1),\ z_{1,1}=0 \text{ and } z_{0,1}>0\}.
\end{aligned}
$$

Notice that, for any $i \in \{1,\dots,4\}$, any variable in the set V_i is associated, through the probability constraints (27), with a unique variable in the set $\bar{V}_i$. Moreover, sets $\bar{V}_1,\dots,\bar{V}_4$ are pairwise disjoint. Therefore for any adjustment of the values of variables in the sets $V_1,\dots,V_4$ we can adjust the values of variables in the sets $\bar{V}_1,\dots,\bar{V}_4$ and maintain the probability constraints. Since both the objective function and the BIC constraints only involve variables in the sets $V_1,\dots,V_4$, we can restrict attention to the variables in these sets.

Suppose that mechanism $(\varrho(x_0,y_0,\boldsymbol{z}))_{(x_0,y_0)\in\{0,1\}^2,\boldsymbol{z}\in S_4(n-1)}$ is optimal. For point 2a of the lemma, notice first that, by the probability constraints, for any $\boldsymbol{z} \in S_4(n-1)$, $\varrho(0,0,\boldsymbol{z}) \le 1/(z_{0,0}+1)$ and $\varrho(0,1,\boldsymbol{z}) \le 1/(z_{0,1}+1)$. Therefore

$$
\sum_{\boldsymbol{z}\in Z_2}\left(\frac{1-Q}{Q}\right)U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_3}\left(\frac{Q}{1-Q}\right)U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_4}U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) \le
$$
$$
\sum_{\boldsymbol{z}\in Z_2}\left(\frac{1-Q}{Q}\right)U(\boldsymbol{z})\frac{1}{z_{0,1}+1} + \sum_{\boldsymbol{z}\in Z_3}\left(\frac{Q}{1-Q}\right)U(\boldsymbol{z})\frac{1}{z_{0,0}+1} + \sum_{\boldsymbol{z}\in Z_4}U(\boldsymbol{z})\frac{1}{z_{0,1}+1}
$$

and the equality is attained when $\varrho(0,0,\boldsymbol{z}) = 1/(z_{0,0}+1)$, for all $\boldsymbol{z} \in Z_3$, and $\varrho(0,1,\boldsymbol{z}) = 1/(z_{0,1}+1)$, for all $\boldsymbol{z} \in Z_2 \cup Z_4$. Next we observe that

$$\sum_{\boldsymbol{z}\in Z_2} U(\boldsymbol{z})\frac{1}{z_{0,1}+1} = \frac{1}{n2^{n-1}}\left(\frac{2^n-(1+Q)^n-(2-Q)^n+1}{1-Q}\right) \tag{28}$$

$$\sum_{\boldsymbol{z}\in Z_3} U(\boldsymbol{z})\frac{1}{z_{0,0}+1} = \frac{1}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{Q}\right) \tag{29}$$

$$\sum_{\boldsymbol{z}\in Z_4} U(\boldsymbol{z})\frac{1}{z_{0,1}+1} = \frac{1}{n2^{n-1}}\left(\frac{(2-Q)^n+Q^n-2}{1-Q}\right) \tag{30}$$

so that

$$\begin{aligned}
&\sum_{\boldsymbol{z}\in Z_2}\left(\frac{1-Q}{Q}\right)U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_3}\left(\frac{Q}{1-Q}\right)U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_4}U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})\\
\leq&\frac{1}{n2^{n-1}}\left(\frac{2^n-(1+Q)^n-(2-Q)^n+1}{Q}+\frac{1-Q^n-(1-Q)^n}{1-Q}+\frac{(2-Q)^n+Q^n-2}{(1-Q)}\right)\\
=&\frac{1}{n2^{n-1}}\left(\frac{(2-Q)^n(2Q-1)-Q(1-Q)^n-(1-Q)(1+Q)^n+2^n(1-Q)-2Q+1}{Q(1-Q)}\right).
\end{aligned}$$

Since, for $Q\in(0,1)$ and $n\geq 2$,

$$2^{n-1}-\frac{1-(1+Q)^{n-1}}{1-(1+Q)} = 1+\sum_{i=0}^{n-2}2^i-\sum_{i=0}^{n-2}(1+Q)^i \geq 1,$$

and

$$2^{n-1}-(1-Q)^{n-1}-(1+Q)^{n-1}\geq 0,$$

as for $Q\in(0,1)$ and $n\geq 2$, $(1-Q)^{n-1}+(1+Q)^{n-1}\leq((1-Q)+(1+Q))^{n-1}=2^{n-1}$, so

$$\begin{aligned}
&\frac{1}{n2^{n-1}}\left(\frac{(2-Q)^n(2Q-1)-Q(1-Q)^n-(1-Q)(1+Q)^n+2^n(1-Q)-2Q+1}{Q(1-Q)}\right)-D\\
&\quad=\frac{1}{n2^{n-1}}\left(\frac{Q2^n+1-Q(1-Q)^n-(1+Q)^n+Q(1+Q)^n-Q-2^nQ^2}{Q(1-Q)}\right)\\
&\quad=\frac{1}{n2^{n-1}}\left(\frac{Q(1-Q)2^n+1-Q-Q(1-Q)^n-(1-Q)(1+Q)^n}{Q(1-Q)}\right)\\
&\quad=\frac{1}{n2^{n-1}}\left(\frac{Q2^n+1-Q(1-Q)^{n-1}-(1+Q)^n)}{Q}\right)\\
&\quad=\frac{1}{n2^{n-1}}\left(\frac{Q2^{n-1}+Q2^{n-1}+1-Q(1-Q)^{n-1}-(1+Q)(1+Q)^{n-1})}{Q}\right)\\
&\quad=\frac{1}{n2^{n-1}}\left(\frac{Q2^{n-1}+1-(1+Q)^{n-1}+Q\left(2^{n-1}-(1-Q)^{n-1}-(1+Q)^{n-1}\right)}{Q}\right)\\
&\quad=\frac{1}{n2^{n-1}}\left(2^{n-1}-\frac{1-(1+Q)^{n-1}}{1-(1+Q)}+\left(2^{n-1}-(1-Q)^{n-1}-(1+Q)^{n-1}\right)\right)>0.
\end{aligned}$$

Hence

$$\sum_{\boldsymbol{z}\in Z_2}\left(\frac{1-Q}{Q}\right)U(\boldsymbol{z})\frac{1}{z_{0,1}+1}+\sum_{\boldsymbol{z}\in Z_3}\left(\frac{Q}{1-Q}\right)U(\boldsymbol{z})\frac{1}{z_{0,0}+1}+\sum_{\boldsymbol{z}\in Z_4}U(\boldsymbol{z})\frac{1}{z_{0,1}+1}>D. \tag{31}$$

Now assume, to the contrary, that there exists $\boldsymbol{z} \in S_4(n-1)$ with $z_{0,1} = 0$ and $z_{1,1} > 0$ (i.e. $\boldsymbol{z} \in Z_1$) such that $\varrho(0, 0, \boldsymbol{z}) > 0$. By the BIC constraint (24) and by (31) either

(i) there exists $\boldsymbol{z'} \in Z_2$ with $\varrho(0, 1, \boldsymbol{z'}) < 1/(z_{0,1} + 1)$, or

(ii) there exists $\boldsymbol{z'} \in Z_3$ with $\varrho(0, 0, \boldsymbol{z'}) < 1/(z_{0,0} + 1)$, or

(iii) there exists $\boldsymbol{z'} \in Z_4$ with $\varrho(0, 1, \boldsymbol{z'}) < 1/(z_{0,1} + 1)$.

Suppose case (i) holds. Then decreasing the value of $\varrho(0, 0, \boldsymbol{z})$ by ε (and adjusting the value of the corresponding variable in $\bar{V}_1$ accordingly) and increasing the value of $\varrho(0, 1, \boldsymbol{z'})$ by $\varepsilon QU(\boldsymbol{z})/((1-Q)U(\boldsymbol{z'}))$ (and adjusting the value of the corresponding variable in $\bar{V}_2$ accordingly), maintains the BIC constraint and changes the value of the objective function by

$$\varepsilon U(\boldsymbol{z}) \left(\frac{r(1-r)(1-2p)}{1-Q} - (1-p-r) \right),$$

which, by (26), is greater than 0. Hence the adjustment increases the value of the objective function and maintains the constraints, a contradiction.

Suppose case (ii) holds. Then decreasing the value of $\varrho(0, 0, \boldsymbol{z})$ by ε (and adjusting the value of the corresponding variable in $\bar{V}_1$ accordingly) and increasing the value of $\varrho(0, 0, \boldsymbol{z'})$ by $\varepsilon(1-Q)U(\boldsymbol{z})/(QU(\boldsymbol{z'}))$ (and adjusting the value of the corresponding variable in $\bar{V}_3$ accordingly), maintains the BIC constraint and changes the value of the objective function by

$$\varepsilon U(\boldsymbol{z}) \left(\frac{r(1-r)(1-2p)}{Q} - (1-p-r) \right),$$

which, by (26), is greater than 0. Hence the adjustment increases the value of the objective function and maintains the constraints, a contradiction.

Lastly, suppose case (iii) holds. Then decreasing the value of $\varrho(0, 0, \boldsymbol{z})$ by ε (and adjusting the value of the corresponding variable in $\bar{V}_1$ accordingly) and increasing the value of $\varrho(0, 1, \boldsymbol{z'})$ by $\varepsilon U(\boldsymbol{z})/U(\boldsymbol{z'})$ (and adjusting the value of the corresponding variable in $\bar{V}_4$ accordingly), maintains the BIC constraint and changes the value of the objective function by $\varepsilon U(\boldsymbol{z})(r - p - (1-p-r))$, which, by (26), is greater than 0. Hence the adjustment increases the value of the objective function and maintains the constraints, a contradiction.

As we reach a contradiction in all three cases, we conclude that for all $\boldsymbol{z} \in Z_1$, $\varrho(0, 0, \boldsymbol{z}) = 0$ and point 2a of the Lemma holds. By point 2a and the probability constraints (27), for all $\boldsymbol{z} \in Z_1$, $\varrho(1, 1, \boldsymbol{z}_{-(0,0),(1,1)}, z_{0,0}+1, z_{1,1}-1) = 1/(z_{1,1}+1)$. Hence point 2b follows.

To prove point 3a of the Lemma recall that, by the probability constraints, for any $\boldsymbol{z} \in S_4(n-1)$, $\varrho(0, 1, \boldsymbol{z}) \leq 1/(z_{0,1} + 1)$. Therefore

$$\sum_{\boldsymbol{z} \in Z_4} U(\boldsymbol{z}) \varrho(0, 1, \boldsymbol{z}) \leq \sum_{\boldsymbol{z} \in Z_4} U(\boldsymbol{z}) \frac{1}{z_{0,1} + 1}$$

and the equality is attained when $\varrho(0, 1, \boldsymbol{z}) = 1/(z_{0,1} + 1)$, for all $\boldsymbol{z} \in Z_4$. Notice that

$$2^n(1-Q) - (2-Q)^n + Q = Q + \sum_{i=0}^{n} \binom{n}{i}(1-Q) - \sum_{i=0}^{n} \binom{n}{i}(1-Q)^i = \sum_{i=1}^{n} \binom{n}{i} \left(1 - Q - (1-Q)^i\right) > 0, \tag{32}$$

as for $Q \in (0,1)$, $n \geq 2$, and for all $i \in \{1, \ldots, n-2\}$, $1 - Q \geq (1-Q)^i$ with equality only if $i = 1$. In addition, for any $Q \in (0,1)$ and $n \geq 2$, $Q^{n-1} < 1$. Therefore, using Equation (30), for any $Q \in (0,1)$,

$$\begin{aligned} D &- \frac{1}{n2^{n-1}} \left(\frac{(2-Q)^n + Q^n - 2}{1-Q} \right) \\ &= \frac{1}{n2^{n-1}} \left(\frac{2^n(1-Q)^2 + Q(1-Q^n) - (2-Q)^n(1-Q)}{Q(1-Q)} \right) \\ &= \frac{1}{n2^{n-1}} \left(\frac{(1-Q)\left(2^n(1-Q) - (2-Q)^n + Q\right) + Q^2(1-Q^{n-1})}{Q(1-Q)} \right) > 0 \end{aligned}$$

and, consequently,

$$\sum_{\boldsymbol{z} \in Z_4} U(\boldsymbol{z}) \frac{1}{z_{0,1}+1} < D. \tag{33}$$

Now assume, by contradiction, that there exists $\boldsymbol{z} \in S_4(n-1)$ with $z_{1,1} = 0$ and $z_{1,0} > 0$ (i.e. $\boldsymbol{z} \in Z_4$), and such that $\varrho(0,1,\boldsymbol{z}) < 1/(z_{0,1}+1)$. By the BIC constraint (24), point 2a of the lemma, and by (33) either

(i) there exists $\boldsymbol{z'} \in Z_2$ with $\varrho(0,1,\boldsymbol{z'}) > 0$, or

(ii) there exists $\boldsymbol{z'} \in Z_3$ with $\varrho(0,0,\boldsymbol{z'}) > 0$.

Suppose case (i) holds. Then increasing the value of $\varrho(0,1,\boldsymbol{z})$ by ε (and adjusting the value of the corresponding variable in $\bar{V}_4$ accordingly) and decreasing the value of $\varrho(0,1,\boldsymbol{z'})$ by $\varepsilon Q U(\boldsymbol{z})/((1-Q)U(\boldsymbol{z'}))$ (and adjusting the value of the corresponding variable in $\bar{V}_2$ accordingly), maintains the BIC constraint and changes the value of the objective function by

$$\varepsilon U(\boldsymbol{z}) \left(r - p - \frac{r(1-r)(1-2p)}{1-Q} \right),$$

which, by (26), is greater than 0. Hence the adjustment increases the value of the objective function and maintains the constraints.

Suppose case (ii) holds. Then increasing the value of $\varrho(0,1,\boldsymbol{z})$ by ε (and adjusting the value of the corresponding variable in $\bar{V}_4$ accordingly) and decreasing the value of $\varrho(0,0,\boldsymbol{z'})$ by $\varepsilon(1-Q)U(\boldsymbol{z})/(QU(\boldsymbol{z'}))$ (and adjusting the value of the corresponding variable in $\bar{V}_3$ accordingly), maintains the BIC constraint and changes the value of the objective function by

$$\varepsilon U(\boldsymbol{z}) \left(r - p - \frac{r(1-r)(1-2p)}{Q} \right),$$

which, by (26), is greater than 0. Hence the adjustment increases the value of the objective function and maintains the constraints.

As we reach a contradiction in both cases, we conclude that for all $\boldsymbol{z} \in Z_4$, $\varrho(0,1,\boldsymbol{z}) = 1/(z_{0,1}+1)$ and that point 3a of the Lemma holds.

By point 3a and the probability constraints (27), for all $\boldsymbol{z} \in Z_4$, $\varrho(1,0,\boldsymbol{z}_{-(0,1),(1,0)}, z_{0,1}+1, z_{1,0}-1) = 0$. Hence point 3b follows.

Using points 2a and 3a of the lemma, the BIC constraint (24) can be rewritten as

$$\sum_{\boldsymbol{z} \in Z_2} \left(\frac{1-Q}{Q} \right) U(\boldsymbol{z}) \varrho(0,1,\boldsymbol{z}) + \sum_{\boldsymbol{z} \in Z_3} \left(\frac{Q}{1-Q} \right) U(\boldsymbol{z}) \varrho(0,0,\boldsymbol{z}) = D', \tag{34}$$

where

$$D' = D - \sum_{\boldsymbol{z}\in Z_4} U(\boldsymbol{z})\frac{1}{z_{0,1}+1} = \frac{1}{n2^{n-1}}\left(\frac{2^n(1-Q)^2 + Q(1-Q^n) - (2-Q)^n(1-Q)}{Q(1-Q)}\right),$$

and the objective function can be rewritten as

$$\frac{r(1-r)(2p-1)n}{2}\left(-\sum_{\boldsymbol{z}\in Z_2} U(\boldsymbol{z})\left(\frac{1}{Q}\right)\varrho(0,1,\boldsymbol{z}) - \sum_{\boldsymbol{z}\in Z_3} U(\boldsymbol{z})\left(\frac{1}{1-Q}\right)\varrho(0,0,\boldsymbol{z})\right) + C'', \tag{35}$$

where, using Equation (30),

$$C'' = C' + \frac{n(r-p)}{2}\sum_{\boldsymbol{z}\in Z_4} U(\boldsymbol{z})\left(\frac{1}{z_{0,1}+1}\right) = C' + \frac{1}{2^n}\left(\frac{(r-p)\left((2-Q)^n + Q^n - 2\right)}{1-Q}\right). \tag{36}$$

Using (34), the objective function can be further rewritten as

$$\left(\frac{r(1-r)(2p-1)n}{2(1-Q)}\right)\frac{2Q-1}{1-Q}\sum_{\boldsymbol{z}\in Z_3} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + C'' - \left(\frac{r(1-r)(2p-1)n}{2(1-Q)}\right)D'. \tag{37}$$

Since $1/2 < p < 1$ and $1/2 < Q < 1$, $2p-1 > 0$, and $(2Q-1)/(1-Q) > 0$. Hence to maximize the value of the objective function we need to maximize $\sum_{\boldsymbol{z}\in Z_3} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z})$.

For point 4a of the lemma suppose that $Q < \varphi_n^{(-1)}(2^n)$. Since

$$\sum_{\boldsymbol{z}\in Z_3} U(\boldsymbol{z})\frac{1}{z_{0,0}+1} = \frac{1}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{Q}\right),$$

we have

$$D' - \sum_{\boldsymbol{z}\in Z_3}\left(\frac{Q}{1-Q}\right)U(\boldsymbol{z})\frac{1}{z_{0,0}+1} = \frac{1}{n2^{n-1}}\left(\frac{2^n(1-Q) + Q(1-Q)^{n-1} - (2-Q)^n}{Q}\right).$$

If $Q < \varphi_n^{(-1)}(2^n)$ then, by Fact 1, the above expression is positive and hence

$$D' > \sum_{\boldsymbol{z}\in Z_3}\left(\frac{Q}{1-Q}\right)U(\boldsymbol{z})\frac{1}{z_{0,0}+1}. \tag{38}$$

By (37) the value of the objective function increases as the value of $\varrho(0,0,\boldsymbol{z})$ increases, for all $\boldsymbol{z}\in Z_3$. By (38) the BIC constraints are satisfied when $\varrho(0,0,\boldsymbol{z}) = 1/(z_{0,0}+1)$, taking its maximal value, for all $\boldsymbol{z}\in Z_3$. Hence point 4a of the Lemma holds. By point 4a and the probability constraints (27), for all $\boldsymbol{z}\in Z_3$, $\varrho(1,0,\boldsymbol{z}_{-(0,0),(1,0)}, z_{0,0}+1, z_{1,0}-1) = 0$. Hence point 4b follows. Point 4c of the lemma follows immediately by point 4a of the Lemma together with (34).

To prove point 5a of the Lemma suppose that $Q \geq \varphi_n^{(-1)}(2^n)$. Then

$$D' \leq \sum_{\boldsymbol{z}\in Z_3}\left(\frac{Q}{1-Q}\right)U(\boldsymbol{z})\frac{1}{z_{0,0}+1}. \tag{39}$$

By (37) the value of the objective function increases as the value $\varrho(0,0,\boldsymbol{z})$ increases, for all $\boldsymbol{z}\in Z_3$. By (39) the BIC constraints are not satisfied when $\varrho(0,0,\boldsymbol{z}) = 1/(z_{0,0}+1)$, taking its maximal value, for all $\boldsymbol{z} \in Z_3$. Hence for all $\boldsymbol{z} \in Z_2$, $\varrho(0,1,\boldsymbol{z}) = 0$, and so point 5b of the Lemma holds. By point 5b, for all $\boldsymbol{z} \in Z_2$, $\varrho(0,1,\boldsymbol{z}_{-(0,1),(1,1)}, z_{0,1}-1, z_{1,1}+1) = 0$. This, together with the probability constraints (27), yields point 5a. Point 5c of the Lemma follows immediately by point 5b of the Lemma together with (34).

To check that any mechanism that satisfies the characterization given in the lemma is optimal under the relaxed optimization problem notice that the characterization fully determines the values of the probabilities ϱ except for the values in sets V_2 and $\bar{V}_2$, when $Q \geq \varphi_n^{(-1)}(2^n)$, and V_3 and $\bar{V}_3$, when $Q < \varphi_n^{(-1)}(2^n)$. Since the value of the objective function given by (37) only depends on the sum $\sum_{\boldsymbol{z}\in Z_3} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z})$ of the values of the variables in V_3, which is fixed by points 4a and 5c, any mechanism satisfying the characterization given in the Lemma is optimal. □

We now check that the BIC constraint of the high type is always satisfied at an optimal BIC mechanism of the relaxed problem.

Lemma 5. *Mechanism* $(\varrho(x_0,y_0,\boldsymbol{z}))_{(x_0,y_0)\in\{0,1\}^2,\boldsymbol{z}\in S_4(n-1)}$ *is optimal under the original problem if and only if it is optimal under the relaxed problem.*

Proof. Let $(\varrho(x_0,y_0,\boldsymbol{z}))_{(x_0,y_0)\in\{0,1\}^2,\boldsymbol{z}\in S_4(n-1)}$ be an optimal mechanism under the relaxed problem. By the characterization given in Lemma 4, the BIC constraint (20) for $x_0 = 1$ can be rewritten as

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\frac{1}{z_{0,1}+1} +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=z_{1,0}=0\\ z_{0,0}>0}} \left(\frac{2Q-1}{Q}\right) U(\boldsymbol{z})\frac{1}{z_{0,1}+1} \leq \frac{1}{n2^{n-1}}\left(\frac{2^nQ - 1 - (2Q-1)(1-Q)^{n-1}}{Q}\right).$$

Since

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\frac{1}{z_{0,1}+1} = \frac{1}{n2^{n-1}}\left(\frac{(2-Q)^n + Q^n - 2}{1-Q}\right)$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0\\ z_{0,0}>0}} U(\boldsymbol{z})\frac{1}{z_{0,1}+1} = \frac{1}{n2^{n-1}}\left(\frac{1 - Q^n - (1-Q)^n}{1-Q}\right),$$

the BIC constraint can be rewritten as

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z})$$
$$\leq \frac{1}{n2^{n-1}}\left(\frac{2^n(1-Q) - (2-Q)^n - Q^{n-1}(1-Q) + 1}{1-Q}\right). \quad (40)$$

Suppose that $Q < \varphi_n^{(-1)}(2^n)$. By Lemma 4,

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) = \frac{1}{n2^{n-1}}\left(\frac{2^n(1-Q) + Q(1-Q)^{n-1} - (2-Q)^n}{1-Q}\right).$$

Thus (40) can be rewritten as

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) \leq \frac{1}{n2^{n-1}}\left(\frac{1-Q(1-Q)^{n-1}-(1-Q)Q^{n-1}}{1-Q}\right). \quad (41)$$

By Lemma 4, for any $\boldsymbol{z} \in S_4(n-1)$ with $z_{1,1} = z_{0,1} = 0$ and $z_{1,0} > 0$, $\varrho(0,0,\boldsymbol{z}) = 1/(z_{0,0}+1)$. Since

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0\\ z_{1,0}>0}} U(\boldsymbol{z})\frac{1}{z_{0,0}+1} = \frac{1}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{Q}\right),$$

so the LHS of (41) is equal to

$$\frac{1}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{Q}\right).$$

Subtracting this from the RHS we obtain

$$\begin{aligned}
&\frac{1}{n2^{n-1}}\left(\frac{1-Q(1-Q)^{n-1}-(1-Q)Q^{n-1}}{1-Q}\right) - \frac{1}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{Q}\right) = \\
&\frac{1}{n2^{n-1}}\left(\frac{Q-Q^2(1-Q)^{n-1}-(1-Q)+(1-Q)^{n+1}}{Q(1-Q)}\right) \\
&\frac{1}{n2^{n-1}}\left(\frac{2Q-1-(1-Q)^{n-1}\left(Q^2-(1-Q)^2\right)}{Q(1-Q)}\right) \\
&\frac{1}{n2^{n-1}}\left(\frac{(2Q-1)\left(1-(1-Q)^{n-1}\right)}{Q(1-Q)}\right),
\end{aligned}$$

which is positive for any $Q \in (1/2, 1)$ and $n \geq 2$. Thus the BIC constraint is satisfied.

Suppose that $Q \geq \varphi_n^{(-1)}(2^n)$. By Lemma 4, for any $\boldsymbol{z} \in S_4(n-1)$ with $z_{1,1} > 0$, $\varrho(0,1,\boldsymbol{z}) = 0$ and

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,1}=0,\ z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) = \frac{1}{n2^{n-1}}\left(\frac{2^n(1-Q)^2+Q(1-Q^n)-(2-Q)^n(1-Q)}{Q^2}\right),$$

which is also equal to the LHS of the constraint. Subtracting this from the RHS we obtain

$$\frac{1}{n2^{n-1}}\left(\frac{(2Q-1)\left(2^n(1-Q)-(2-Q)^n+Q\right)}{Q^2(1-Q)}\right).$$

By (32), $2^n(1-Q)-(2-Q)^n+Q > 0$ for $n \geq 2$ and $Q \in (1/2, 1)$. In addition $2Q-1 > 0$ for $Q \in (1/2, 1)$. Hence the difference RHS minus LHS is positive and the BIC constraint is satisfied. □

This completes the proof of the theorem.

B Online Appendix 1: Proofs

B.1 Proof of Proposition 1

We provide computations for the values of the planner's objective function under the optimal DSIC and BIC mechanisms.

Lemma 6. *The probability of selecting a high quality project under an optimal DSIC mechanism is*

$$\Pi^{\mathsf{D}}(p,r) = r - \frac{2r-1}{2^n}.$$

Proof. Since DSIC mechanisms are independent of agents' reports, the following mechanism

$$\varrho^{\mathsf{D}}(x_0,y_0,\boldsymbol{z}) = \begin{cases} \frac{1}{z_{0,1}+z_{1,1}+1}, & \text{if } y_0 = 1 \text{ and } z_{x_0,1} = z_{0,1}+z_{1,1}, \\ \frac{1}{n}, & \text{if } y_0 = 0 \text{ and } z_{0,0}+z_{1,0} = n-1, \\ 0, & \text{otherwise,} \end{cases}$$

is an optimal DSIC mechanism.

By (8), the value of the objective function under ϱ^{D} is equal to

$$\Pi^{\mathsf{D}}(p,r) = \frac{n}{2} \sum_{(x_0,y_0)\in\{0,1\}^2} \sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\zeta^1(x_0,y_0)\varrho^{\mathsf{D}}(x_0,y_0,\boldsymbol{z}) = r - \frac{2r-1}{2^n}. \quad (42)$$

□

Lemma 7. *The probability of selecting a high quality project under an optimal anonymous BIC mechanism is*

$$\Pi^{\mathsf{B}}(p,r) = r - \frac{2r-1}{2^n}\left(1 - \left(\frac{r(1-r)(2p-1)^2}{Q(1-Q)}\right)\min\left(1-(1-Q)^{n-1}, \frac{2^n(1-Q)-(2-Q)^n+Q}{Q}\right)\right)$$

Proof. By (37), the value of the objective function under any optimal anonymous BIC mechanism, $\varrho \in R^{\mathsf{B}}$, is

$$\Pi^{\mathsf{B}}(p,r) = C'' - \left(\frac{r(1-r)(2p-1)n}{2(1-Q)}\right)\left(D' - \frac{2Q-1}{1-Q} \sum_{\substack{\boldsymbol{z}\in S_4(n-1) \\ z_{0,1}=z_{1,1}=0,\, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z})\right),$$

where

$$D' = \frac{1}{n2^{n-1}}\left(\frac{2^n(1-Q)^2+Q(1-Q^n)-(2-Q)^n(1-Q)}{Q(1-Q)}\right),$$

by (18), (23), and (36),

$$C'' = \frac{1}{2^n}\left(\frac{pr(1-Q)2^n - r(1-r)(2p-1)((2-Q)^n+Q^n)+Q(p-r)}{Q(1-Q)}\right) \quad (43)$$

and, by points 4a and 5c of Lemma 4 and (29),

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,1}=0,\, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) = \left\{ \begin{array}{ll} \frac{1}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{Q}\right), & \text{if } Q < Q^*_n \\ \frac{1}{n2^{n-1}}\left(\frac{2^n(1-Q)^2+Q(1-Q^n)-(2-Q)^n(1-Q)}{Q^2}\right), & \text{if } Q \geq Q^*_n. \end{array} \right\}$$

$$= \left\{ \begin{array}{ll} \frac{1}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{Q}\right), & \text{if } \frac{1-Q}{Q}D' > \frac{1}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{Q}\right) \\ \frac{1-Q}{Q}D', & \text{if } \frac{1-Q}{Q}D' \leq \frac{1}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{Q}\right). \end{array} \right\}$$

$$= \min\left(\frac{1}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{Q}\right), \frac{1-Q}{Q}D'\right)$$

$$= \left(\frac{1-Q}{Q}\right)\left(D' - \frac{1}{n2^{n-1}}\frac{\max\left(2^n(1-Q)+Q(1-Q)^{n-1}-(2-Q)^n, 0\right)}{Q}\right).$$

Therefore the value of the objective function is

$$\Pi^{\mathsf{B}}(p,r) = r - \frac{2r-1}{2^n}\left(\frac{1}{Q(1-Q)}\right)\left(p(1-p) + \left(\frac{r(1-r)(2p-1)^2}{Q}\right)\max\left(Q(1-Q)^{n-1}, (2-Q)^n - 2^n(1-Q)\right)\right). \tag{44}$$

Since $Q(1-Q) = p(1-p) + r(1-r)(2p-1)^2$, this can be further rewritten as

$$\Pi^{\mathsf{B}}(p,r) = r - \frac{2r-1}{2^n}\left(1 - \left(\frac{r(1-r)(2p-1)^2}{Q(1-Q)}\right)\min\left(1-(1-Q)^{n-1}, \frac{2^n(1-Q)-(2-Q)^n+Q}{Q}\right)\right)$$

□

B.2 Proof of Proposition 2

We first provide formulas for the objective function and the BIC constraints. Adopting the same notations as in Section 4, we can write the objective function of the social planner as

$$\Pi = \frac{1}{2}\left(\frac{1}{2} + (1-q)(1-q(2r-1)) + a_1\pi_1 + a_2\pi_2 + a_3\pi_3 + a_4\pi_4 + a_5\pi_5 + a_6\pi_6\right).$$

We stress that in the case of peer evaluation, x_0 denotes the signal received by agent 1 about the type of agent 0 and x_1 denotes the signal received by agent 0 about the type of agent 1. So (x_0, y_0) is the pair of reports on agent 0 and (x_1, y_1) is the pair of reports on agent 1. Therefore the formula for the objective function remains the same as in the benchmark model. As in the benchmark model, the optimal DSIC mechanism selects the agent with the highest report of the external referee when the two agents receive different reports and each agent with equal probability when the reports are identical,

$$\pi_1 = \pi_3 = \pi_6 = 1, \pi_4 = 0, \pi_2 = \pi_5 = \frac{1}{2}.$$

Let $Q = qr + (1-q)(1-r)$ denote, as in the benchmark model, the probability that the two signals X_i and Y_i are equal. Furthermore, define

$$\begin{aligned} R_1 &= pqr + (1-p)(1-q)(1-r), \\ R_2 &= p(1-q)r + (1-p)q(1-r), \\ R_3 &= pq(1-r) + (1-p)(1-q)r, \\ R_4 &= p(1-q)(1-r) + (1-p)qr \end{aligned}$$

With these notations in hand, the four BIC constraints are given by **Type** $(V_i, X_{i\oplus 1}) = (0,1)$

$$\begin{aligned} &-(R_1Q + R_4(1-Q))\pi_1 - (R_2+R_1)(1-Q)\pi_2 + (R_1Q - R_3(1-Q))\pi_3 \\ &-(R_2Q + R_3(1-Q))\pi_4 - (R_3+R_4)Q\pi_5 - (R_2Q - R_4(1-Q))\pi_6 \\ &+\frac{1}{2}[(R_1+R_2)(1-Q) + (R_3+R_4)Q + 2R_2Q + 2R_3(1-Q)] \geq 0. \end{aligned} \tag{45}$$

Type $(V_i, X_{i\oplus 1}) = (0,0)$

$$\begin{aligned} &(R_1(1-Q) + R_4Q)\pi_1 + (R_2+R_1)Q\pi_2 - (R_1(1-Q) - R_3Q)\pi_3 \\ &+(R_2(1-Q) + R_3Q)\pi_4 + (R_3+R_4)(1-Q)\pi_5 + (R_2(1-Q) - R_4Q)\pi_6 \\ &-\frac{1}{2}[(R_1+R_2)Q + (R_3+R_4)(1-Q) + 2R_2(1-Q) + 2R_3Q] \geq 0. \end{aligned} \tag{46}$$

Type $(V_i, X_{i\oplus 1}) = (1,1)$

$$\begin{aligned} &-(R_1(1-Q) + R_4Q)\pi_1 - (R_3+R_4)(1-Q)\pi_2 - (R_2(1-Q) - R_4Q)\pi_3 \\ &-((R_2(1-Q) + R_3Q))\pi_4 - (R_1+R_2)Q\pi_5 + (R_1(1-Q) - R_3Q)\pi_6 \\ &+\frac{1}{2}[(R_1+R_2)Q + (R_3+R_4)(1-Q) + 2R_2(1-Q) + 2R_3Q] \geq 0. \end{aligned} \tag{47}$$

Type $(V_i, X_{i\oplus 1}) = (1,0)$

$$\begin{aligned} &(R_1Q + R_4(1-Q))\pi_1 + (R_3+R_4)Q\pi_2 + (R_2Q - R_4(1-Q))\pi_3 \\ &+(R_2Q + R_3(1-Q))\pi_4 + (R_1+R_2)(1-Q)\pi_5 - (R_1Q - R_3(1-Q))\pi_6 \\ &-\frac{1}{2}[(R_1+R_2)(1-Q) + (R_3+R_4)Q + 2R_2Q + 2R_3(1-Q)] \geq 0. \end{aligned} \tag{48}$$

We consider the relaxed problem where the BIC constraints (46) and (48) are ignored and rewrite the BIC constraints (45) and (47) as

$$\begin{aligned} b_1\pi_1 + b_2\pi_2 + b_3\pi_3 + b_4\pi_4 + b_5\pi_5 + b_6\pi_6 + B &\geq 0, \\ c_1\pi_1 + c_2\pi_2 + c_3\pi_3 + c_4\pi_4 + c_5\pi_5 + c_6\pi_6 + C &\geq 0. \end{aligned}$$

The problem is then a linear relaxation of a knapsack problem with two constraints, and we define the efficiency indices $\beta_i = -a_i/b_i$ and $\gamma_i = -a_i/c_i$ We easily check that $b_i \leq 0$ for $i \in \{1,2,4,5,6\}$, $b_3 \geq 0$, $c_i \leq 0$ for $i \in \{1,2,4,5\}$ and $c_6 \geq 0$. The coefficient c_3 is either positive or negative depending on the sign of $p(1-q)^2 - q^2(1-p)$. The following claim ranks the values β_1, β_2, β_4, β_5 and β_6 and γ_1, $gamma_2$, γ_3, γ_4 and γ_5.

Claim 1. *We have* $\beta_6 > \beta_5 > \beta_2 > \beta_4$*,* $\beta_1 > \beta_5$*,* $\gamma_1 > \gamma_2 > \gamma_5$*,* $\gamma_2 > \gamma_4$*,* $\gamma_3 > \gamma_2$*.*

Proof. We first note that $\beta_6 = (1-q)/(p-q) \geq 1$ and $a_5 = q^2 r(1-r) - (1-q)^2 r(1-r) \leq pqr(1-r) < (p(1-r)+r(1-p))(qr+(1-q)(1-r)) = b_5$, $\beta_5 < 1 < \beta_6$.

Next, to prove $\beta_5 > \beta_2$, because $a_2 = a_5$, it is sufficient to prove $b_2 > b_5$ or

$$pr + (1-p)(1-r) > p(1-r) + r(1-p), \tag{49}$$

an inequality which is always verified.

To prove $\beta_2 > \beta_4$, as $a_2 = a_4 r/(1-r)$, it is enough to show

$$r\left(R_2(1-Q) + R_3 Q\right) - (1-r)Q\left(R_1 + R_2\right) > 0. \tag{50}$$

Now, observe that $r(1-Q) - (1-r)Q = (1-q)(2r-1) > 0$ so that $rR_2(1-Q) > (1-r)R_2 Q$. In addition $rR_3 - (1-r)R_1 = (1-p)(1-q)(2r-1) \geq 0$ so that $rR_3 Q - (1-r)R_1 Q \geq 0$, establishing inequality (50).

Next, note that

$$\beta_1 = \frac{a_1}{b_1} = \frac{q^2 r^2 - (1-q)^2(1-r)^2}{R_1 Q + R_4(1-Q)} > \frac{q^2 r^2 - (1-q)^2(1-r)^2}{R_1 Q + R_4 Q}$$

while

$$\beta_5 = \frac{q^2 r(1-r) - (1-q)^2 r(1-r)}{R_3 Q + R_4 Q}.$$

In order to prove $\beta_1 > \beta_5$, it is thus sufficient to show

$$(q^2 r^2 - (1-q)^2(1-r)^2)(R_3 + R_4) > (q^2 r(1-r) - (1-q)^2 r(1-r))(R_1 + R_4). \tag{51}$$

Now,

$$\begin{aligned}\left(q^2 r^2 - (1-q)^2(1-r)^2\right) R_3 - \left(q^2 r(1-r) - (1-q)^2 r(1-r)\right) R_1 \\ = (1-p)(1-q)r(2r-1)(2q-1) > 0\end{aligned}$$

and as $q^2 r^2 - (1-q)^2(1-r)^2 > q^2 r(1-r) - (1-q)^2 r(1-r)$,

$$\left(q^2 r^2 - (1-q)^2(1-r)^2\right) R_4 - \left(q^2 r(1-r) - (1-q)^2 r(1-r)\right) R_4 > 0,$$

establishing equality (51).

Next, to prove $\gamma_1 > \gamma_2$, note that

$$\gamma_1 = \frac{a_1}{R_1(1-Q) + R_4 Q} > \frac{a_1}{R_1 Q + R_4 Q} \quad \text{and} \quad \gamma_2 = \frac{a_2}{R_1 Q + R_2 Q}.$$

Now $a_1 > a_2$ and $R_2 - R_4 = (2r-1)(p-q) \geq 0$, establishing the result.

To prove $\gamma_2 > \gamma_5$, recall that $a_2 = a_5$ and note that, as $Q > 1-Q$, $c_5 > b_5 > b_2 > c_2$.

Next, to prove $\gamma_2 > \gamma_4$, it is sufficient to note that $c_2 < b_2$ and $c_4 = R_3(1-Q) + R_2(1-Q) < R_2 Q + R_2(1-Q) = b_4$ as $Q > 1/2$ and $R_3 > R_2$.

To prove $\gamma_3 > \gamma_2$ when $c_3 \leq 0$, note that

$$\gamma_3 = \frac{q(1-q)}{p(1-q)^2 - (1-p)q^2} \text{ while } \gamma_2 = \frac{(2q-1)r(1-r)}{[q(1-r)+r(1-q)][p(1-r)+r(1-p)]}.$$

A sufficient condition for $\gamma_3 > \gamma_2$ is:

$$q(1-q)[q(1-p)+p(1-q)]r(1-r) \geq (2q-1)r(1-r)[p(1-q)^2-(1-p)q^2]. \quad (52)$$

Note that inequality (52) is always satisfied as $qp(1-q)^2 > (2q-1)p(1-q)^2$.

If $c_3 > 0$, $\gamma_3 = +\infty > \gamma_2$. □

Proof of Proposition 2. We use Claim 1 to compute the solution of the knapsack problem. We note that π_1, π_3, π_6 must be larger than π_2 and π_5, and that $\pi_2 \geq \pi_4$. Either the three probabilities π_1, π_3 and π_6 are equal to one, or the three probabilities π_2, π_4, and π_5 are equal to zero. It is easy to check that when $\pi_2 = \pi_4 = \pi_5 = 0$, the two BIC constraints (45) and (47) are slack, so that an increase in any of the probabilities π_1, π_3 or π_6 would result in an increase in welfare. Hence one of the three probabilities π_2, π_4, π_5 has to be different from 0, and we conclude that at the optimum $\pi_1 = \pi_3 = \pi_6 = 1$. In addition, we know that either $\pi_4 = 0$ or $\pi_2 = 1$. However, if $\pi_1 = \pi_2 = \pi_3 = \pi_6 = 1$, the BIC constraint (45) cannot be satisfied, so that $\pi_2 < 1$ and hence $\pi_4 = 0$. But now, given that $\pi_1 = \pi_3 = \pi_6 = 1$ and $\pi_4 = 0$, the two BIC constraints (45) and (47) can only be satisfied when $\pi_2 = \pi_5 = 1/2$, so that the optimal BIC mechanism is equal to the optimal DSIC mechanism. □

B.3 Proof of Proposition 3

In this section we prove Proposition 3 that compares optimal BIC mechanisms in self-evaluation and peer-evaluation scenarios. To this end, we first obtain a full characterization of optimal BIC mechanisms for the artificial peer evaluation scenario, where agents do not receive any signal about their own type, but only a signal on the quality of their competitor. Then we use this characterization to compare planner's payoffs in the two scenarios.

As in the case of the benchmark model, we consider the representation of the anonymous mechanisms based on the function $\varrho : \{0,1\}^2 \times S_4(n-1) \to [0,1]$, which determines probability of the project being selected based on the scores it *received* and on the numbers of possible scores among the remaining projects. We stress that in the case of peer evaluation, x_0 denotes the signal received by agent $n-1$ about the type of agent 0 so (x_0, y_0) is the pair of reports on agent 0. Therefore the formula for the objective function remains the same as in the benchmark model.

The formulation of the objective function (8) and the probability constraints (11) remains like in the benchmark model. There are $4n$ BIC constraints, one for every quadruple (i, a, b, b') where $i \in \{0, \ldots, n-1\}$, $a \in \{0,1\}$, and $b \neq b'$. Since the BIC constraints are independent across agents and given the form of the objective function, we can restrict attention to the BIC constraints for agent 0. There are 4 such BIC constraints. Given (v_0, x_1, x_1') with $v_0 \in \{0,1\}$ and $x_1 \neq x_1'$ the corresponding BIC constraint is

$$\mathbf{E}(\pi_0(x_1, \boldsymbol{X}_{-1}, \boldsymbol{Y}) \mid X_1 = x_1, v_0 = a) - \mathbf{E}(\pi_0(x_1', \boldsymbol{X}_{-1}, \boldsymbol{Y}) \mid X_1 = x_1, v_0 = a) \geq 0.$$

The LHS of the constraint can be rewritten as

$$\sum_{\boldsymbol{t}\in\{0,1\}^N} \sum_{\boldsymbol{x}_{-1}\in\{0,1\}^{N\setminus\{1\}}} \sum_{\boldsymbol{y}\in\{0,1\}^N} \mathbf{Pr}(\boldsymbol{\Theta} = \boldsymbol{t}, \boldsymbol{X}_{-1} = \boldsymbol{x}_{-1}, \boldsymbol{Y} = \boldsymbol{y} \mid X_1 = x_1, v_0 = a)(\pi_0(x_1, \boldsymbol{x}_{-1}, \boldsymbol{y}) - \pi_0(x_1', \boldsymbol{x}_{-1}, \boldsymbol{y})) =$$

$$\sum_{\boldsymbol{x}_{-1}\in\{0,1\}^{N\setminus\{1\}}} \sum_{\boldsymbol{y}\in\{0,1\}^N} \left(\sum_{\boldsymbol{t}\in\{0,1\}^N} \mathbf{Pr}(\boldsymbol{\Theta} = \boldsymbol{t}, \boldsymbol{X}_{-1} = \boldsymbol{x}_{-1}, \boldsymbol{Y} = \boldsymbol{y} \mid X_1 = x_1, v_0 = a) \right) (\pi_0(x_1, \boldsymbol{x}_{-1}, \boldsymbol{y}) - \pi_0(x_1', \boldsymbol{x}_{-1}, \boldsymbol{y})),$$

where

$$\begin{aligned}
&\mathbf{Pr}(\boldsymbol{\Theta} = \boldsymbol{t}, \boldsymbol{X}_{-1} = \boldsymbol{x}_{-1}, \boldsymbol{Y} = \boldsymbol{y} \mid X_1 = x_1, v_0 = a)\\
&= \mathbf{Pr}(\Theta_0 = t_0, X_0 = x_0, Y_0 = y_0 \mid v_0 = a)\mathbf{Pr}(\Theta_1 = t_1, Y_1 = y_1 \mid X_1 = x_1)\mathbf{Pr}(\boldsymbol{\Theta}_{-0,1} = \boldsymbol{t}_{-0,1}, \boldsymbol{X}_{-0,1} = \boldsymbol{x}_{-0,1}, \boldsymbol{Y}_{-0,1} = \boldsymbol{y}_{-0,1})\\
&= \frac{\mathbf{Pr}(\Theta_0 = t_0, X_0 = x_0, Y_0 = y_0, v_0 = a)}{\mathbf{Pr}(v_0 = a)} \frac{\mathbf{Pr}(\Theta_1 = t_1, X_1 = x_1, Y_1 = y_1)}{\mathbf{Pr}(X_1 = x_1)} \mathbf{Pr}(\boldsymbol{\Theta}_{-0,1} = \boldsymbol{t}_{-0,1}, \boldsymbol{X}_{-0,1} = \boldsymbol{x}_{-0,1}, \boldsymbol{Y}_{-0,1} = \boldsymbol{y}_{-0,1}) =\\
&= \frac{\mathbf{Pr}(V_0 = a, X_0 = x_0, Y_0 = y_0 \mid \Theta_0 = t_0)\mathbf{Pr}(\Theta_0 = t_0)}{\mathbf{Pr}(v_0 = a)} \frac{1}{\mathbf{Pr}(X_1 = x_1)} \mathbf{Pr}(\boldsymbol{\Theta} = \boldsymbol{t}_{-0}, \boldsymbol{X}_{-0} = \boldsymbol{x}_{-0}, \boldsymbol{Y}_{-0} = \boldsymbol{y}_{-0})\\
&= 2\mathbf{Pr}(v_0 = a \mid \Theta_0 = t_0)\mathbf{Pr}(X_0 = x_0, Y_0 = y_0 \mid \Theta_0 = t_0)\mathbf{Pr}(\boldsymbol{\Theta} = \boldsymbol{t}_{-0}, \boldsymbol{X}_{-0} = \boldsymbol{x}_{-0}, \boldsymbol{Y}_{-0} = \boldsymbol{y}_{-0})\\
&= 2\mathbf{Pr}(v_0 = a \mid \Theta_0 = t_0) \frac{\mathbf{Pr}(X_0 = x_0, Y_0 = y_0, \Theta_0 = t_0)}{\mathbf{Pr}(\Theta_0 = t_0)} \mathbf{Pr}(\boldsymbol{\Theta} = \boldsymbol{t}_{-0}, \boldsymbol{X}_{-0} = \boldsymbol{x}_{-0}, \boldsymbol{Y}_{-0} = \boldsymbol{y}_{-0})\\
&= 4\mathbf{Pr}(v_0 = a \mid \Theta_0 = t_0)\mathbf{Pr}(\boldsymbol{\Theta} = \boldsymbol{t}, \boldsymbol{X} = \boldsymbol{x}, \boldsymbol{Y} = \boldsymbol{y})\\
&= 4\mathbf{Pr}(v_0 = a \mid \Theta_0 = t_0) \prod_{j\in N} \mathbf{Pr}(\Theta_j = t_j, X_j = x_j, Y_j = y_j)\\
&= 4\mathbf{Pr}(v_0 = a \mid \Theta_0 = t_0) \prod_{j\in N} \mathbf{Pr}(X_j = x_j, Y_j = y_j \mid \Theta_j = t_j)\mathbf{Pr}(\Theta_j = t_j)\\
&= 4\mathbf{Pr}(v_0 = a \mid \Theta_0 = t_0) \prod_{j\in N} \frac{\mathbf{Pr}(X_j = x_j, Y_j = y_j \mid \Theta_j = t_j)}{2},
\end{aligned}$$

as

$$\begin{aligned}
\mathbf{Pr}(\Theta_0 = t_0) &= \frac{1}{2},\\
\mathbf{Pr}(v_0 = a) &= \mathbf{Pr}(v_0 = a, \Theta_1 = 0) + \mathbf{Pr}(v_0 = a, \Theta_1 = 1)\\
&= \mathbf{Pr}(v_0 = a \mid \Theta_1 = 1)\mathbf{Pr}(\Theta_1 = 1) + \mathbf{Pr}(v_0 = a \mid \Theta_1 = 1)\mathbf{Pr}(\Theta_1 = 1)\\
&= \frac{1}{2}(p + 1 - p) = \frac{1}{2},\\
\mathbf{Pr}(X_1 = x_1) &= \mathbf{Pr}(X_1 = x_1, \Theta_1 = 0) + \mathbf{Pr}(X_1 = x_1, \Theta_1 = 1)\\
&= \mathbf{Pr}(X_1 = x_1 \mid \Theta_1 = 1)\mathbf{Pr}(\Theta_1 = 1) + \mathbf{Pr}(X_1 = x_1 \mid \Theta_1 = 1)\mathbf{Pr}(\Theta_1 = 1)\\
&= \frac{1}{2}(p + 1 - p) = \frac{1}{2}.
\end{aligned}$$

Let

$$\eta^t(a) = \mathbf{Pr}(v_0 = a \mid \Theta_0 = t) = [a = t]p + [a \neq t](1 - p).$$

Then

$$\sum_{\boldsymbol{t}\in\{0,1\}^N} \mathbf{Pr}(\boldsymbol{\Theta}=\boldsymbol{t}, \boldsymbol{X}_{-1}=\boldsymbol{x}_{-1}, \boldsymbol{Y}=\boldsymbol{y} \mid X_1=x_1, v_0=a) =$$

$$4\sum_{\boldsymbol{t}\in\{0,1\}^N} \mathbf{Pr}(v_0=a \mid \Theta_0=t_0)\prod_{j\in N}\frac{\mathbf{Pr}(X_j=x_j, Y_j=y_j \mid \Theta_j=t_j)}{2} =$$

$$2\left(\mathbf{Pr}(v_0=a \mid \Theta_0=0)\mathbf{Pr}(X_j=x_j,Y_j=y_j \mid \Theta_0=0)+\mathbf{Pr}(v_0=a \mid \Theta_0=1)\mathbf{Pr}(X_j=x_j,Y_j=y_j \mid \Theta_0=1)\right)$$
$$\prod_{j\in N\setminus\{0\}}\frac{\mathbf{Pr}(X_j=x_j,Y_j=y_j \mid \Theta_j=0)+\mathbf{Pr}(X_j=x_j,Y_j=y_j \mid \Theta_j=1)}{2} =$$

$$2\left(\eta^0(a)\xi_0^0(\boldsymbol{x},\boldsymbol{y})+\eta^1(a)\xi_0^1(\boldsymbol{x},\boldsymbol{y})\right)T_0(\boldsymbol{x},\boldsymbol{y}),$$

and the BIC constraints can be rewritten as

$$\sum_{\boldsymbol{x}_{-1}\in\{0,1\}^{N\setminus\{1\}}}\sum_{\boldsymbol{y}\in\{0,1\}^N}\left(\eta^0(a)\xi_0^0(\boldsymbol{x},\boldsymbol{y})+\eta^1(a)\xi_0^1(\boldsymbol{x},\boldsymbol{y})\right)T_0(\boldsymbol{x},\boldsymbol{y})\left(\pi_0(\boldsymbol{x},\boldsymbol{y})-\pi_0(x_1',\boldsymbol{x}_{-1},\boldsymbol{y})\right)\geq 0.$$

Notice that for any $\boldsymbol{x}_{-0}\in\{0,1\}^{N\setminus\{0\}}$ and any $\boldsymbol{y}_{-0}\in\{0,1\}^{N\setminus\{0\}}$, $|\{i\in N\setminus\{0\}: x_i = x_1 \text{ and } y_i = y_1\}|\geq 1$. Moreover, for any $y_1\in\{0,1\}$ and any $\boldsymbol{z}\in S_4(n-1)$ with $z_{x_1,y_1}>0$ there are

$$\binom{n-2}{\boldsymbol{z}_{-(x_1,y_1)}, z_{x_1y_1}-1} = \left(\frac{z_{x_1y_1}}{n-1}\right)\binom{n-1}{z_{0,0},z_{0,1},z_{1,0},z_{1,1}}$$

vectors $(\boldsymbol{x}_{-0},\boldsymbol{y}_{-0})$ such that for all $(b,b')\in\{0,1\}^2$, $|\{i\in N\setminus\{0\}: x_i=b \text{ and } y_i = b'\}| = z_{b,b'}$. Therefore we can rewrite the BIC constraints as

$$\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{y_1\in\{0,1\}}\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{x_1y_1}>0}} z_{x_1y_1}U(\boldsymbol{z})\alpha(a,x_0,y_0)\Big(\varrho(x_0,y_0,\boldsymbol{z})$$
$$-\varrho(x_0,y_0,\boldsymbol{z}_{-(x_1,y_1),(x_1',y_1)},z_{x_1,y_1}-1,z_{x_1',y_1}+1)\Big)\geq 0,$$

where

$$\alpha(a,x_0,y_0)=\eta^0(a)\zeta^0(x_0,y_0)+\eta^1(a)\zeta^1(x_0,y_0).$$

The LHS of the constraint can be further rewritten as

$$\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{y_1\in\{0,1\}}\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{x_1y_1}>0}} z_{x_1y_1}U(\boldsymbol{z})\alpha(a,x_0,y_0)\varrho(x_0,y_0,\boldsymbol{z})$$
$$-\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{y_1\in\{0,1\}}\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{x_1y_1}>0}} z_{x_1y_1}U(\boldsymbol{z})\alpha(a,x_0,y_0)\varrho(x_0,y_0,\boldsymbol{z}_{-(x_1,y_1),(x_1',y_1)},z_{x_1,y_1}-1,z_{x_1',y_1}+1)$$
$$=\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{y_1\in\{0,1\}}\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{x_1y_1}>0}} z_{x_1y_1}U(\boldsymbol{z})\alpha(a,x_0,y_0)\varrho(x_0,y_0,\boldsymbol{z})$$
$$-\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{y_1\in\{0,1\}}\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{x_1'y_1}>0}} (z_{x_1y_1}+1)U(\boldsymbol{z}_{-(x_1,y_1),(x_1',y_1)},z_{x_1,y_1}+1,z_{x_1',y_1}-1)\alpha(a,x_0,y_0)\varrho(x_0,y_0,\boldsymbol{z}).$$

Since

$$U(\boldsymbol{z}_{-(x_1,y_1),(x_1',y_1)},z_{x_1,y_1}+1,z_{x_1',y_1}-1)=\frac{z_{x_1'y_1}}{z_{x_1y_1}+1}\frac{[x_1=y_1]Q+[x_1\neq y_1](1-Q)}{[x_1\neq y_1]Q+[x_1=y_1](1-Q)}U(\boldsymbol{z})$$

so this can be further rewritten as

$$\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{y_1\in\{0,1\}}\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{x_1y_1}>0}} z_{x_1y_1}U(\boldsymbol{z})\alpha(a,x_0,y_0)\varrho(x_0,y_0,\boldsymbol{z})$$
$$-\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{y_1\in\{0,1\}}\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{x_1'y_1}>0}} z_{x_1'y_1}\frac{[x_1=y_1]Q+[x_1\neq y_1](1-Q)}{[x_1\neq y_1]Q+[x_1=y_1](1-Q)}U(\boldsymbol{z})\alpha(a,x_0,y_0)\varrho(x_0,y_0,\boldsymbol{z})$$
$$=\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{y_1\in\{0,1\}}\sum_{\boldsymbol{z}\in S_4(n-1)} z_{x_1y_1}U(\boldsymbol{z})\alpha(a,x_0,y_0)\varrho(x_0,y_0,\boldsymbol{z})$$
$$-\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{y_1\in\{0,1\}}\sum_{\boldsymbol{z}\in S_4(n-1)} z_{x_1'y_1}\frac{[x_1=y_1]Q+[x_1\neq y_1](1-Q)}{[x_1\neq y_1]Q+[x_1=y_1](1-Q)}U(\boldsymbol{z})\alpha(a,x_0,y_0)\varrho(x_0,y_0,\boldsymbol{z})$$
$$=\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{y_1\in\{0,1\}}\sum_{\boldsymbol{z}\in S_4(n-1)}\left(z_{x_1y_1}-z_{x_1'y_1}\frac{[x_1=y_1]Q+[x_1\neq y_1](1-Q)}{[x_1\neq y_1]Q+[x_1=y_1](1-Q)}\right)U(\boldsymbol{z})\alpha(a,x_0,y_0)\varrho(x_0,y_0,\boldsymbol{z})$$
$$=\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{\boldsymbol{z}\in S_4(n-1)}U(\boldsymbol{z})\alpha(a,x_0,y_0)\varrho(x_0,y_0,\boldsymbol{z})\sum_{y_1\in\{0,1\}}\left(z_{x_1y_1}-z_{x_1'y_1}\frac{[x_1=y_1]Q+[x_1\neq y_1](1-Q)}{[x_1\neq y_1]Q+[x_1=y_1](1-Q)}\right)$$
$$=\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{\boldsymbol{z}\in S_4(n-1)}U(\boldsymbol{z})\alpha(a,x_0,y_0)\varrho(x_0,y_0,\boldsymbol{z})\sum_{y_1\in\{0,1\}}\left(z_{x_1y_1}+z_{x_1'y_1}-\frac{z_{x_1'y_1}}{[x_1\neq y_1]Q+[x_1=y_1](1-Q)}\right)$$
$$=\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{\boldsymbol{z}\in S_4(n-1)}U(\boldsymbol{z})\alpha(a,x_0,y_0)\varrho(x_0,y_0,\boldsymbol{z})\left(n-1-\sum_{y_1\in\{0,1\}}\frac{z_{x_1'y_1}}{[x_1\neq y_1]Q+[x_1=y_1](1-Q)}\right)$$
$$=\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{\boldsymbol{z}\in S_4(n-1)}U(\boldsymbol{z})\frac{\alpha(a,x_0,y_0)}{Q(1-Q)}\varrho(x_0,y_0,\boldsymbol{z})$$
$$\left((n-1)Q(1-Q)-\sum_{y_1\in\{0,1\}}z_{x_1'y_1}\left([x_1=y_1]Q+[x_1\neq y_1](1-Q)\right)\right).$$

Thus, multiplying both sides by $Q(1-Q)$, the BIC constraint can be rewritten as

$$\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{\boldsymbol{z}\in S_4(n-1)}U(\boldsymbol{z})\alpha(a,x_0,y_0)\varrho(x_0,y_0,\boldsymbol{z})$$
$$\left((n-1)Q(1-Q)-\sum_{y_1\in\{0,1\}}z_{x_1'y_1}\left([x_1=y_1]Q+[x_1\neq y_1](1-Q)\right)\right)\geq 0$$

and, further, as

$$\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{\boldsymbol{z}\in S_4(n-1)}U(\boldsymbol{z})\alpha(a,x_0,y_0)\beta(x_1,x_1',\boldsymbol{z})\varrho(x_0,y_0,\boldsymbol{z})\geq 0, \tag{53}$$

where

$$\beta(x_1,x_1',\boldsymbol{z})=\left((n-1)Q(1-Q)-\sum_{y_1\in\{0,1\}}z_{x_1'y_1}\left([x_1=y_1]Q+[x_1\neq y_1](1-Q)\right)\right). \tag{54}$$

Adding pairs of BIC constraints with the same values of (x_0,x_0') and different values of a we obtain a relaxed optimization problem with two constraints, one for each

$(x_0, x'_0) \in \{0,1\}^2$ with $x_0 \neq x'_0$,

$$\sum_{(x_0,y_0)\in\{0,1\}^2} \sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\zeta(x_0,y_0)\beta(x_1,x'_1,\boldsymbol{z})\varrho(x_0,y_0,\boldsymbol{z}) \geq 0.$$

We solve the relaxed optimization problem, following similar steps to those used to find optimal BIC mechanisms in the case of the benchmark model. The further analysis is divided in three parts. First we consider a further relaxed optimization problem where the BIC constraint for $x_1 = 0$ and $x'_1 = 1$ is dropped and we fully characterize optimal mechanisms for this problem. Next we show that the optimal mechanisms satisfy the BIC constraint for $x_1 = 0$ and $x'_1 = 1$. Hence we obtain a full characterization of the optimal mechanisms for the unrelaxed optimization problem.

Lemma 8. *For any optimal mechanism $(\varrho(x_0,y_0,\boldsymbol{z}))_{(x_0,y_0)\in\{0,1\}^2,\boldsymbol{z}\in S_4(n-1)}$ under the relaxed optimization problem without the BIC constraint for $x_1 = 0$ and $x'_1 = 1$, any $x_0 \in \{0,1\}$, and any $\boldsymbol{z} \in S_4(n-1)$, if $z_{x_0,1} > 0$ then $\varrho(x_0,0,\boldsymbol{z}) = 0$.*

Proof. Let $(\varrho(x_0,y_0,\boldsymbol{z}))_{(x_0,y_0)\in\{0,1\}^2,\boldsymbol{z}\in S_4(n-1)}$ be an optimal mechanism of the relaxed optimization problem. Assume, to the contrary, that there exists $x_0 \in \{0,1\}$ and $\boldsymbol{z} \in S_4(n-1)$ with $z_{x_0,1} > 0$ such that $\varrho(x_0,0,\boldsymbol{z}) > 0$.

Consider a modification to the mechanism where $\varrho(x_0,0,\boldsymbol{z})$ is decreased by $\varepsilon/(z_{x_0,0}+1)$ and $\varrho(x_0,1,z_{-(x_0,0),(x_0,1)}z_{x_0,0}+1, z_{x_0,1}-1)$ is increased by $\varepsilon/z_{x_0,1}$, where $\varepsilon \in (0,(z_{x_0,0}+1)\varrho(x_0,0,\boldsymbol{z}))$. This modification maintains the probability constraint (11). Moreover, for any $A \in \mathbb{R}$ and $B \in \mathbb{R}$, the change in the sum

$$U(\boldsymbol{z})A\varrho(x_0,0,\boldsymbol{z}) + U(\boldsymbol{z}_{-(x_0,0),(x_0,1)}, z_{x_0,0}+1, z_{x_0,1}-1)B\varrho(x_0,1,\boldsymbol{z}_{-(x_0,0),(x_0,1)}, z_{x_0,0}+1, z_{x_0,1}-1)$$

resulting from this modification is equal to, in the case of $x_0 = 0$,

$$-U(\boldsymbol{z})\frac{\varepsilon}{z_{0,0}+1}A + U(\boldsymbol{z}_{-(0,0),(0,1)}, z_{0,0}+1, z_{0,1}-1)\frac{\varepsilon}{z_{0,1}}B =$$
$$\frac{1}{n2^{n-1}}\binom{n}{\boldsymbol{z}_{-(0,0),(0,1)}, z_{0,0}+1, z_{0,1}} Q^{z_{0,0}+z_{1,1}+1}(1-Q)^{z_{0,1}+z_{1,0}}\varepsilon\left(\frac{B}{1-Q} - \frac{A}{Q}\right) = \left(\frac{BQ - A(1-Q)}{Q(1-Q)}\right)$$

and, in the case of $x_0 = 1$,

$$-U(\boldsymbol{z})\frac{\varepsilon}{z_{1,0}+1}A + U(\boldsymbol{z}_{-(1,0),(1,1)}, z_{1,0}+1, z_{1,1}-1)\frac{\varepsilon}{z_{1,1}}B =$$
$$\frac{1}{n}\binom{n}{\boldsymbol{z}_{-(1,0),(1,1)}, z_{1,0}+1, z_{1,1}} Q^{z_{0,0}+z_{1,1}}(1-Q)^{z_{0,1}+z_{1,0}+1}\varepsilon\left(\frac{B}{Q} - \frac{A}{1-Q}\right) = \left(\frac{B(1-Q) - AQ}{Q(1-Q)}\right).$$

Thus the sign of the change is the same as the sign of $BQ - A(1-Q)$, in the case of $x_0 = 0$, and as the sign of $B(1-Q) - AQ$, in the case of $x_0 = 1$.

Consider the change to the value of the objective function resulting from the change to the mechanism. In the case of $x_0 = 0$ we have $A = \zeta^1(0,0) = (1-q)(1-r)$, $B = \zeta^1(0,1) = (1-q)r$, and

$$BQ - A(1-Q) = q(2r-1)(1-q) > 0,$$

as $1/2 < r \leq q < 1$. In the case of $x_0 = 1$ we have $A = \zeta^1(1,0) = q(1-r)$, $B = \zeta^1(1,1) = qr$, and again

$$B(1-Q) - AQ = q(2r-1)(1-q) > 0.$$

Thus the change results in an increase in the value of the objective function.

Consider now the BIC constraints and the change to the value of the LHS of a constraint resulting from the change to the mechanism. Notice that, by (54), for any $x_1, x_1' \in \{0,1\}$ such that $x_1 \neq x_1'$,

$$\begin{aligned}\beta(x_1, x_1', \boldsymbol{z}_{-(0,0),(0,1)}, z_{0,0}+1, z_{0,1}-1) &= \beta(x_1, x_1', \boldsymbol{z}) + [x_1' = 0](2Q-1),\\ \beta(x_1, x_1', \boldsymbol{z}_{-(1,0),(1,1)}, z_{1,0}+1, z_{1,1}-1) &= \beta(x_1, x_1', \boldsymbol{z}) - [x_1' = 1](2Q-1).\end{aligned} \tag{55}$$

Suppose that $x_0 = 0$. Since $\zeta(0,0) = Q$ so

$$A(1-Q) = Q(1-Q)\beta(x_1, x_1', \boldsymbol{z})$$

and, since $\zeta(0,1) = 1-Q$ so, using (55),

$$BQ = Q(1-Q)\beta(\boldsymbol{z}_{-(0,0),(0,1)}, z_{0,0}+1, z_{0,1}-1) = Q(1-Q)(\beta(x_1, x_1', \boldsymbol{z}) + [x_1' = 0](2Q-1)).$$

Thus

$$BQ - A(1-Q) = [x_1' = 0](2Q-1) \geq 0,$$

as $Q > 1/2$. Thus the LHS of the BIC constraint weakly increases and the constraint remains satisfied. Suppose that $x_0 = 1$. Since $\zeta(1,0) = 1-Q$ so

$$AQ = Q(1-Q)\beta(x_1, x_1', \boldsymbol{z})$$

and, since $\zeta(1,1) = Q$ so, using (55),

$$B(1-Q) = Q(1-Q)\beta(\boldsymbol{z}_{-(1,0),(1,1)}, z_{1,0}+1, z_{1,1}-1) = Q(1-Q)\left(\beta(x_1, x_1', \boldsymbol{z}) - [x_1' = 1](2Q-1)\right).$$

Thus

$$B(1-Q) - AQ = -[x_1' = 1](2Q-1),$$

which, in the case of $x_1' = 0$, is equal to 0. Hence in this case the LHS of the BIC constraint remains unchanged.

This shows that changing the mechanism increases the value of the objective function and maintains the BIC constraint. A contradiction with the assumption of optimality of the mechanism. Thus the lemma is satisfied. □

By Lemma (8), for any $x_0 \in \{0,1\}$, either $\varrho(x_0, 0, \boldsymbol{z}) = 0$ or $z_{x_0,1} = 0$. Therefore, depending on the value of x_0, y_0, and $\boldsymbol{z}$, the probability constraint (11) can be rewritten as follows:

$$
\begin{aligned}
(z_{0,0}+1)\varrho(0,0,\boldsymbol{z}) + z_{1,1}\varrho(1,1,\boldsymbol{z}_{-(0,0),(1,1)}, z_{0,0}+1, z_{1,1}-1) &= 1, \text{ if } z_{0,1}=0 \text{ and } z_{1,1}>0\\
(z_{0,0}+1)\varrho(0,0,\boldsymbol{z}) + z_{1,0}\varrho(1,0,\boldsymbol{z}_{-(0,0),(1,0)}, z_{0,0}+1, z_{1,0}-1) &= 1, \text{ if } z_{0,1}=0,\ z_{1,1}=0, \text{ and } z_{1,0}>0\\
(z_{0,0}+1)\varrho(0,0,\boldsymbol{z}) &= 1, \text{ if } z_{0,1}=0,\ z_{1,1}=0, \text{ and } z_{1,0}=0\\
(z_{0,1}+1)\varrho(0,1,\boldsymbol{z}) + z_{1,1}\varrho(1,1,\boldsymbol{z}_{-(0,1),(1,1)}, z_{0,1}+1, z_{1,1}-1) &= 1, \text{ if } z_{1,1}>0\\
(z_{0,1}+1)\varrho(0,1,\boldsymbol{z}) + z_{1,0}\varrho(1,0,\boldsymbol{z}_{-(0,1),(1,0)}, z_{0,1}+1, z_{1,0}-1) &= 1, \text{ if } z_{1,1}=0 \text{ and } z_{1,0}>0\\
(z_{0,1}+1)\varrho(0,1,\boldsymbol{z}) &= 1, \text{ if } z_{1,1}=0 \text{ and } z_{1,0}=0\\
(z_{1,0}+1)\varrho(1,0,\boldsymbol{z}) + z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,1),(1,0)}, z_{0,1}-1, z_{1,0}+1) &= 1, \text{ if } z_{1,1}=0 \text{ and } z_{0,1}>0\\
(z_{1,0}+1)\varrho(1,0,\boldsymbol{z}) + z_{0,0}\varrho(0,0,\boldsymbol{z}_{-(0,0),(1,0)}, z_{0,0}-1, z_{1,0}+1) &= 1, \text{ if } z_{1,1}=0,\ z_{0,1}=0, \text{ and } z_{0,0}>0\\
(z_{1,0}+1)\varrho(1,0,\boldsymbol{z}) &= 1, \text{ if } z_{1,1}=0,\ z_{0,1}=0, \text{ and } z_{0,0}=0\\
(z_{1,1}+1)\varrho(1,1,\boldsymbol{z}) + z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,1),(1,1)}, z_{0,1}-1, z_{1,1}+1) &= 1, \text{ if } z_{0,1}>0\\
(z_{1,1}+1)\varrho(1,1,\boldsymbol{z}) + z_{0,0}\varrho(0,0,\boldsymbol{z}_{-(0,0),(1,1)}, z_{0,0}-1, z_{1,1}+1) &= 1, \text{ if } z_{0,1}=0 \text{ and } z_{0,0}>0\\
(z_{1,1}+1)\varrho(1,1,\boldsymbol{z}) &= 1, \text{ if } z_{0,1}=0 \text{ and } z_{0,0}=0.
\end{aligned}
\tag{56}
$$

Using (56), we rewrite the objective function as

$$
\begin{aligned}
\frac{n}{2}\Bigg(&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0, z_{0,1}>0}} (q-r)U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}>0}} \frac{r(1-r)(2q-1)}{1-Q}U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,1}=0, z_{0,0}>0}} \frac{r(1-r)(2q-1)}{Q}U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0, z_{0,0}>0}} (q+r-1)U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z})\Bigg) + C'
\end{aligned}
\tag{57}
$$

where constant C' is equal to

$$
\begin{aligned}
\frac{n}{2}\Bigg(&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}>0}} \frac{Q}{1-Q}\frac{U(\boldsymbol{z})}{z_{1,1}+1}(1-q)r + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0, z_{0,0}>0}} \frac{U(\boldsymbol{z})}{z_{1,1}+1}(1-q)(1-r) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0, z_{0,1}>0}} \frac{U(\boldsymbol{z})}{z_{1,0}+1}(1-q)r + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,1}=0, z_{0,0}>0}} \frac{1-Q}{Q}\frac{U(\boldsymbol{z})}{z_{1,0}+1}(1-q)(1-r) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,0}=z_{0,1}=0}} \frac{U(\boldsymbol{z})}{z_{1,1}+1}qr + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,0}=z_{0,1}=z_{1,1}=0}} \frac{U(\boldsymbol{z})}{n}q(1-r) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,0}=z_{1,1}=0}} \frac{U(\boldsymbol{z})}{z_{0,1}+1}(1-q)r + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,0}=z_{1,1}=0}} \frac{U(\boldsymbol{z})}{n}(1-q)(1-r)\Bigg).
\end{aligned}
\tag{58}
$$

Similarly, we rewrite the BIC constraint for $x_1 = 1$ and $x'_1 = 0$ as

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0, z_{1,1}=0}} U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0}} \left(\frac{Q}{1-Q}\right) U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{0,0}>0}} \left(\frac{1-Q}{Q}\right) U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) \tag{59}$$
$$\leq \frac{1}{n2^{n-1}}\left(\frac{2^n Q^2 - (2Q-1)(1+Q)^n + Q - 1}{Q(1-Q)}\right)$$

and the BIC constraint for $x_1 = 0$ and $x'_1 = 1$ as

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0, z_{1,1}=0}} U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0}} U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{0,0}>0}} U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) \tag{60}$$
$$\geq \frac{1}{n2^{n-1}}\left(\frac{(1+Q)^n(2Q-1) - Q^2 2^n - Q + 1}{Q(1-Q)}\right)$$

In the following analysis we will use sets or report statistics defined below:

$$\begin{aligned}
Z_1 &= \{\boldsymbol{z} \in S_4(n-1) : z_{0,1} > 0 \text{ and } z_{1,1} = 0\}\\
Z_2 &= \{\boldsymbol{z} \in S_4(n-1) : z_{0,1} > 0\}\\
Z_3 &= \{\boldsymbol{z} \in S_4(n-1) : z_{0,1} = z_{1,1} = 0,\ z_{0,0} > 0\}\\
Z_4 &= \{\boldsymbol{z} \in S_4(n-1) : z_{0,1} = 0 \text{ and } z_{0,0} > 0\}.
\end{aligned}$$

These sets are clearly pairwise disjoint. Set Z_1 consists of reports on $n-1$ agents where at least one agent received high report from the non-expert and there is no agent who received two high reports. Set Z_2 consists of reports on $n-1$ agents where at least one agent received low report from an expert and hight report from the non-expert. Set Z_3 consists of reports on $n-1$ agents where no agent received a high report from the non-expert and at least one agent received two low reports. Lastly, set Z_4 consists of reports on $n-1$ agents where no agent received high report from an expert and received high report from the non-expert and there exists an agent who received two low reports. We will also refer to the following sets of variables:

$$\begin{aligned}
V_1 &= \{\varrho(1,0,\boldsymbol{z}) : \boldsymbol{z} \in Z_1\} & V_3 &= \{\varrho(1,0,\boldsymbol{z}) : \boldsymbol{z} \in Z_3\}\\
V_2 &= \{\varrho(1,1,\boldsymbol{z}) : \boldsymbol{z} \in Z_2\} & V_4 &= \{\varrho(1,1,\boldsymbol{z}) : \boldsymbol{z} \in Z_4\}
\end{aligned}$$

$$\begin{aligned}
\bar{V}_1 &= \{\varrho(0,1,\boldsymbol{z}) : \boldsymbol{z} \in S_4(n-1),\ z_{1,0} > 0 \text{ and } z_{1,1} = 0\}\\
\bar{V}_2 &= \{\varrho(0,1,\boldsymbol{z}) : \boldsymbol{z} \in S_4(n-1),\ z_{1,1} > 0\}\\
\bar{V}_3 &= \{\varrho(0,0,\boldsymbol{z}) : \boldsymbol{z} \in S_4(n-1),\ z_{0,1} = z_{1,1} = 0,\ z_{1,0} > 0\}\\
\bar{V}_4 &= \{\varrho(0,0,\boldsymbol{z}) : \boldsymbol{z} \in S_4(n-1),\ z_{0,1} = 0 \text{ and } z_{1,1} > 0\}.
\end{aligned}$$

Notice that, for any $i \in \{1, \ldots, 4\}$, any variable in set V_i is associated, through the probability constraints (56), with exactly one other variable and that variable is in the set $\bar{V}_i$. We will call this associated variable the twin variable. Notice that all these sets are pairwise disjoint. Hence for any adjustment of the values of variables in sets $V_1, \ldots, V_4$ we can adjust the values of their twin variables in sets $\bar{V}_1, \ldots, \bar{V}_4$ and maintain the probability constraints and we can consider adjustments of the values of variables in the sets $V_1, \ldots, V_4$ independently. Therefore, since both the objective function and the BIC constraints contain only the variables from sets $V_1, \ldots, V_4$, we can restrict attention to these variables and consider adjustments that maintain the BIC constraints and the constraint of non-negativity of the variables.

We first show that the BIC constraint for $x_1 = 0$ and $x_1' = 1$ is satisfied with equality at any optimum of the relaxed optimization problem.

Lemma 9. *At any optimum of the relaxed optimization problem without the BIC constraint for $x_1 = 0$ and $x_1' = 1$, the BIC constraint for $x_1 = 1$ and $x_1' = 0$ is satisfied with equality.*

Proof. Consider the relaxed optimization problem without the BIC constraint for $x_1 = 0$ and $x_1' = 1$. We show first that at any optimum either the BIC constraint for $x_1 = 1$ and $x_1' = 0$ is satisfied with equality or, for all $\boldsymbol{z} \in Z_1 \cup Z_3$, $\varrho(1, 0, \boldsymbol{z}) = 1/(z_{1,0}+1)$, and for all $\boldsymbol{z} \in Z_2 \cup Z_4$, $\varrho(1, 1, \boldsymbol{z}) = 1/(z_{1,1} + 1)$. This follows because the coefficients at $\varrho(1, 0, \boldsymbol{z})$ and $\varrho(1, 1, \boldsymbol{z})$ in the objective function (57) are all positive (as $1/2 < r \leq q < 1$). Hence, increasing the value of $\varrho(1, 0, \boldsymbol{z})$ or $\varrho(1, 1, \boldsymbol{z})$ by sufficiently small amount and decreasing the value of the corresponding twin variable by the right amount, to maintain the probability constraint, maintains the BIC constraints and increases the value of the objective function. Since the objective function is at an optimum, this is not possible. Hence the claim follows.

Second, we show that if for all $\boldsymbol{z} \in Z_1 \cup Z_3$, $\varrho(1, 0, \boldsymbol{z}) = 1/(z_{1,0} + 1)$, and for all $\boldsymbol{z} \in Z_2 \cup Z_4$, $\varrho(1, 1, \boldsymbol{z}) = 1/(z_{1,1} + 1)$, then the BIC constraint is not satisfied. For in this case the LHS of the BIC constraint is equal to

$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}>0, z_{1,1}=0}} U(\boldsymbol{z}) \frac{1}{z_{1,0}+1} + \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}>0}} \left(\frac{Q}{1-Q}\right) U(\boldsymbol{z}) \frac{1}{z_{1,1}+1} +$$
$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{0,0}>0}} \left(\frac{1-Q}{Q}\right) U(\boldsymbol{z}) \frac{1}{z_{1,0}+1} + \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} U(\boldsymbol{z}) \frac{1}{z_{1,1}+1}.$$

Since

$$\sum_{\boldsymbol{z} \in Z_2} U(\boldsymbol{z}) \frac{1}{z_{1,1}+1} = \frac{1}{n2^{n-1}} \left(\frac{2^n - (1+Q)^n - (2-Q)^n + 1}{Q}\right),$$
$$\sum_{\boldsymbol{z} \in Z_3} U(\boldsymbol{z}) \frac{1}{z_{1,0}+1} = \frac{1}{n2^{n-1}} \left(\frac{1 - Q^n - (1-Q)^n}{1-Q}\right),$$
$$\sum_{\boldsymbol{z} \in Z_4} U(\boldsymbol{z}) \frac{1}{z_{1,1}+1} = \frac{1}{n2^{n-1}} \left(\frac{(1+Q)^n + (1-Q)^n - 2}{Q}\right),$$

so the LHS of the BIC constraint is greater or equal to

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0}} \left(\frac{Q}{1-Q}\right) U(\boldsymbol{z})\frac{1}{z_{1,1}+1} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{0,0}>0}} \left(\frac{1-Q}{Q}\right) U(\boldsymbol{z})\frac{1}{z_{1,0}+1} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} U(\boldsymbol{z})\frac{1}{z_{1,1}+1}$$

$$= \frac{Q}{1-Q}\frac{1}{n2^{n-1}}\left(\frac{2^n-(1+Q)^n-(2-Q)^n+1}{Q}\right) + \frac{1-Q}{Q}\frac{1}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{1-Q}\right) + \frac{1}{n2^{n-1}}\left(\frac{(1+Q)^n+(1-Q)^n-2}{Q}\right).$$

Subtracting the RHS of the BIC constraint from it we obtain

$$\frac{1}{n2^{n-1}}\left(\frac{(1-Q)2^n-(2-Q)^n-(1-Q)Q^{n-1}+1}{1-Q}\right).$$

Since

$$(1-Q)2^n-(2-Q)^n-(1-Q)Q^{n-1}+1 = (1-Q)2^n-(2-Q)^n+Q+(1-Q)(1-Q^{n-1}) > 0,$$

as, for $Q \in (1/2, 1)$ and $n \geq 2$,

$$(1-Q)2^n-(2-Q)^n+Q = Q+(1-Q)\sum_{j=0}^{n}\binom{n}{j}+\sum_{j=0}^{n}\binom{n}{j}(1-Q)^j = \sum_{j=1}^{n}\binom{n}{j}\left(1-Q-(1-Q)^j\right) \geq 0$$

and

$$(1-Q)(1-Q^{n-1}) > 0,$$

so

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0}} \left(\frac{Q}{1-Q}\right) U(\boldsymbol{z})\frac{1}{z_{1,1}+1} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{0,0}>0}} \left(\frac{1-Q}{Q}\right) U(\boldsymbol{z})\frac{1}{z_{1,0}+1} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} U(\boldsymbol{z})\frac{1}{z_{1,1}+1}$$

$$> \frac{1}{n2^{n-1}}\left(\frac{2^nQ^2-(2Q-1)(1+Q)^n+Q-1}{Q(1-Q)}\right). \tag{61}$$

Hence the LHS of the BIC constraint is strictly greater than the RHS of the BIC constraint and so the constraint is not satisfied.

The lemma follows immediately from the two points shown above. □

Next we provide a complete characterization of the optimal mechanisms of the relaxed optimization problem.

Lemma 10. *Consider the relaxed optimization problem without the BIC constraint for $x_1 = 0$ and $x_1' = 1$. If Mechanism $(\varrho(x_0, y_0, \boldsymbol{z}))_{(x_0,y_0)\in\{0,1\}^2, \boldsymbol{z}\in S_4(n-1)}$ is optimal then for any $\boldsymbol{z} \in S_4(n-1)$:*

1. *If $z_{0,1} = 0$ then*
   (a) *if $z_{0,0} > 0$ then $\varrho(1, 1, \boldsymbol{z}) = \frac{1}{z_{1,1}+1}$,*
   (b) *if $z_{1,1} > 0$ then $\varrho(0, 0, \boldsymbol{z}) = 0$.*
2. *If $z_{1,1} = 0$ then*

(a) *if* $z_{0,1} > 0$ *then* $\varrho(1,0,\boldsymbol{z}) = 0$,

(b) *if* $z_{1,0} > 0$ *then* $\varrho(0,1,\boldsymbol{z}) = \frac{1}{z_{0,1}+1}$.

3. *If* $z_{0,1} = z_{1,1} = 0$ *then*

(a) *if* $z_{0,0} > 0$ *then* $\varrho(1,0,\boldsymbol{z}) = \frac{1}{z_{1,0}+1}$,

(b) *if* $z_{1,0} > 0$ *then* $\varrho(0,0,\boldsymbol{z}) = 0$.

4. $\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}>0}} U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) = \frac{1}{n2^{n-1}}\left(\frac{Q^2 2^n + (1-Q)Q^n - Q(1+Q)^n}{Q^2}\right)$.

Proof. Let

$$D = \frac{1}{n2^{n-1}}\left(\frac{2^n Q^2 - (2Q-1)(1+Q)^n + Q - 1}{Q(1-Q)}\right).$$

By Lemma 9 and (59) the BIC constraint for $x_1 = 1$ and $x_1' = 0$ is

$$\begin{aligned}&\sum_{\boldsymbol{z}\in Z_1} U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_2}\left(\frac{Q}{1-Q}\right)U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) + \\ &\sum_{\boldsymbol{z}\in Z_3}\left(\frac{1-Q}{Q}\right)U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_4} U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) = D\end{aligned} \tag{62}$$

and the objective function (57) can be rewritten as

$$\begin{aligned}\frac{n}{2}\Bigg(&\sum_{\boldsymbol{z}\in Z_1}(q-r)U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_2}\frac{r(1-r)(2q-1)}{1-Q}U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) + \\ &\sum_{\boldsymbol{z}\in Z_3}\frac{r(1-r)(2q-1)}{Q}U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_4}(q+r-1)U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z})\Bigg) + C'\end{aligned} \tag{63}$$

where C' is a constant.

Since

$$(q+r-1)(1-Q) - (2q-1)(1-r)r = (2q-1)(1-r)r - (q-r)Q = q(1-q)(2r-1) > 0$$

(for $1/2 < r \leq q < 1$) and $Q > 1 - Q$ so

$$q + r - 1 > \frac{r(1-r)(2q-1)}{1-Q} > \frac{r(1-r)(2q-1)}{Q} > q - r. \tag{64}$$

Suppose that mechanism $(\varrho(x_0,y_0,\boldsymbol{z}))_{(x_0,y_0)\in\{0,1\}^2,\boldsymbol{z}\in S_4(n-1)}$ is optimal. For point 1a of the lemma assume, to the contrary, that there exists $\boldsymbol{z}\in S_4(n-1)$ with $z_{0,1} = 0$ and $z_{0,0} > 0$ (i.e. $\boldsymbol{z}\in Z_4$) such that $\varrho(1,1,\boldsymbol{z}) < 1/(z_{1,1}+1)$. Then

$$\sum_{\boldsymbol{z}\in Z_4} U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) < \sum_{\boldsymbol{z}\in Z_4} U(\boldsymbol{z})\frac{1}{z_{1,1}+1}.$$

Since

$$\sum_{\boldsymbol{z}\in Z_4} U(\boldsymbol{z})\frac{1}{z_{1,1}+1} = \frac{1}{n2^{n-1}}\left(\frac{(1+Q)^n + (1-Q)^n - 2}{Q}\right),$$

$$D - \frac{1}{n2^{n-1}}\left(\frac{(1+Q)^n + (1-Q)^n - 2}{Q}\right) = \frac{1}{n2^{n-1}}\left(\frac{Q^2 2^n - (1-Q)^{n+1} - Q(1+Q)^n + 1 - Q}{Q(1-Q)}\right),$$

$$Q^2 2^n - (1-Q)^{n+1} - Q(1+Q)^n + 1 - Q = Q(Q2^n - (1+Q)^n) + (1-Q)^2 - (1-Q)^{n+1},$$

and, for $Q \in (1/2, 1)$ and $n \geq 2$, we have $(1-Q)^2 > (1-Q)^{n+1}$ and

$$Q2^n - (1+Q)^n + 1 - Q = Q\sum_{j=0}^{n}\binom{n}{j} - \sum_{j=0}^{n}\binom{n}{j}Q^j + 1 - Q = \sum_{j=1}^{n}\binom{n}{j}\left(Q - Q^j\right) > 0, \quad (65)$$

so

$$\sum_{\boldsymbol{z} \in Z_4} U(\boldsymbol{z})\varrho(1, 1, \boldsymbol{z}) < D.$$

Hence, by point (1a) of the lemma, either (i) there exists $\boldsymbol{z'} \in Z_1$ with $\varrho(1, 0, \boldsymbol{z'}) > 0$, or (ii) there exists $\boldsymbol{z'} \in Z_2$ with $\varrho(1, 1, \boldsymbol{z'}) > 0$, or (iii) there exists $\boldsymbol{z'} \in Z_3$ with $\varrho(1, 0, \boldsymbol{z'}) > 0$. Suppose case (i) holds. Then increasing the value of $\varrho(1, 1, \boldsymbol{z})$ by ε (and adjusting the value of the corresponding twin variable in $\bar{V}_4$ accordingly) and decreasing the value of $\varrho(1, 0, \boldsymbol{z'})$ by $\varepsilon U(\boldsymbol{z})/U(\boldsymbol{z'})$ (and adjusting the value of the corresponding twin variable in $\bar{V}_1$ accordingly), where $\varepsilon > 0$ is sufficiently small to maintain the probability constraints and the non-negativity constraints, maintains the BIC constraint and increases the value of the objective function as, by (64), $q + r - 1 > q - r$. Hence we get a contradiction with the assumption of optimality of the mechanism. Suppose case (ii) holds. Then increasing the value of $\varrho(1, 1, \boldsymbol{z})$ by ε (and adjusting the value of the corresponding twin variable in $\bar{V}_4$ accordingly) and decreasing the value of $\varrho(1, 1, \boldsymbol{z'})$ by $\varepsilon(1-Q)U(\boldsymbol{z})/(QU(\boldsymbol{z'}))$ (and adjusting the value of the corresponding twin variable in $\bar{V}_2$ accordingly), where $\varepsilon > 0$ is sufficiently small to maintain the probability constraints and the non-negativity constraints, maintains the BIC constraint and changes the value of the objective function by

$$\varepsilon U(\boldsymbol{z})\left(q + r - 1 - \frac{r(1-r)(2q-1)}{Q}\right),$$

which, by (64), is greater than 0. Hence the adjustment increases the value of the objective function and maintains the constraints, a contradiction with the assumption of optimality of the mechanism. Lastly, suppose case (iii) holds. Then increasing the value of $\varrho(1, 1, \boldsymbol{z})$ by ε (and adjusting the value of the corresponding twin variable in $\bar{V}_4$ accordingly) and decreasing the value of $\varrho(1, 0, \boldsymbol{z'})$ by $\varepsilon QU(\boldsymbol{z})/((1-Q)U(\boldsymbol{z'}))$ (and adjusting the value of the corresponding twin variable in $\bar{V}_3$ accordingly), where $\varepsilon > 0$ is sufficiently small to maintain the probability constraints and the non-negativity constraints, maintains the BIC constraint and changes the value of the objective function by

$$\varepsilon U(\boldsymbol{z})\left(q + r - 1 - \frac{r(1-r)(2q-1)}{1-Q}\right),$$

which, by (64), is greater than 0. Hence the adjustment increases the value of the objective function and maintains the constraints, a contradiction with the assumption of optimality of the mechanism. Since in all the three cases we arrive at contradiction, it must be that for all $\boldsymbol{z} \in Z_4$, $\varrho(1, 1, \boldsymbol{z}) = 1/(z_{1,1}+1)$ and that point 1a of the lemma holds. By point 1a and the probability constraints (56), for all $\boldsymbol{z} \in Z_4$, $\varrho(0, 0, \boldsymbol{z}_{-(0,0),(1,1)}, z_{0,0} - 1, z_{1,1} + 1) = 0$. Hence point 1b of the lemma follows.

For point 2a of the lemma, assume, to the contrary, that there exists $\boldsymbol{z} \in S_4(n-1)$ with $z_{1,1} = 0$ and $z_{0,1} > 0$ (i.e. $\boldsymbol{z} \in Z_1$), and such that $\varrho(1, 0, \boldsymbol{z}) > 0$. By the BIC constraint (62) together with the inequality (61) and with point 1a of the lemma, either (i) there exists $\boldsymbol{z'} \in Z_2$ with $\varrho(1,1,\boldsymbol{z'}) < 1/(z_{1,1}+1)$, or (ii) there exists $\boldsymbol{z'} \in Z_3$ with $\varrho(1,0,\boldsymbol{z'}) < 1/(z_{1,0}+1)$. Suppose case (i) holds. Then decreasing the value of $\varrho(1,0,\boldsymbol{z})$ by ε (and adjusting the value of the corresponding twin variable in $\bar{V}_1$ accordingly) and increasing the value of $\varrho(1,1,\boldsymbol{z'})$ by $\varepsilon(1-Q)U(\boldsymbol{z})/(QU(\boldsymbol{z'}))$ (and adjusting the value of the corresponding twin variable in $\bar{V}_2$ accordingly), where $\varepsilon > 0$ is sufficiently small to maintain the probability constraints and the non-negativity constraints, maintains the BIC constraint and changes the value of the objective function by

$$\varepsilon U(\boldsymbol{z})\left(\frac{r(1-r)(2q-1)}{Q} - (q-r)\right),$$

which, by (64), is greater than 0. Hence the adjustment increases the value of the objective function and maintains the constraints, a contradiction with the assumption of optimality of the mechanism. Suppose case (ii) holds. Then decreasing the value of $\varrho(1,0,\boldsymbol{z})$ by ε (and adjusting the value of the corresponding twin variable in $\bar{V}_1$ accordingly) and increasing the value of $\varrho(1,0,\boldsymbol{z'})$ by $\varepsilon QU(\boldsymbol{z})/((1-Q)U(\boldsymbol{z'}))$ (and adjusting the value of the corresponding twin variable in $\bar{V}_3$ accordingly), where $\varepsilon > 0$ is sufficiently small to maintain the probability constraints and the non-negativity constraints, maintains the BIC constraint and changes the value of the objective function by

$$\varepsilon U(\boldsymbol{z})\left(\frac{r(1-r)(2q-1)}{1-Q} - (q-r)\right),$$

which, by (64), is greater than 0. Hence the adjustment increases the value of the objective function and maintains the constraints, a contradiction with optimality of the mechanism. Since in both cases we arrive at contradiction, it must be that for all $\boldsymbol{z} \in Z_1$, $\varrho(1,0,\boldsymbol{z}) = 0$ and that point 2a of the lemma holds. By point 2a and the probability constraints (56), for all $\boldsymbol{z} \in Z_1$, $\varrho(0,1,\boldsymbol{z}_{-(0,1),(1,0)}, z_{0,1}-1, z_{1,0}+1) = 1/(z_{0,1}+1)$. Hence point 2b of the lemma follows.

Using points 1a and 2a of the lemma, the BIC constraint (62) can be rewritten as

$$\sum_{\boldsymbol{z}\in Z_2}\left(\frac{Q}{1-Q}\right)U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) + \sum_{\boldsymbol{z}\in Z_3}\left(\frac{1-Q}{Q}\right)U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) = D' \tag{66}$$

where

$$D' = D - \sum_{\boldsymbol{z}\in Z_4} U(\boldsymbol{z})\frac{1}{z_{1,1}+1} = \frac{1}{n2^{n-1}}\left(\frac{Q^2 2^n - (1-Q)^{n+1} - Q(1+Q)^n + 1 - Q}{Q(1-Q)}\right),$$

and the objective function can be rewritten as

$$\frac{nr(1-r)(2q-1)}{2}\left(\frac{1}{1-Q}\sum_{\boldsymbol{z}\in Z_2} U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) + \frac{1}{Q}\sum_{\boldsymbol{z}\in Z_3} U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z})\right) + C'' \tag{67}$$

where

$$C'' = C' + \frac{n}{2}(q+r-1)\sum_{\boldsymbol{z}\in Z_4} U(\boldsymbol{z})\frac{1}{z_{1,1}+1}. \tag{68}$$

Using (66), the objective function can be further rewritten as

$$\frac{nr(1-r)(2q-1)}{2}\frac{2Q-1}{Q^2}\sum_{\boldsymbol{z}\in Z_3} U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + C'' + \frac{nr(1-r)(2q-1)}{2Q}D'.$$

Since $1/2 < Q < 1$ so $(2Q-1)/Q^2 > 0$ and to maximize the value of the objective function we need to maximize $\sum_{\boldsymbol{z} \in Z_3} U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z})$.

Since

$$\sum_{\boldsymbol{z} \in Z_3} U(\boldsymbol{z}) \frac{1}{z_{1,0}+1} = \frac{1}{n 2^{n-1}} \left(\frac{1 - Q^n - (1-Q)^n}{1-Q} \right)$$

so

$$D' - \frac{1-Q}{Q} \sum_{\boldsymbol{z} \in Z_3} U(\boldsymbol{z}) \frac{1}{z_{1,0}+1} = \frac{1}{n 2^{n-1}} \left(\frac{Q^2 2^n + (1-Q)Q^n - Q(1+Q)^n}{Q(1-Q)} \right).$$

Since $n \geq 2$ and $Q \in (1/2, 1)$ so

$$\begin{aligned} Q2^n + (1-Q)Q^{n-1} - (1+Q)^n &= Q \sum_{j=0}^{n} \binom{n}{j} + (1-Q)Q^{n-1} - \sum_{j=0}^{n} \binom{n}{j} Q^j \\ &= (1-Q)Q^{n-1} + \sum_{j=0}^{n} \binom{n}{j} \left(Q - Q^j\right) \\ &= (1-Q)Q^{n-1} + Q - 1 + Q - Q^n + \sum_{j=1}^{n-1} \binom{n}{j} \left(Q - Q^j\right) \\ &= (2Q-1)\left(1 - Q^{n-1}\right) + \sum_{j=1}^{n-1} \binom{n}{j} \left(Q - Q^j\right) > 0 \end{aligned}$$

and

$$D' - \frac{1-Q}{Q} \sum_{\boldsymbol{z} \in Z_3} U(\boldsymbol{z}) \frac{1}{z_{1,0}+1} > 0.$$

By the probability constraints (56), $\varrho(1,0,\boldsymbol{z}) \leq 1/(z_{1,0}+1)$. Hence setting the value of all variables $\varrho(1,0,\boldsymbol{z})$ with $\boldsymbol{z} \in Z_3$ (i.e. $z_{0,1} = z_{1,1} = 0$ and $z_{0,0} > 0$) to $1/(z_{1,0}+1)$ maximises the value of the objective function and allows for satisfying the BIC constraint by setting the values of variables $\varrho(1,1,\boldsymbol{z})$ with $\boldsymbol{z} \in Z_2$ to any values in $[0, 1/(z_{1,1}+1)]$ such that

$$\sum_{\boldsymbol{z} \in Z_2} U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) = \frac{1}{n 2^{n-1}} \left(\frac{Q^2 2^n + (1-Q)Q^n - Q(1+Q)^n}{Q^2} \right).$$

This shows points 3a and 4 of the lemma. By point 3a and the probability constraints (56), for all $\boldsymbol{z} \in Z_3$, $\varrho(0,0,\boldsymbol{z}_{-(0,0),(1,0)}, z_{0,0}-1, z_{1,0}+1) = 0$. Hence point 3b of the lemma follows. □

Lemma 11. *An upper bound on the probability of selecting a high quality project under an optimal anonymous BIC mechanism with peer reports is*

$$\overline{\Pi^{\mathsf{pB}}}(p,r) = r - \frac{2r-1}{2^n} \left(1 - \frac{r(1-r)(2p-1)^2(1-Q^{n-1})}{Q(1-Q)} \right).$$

Proof. By (67), the value of the objective function under any optimal anonymous BIC mechanism, $\varrho \in R^{\mathsf{pB}}$, is

$$\Pi^{\mathsf{pB}}(p,r) = C'' + \left(\frac{r(1-r)(2p-1)n}{2Q} \right) \left(D' + \frac{2Q-1}{Q} \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ z_{0,1}=z_{1,1}=0,\ z_{0,0}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) \right),$$

where

$$D' = \frac{1}{n2^{n-1}}\left(\frac{Q^2 2^n - (1-Q)^{n+1} - Q(1+Q)^n + 1 - Q}{Q(1-Q)}\right),$$

by (58), and (68),

$$C'' = \frac{1}{2^n}\left(\frac{(1-p)rQ2^n + r(1-r)(2p-1)((1+Q)^n + (1-Q)^n) - (1-Q)(p+r-1)}{Q(1-Q)}\right)$$

and, by point 4 of Lemma 10,

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,1}=0,\ z_{0,0}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) = \frac{1}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{1-Q}\right).$$

Therefore the value of the objective function is

$$\begin{aligned}
\Pi^{\mathsf{pB}}(p,r) =& \frac{1}{2^n}\left(\frac{(1-p)rQ2^n + r(1-r)(2p-1)((1+Q)^n + (1-Q)^n) - (1-Q)(p+r-1)}{Q(1-Q)}\right)\\
&+ \frac{1}{2^n}\left(\frac{r(1-r)(2p-1)(Q^2 2^n - (1-Q)^{n+1} - Q(1+Q)^n + 1 - Q)}{Q^2(1-Q)}\right)\\
&+ \frac{1}{2^n}\left(\frac{r(1-r)(2p-1)(2Q-1)\left(1-Q^n-(1-Q)^n\right)}{Q^2(1-Q)}\right)\\
=& \frac{1}{2^n}\left(\frac{(1-p)rQ2^n - (1-Q)(p+r-1)}{Q(1-Q)}\right) + \frac{1}{2^n}\left(\frac{r(1-r)(2p-1)(Q^2 2^n + 1 - Q)}{Q^2(1-Q)}\right)\\
&+ \frac{1}{2^n}\left(\frac{r(1-r)(2p-1)(2Q-1)\left(1-Q^n\right)}{Q^2(1-Q)}\right)\\
=& r + \frac{1}{2^n}\left(\frac{-p(1-p)(2r-1)}{Q(1-Q)}\right)\\
&- \frac{1}{2^n}\left(\frac{r(1-r)(2p-1)(2Q-1)Q^n}{Q^2(1-Q)}\right)\\
=& r - \frac{2r-1}{2^n}\left(1 - \frac{r(1-r)(2p-1)^2(1-Q^{n-1})}{Q(1-Q)}\right).
\end{aligned}$$

□

Proof of Proposition 3. The formula for upper bound on the the probability of selecting a high quality project in the mechanism follows directly from Lemma 11.

To show that $\Pi^{\mathsf{B}}(p,r) \geq \Pi^{\mathsf{pB}}(p,r)$ when $r \in (1/2, 2)$ and $p \in (r, 1)$, with strict inequality when $n \geq 3$, it is enough to show that under these assumptions,

$$\left(\frac{(2p-1)^2}{Q(1-Q)}\right)\min\left(1-(1-Q)^{n-1}, \frac{2^n(1-Q) - (2-Q)^n + Q}{Q}\right) > \frac{(2p-1)^2(1-Q^{n-1})}{Q(1-Q)}.$$

The inequality holds if

$$\min\left(1-(1-Q)^{n-1}, \frac{2^n(1-Q) - (2-Q)^n + Q}{Q}\right) > 1 - Q^{n-1}.$$

If $r > 1/2$ and $p \in (0,1)$ then $Q > 1/2$ and $Q > 1-Q$. Thus $1-(1-Q)^{n-1} > 1-Q^{n-1}$. It remains to show that

$$2^n(1-Q) - (2-Q)^n + Q \geq Q - Q^n. \tag{69}$$

It is immediate to see that this holds with equality when $n = 2$. Suppose that $n = 3$. Let $\varphi(x) = 2^n(1-x) - (2-x)^n + x^n$. Since

$$\varphi''(x) = n(n-1)(x^{n-2} - (2-x)^{n-2}) \leq 0$$

when $x \in [1/2, 1]$ and $n \geq 3$ with equality when $x = 1$ only, so $\varphi'(x)$ is strictly decreasing on $[1/2, 1]$ and

$$\varphi'(x) = n(x^{n-1}+(2-x)^{n-1})-2^n \leq \varphi'\left(\frac{1}{2}\right) = n\left(\left(\frac{3}{2}\right)^{n-1} + \left(\frac{1}{2}\right)^{n-1}\right)-2^n < 2n-2^n < 0$$

when $x \in [1/2, 1]$ and $n \geq 3$ with equality only when $x = 1$. Hence $\varphi(x)$ is strictly decreasing on $[1/2, 1]$ and so

$$\varphi(Q) = 2^n(1-Q) - (2-Q)^n + Q^n > \varphi(1) = 0,$$

for any $Q \in (1/2, 1)$ and $n \geq 3$. Since $Q \in (1/2, 1)$ when $r \in (1/2, 1)$ and $p \in [r, 1)$ so Inequality (69) follows. □

B.4 Proof of Proposition 4.

Proof. The four EPIC constraints are given by:
Incentive constraint $IC(0,0)$

$$\begin{aligned} 0 \leq\ & Q^2[\pi(00,00) - \pi(10,00)] + Q(1-Q)[\pi(00,10) - \pi(10,10)] \\ & + Q(1-Q)[\pi(00,10) - \pi(10,10)] + (1-Q)^2[\pi(00,11) - \pi(10,11)] \end{aligned}$$

Incentive constraint $IC(0,1)$

$$\begin{aligned} 0 \leq\ & Q^2[\pi(01,01) - \pi(11,01)] + Q(1-Q)[\pi(01,00) - \pi(11,00)] \\ & + Q(1-Q)[\pi(01,11) - \pi(11,11)] + (1-Q)^2[\pi(01,10) - \pi(11,10)] \end{aligned}$$

Incentive constraint $IC(1,0)$

$$\begin{aligned} 0 \leq\ & Q^2[\pi(10,10) - \pi(00,10)] + Q(1-Q)[\pi(10,11) - \pi(00,11)] \\ & + Q(1-Q)[\pi(10,00) - \pi(00,00)] + (1-Q)^2[\pi(10,01) - \pi(00,01)] \end{aligned}$$

Incentive constraint $IC(1,1)$

$$\begin{aligned} 0 \leq\ & Q^2[\pi(11,11) - \pi(01,11)] + Q(1-Q)[\pi(11,10) - \pi(01,10)] \\ & + Q(1-Q)[\pi(11,01) - \pi(01,01)] + (1-Q)^2[\pi(11,00) - \pi(01,00)] \end{aligned}$$

We solve the relaxed problem where we maximize the expected utility of the planner:

$$\begin{aligned} \Pi = & \frac{1}{2}((1-p)(1-r)Q(\pi(00,00) + \pi(01,01)) + (1-p)(1-r)(1-Q)(\pi(00,01) + \pi(01,00)) \\ & + p(1-r)Q(\pi(10,00) + \pi(11,01)) + p(1-r)(1-Q)(\pi(10,01) + \pi(11,00)) \\ & + (1-p)rQ(\pi(00,10) + \pi(01,11)) + (1-p)r(1-Q)(\pi(00,11) + \pi(01,10)) \\ & + prQ(\pi(10,10) + \pi(11,11)) + pr(1-Q)(\pi(10,11) + \pi(11,10))) \end{aligned}$$

subject to the two incentive constraints of the low type $IC(0,0)$ and $IC(0,1)$.

Using non-wastefulness and symmetry, we can simplify the problem by only considering the six probabilities:

$$\pi_1 = \pi(10,10), \pi_2 = \pi(10,00), \pi_3 = \pi(00,10), \pi_4 = \pi(10,01); \pi_5 = \pi(10,11), \pi_6 = \pi(11,10).$$

And the problem becomes:

$$\begin{array}{cc} \max\limits_{\pi_i} & \frac{1}{2} + (1-p)(1-p(2r-1)) + \sum a_i\pi_i \\ s.to & \frac{1}{2} - \sum b_i\pi_i \geq 0 \\ & 1 - Q - \sum c_i\pi_i \geq 0 \\ & 0 \leq \pi_i \leq 1 \end{array}$$

where

$$\begin{aligned} a_1 = \left(p^2r^2 - (1-p)^2(1-r)^2\right), a_2 = a_5 = \left(p^2r(1-r) - (1-p)^2r(1-r)\right), \\ a_3 = a_6 = \left(r^2p(1-p) - (1-r)^2p(1-p)\right), a_4 = \left(p^2(1-r)^2 - (1-p)^2r^2\right) \\ b_1 = Q(1-Q), b_2 = Q^2, b_3 = 0, b_4 = Q(1-Q), b_5 = (1-Q)^2, b_6 = 0 \\ c_1 = Q^2, c_2 = Q(1-Q), c_3 = 0, c_4 = (1-Q)^2, c_5 = Q(1-Q), c_6 = -(2Q-1) \end{aligned}$$

This is a linear relaxation of a knapsack problem with two constraints. As $b_3 = b_6 = 0$, $c_3 = 0, c_6 < 0$, we can set $\pi_3 = \pi_6 = 1$After this transformation, the two constraints become

$$\begin{aligned} 0 &\leq \frac{1}{2} - b_1\pi - b_2\pi_2 - b_4\pi_4 - b_5\pi_5, \\ 0 &\leq Q - c_1\pi_1 - c_2\pi_2 - c_4\pi_4 - c_5\pi_5 \end{aligned}$$

Let $\beta_i = -\frac{a_i}{b_i}$ and $\gamma_i = -\frac{a_i}{c_i}$ for $i = 1,2,4,5$ denote the efficiency indices. Using the computations of the BIC case, we obtain the following rankings

$$\beta_1 > \beta_5 > \beta_2 > \beta_4,$$

$$\gamma_1 > \gamma_5 = \gamma_2 > \gamma_4.$$

It is easy to check that $\pi_1 = 1$ and $\pi_4 = 0$. Plugging in the constraints, we get

$$\begin{aligned} 0 &\leq \frac{1}{2} - Q(1-Q) - \pi_2Q^2 - \pi_5(1-Q)^2, \\ 0 &\leq Q(1-Q)(1-(\pi_2+\pi_5)) \end{aligned}$$

Now, we argue that at the optimum $\pi_2 + \pi_5 = 1$. Suppose not, i.e. $\pi_2 + \pi_5 < 1$. if $\pi_5 > \pi_2$, then one can increase the planner's payoff by increasing π_2 and π_5 by ϵ while keeping the incentive constraint satisfied. If $\pi_5 < \pi_2$, then consider a switch to $\pi_2' = \pi_5 + \epsilon, \pi_5' = \pi_2 + \epsilon$. This results in an increase in the planner's payoff while keeping the incentive constraints satisfied for ϵ sufficiently small. Now, we conclude that $\pi_2 + \pi_5 = 1$, which implies that the planner's payoff is the same as when $\pi_2 = \pi_5 = \frac{1}{2}$ (because the coefficients of π_2 and π_5 in the planner's objective function are identical). $\square$

B.5 Proof of Proposition 5.

Proof. We start by rewriting the EPIC constraints. There is one constraint for every agent $i \in N$, every signal value $x_i \in \{0,1\}$, and every tuple of signals of other agents, $\boldsymbol{x}_{-i} \in \{0,1\}^{N\setminus\{i\}}$, $n2^n$ constraints in total. We restrict attention to the EPIC constraints for agent 0. There are 2^n such constraints. Given $(x_0, x_0') \in \{0,1\}^2$ with $x_0 \neq x_0'$ and $\boldsymbol{x}_{-0} \in \{0,1\}^{N\setminus\{0\}}$ the corresponding EPIC constraint is

$$\sum_{\boldsymbol{y}\in\{0,1\}^N} \mathbf{Pr}(\boldsymbol{Y} = \boldsymbol{y} \mid \boldsymbol{X} = \boldsymbol{x}) \left(\pi_0(\boldsymbol{x}, \boldsymbol{y}) - \pi_0(x_0', \boldsymbol{x}_{-0}, \boldsymbol{y})\right) \geq 0.$$

Given $(x_0, x_0') \in \{0,1\}^2$ with $x_0 \neq x_0'$ and $\boldsymbol{x}_{-0} \in \{0,1\}^{N\setminus\{0\}}$ Since

$$\mathbf{Pr}(\boldsymbol{Y} = \boldsymbol{y} \mid \boldsymbol{X} = \boldsymbol{x}) = \frac{\prod_{i\in N}\left([x_i = y_i]Q + [x_i \neq y_i](1-Q)\right)}{2^n} \qquad = T_i(\boldsymbol{x}, \boldsymbol{y})\zeta(x_0, y_0),$$

where ζ and T_i are defined in the paper, the constraint can be rewritten as

$$\sum_{\boldsymbol{y}\in\{0,1\}^N} T_i(\boldsymbol{x}, \boldsymbol{y})\zeta(x_0, y_0) \left(\pi_0(\boldsymbol{x}, \boldsymbol{y}) - \pi_0(x_0', \boldsymbol{x}_{-0}, \boldsymbol{y})\right) \geq 0$$

Since the mechanism is anonymous, the probability of choosing agent 0 depends only on the reports received by 0 and the numbers of reports, $(z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}) \in S_4(n-1)$, received by the remaining agents. Since the LHS of every constraint takes a sum over all possible reports of the referee, every constraint for given $x_0 \in \{0,1\}$ and $\boldsymbol{x}_{-0} \in \{0,1\}^{N\setminus\{0\}}$ depends on the number of values 1 and the number of values 0 in $\boldsymbol{x}_{-i}$. Hence, under the representation of anonymous mechanisms by a function $\varrho\{0,1\}^2 \times S_4(n-1) \times [0,1]$, there is one EPIC constraint for every signal value $x_0 \in \{0,1\}$ and every number, $k \in \{0,\dots,n-1\}$, of values 1 among the reports of the remaining agents. There are $2n$ constraints in total. The constraint for given $x_0 =\in \{0,1\}$ and $k \in \{0,\dots,n-1\}$ is

$$\sum_{y_0\in\{0,1\}} \sum_{\substack{(z_{0,0},z_{0,1})\in S_2(n-k-1)\\(z_{1,0},z_{1,1})\in S_2(k)}} \binom{n-k-1}{z_{0,0}, z_{0,1}}\binom{k}{z_{1,0}, z_{1,1}}$$
$$\frac{Q^{z_{0,0}+z_{1,1}}(1-Q)^{z_{0,1}+z_{1,0}}}{2^{n-1}}\zeta(x_0, y_0)\left(\varrho(x_0, y_0, \boldsymbol{z}) - \varrho(x_0', y_0, \boldsymbol{z})\right) \geq 0,$$

where $x_0' = 1 - x_0$. Since

$$\binom{n-k}{z_{0,0}, z_{0,1}}\binom{k}{z_{1,0}, z_{1,1}} = \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} \Big/ \binom{n}{k}$$

so, multiplying both sides by $\binom{n}{k}$, the constraint can be further rewritten as

$$\sum_{y_0\in\{0,1\}} \sum_{\substack{(z_{0,0},z_{0,1})\in S_2(n-k-1)\\(z_{1,0},z_{1,1})\in S_2(k)}} U(\boldsymbol{z})\zeta(x_0, y_0)\left(\varrho(x_0, y_0, \boldsymbol{z}) - \varrho(x_0', y_0, \boldsymbol{z})\right) \geq 0,$$

Obviously, EPIC constraints are stronger than BIC constraint and since, by Theorem 1, all optimal BIC mechanisms are lexicographic so we can restrict attention to lexicographic EPIC mechanisms. Moreover, since any optimal BIC mechanism satisfies the BIC constraint for the low type with equality and since (as we showed above) the BIC constraint for the low type is a weighted sum of the EPIC constraints for the low type so we can restrict attention to lexicographic EPIC mechanism that satisfy the constraints for the low type with equality.

In all lexicographic mechanisms $\varrho(x_0, 0, \boldsymbol{z}) = 0$ if $z_{0,1} + z_{1,1} > 0$ (if there are agents receiving high report from the external referee, no agent receiving low report from the external referee is chosen by the mechanism). Using that together with the probability constraint (11), in the case of $x_0 = 0$ the constraint for given $k \in \{0, \dots, n-1\}$ can be rewritten as

$$\sum_{\substack{(z_{0,0},z_{0,1})\in S_2(n-k-1)\\(z_{1,0},z_{1,1})\in S_2(k)\\ \text{s.t. } z_{1,1}>0}} (1-Q)U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) + \sum_{\substack{(z_{0,0},z_{0,1})\in S_2(n-k-1)\\(z_{1,0},z_{1,1})\in S_2(k)\\ \text{s.t. } z_{0,1}=z_{1,1}=0,\ z_{1,0}>0}} QU(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) +$$
$$\left(\frac{1-Q}{Q}\right) \sum_{\substack{(z_{0,0},z_{0,1})\in S_2(n-k-2)\\(z_{1,0},z_{1,1})\in S_2(k+1)\\ \text{s.t. } z_{1,1}>0}} (1-Q)U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) +$$
$$\left(\frac{Q}{1-Q}\right) \sum_{\substack{(z_{0,0},z_{0,1})\in S_2(n-k-2)\\(z_{1,0},z_{1,1})\in S_2(k+1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0,\ z_{1,0}>0}} QU(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) = D_k$$

where

$$D_k = \binom{n}{k+1}\left(\frac{1}{n2^{n-1}}\right)\left(\frac{(1-Q)\left(1-(1-Q)^{k+1}+Q^{n-k+1}(1-Q)^{k-1}\right)}{Q}\right) - \binom{n}{k}\frac{[k>0]}{n2^{n-1}}\left(1-Q^{n-k}\right)(1-Q)^k - \frac{[k=0]}{n2^{n-1}}.$$

Given $k \in \{0, \dots, n-1\}$ let

$$X_k = \sum_{\substack{(z_{0,0},z_{0,1})\in S_2(n-k-1)\\(z_{1,0},z_{1,1})\in S_2(k)\\ \text{s.t. } z_{1,1}>0}} (1-Q)U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})$$
$$Y_k = \sum_{\substack{(z_{0,0},z_{0,1})\in S_2(n-k-1)\\(z_{1,0},z_{1,1})\in S_2(k)\\ \text{s.t. } z_{0,1}=z_{1,1}=0,\ z_{1,0}>0}} QU(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}).$$

Using that the constraint for $x_0 = 0$ and $k \in \{0, \dots, n-2\}$ can be rewritten as

$$X_k + Y_k + \left(\frac{1-Q}{Q}\right)X_{k+1} + \left(\frac{Q}{1-Q}\right)Y_{k+1} = D_k$$

and the constraint for $k = n-1$ can be rewritten as

$$X_k + Y_k = D_k.$$

In addition, $X_0 = 0$ and $Y_0 = 0$.

Suppose that ϱ is an optimal BIC mechanism. We will show that it does not satisfy at least one of the EPIC constraints with equality.

Suppose that $Q < \varphi_n^{(-1)}(2^n)$. By point 4a of Lemma 4, if $z_{1,1} = z_{0,1} = 0$ and $z_{1,0} > 0$ then $\varrho(0,0,\boldsymbol{z}) = \frac{1}{z_{0,0}+1}$. Hence

$$Y_k = B_k = \sum_{\substack{(z_{0,0},z_{0,1})\in S_2(n-k-1)\\ (z_{1,0},z_{1,1})\in S_2(k)\\ \text{s.t. } z_{0,1}=z_{1,1}=0,\ z_{1,0}>0}} QU(\boldsymbol{z})\frac{1}{z_{0,0}+1} = \binom{n}{k}\frac{[k>0]}{n2^{n-1}}(1-(1-Q)^{n-k})(1-Q)^k$$

and the EPIC constraints for $x_0 = 0$ and $k \in \{0,\dots,n-2\}$ can be rewritten as

$$X_k + \left(\frac{1-Q}{Q}\right) X_{k+1} = E_k.$$

and the constraint for $k = n-1$ can be rewritten as

$$X_k = E_k,$$

where

$$E_k = D_k - B_k - \left(\frac{Q}{1-Q}\right) B_{k+1}$$

and $X_0 = 0$.

Solving the system of equations determined by the constraints (starting from X_{n-1} and deriving subsequent X_k for decreasing k) we obtain, for $k \in \{0,\dots,n-1\}$

$$X_k = \sum_{i=0}^{n-k-1}(-1)^i\left(\frac{1-Q}{Q}\right)^i E_{k+i}.$$

Since $X_0 = 0$ so the solution is valid only if

$$\sum_{i=0}^{n-1}(-1)^i\left(\frac{1-Q}{Q}\right)^i E_i = 0. \tag{70}$$

Using binomial formula,

$$\sum_{i=0}^{n-1}(-1)^i\left(\frac{1-Q}{Q}\right)^i E_i = \frac{1}{n2^{n-1}}\left(\frac{2Q-1}{(1-Q)^2}\right)W(n)$$

where

$$W(n) = \left(Q^n - \left(\frac{Q^2}{2Q-1}\right)\left(\frac{2Q-1}{Q}\right)^n + \left(1-\frac{(1-Q)^2}{Q}\right)^n - \left(\frac{(1-Q)(2Q-1)}{Q}\right)^n + (1-Q)^n - 1\right).$$

Since

$$\frac{Q^2}{2Q-1} = 1 + \frac{(1-Q)^2}{2Q-1}$$

so, since $Q \in (1/2, 1)$,

$$W(n) < V(n) = \left(Q^n - \left(\frac{2Q-1}{Q}\right)^n + \left(1-\frac{(1-Q)^2}{Q}\right)^n - \left(\frac{(1-Q)(2Q-1)}{Q}\right)^n + (1-Q)^n - 1\right).$$

Using the fact that, for any $r \in \mathbb{R}$,

$$r^n = 1 - (1-r)\sum_{i=0}^{n} r^i,$$

$$V(n) = \sum_{i=0}^{n-1}\Bigg(\left(\frac{1-Q}{Q}\right)\left(\frac{2Q-1}{Q}\right)^i + \left(Q + \frac{(1-Q)^2}{Q}\right)\left(\frac{(1-Q)(2Q-1)}{Q}\right)^i - (1-Q)Q^i - Q(1-Q)^i - \left(\frac{(1-Q)^2}{Q}\right)\left(1 - \frac{(1-Q)^2}{Q}\right)^i\Bigg)$$

Since $V(2) = -(1-Q)^4/Q^2$ so, for $n \geq 2$,

$$V(n) = \sum_{i=2}^{n-1}\Bigg(\left(\frac{1-Q}{Q}\right)\left(\frac{2Q-1}{Q}\right)^i + \left(Q + \frac{(1-Q)^2}{Q}\right)\left(\frac{(1-Q)(2Q-1)}{Q}\right)^i - (1-Q)Q^i - Q(1-Q)^i - \left(\frac{(1-Q)^2}{Q}\right)\left(1 - \frac{(1-Q)^2}{Q}\right)^i\Bigg) - \frac{(1-Q)^4}{Q^2}$$

Since

$$\frac{1-Q}{Q} = 1 - Q + \frac{(1-Q)^2}{Q}, \quad \frac{2Q-1}{Q} = Q - \frac{(1-Q)^2}{Q}, \text{ and}$$
$$\frac{(1-Q)(2Q-1)}{Q} = 1 - Q - \frac{(1-Q)^2}{Q}$$

so, in the case of $Q \in (1/2, 1)$ and $n \geq 2$,

$$V(n) < \sum_{i=2}^{n-1}\Bigg(\left(\frac{(1-Q)^2}{Q}\right)\left(\frac{2Q-1}{Q}\right)^i + \left(\frac{(1-Q)^2}{Q}\right)\left(\frac{(1-Q)(2Q-1)}{Q}\right)^i - \left(\frac{(1-Q)^2}{Q}\right)\left(1 - \frac{(1-Q)^2}{Q}\right)^i\Bigg).$$

Since

$$\frac{2Q-1}{Q} + \frac{(1-Q)(2Q-1)}{Q} < 1 - \frac{(1-Q)^2}{Q}$$

and, for $i \geq 1$,

$$\left(\frac{2Q-1}{Q}\right)^i + \left(\frac{(1-Q)(2Q-1)}{Q}\right)^i \leq \left(\frac{2Q-1}{Q} + \frac{(1-Q)(2Q-1)}{Q}\right)^i$$

so $V(n) < 0$. Hence, for any $Q \in (1/2, 1)$ and $n \geq 2$, $W(n) < 0$ and, consequently, (70) is not satisfied for any $Q \in (1/2, 1)$. Thus in the case of $Q < \varphi_n^{(-1)}(2^n)$ (and $Q \in (1/2, 1)$) no optimal BIC mechanisms satisfies the EPIC constraints.

Suppose that $Q \geq \varphi_n^{(-1)}(2^n)$. By point 5b of Lemma 4, if $z_{1,1} > 0$ then $\varrho(0, 1, \boldsymbol{z}) = 0$. Hence $X_k = 0$, for all $k \in \{0, \dots, n-1\}$, and the EPIC constraints for $x_0 = 0$ and $k \in \{0, \dots, n-2\}$ can be rewritten as

$$Y_k + \left(\frac{Q}{1-Q}\right) Y_{k+1} = D_k.$$

and the constraint for $k = n-1$ can be rewritten as

$$Y_k = D_k,$$

where $Y_0 = 0$.

Solving the system of equations determined by the constraints (starting from Y_{n-1} and deriving subsequent Y_k for decreasing k) we obtain, for $k \in \{0, \dots, n-1\}$

$$Y_k = \sum_{i=0}^{n-k-1} (-1)^i \left(\frac{Q}{1-Q}\right)^i D_{k+i}.$$

Since $Y_0 = 0$ so the solution is valid only if

$$\sum_{i=0}^{n-1} (-1)^i \left(\frac{Q}{1-Q}\right)^i D_i = 0. \tag{71}$$

Using binomial formula,

$$\sum_{i=0}^{n-1} (-1)^i \left(\frac{Q}{1-Q}\right)^i D_i = \frac{1}{n2^{n-1}} \left(\frac{2Q-1}{Q^2}\right) W(n)$$

where

$$W(n) = (-1)^{n+1} \frac{(2Q-1)^{n-1}}{(1-Q)^{n-2}} - (1-Q)^n.$$

If n is even then, since $Q \in (1/2, 1)$ and $n \geq 2$, $W(n) < 0$ and, consequently, (71) is not satisfied for any $Q \in (1/2, 1)$. If n is odd, i.e. $n \geq 3$ and odd, then $W(n) = 0$ if and only if $(2Q-1)^{n-1} = (1-Q)^{2(n-1)}$. Hence $W(n) = 0$ if and only if

$$2Q - 1 = (1-Q)^2$$

which, in the case of $Q \in (1/2, 1)$ holds for $Q = 2 - \sqrt{2}$. By Fact 1, $\varphi_n^{(-1)}(2^n) > 2/3$ and, since $2 - \sqrt{2} < 2/3$, $\varphi_n^{(-1)}(2^n) > 2 - \sqrt{2}$. Hence for all $\varphi_n^{(-1)}(2^n) \leq Q < 1$, $W(n) \neq 0$ and any odd $n \geq 3$, $W(n) \neq 0$ and, consequently, (71) is not satisfied. Thus in the case of $Q \geq \varphi_n^{(-1)}(2^n)$ (and $Q \in (1/2, 1)$) no optimal BIC mechanisms satisfies the EPIC constraints. □

C Online Appendix 2: Robustness checks

C.1 Robustness checks

We assume $n = 2$ and run four robustness checks. We first consider a non-symmetric structure where the probability of high and low type agents are different. We then extend the model by allowing for more than two signals for the referees. Finally, we show that there is no loss of generality in assuming anonymity and non-wastefulness of the selection mechanism.

C.2 Non-symmetric type structure

We first check whether our results extend to a model where the signal and type structures are non-symmetric. While there are many ways to generalize the model, we focus on a situation where the probability of the two types is not fixed at 1/2, but can vary. We let $\rho \in (0, 1)$ denote the probability that the candidate is a high type.

Define

$$f(\rho) = \frac{\rho^2}{\rho^2 + (1-\rho)^2}$$

Then, f is an increasing function with $f(0) = 0, f(1/2) = 1/2$ and $f(1) = 1$. We show that as long as $p > f(\rho)$,[13] the analysis of the benchmark model generalizes: the optimal mechanism for the planner is a lexicographic mechanism where the candidate is selected according to the reports of the referees and ties are broken using self-reports when the two reports of the referees are identical. Interestingly, this mechanism becomes infeasible when the inequality is reversed. In that case, the construction of a compensating mechanism, where an increase in the probability of selecting the candidate reporting the high type when the two referees report a high type is compensated by a decrease in the probability of selecting a candidate reporting a high type when the two types are low, becomes too costly.

Formally, as in the benchmark model, using the no-waste and anonymity assumptions, an optimal mechanism is characterized by the six probabilities:

$$\pi_1 = \pi(10, 10), \pi_2 = \pi(10, 00), \pi_3 = \pi(00, 10),$$
$$\pi_4 = \pi(10, 01), \pi_5 = \pi(10, 11), \pi_6 = \pi(11, 10).$$

Proposition 6. *Whenever the referees receive different signals, the optimal mechanism selects the candidate with the high signal:* $\pi_1 = 1, \pi_3 = 1, \pi_4 = 0$ *and* $\pi_6 = 1$.

1. *If* $p > f(\rho)$ *then the optimal mechanism selects*

$$\pi_5 = 1, \pi_2 = \frac{(1-\rho)^2 p - \rho^2(1-p)}{[(1-\rho)r + \rho(1-r)][(1-\rho)pr + \rho(1-p)(1-r)]}$$

2. *If* $p < f(\rho)$ *then the optimal mechanism selects*

$$\pi_5 = \pi_2 = \frac{1}{2}.$$

[13]Note that since $f(1/2) = 1/2$ and $p > 1/2$, this inequality was satisfied in the benchmark model.

Proof of Proposition 6. We define

$$\begin{aligned}
Q_{00} &= (1-\rho)pr + \rho(1-p)(1-r) \\
Q_{01} &= (1-\rho)p(1-r) + \rho(1-p)r \\
Q_{10} &= (1-\rho)(1-p)r + \rho p(1-r) \\
Q_{11} &= (1-\rho)(1-p)(1-r) + \rho pr
\end{aligned}$$

The expected value of the planner is

$$V = A + a_1\pi_1 + a_2\pi_2 + a_3\pi_3 + a_4\pi_4 + a_5\pi_5 + a_6\pi_6,$$

where

$$\begin{aligned}
a_1 &= prQ_{00} - (1-p)(1-r)Q_{11}, \\
a_2 &= p(1-r)Q_{00} - (1-p)(1-r)Q_{10}, \\
a_3 &= (1-p)rQ_{00} - (1-p)(1-r)Q_{01}, \\
a_4 &= p(1-r)Q_{01} - (1-p)rQ_{10}, \\
a_5 &= prQ_{01} - (1-p)rQ_{11}, \\
a_6 &= prQ_{10} - p(1-r)Q_{11}
\end{aligned}$$

The incentive constraint of the low type is given by

$$B - b_1\pi_1 - b_2\pi_2 - b_3\pi_3 - b_4\pi_4 - b_5\pi_5 - b_6\pi_6 \geq 0$$

with

$$\begin{aligned}
B &= Q_{00}[\frac{Q_{00}}{2} + Q_{01} + \frac{Q_{10}}{2}] + Q_{01}[\frac{Q_{01}}{2} + Q_{10} + \frac{Q_{11}}{2}], \\
b_1 &= Q_{00}(Q_{01} + Q_{11}), \\
b_2 &= Q_{00}(Q_{00} + Q_{10}), \\
b_3 &= 0, \\
b_4 &= Q_{01}(Q_{00} + Q_{10}) \\
b_5 &= Q_{01}(Q_{01} + Q_{11}) \\
b_6 &= Q_{01}Q_{10} - Q_{00}Q_{11}
\end{aligned}$$

and the incentive constraint of the high type is given by

$$C + c_1\pi_1 + c_2\pi_2 + c_3\pi_3 + c_4\pi_4 + c_5\pi_5 + c_6\pi_6 \geq 0$$

with

$$
\begin{aligned}
C &= Q_{10}[-\frac{Q_{00}}{2} - Q_{01} - \frac{Q_{10}}{2}] + Q_{11}[-\frac{Q_{01}}{2} - Q_{10} - \frac{Q_{11}}{2}], \\
c_1 &= Q_{11}(Q_{10} + Q_{00}), \\
c_2 &= Q_{10}(Q_{00} + Q_{10}), \\
c_3 &= Q_{01}Q_{10} - Q_{00}Q_{11}, \\
c_4 &= Q_{10}(Q_{01} + Q_{10}) \\
c_5 &= Q_{11}(Q_{01} + Q_{11}) \\
c_6 &= 0
\end{aligned}
$$

Now, we immediately obtain the following inequalities:

$$
\begin{aligned}
Q_{00}Q_{11} - Q_{01}Q_{10} = \rho(1-\rho)(2p-1)(2r-1) &> 0, \\
rQ_{10} - (1-r)Q_{11} = (1-\rho)(1-p)(2r-1) &> 0, \\
rQ_{00} - (1-r)Q_{01} = (1-\rho)p(2r-1) &> 0.
\end{aligned}
$$

We first consider the relaxed problem where the planner only faces the constraint of the low type. Clearly as $b_3 = 0$ and $b_6 < 0$, at the optimum, the planner chooses $\pi_3 = \pi_6 = 1$. For the four remaining probabilities we observe that

$$\frac{a_1}{b_1} > \frac{a_5}{b_5}, \frac{a_2}{b_2} > \frac{a_4}{b_4}.$$

The only unresolved comparison is the comparison between $\frac{a_5}{b_5}$ and $\frac{a_2}{b_2}$ and we find that $\frac{a_5}{b_5} > \frac{a_2}{b_2}$ if and only if

$$p > \frac{\rho^2}{(1-\rho)^2 + \rho^2}.$$

In that case, the planner optimally chooses to increase π_5 to 1 and lower π_2 so that the incentive constraint of the low type is satisfied. It is easy to check that the incentive constraint of the high type is also satisfied.

Next, suppose that $(1-\rho)^2 p - \rho^2(1-p) < 0.$. Suppose first that the incentive constraint of the high type is slack. Then the planner must optimally choose $\pi_2 = 1$ and $\pi_5 < \frac{1}{2}$ to satisfy the incentive constraint fo the low type. However, the incentive constraint of the high type will be violated. Suppose next that the incentive constraint of the low type is slack and consider the relaxed problem where the planner only faces the incentive constraint of the high type. It is easy to check that the planner optimally chooses $\pi_3 = \pi_6 = 1$. In addition,

$$\frac{a_1}{c_1} > \frac{a_5}{c_5}, \frac{a_2}{c_2} > \frac{a_4}{c_4}.$$

And, if $b_2 < b_5$, then necessarily $c_2 < c_5$, so that the planner optimally selects $\pi_2 = 1$ and π_5 to satisfy the incentive constraint of the high type. However, the incentive constraint of the low type will then be violated.

We conclude that both incentive constraints must be binding at the optimum. It is easy to check that the optimal mechanism when both incentive constraints are binding is the optimal DIC mechanism where $\pi_5 = \pi_2 = \frac{1}{2}$. □

C.3 Multiple signals for the referees

In the second robustness check, we let the number of signal values increase. Due to the significantly higher complexity of the problem (in particular the increase of the number of BIC constraints) we restrict attention to scenarios where there are two agents with the same binary signals as in the benchmark model, while the set of signals of the referee is $S = \{s_1, \dots, s_m\}$ where $0 = s_1 < \dots < s_m = 1$ and $m \geq 2$. We restrict attention to the case of $n = 2$ agents. Like in the benchmark model, we assume that $p > 1/2$. In the case of signals of the referee, we assume that for each $i \in \{0,1\}$, $\mathbf{Pr}(Y_i = y \mid \Theta_i = 1)$ is strictly increasing in y and $\mathbf{Pr}(Y_i = y \mid \Theta_i = 0)$ is strictly decreasing in y. Like in the benchmark model, we assume that for each agent i and each type $t \in \{0,1\}$, random variables $X_i \mid \Theta_i = t$ and $Y_i \mid \Theta_i = t$ are independent, i.e. for all $x \in \{0,1\}$ and $y \in S$, $\mathbf{Pr}(X_i = x, Y_i = y \mid \Theta_i = t) = \mathbf{Pr}(X_i = x \mid \Theta_i = t)\mathbf{Pr}(Y_i = y \mid \Theta_i = t)$. Like in the benchmark model, we assume anonymity across the agents, i.e. for all $x \in \{0,1\}$, $y \in S$, and $t \in \{0,1\}$, $\mathbf{Pr}(X_0 = x \mid \Theta_0 = t) = \mathbf{Pr}(X_1 = x \mid \Theta_1 = t)$ and $\mathbf{Pr}(Y_0 = y \mid \Theta_0 = t) = \mathbf{Pr}(Y_1 = y \mid \Theta_1 = t)$.

Like in the case of two agents in the benchmark model, an optimal DSIC mechanism selects the agent with higher report from the referee and, in the case of a tie in these reports, selects each of the agents with probability 1/2. Consider a mechanism defined as follows. Let s^* be the minimal signal value in S such that

$$\sum_{\substack{y \in S \\ y > s^*}} \mathbf{Pr}(Y_0 = y)\mathbf{Pr}(X_0 = 0, Y_0 = y) < \frac{1}{2} \sum_{y \in S} \mathbf{Pr}(Y_0 = y)\mathbf{Pr}(X_0 = 0, Y_0 = y),$$

and let

$$q = \frac{\frac{1}{2}\sum_{y \in S} \mathbf{Pr}(Y_0 = y)\mathbf{Pr}(X_0 = 0, Y_0 = y) - \sum_{\substack{y \in S \\ y > s^*}} \mathbf{Pr}(Y_0 = y)\mathbf{Pr}(X_0 = 0, Y_0 = y)}{\mathbf{Pr}(Y_0 = s^*)\mathbf{Pr}(X_0 = 0, Y_0 = s^*)}.$$

Let

$$\pi^{\mathrm{L}}(x_0, x_1, y_0, y_1) = \begin{cases} 1, & \text{if } y_0 > y_1, \\ 1, & \text{if } y_0 = y_1 > s^* \text{ and } x_0 > x_1, \\ q, & \text{if } y_0 = y_1 = s^* \text{ and } x_0 > x_1, \\ 0, & \text{if } y_0 = y_1 < s^* \text{ and } x_0 > x_1, \end{cases}$$

and the remaining values of π^{L} are determined by the anonymity and the no-waste properties. The mechanism is a generalization of the lexicographic mechanism for two agents and $m = 2$ signal values of the referee to $m \geq 2$ signal values of the referee. The mechanism follows the reports of the referees and, in case of a tie in these reports, uses the reports of the agents to select an agent. If the reports of the referee are above the threshold s^* the mechanism follows the reports of the agents and selects the agent with the higher self-report. If the reports are below the threshold s^* then the mechanism goes against the reports of the agents and selects the agent with lower self-report.

It is elementary to verify that π^{L} satisfies both BIC constraints, the one for the low type with equality and the one for the hight type with a slack. The following fact states that if

$$\frac{p}{1-p} > \frac{\mathbf{Pr}(Y_0 = s_m \mid \Theta_0 = 1)}{\mathbf{Pr}(Y_0 = s_m \mid \Theta_0 = 0)} \frac{\mathbf{Pr}(Y_0 = s_{m-1} \mid \Theta_0 = 1)}{\mathbf{Pr}(Y_0 = s_{m-1} \mid \Theta_0 = 0)} \tag{72}$$

then the mechanism has strictly higher value to the social planner than an optimal DSIC mechanism. Notice that (72) is satisfied by the benchmark model, as the RHS is equal to 1 when $m = 2$, while the LHS > 1 due to the assumption that $p > 1/2$.

Proposition 7. *If* (72) *is satisfied then* π^{L} *has strictly greater value to the social planner than an optimal DSIC.*

Proof of Proposition 7. The difference in social planner's payoff from mechanism π and a DSIC mechanism is equal to

$$\Delta = 2\left(\sum_{\substack{y\in S\\ y>s^*}} b(1,y,0,y) + qb(1,s^*,0,s^*) - \frac{1}{2}\sum_{y\in S} b(1,y,0,y)\right),$$

where

$$b(x_0,y_0,x_1,y_1) = \mathbf{Pr}(X_0=x_0,Y_0=y_0)\mathbf{Pr}(X_1=x_1,Y_1=y_1)d(x_0,y_0,x_1,y_1),$$

and

$$d(x_0,y_0,x_1,y_1) = \mathbf{Pr}(\Theta_0=1 \mid X_0=x_0,Y_0=y_0) - \mathbf{Pr}(\Theta_1=1 \mid X_1=x_1,Y_1=y_1).$$

Using the anonymity and independence assumptions,

$$\begin{aligned} b(1,y,0,y) &= \frac{1}{4}\mathbf{Pr}(Y_0=y \mid \Theta_0=0)\mathbf{Pr}(Y_0=y \mid \Theta_0=1)(2p-1) \\ &= P(Y_0=y)P(X_0=0,Y_0=y)\sigma(y), \end{aligned}$$

where

$$\sigma(y) = \frac{2p-1}{1+(1-p)\frac{\mathbf{Pr}(Y_0=y|\Theta_0=1)}{\mathbf{Pr}(Y_0=y|\Theta_0=0)} + p\frac{\mathbf{Pr}(Y_0=y|\Theta_0=0)}{\mathbf{Pr}(Y_0=y|\Theta_0=1)}}.$$

Let $h(y) = \mathbf{Pr}(Y_0=y \mid \Theta_0=1)/\mathbf{Pr}(Y_0=y \mid \Theta_0=0)$. Since $\mathbf{Pr}(Y_0=y \mid \Theta_0=1)$ is strictly increasing in y and $\mathbf{Pr}(Y_0=y \mid \Theta_0=0)$ is strictly decreasing in y so $h(y)$ is strictly increasing in y. Notice that, for $y > y'$,

$$p\frac{1}{h(y)} + (1-p)h(y) - p\frac{1}{h(y')} - (1-p)h(y') = (h(y)-h(y'))\left(1-p-\frac{p}{h(y)h(y')}\right).$$

Since $y > y'$ and h is increasing so this is negative when

$$\frac{p}{1-p} > h(y)h(y') = \frac{\mathbf{Pr}(Y_0=y \mid \Theta_0=1)}{\mathbf{Pr}(Y_0=y \mid \Theta_0=0)}\frac{\mathbf{Pr}(Y_0=y' \mid \Theta_0=1)}{\mathbf{Pr}(Y_0=y' \mid \Theta_0=0)},$$

in which case $\sigma(y) > \sigma(y')$. Since h is increasing so $h(y)h(y')$ takes its maximal value when $y = s_m$ and $y' = s_{m-1}$. This if

$$\frac{p}{1-p} > \frac{\mathbf{Pr}(Y_0=s_m \mid \Theta_0=1)}{\mathbf{Pr}(Y_0=s_m \mid \Theta_0=0)}\frac{\mathbf{Pr}(Y_0=s_{m-1} \mid \Theta_0=1)}{\mathbf{Pr}(Y_0=s_{m-1} \mid \Theta_0=0)}$$

then $\sigma(y)$ is strictly increasing in y.

Since

$$\sum_{\substack{y\in S\\ y>s^*}} \mathbf{Pr}(Y_0=y)\mathbf{Pr}(X_0=0,Y_0=y) + q\mathbf{Pr}(Y_0=s^*)\mathbf{Pr}(X_0=0,Y_0=s^*) = \frac{1}{2}\sum_{y\in S}\mathbf{Pr}(Y_0=y)\mathbf{Pr}(X_0=0,Y_0=y)$$

and $\sigma(y)$ is strictly increasing in y so

$$\sum_{\substack{y\in S\\ y>s^*}} \mathbf{Pr}(Y_0 = y)\mathbf{Pr}(X_0 = 0, Y_0 = y)\sigma(y)$$
$$+ q\mathbf{Pr}(Y_0 = s^*)\mathbf{Pr}(X_0 = 0, Y_0 = s^*)\sigma(s^*) >$$
$$\frac{1}{2}\sum_{y\in S} \mathbf{Pr}(Y_0 = y)\mathbf{Pr}(X_0 = 0, Y_0 = y)\sigma(y).$$

□

Proposition 7 shows that if p is high enough, it is possible to design a BIC mechanism that has a greater value to the social planner than an optimal DSIC mechanism by changing the probabilities of selecting an agent with higher self report when reports of the referee are the same for both agents. Because $\sigma(y)$ defined in the proof of the proposition is strictly increasing, mechanism π^{L} is optimal in the class of such mechanisms. The problem with this class of mechanisms is that the gain in value happens in the cases when the two signals of the referee are equal – an event the probability of which goes to 0 when the number of signals increases. Can there be BIC mechanisms which improve the value even further and select agents on the basis of their self reports overriding the reports of the referee? We give below an example that shows that such mechanisms are indeed possible.

Example 2. *Let $m = 4$, and for all $i \in \{0, 1\}$, let*

$$\mathbf{Pr}(Y_i = s_1 \mid \Theta_i = 1) = \frac{1}{18}, \qquad \mathbf{Pr}(Y_i = s_3 \mid \Theta_i = 1) = \frac{5}{18},$$
$$\mathbf{Pr}(Y_i = s_2 \mid \Theta_i = 1) = \frac{2}{18}, \qquad \mathbf{Pr}(Y_i = s_4 \mid \Theta_i = 1) = \frac{10}{18},$$

and for all $j \in \{1, 2, 3, 4\}$, $\mathbf{Pr}(Y_i = s_j \mid \Theta_i = 0) = \mathbf{Pr}(Y_i = s_{5-j} \mid \Theta_i = 1)$. Let $p > 25/26$. Notice the the probabilities satisfy condition (72).

The utility of the social planner from a DSIC mechanism is equal to $149/216$. Mechanism π^{L} in this case is defined by $s^* = s_1$ and $\pi^{\mathrm{L}}(1, 0, s_1, s_1) = q = (99p - 74)/1296 \in (551/33696, 650/33696)$. The utility of the social planner from mechanism π^{L} is $149/216 + 5(2p - 1)(99p + 1222)/419904 > 149/216$ when $p > 25/26$.

Consider a mechanism π' defined for (x_0, x_1, y_0, y_1) such that either $x_0 > x_1$ or $x_0 = x_1$ and $y_0 > y_1$ as follows

$$\pi'(x_0, x_1, y_0, y_1) = \begin{cases} 1, & \text{if } (x_0, x_1, y_0, y_1) = (1, 0, s_3, s_4) \\ \frac{612589}{370656} - \frac{7}{9p+1}, & \text{if } (x_0, x_1, y_0, y_1) = (1, 0, s_2, s_2) \\ 0, & \text{if } (x_0, x_1, y_0, y_1) = (1, 0, s_1, s_1) \\ \pi^{\mathrm{L}}(x_0, x_1, y_0, y_1), & \text{otherwise,} \end{cases}$$

and the remaining values of π' are determined by the anonymity and the no-waste properties. It is elementary to verify that π' satisfies the BIC constraint for the low type with equality and the BIC constraint for the high type with a slack. The utility of the social planner from mechanism π' exceeds the utility from π^{L} by $5(176687 - 1354637p + 1969064p^2 - 612612p^3)/(13343616(9p+1))$, which is positive for all $p \in (25/26, 1)$. Given that mechanism π^{L} is optimal in the class of mechanism that follow reports of the agents in the cases of tying reports of the referee only, the value of mechanism π' exceeds the value of all mechanisms in that class.

C.4 Non-anonymous mechanisms

Suppose that the planner is not constrained to choose anonymous mechanisms, and may treat the two agents differently so that $\pi_0(x_0x_1, y_0y_1) \neq \pi_1(x_1x_0, y_1y_0)$. Assuming that there is no waste, we show that the social planner cannot do better than in the optimal anonymous BIC mechanism, so that there is no loss of generality in focusing attention to anonymous mechanisms.

As the mechanism does not involve any waste, for any report $(\boldsymbol{x}, \boldsymbol{y})$, we have

$$\pi_0(\boldsymbol{x}, \boldsymbol{y}) + \pi_1(\boldsymbol{x}, \boldsymbol{y}) = 1.$$

This allows us to focus attention on the probability of assigning the object to agent 0, and we drop the subscript and denote $\pi(\boldsymbol{x}, \boldsymbol{y})$ the sixteen probabilities that characterize the mechanism.

Now, we write the social planner's objective function as

$$\begin{aligned}\Pi = 1 &- C\pi(00,01) - B\pi(01,00) - A\pi(01,01) + C\pi(00,10) - D\pi(01,10) - B\pi(01,11) \\ &+ B\pi(10,00) + D\pi(10,01) - C\pi(11,01) + A\pi(10,10) + B\pi(10,11) + C\pi(11,10)\end{aligned}$$

where

$$\begin{aligned}A &= p^2r^2 - (1-p)^2(1-r)^2 \\ B &= p^2r(1-r) - (1-p)^2r(1-r) \\ C &= r^2p(1-p) - (1-r)^2p(1-p) \\ D &= p^2(1-r)^2 - (1-p)^2r^2.\end{aligned}$$

It is easy to check that for $p < 1$ and $r > 1/2$, $A > B$, $B > C$ and $B > D$.

We consider the relaxed problem where the only constraints are the constraints of the agent of type 0 and write down the BIC constraints for agents 0 and 1 (which are now different as the two agents need not be treated anonymousally), giving

$$\begin{aligned}&Q^2(\pi(00,00) - \pi(10,00) + \pi(01,01) - \pi(11,01)) + \\ &(1-Q)Q(\pi(00,01) - \pi(10,01) + \pi(01,00) - \pi(11,00)) + \\ &Q(1-Q)(\pi(00,10) - \pi(10,10) + \pi(01,11) - \pi(11,11)) + \\ &(1-Q)^2(\pi(00,11) - \pi(10,11) + \pi(01,10) - \pi(11,10)) \geq 0,\end{aligned}$$

$$\begin{aligned}&Q^2(\pi(01,00) - \pi(00,00) + \pi(11,10) - \pi(10,10)] + \\ &(1-Q)Q(\pi(01,10) - \pi(00,10) + \pi(11,00) - \pi(10,00)) + \\ &Q(1-Q)(\pi(01,01) - \pi(00,01) + \pi(11,11) - \pi(10,11)) + \\ &(1-Q)^2(\pi(01,11) - \pi(00,11) + \pi(11,01) - \pi(10,01)) \geq 0.\end{aligned}$$

Proposition 8. *Suppose that $n = 2$. The value of the optimal BIC mechanism is equal to the value of the optimal anonymous BIC mechanism.*

Proof of Proposition 8. The We consider the relaxed problem where the only constraint is the sum of the constraints of the two agents:

$$\begin{aligned}&Q\pi(01,00) + Q\pi(01,01) + (1-Q)\pi(01,10) + (1-Q)\pi(01,11) \\ &-Q\pi(10,00) - (1-Q)\pi(10,01) - (2Q-1)\pi(11,01) - Q\pi(10,10) \\ &\qquad -(1-Q)\pi(10,11) + (2Q-1)\pi(11,10) \geq 0.\end{aligned} \tag{73}$$

We easily check that at the optimum, the constraint (73) is satisfied with equality. Furthermore, at the optimum we must have $\pi(11,10) = 1$, $\pi(11,01) = 0$, $\pi(00,01) = 0$, and $\pi(00,10) = 1$. For the eight remaining probabilities, we compute the index (the ratio between the linear coefficient in the social planner's objective function and in the BIC constraint) and obtain

probability	index	probability	index
$\pi(01,00)$	$\frac{B}{Q}$	$\pi(01,01)$	$\frac{A}{Q}$
$\pi(01,10)$	$\frac{D}{1-Q}$	$\pi(01,11)$	$\frac{B}{1-Q}$
$\pi(10,00)$	$\frac{B}{Q}$	$\pi(10,01)$	$\frac{D}{1-Q}$
$\pi(10,10)$	$\frac{A}{Q}$	$\pi(10,11)$	$\frac{B}{1-Q}$

We compute

$$A(1-Q) - BQ = p(1-p)(2r-1)(p+r-1) > 0$$

and

$$B(1-Q) - DQ = p(1-p)(2r-1)(p-r) > 0$$

so that we obtain the following ranking of indices: $\frac{A}{Q} > \frac{B}{1-Q} > \frac{B}{Q} > \frac{D}{1-Q}$.

We now establish the following simple claim:

Claim 2. *For any i, j, if $\beta_i > \beta_j$, then at the optimum $\pi_i \geq \pi_j$. In addition, either $\pi_i = 1$ or $\pi_j = 0$.*

Proof. Suppose that $\beta_i > \beta_j$ and that $\pi_i < \pi_j$. Consider a small increase in π_i by ϵ and a decrease in π_j by δ such that the BIC constraint remains unchanged:

$$b_i\epsilon - b_j\delta = 0.$$

This results in a change in the objective function of the social planner

$$\Delta\Pi = a_i\epsilon - a_j\delta = (\beta_i - \beta_j)\epsilon > 0.$$

resulting in a contradiction that proves the first statement of the claim. Next suppose that $\pi_i \geq \pi_j$ and π_i and π_j are both interior values in $(0,1)$. The same steps show that a small increase in π_i and a small decrease in π_j result in an increase in welfare. □

Now, for all probabilities with positive coefficient in the value function, by Claim 2 we obtain:

$$\pi(10,10) > \pi(10,11) > \pi(10,00) > \pi(10,01).$$

Similarly, for all probabilities with negative coefficients in the value function,

$$\pi(01,10) > \pi(01,00) > \pi(01,11) > \pi(01,01),$$

where in each ranking there is at most one probability which is strictly interior. In addition, we check that the interior probability in each of the two rankings must correspond to the same index, since otherwise there is a possible improvement by transferring probabilities. We check that the interior probabilities which are chosen to equalize the BIC constraints are, as in the anonymous mechanism, the probabilities $\pi(10,00)$ and $\pi(01,00)$ which must satisfy:

$$\pi(01,00) - \pi(10,00) = \frac{1-Q}{Q}.$$

As the objective function of the planner only depends on the difference $\pi(01,00) - \pi(10,00)$, any probability choice which results in the same difference $\pi(01,00)-\pi(10,00)$ gives rise to the same value for the planner. Finally, note that in the optimal anonymous BIC mechanism, $\pi(10,00) = 1 - 1/(2Q)$, so that $\pi(01,00) - \pi(10,00) = (1-Q)/Q$, and hence the value of the relaxed problem is equal to the value of the optimal anonymous BIC mechanism. □

C.5 Wasteful mechanisms

Suppose that the planner is not forced to select the project of one of the two agents. A *mechanism with waste* is a mechanism assigning to each report $(\boldsymbol{x}, \boldsymbol{y})$ two nonnegative numbers $\pi_0(\boldsymbol{x}, \boldsymbol{y})$ and $\pi_1(\boldsymbol{x}, \boldsymbol{y})$ such that

$$\pi_0(\boldsymbol{x}, \boldsymbol{y}) + \pi_1(\boldsymbol{x}, \boldsymbol{y}) \leq 1.$$

Allowing for waste but keeping anonymity, we write the objective function of the social planner as

$$\begin{aligned}\Pi = &(1-p)(1-r)Q(\pi(00,00) + \pi(01,01)) + (1-p)(1-r)(1-Q)(\pi(00,01) + \pi(01,00)) \\ &+ p(1-r)Q(\pi(10,00) + \pi(11,01)) + p(1-r)(1-Q)(\pi(10,01) + \pi(11,00)) \\ &+ (1-p)rQ(\pi(00,10) + \pi(01,11)) + (1-p)r(1-Q)(\pi(00,11) + \pi(01,10)) \\ &+ prQ(\pi(10,10) + \pi(11,11)) + pr(1-Q)(\pi(10,11) + \pi(11,10))\end{aligned}$$

while the BIC constraint of type 0 is given by

$$\begin{aligned}Q^2(\pi(00,00) - \pi(10,00) + \pi(01,01) - \pi(11,01)) +& \\ Q(1-Q)(\pi(00,01) - \pi(10,01) + \pi(01,00) - \pi(11,00)) +& \\ (1-Q)Q(\pi(00,10) - \pi(10,10) + \pi(01,11) - \pi(11,11)) +& \\ (1-Q)^2(\pi(00,11) - \pi(10,11) + \pi(01,10) - \pi(11,10)) \geq 0.&\end{aligned}$$

Assuming that the mechanism treats the two agents anonymously, we focus attention on the probability of assigning the good to the first agent, and, dispensing with the index, characterize a mechanism with the 16 probabilities $\pi(\boldsymbol{x}, \boldsymbol{y})$ where $\boldsymbol{x} \in \{0,1\}^2$ and $\boldsymbol{y} \in \{0,1\}^2$. The following Proposition shows that the optimal BIC mechanism is identical to the optimal BIC mechanism given in Section 4.

Proposition 9. *Suppose that $n = 2$. The optimal BIC mechanism does not induce any waste: $\pi_0(\boldsymbol{x}, \boldsymbol{y}) + \pi_1(\boldsymbol{x}, \boldsymbol{y}) = 1$ for any report $(\boldsymbol{x}, \boldsymbol{y})$.*

Proof of Proposition 9. Consider the relaxed problem of maximizing the objective function W under the BIC constraint of the agent of type 0. Because the BIC constraint is increasing in any of the eight probabilities $\pi(\mathbf{x}, \mathbf{y})$ where $x_0 = 0$, we must have

$$\pi(00,00) = \pi(00,11) = \frac{1}{2}, \pi(10,10) + \pi(01,01) = 1, \pi(00,01) + \pi(00,10) = 1,$$

$$\pi(01,00) + \pi(10,00) = \pi(01,11) + \pi(10,11) = \pi(01,10) + \pi(10,01) = 1.$$

In addition, we note that any probability transfer from $\pi(00,01)$ to $\pi(00,10)$ and from $\pi(11,01)$ to $\pi(11,10)$ which leaves the BIC constraint satisfied results in an increase in welfare so that $\pi(00,01) = 0$ and $\pi(11,01) = 0$. Next, we compute the indices, corresponding to the ratio of the linear coefficient in Π to the linear coefficient in the BIC constraint, $\beta_i = a_i/b_i$, of the probabilities which remain to be determined:

probability	index	probability	index
$\pi(10,11)$	$\frac{r(1-r)(2p-1)}{1-Q}$	$\pi(10,10)$	$p+r-1$
$\pi(10,01)$	$p-r$	$\pi(10,00)$	$\frac{r(1-r)(2p-1)}{Q}$
$\pi(11,11)$	$\frac{pr}{1-r}$	$\pi(11,00)$	$\frac{p(1-r)}{r}$
$\pi(11,10)$	$\frac{pr}{1-r}$		

We observe that, as $p(1-r)/r > r(1-r)(2p-1)/Q$, the indices of all three probabilities $\pi(11,11)$, $\pi(11,00)$ and $\pi(11,10)$ are greater than the indices of $\pi(10,00)$ and $\pi(10,01)$. Now recall that in the optimal BIC mechanism without waste $\pi(11,11) = \pi(11,00) = 1/2$, $\pi(11,10) = 1$, $\pi(10,00) < 1$ and $\pi(10,01) = 0$. This shows that the optimal BIC mechanism without waste is also optimal when waste is allowed, as any probability transfer to $\pi(10,00)$ or $\pi(10,01)$ which leaves the BIC constraint satisfied must result in a decrease in the objective function of the planner. $\square$

D Supplementary material

The supplementary material contains the detailed derivations of Equations used in the proof of Theorem 1.

D.1 Probability constraints and objective function under anonymous mechanisms

In this section we provide detailed derivations of Equations (11) and (8), expressing probability constraints and the objective function under the representation of anonymous mechanisms, $\varrho : \{0,1\}^2 \times S_4(n-1) \to [0,1]$.

By anonymity constraints, for any $i \in \{0,\ldots,n-1\}$, $j \in \{0,\ldots,n-1\} \setminus \{i\}$, $k \in \{0,\ldots,n-1\} \setminus \{i,j\}$, $\boldsymbol{x} \in \{0,1\}^N$, and $\boldsymbol{y} \in \{0,1\}^N$, $\pi_i(\boldsymbol{x},\boldsymbol{y}) = \pi_0(\boldsymbol{x}_{0\leftrightarrow i}, \boldsymbol{y}_{0\leftrightarrow i})$ and $\pi_k(\boldsymbol{x},\boldsymbol{y}) = \pi_k(\boldsymbol{x}_{i\leftrightarrow j}, \boldsymbol{y}_{i\leftrightarrow j})$. Using that, the probability constraints for each $(\boldsymbol{x},\boldsymbol{y}) \in \{0,1\}^N \times \{0,1\}^N$,

$$\sum_{i\in N} \pi_i(\boldsymbol{x},\boldsymbol{y}) = 1$$

can be rewritten as

$$\sum_{i\in N} \pi_0(\boldsymbol{x}_{0\leftrightarrow i}, \boldsymbol{y}_{0\leftrightarrow i}) = 1,$$

where, given a vector $\boldsymbol{x} = (x_0,\ldots,x_{n-1})$, $i \in \{0,\ldots,n-1\}$, and $j \in \{0,\ldots,n-1\} \setminus \{i\}$, $\boldsymbol{x}_{i\leftrightarrow j}$ is a vector obtained from vector $\boldsymbol{x}$ by swapping the values at positions i and j.

Adding all such probability constraints with the same values of x_0, y_0, and, for all $(b,b') \in \{0,1\}^2$, the same value of $|\{j \in N \setminus \{0\} : (x_j,y_j) = (b,b')\}|$, we obtain

$$(z_{x_0,y_0}+1)C(\boldsymbol{z})\varrho(x_0,y_0,\boldsymbol{z}) + \sum_{(b,b')\in\{0,1\}^2\setminus(x_0,y_0)} z_{b,b'} C(\boldsymbol{z}) \varrho(b,b',\boldsymbol{z}_{-(b,b'),(x_0,y_0)}, z_{b,b'}-1, z_{x_0,y_0}+1) = C(\boldsymbol{z}),$$

where

$$C(\boldsymbol{z}) = \binom{n-1}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} = \frac{(n-1)!}{z_{0,0}!z_{0,1}!z_{1,0}!z_{1,1}!}$$

is the multinomial coefficient equal to the number of ways in which the agents in $N \setminus \{0\}$ can be partition into four groups of sizes $z_{0,0}$, $z_{0,1}$, $z_{1,0}$, and $z_{1,1}$. Dividing both sides by $C(\boldsymbol{z})$ we obtain the probability constraints that have to be satisfied by the selection function ϱ,

$$(z_{x_0,y_0}+1)\varrho(x_0,y_0,\boldsymbol{z}) + \sum_{(b,b')\in\{0,1\}^2\setminus(x_0,y_0)} z_{b,b'} \varrho(b,b',\boldsymbol{z}_{-(b,b'),(x_0,y_0)}, z_{b,b'}-1, z_{x_0,y_0}+1) = 1,$$

which are exactly the probability constraints given in Equation (11).

We now rewrite the objective function under the representation $\varrho : \{0,1\}^2 \times S_4(n-1) \to [0,1]$. Given a mechanism $\pi : \{0,1\}^N \times \{0,1\}^N \to \Delta(N)$, the objective function is

$$\begin{aligned}
&\mathbf{E}\left(\sum_{i\in N} \pi_i(\boldsymbol{X},\boldsymbol{Y})\,[\Theta_i = 1]\right) = \\
&\quad \sum_{i\in N} \sum_{\boldsymbol{x}\in\{0,1\}^N} \sum_{\boldsymbol{y}\in\{0,1\}^N} \mathbf{Pr}(\Theta_i = 1)\mathbf{Pr}(X_i = x_i, Y_i = y_i \mid \Theta_i = 1)\mathbf{Pr}(\boldsymbol{X}_{-i} = \boldsymbol{x}_{-i}, \boldsymbol{Y}_{-i} = \boldsymbol{y}_{-i})\pi_i(\boldsymbol{x},\boldsymbol{y}) = \\
&\quad \frac{1}{2}\sum_{i\in N} \sum_{\boldsymbol{x}\in\{0,1\}^N} \sum_{\boldsymbol{y}\in\{0,1\}^N} \mathbf{Pr}(X_i = x_i, Y_i = y_i \mid \Theta_i = 1)\mathbf{Pr}(\boldsymbol{X}_{-i} = \boldsymbol{x}_{-i}, \boldsymbol{y}_{-i} = \boldsymbol{y}_{-i})\pi_i(\boldsymbol{x},\boldsymbol{y})\,.
\end{aligned}$$

For any $i \in N$, the probability that the agents other than i and the non-expert receive signals $(\boldsymbol{x}_{-i}, \boldsymbol{y}_{-i})$ about the projects in $N \setminus \{i\}$ is equal to

$$\begin{aligned}
&\mathbf{Pr}(\boldsymbol{X}_{-i} = \boldsymbol{x}_{-i}, \boldsymbol{Y}_{-i} = \boldsymbol{y}_{-i}) = T_i(\boldsymbol{x}, \boldsymbol{y}) \\
&= \prod_{j \in N \setminus \{i\}} (\mathbf{Pr}(\Theta_j = 0)\mathbf{Pr}(X_j = x_j, Y_j = y_j \mid \Theta_j = 0) + \mathbf{Pr}(\Theta_j = 1)\mathbf{Pr}(X_j = x_j, Y_j = y_j \mid \Theta_j = 1)) \\
&= \prod_{j \in N \setminus \{i\}} \frac{\mathbf{Pr}(X_j = x_j, Y_j = y_j \mid \Theta_j = 0) + \mathbf{Pr}(X_j = x_j, Y_j = y_j \mid \Theta_j = 1)}{2} \\
&= \prod_{j \in N \setminus \{i\}} \frac{\mathbf{Pr}(X_j = x_j \mid \Theta_j = 0)\mathbf{Pr}(Y_j = y_j \mid \Theta_j = 0) + \mathbf{Pr}(X_j = x_j \mid \Theta_j = 0)\mathbf{Pr}(Y_j = y_j \mid \Theta_j = 1)}{2} \\
&= \prod_{j \in N \setminus \{i\}} \frac{[x_j = y_j](pr + (1-p)(1-r)) + [x_j \neq y_j](p(1-r) + (1-p)r)}{2} \\
&= \prod_{j \in N \setminus \{i\}} \frac{[x_j = y_j]Q + [x_j \neq y_j](1-Q)}{2},
\end{aligned} \tag{74}$$

where, given $a \in \{0, 1\}$, the probability that agent i and the non-expert receive signals (x_i, y_i) about the project i of quality $\Theta_i = a$,

$$\mathbf{Pr}(X_i = x_i, Y_i = y_i \mid \Theta_i = a) = \xi_i^a(\boldsymbol{x}, \boldsymbol{y}) = ([x_i = a]p + [x_i \neq a](1-p))([y_i = a]r + [y_i \neq a](1-r)). \tag{75}$$

Given that, the objective function can be written as

$$\frac{1}{2} \sum_{i \in N} \sum_{\substack{\boldsymbol{x} \in \{0,1\}^N \\ \boldsymbol{y} \in \{0,1\}^N}} T_i(\boldsymbol{x}, \boldsymbol{y}) \xi_i^1(\boldsymbol{x}, \boldsymbol{y}) \pi_i(\boldsymbol{x}, \boldsymbol{y}). \tag{76}$$

Notice that, for any $i \in \{0, \ldots, n-1\}$, $\boldsymbol{x} \in \{0,1\}^N$, and $\boldsymbol{y} \in \{0,1\}^N$, $T_i(\boldsymbol{x}, \boldsymbol{y}) = T_0(\boldsymbol{x}_{0 \leftrightarrow i}, \boldsymbol{y}_{0 \leftrightarrow i})$ and $\xi_i^a(\boldsymbol{x}, \boldsymbol{y}) = \xi_0^a(\boldsymbol{x}_{0 \leftrightarrow i}, \boldsymbol{y}_{0 \leftrightarrow i})$. Therefore, the value of the objective function is equal to

$$\frac{n}{2} \sum_{\substack{\boldsymbol{x} \in \{0,1\}^N \\ \boldsymbol{y} \in \{0,1\}^N}} T_0(\boldsymbol{x}, \boldsymbol{y}) \xi_0^1(\boldsymbol{x}, \boldsymbol{y}) \pi_0(\boldsymbol{x}, \boldsymbol{y}). \tag{77}$$

Using (9) and (14), the objective function given in (77) can be rewritten as

$$\frac{n}{2} \sum_{(x_0, y_0) \in \{0,1\}^2} \sum_{\boldsymbol{z} \in S_4(n-1)} U(\boldsymbol{z}) \zeta^1(x_0, y_0) \varrho(x_0, y_0, \boldsymbol{z}),$$

which is the formulation given in Equation (8).

D.2 BIC constraints under anonymous mechanisms

There is one BIC constraint for every triple $(i, b, b') \in N \times \{0,1\}^2$ with $b \neq b'$. Hence there are $2n$ BIC constraints, overall. Since the BIC constraints are independent across the agents and given the form of the objective function in Equation (77), we can restrict attention to the BIC constraints for agent 0. There are two such BIC constraints. Given $(x_0, x_0') \in \{0, 1\}$ with $x_0 \neq x_0'$ the corresponding BIC constraint is

$$\mathbf{E}\left(\pi_0(x_0, \boldsymbol{X}_{-0}, \boldsymbol{Y}) \mid X_0 = x_0\right) - \mathbf{E}\left(\pi_0(x_0', \boldsymbol{X}_{-0}, \boldsymbol{Y}) \mid X_0 = x_0\right) \geq 0.$$

The LHS of the constraint can be rewritten as

$$\sum_{\boldsymbol{t}\in\{0,1\}^N}\sum_{\boldsymbol{x}_{-0}\in\{0,1\}^{N\setminus\{0\}}}\sum_{\boldsymbol{y}\in\{0,1\}^N}\mathbf{Pr}(\boldsymbol{\Theta}=\boldsymbol{t},\boldsymbol{X}_{-0}=\boldsymbol{x}_{-0},\boldsymbol{Y}=\boldsymbol{y}\mid X_0=x_0)(\pi_0(x_0,\boldsymbol{x}_{-0},\boldsymbol{y})-\pi_0(x_0',\boldsymbol{x}_{-0},\boldsymbol{y}))=$$

$$\sum_{\boldsymbol{x}_{-0}\in\{0,1\}^{N\setminus\{0\}}}\sum_{\boldsymbol{y}\in\{0,1\}^N}\left(\sum_{\boldsymbol{t}\in\{0,1\}^N}\mathbf{Pr}(\boldsymbol{\Theta}=\boldsymbol{t},\boldsymbol{X}_{-0}=\boldsymbol{x}_{-0},\boldsymbol{Y}=\boldsymbol{y}\mid X_0=x_0)\right)(\pi_0(x_0,\boldsymbol{x}_{-0},\boldsymbol{y})-\pi_0(x_0',\boldsymbol{x}_{-0},\boldsymbol{y})),$$

where

$$\begin{aligned}
&\mathbf{Pr}(\boldsymbol{\Theta}=\boldsymbol{t},\boldsymbol{X}_{-0}=\boldsymbol{x}_{-0},\boldsymbol{Y}=\boldsymbol{y}\mid X_0=x_0)\\
&\quad=\mathbf{Pr}(\Theta_0=t_0,Y_0=y_0\mid X_0=x_0)\mathbf{Pr}(\boldsymbol{\Theta}_{-0}=\boldsymbol{t}_{-0},\boldsymbol{X}_{-0}=\boldsymbol{x}_{-0},\boldsymbol{Y}_{-0}=\boldsymbol{y}_{-0})\\
&\quad=\frac{\mathbf{Pr}(\Theta_0=t_0,X_0=x_0,Y_0=y_0)}{\mathbf{Pr}(X_0=x_0)}\mathbf{Pr}(\boldsymbol{\Theta}_{-0}=\boldsymbol{t}_{-0},\boldsymbol{X}_{-0}=\boldsymbol{x}_{-0},\boldsymbol{Y}_{-0}=\boldsymbol{y}_{-0})=\\
&\quad=\frac{1}{\mathbf{Pr}(X_0=x_0)}\mathbf{Pr}(\boldsymbol{\Theta}=\boldsymbol{t},\boldsymbol{X}=\boldsymbol{x},\boldsymbol{Y}=\boldsymbol{y})\\
&\quad=2\prod_{j\in N}\mathbf{Pr}(\Theta_j=t_j,X_j=x_j,Y_j=y_j)\\
&\quad=2\prod_{j\in N}\mathbf{Pr}(X_j=x_j,Y_j=y_j\mid\Theta_j=t_j)\mathbf{Pr}(\Theta_j=t_j)\\
&\quad=2\prod_{j\in N}\frac{\mathbf{Pr}(X_j=x_j,Y_j=y_j\mid\Theta_j=t_j)}{2},
\end{aligned}$$

as

$$\begin{aligned}
\mathbf{Pr}(X_0=x_0)&=\mathbf{Pr}(X_0=x_0,\Theta_0=0)+\mathbf{Pr}(X_0=x_0,\Theta_0=1)\\
&=\mathbf{Pr}(X_0=x_0\mid\Theta_0=0)\mathbf{Pr}(\Theta_0=0)+\mathbf{Pr}(X_0=x_0\mid\Theta_0=1)\mathbf{Pr}(\Theta_0=1)\\
&=\frac{1}{2}(p+1-p)=\frac{1}{2}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{\boldsymbol{t}\in\{0,1\}^N}\mathbf{Pr}(\boldsymbol{\Theta}=\boldsymbol{t},\boldsymbol{X}_{-0}=\boldsymbol{x}_{-0},\boldsymbol{Y}=\boldsymbol{y}\mid X_0=x_0)=\\
&2\sum_{\boldsymbol{t}\in\{0,1\}^N}\prod_{j\in N}\frac{\mathbf{Pr}(X_j=x_j,Y_j=y_j\mid\Theta_j=t_j)}{2}=\\
&2\prod_{j\in N}\frac{\mathbf{Pr}(X_j=x_j,Y_j=y_j\mid\Theta_j=0)+\mathbf{Pr}(X_j=x_j,Y_j=y_j\mid\Theta_j=1)}{2}=\\
&2\left(\xi_0^0(\boldsymbol{x},\boldsymbol{y})+\xi_0^1(\boldsymbol{x},\boldsymbol{y})\right)T_0(\boldsymbol{x},\boldsymbol{y}),
\end{aligned}$$

and the BIC constraints can be rewritten as

$$\sum_{\boldsymbol{x}_{-0}\in\{0,1\}^{N\setminus\{0\}}}\sum_{\boldsymbol{y}\in\{0,1\}^N}\left(\xi_0^0(\boldsymbol{x},\boldsymbol{y})+\xi_0^1(\boldsymbol{x},\boldsymbol{y})\right)T_0(\boldsymbol{x},\boldsymbol{y})\left(\pi_0(\boldsymbol{x},\boldsymbol{y})-\pi_0(x_0',\boldsymbol{x}_{-0},\boldsymbol{y})\right)\geq 0.$$

Using ζ defined in Equation (14) we can rewrite the BIC constraints as

$$\sum_{y_0\in\{0,1\}}\sum_{\boldsymbol{z}\in S_4(n-1)}U(\boldsymbol{z})\zeta(x_0,y_0)\left(\varrho(x_0,y_0,\boldsymbol{z})-\varrho(x_0',y_0,\boldsymbol{z})\right)\geq 0,$$

which is the formulation given in Equation (12).

D.3 Derivation of Equation (17)

In this section we derive formulation of the objective function given in Equation (17).

Using Equations (15) and (16) in the Appendix, the objective function can be rewritten as

$$
\begin{aligned}
\frac{n}{2}\Bigg(& \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\Bigg(\zeta^1(0,0)\varrho(0,0,\boldsymbol{z})+\zeta^1(0,1)\varrho(0,1,\boldsymbol{z})+\\
&\qquad \zeta^1(1,1)\left(\frac{1-z_{0,0}\varrho(0,0,\boldsymbol{z}_{-(0,0),(1,1)},z_{0,0}-1,z_{1,1}+1)-z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,1),(1,1)},z_{0,1}-1,z_{1,1}+1)}{z_{1,1}+1}\right)\Bigg)+\\
& \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0}} U(\boldsymbol{z})\Bigg(\zeta^1(0,0)\varrho(0,0,\boldsymbol{z})+\zeta^1(0,1)\varrho(0,1,\boldsymbol{z})+\\
&\qquad \zeta^1(1,1)\left(\frac{1-z_{0,0}\varrho(0,0,\boldsymbol{z}_{-(0,0),(1,1)},z_{0,0}-1,z_{1,1}+1)-z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,1),(1,1)},z_{0,1}-1,z_{1,1}+1)}{z_{1,1}+1}\right)+\\
&\qquad \zeta^1(1,0)\left(\frac{1-z_{0,0}\varrho(0,0,\boldsymbol{z}_{-(0,0),(1,0)},z_{0,0}-1,z_{1,0}+1)-z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,1),(1,0)},z_{0,1}-1,z_{1,0}+1)}{z_{1,0}+1}\right)\Bigg)\Bigg)
\end{aligned}
$$

and, further, as

$$
\begin{aligned}
\frac{n}{2}\Bigg(& \sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\left(\zeta^1(0,0)\varrho(0,0,\boldsymbol{z})+\zeta^1(0,1)\varrho(0,1,\boldsymbol{z})\right)\\
& -\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z}_{-(0,0),(1,1)},z_{0,0}+1,z_{1,1}-1)\zeta^1(1,1)\frac{(z_{0,0}+1)\varrho(0,0,\boldsymbol{z})}{z_{1,1}}\\
& -\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z}_{-(0,1),(1,1)},z_{0,1}+1,z_{1,1}-1)\zeta^1(1,1)\frac{(z_{0,1}+1)\varrho(0,1,\boldsymbol{z})}{z_{1,1}}\\
& -\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}>0,z_{1,1}=0}} U(\boldsymbol{z}_{-(0,0),(1,0)},z_{0,0}+1,z_{1,0}-1)\zeta^1(1,0)\frac{(z_{0,0}+1)\varrho(0,0,\boldsymbol{z})}{z_{1,0}}\\
& -\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}>0,z_{1,1}=0}} U(\boldsymbol{z}_{-(0,1),(1,0)},z_{0,1}+1,z_{1,0}-1)\zeta^1(1,0)\frac{(z_{0,1}+1)\varrho(0,1,\boldsymbol{z})}{z_{1,0}}\\
& +\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\frac{\zeta^1(1,1)}{z_{1,1}+1}+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0}} U(\boldsymbol{z})\frac{\zeta^1(1,0)}{z_{1,0}+1}.\Bigg).
\end{aligned}
$$

By (9),

$$
\begin{aligned}
U(\boldsymbol{z}_{-(0,0),(1,1)},z_{0,0}+1,z_{1,1}-1) &= \frac{z_{1,1}}{z_{0,0}+1}U(\boldsymbol{z})\\
U(\boldsymbol{z}_{-(0,1),(1,1)},z_{0,1}+1,z_{1,1}-1) &= \frac{1-Q}{Q}\frac{z_{1,1}}{z_{0,1}+1}U(\boldsymbol{z})\\
U(\boldsymbol{z}_{-(0,0),(1,0)},z_{0,0}+1,z_{1,0}-1) &= \frac{Q}{1-Q}\frac{z_{1,0}}{z_{0,0}+1}U(\boldsymbol{z})\\
U(\boldsymbol{z}_{-(0,1),(1,0)},z_{0,1}+1,z_{1,0}-1) &= \frac{z_{1,0}}{z_{0,1}+1}U(\boldsymbol{z})\\
U(\boldsymbol{z}_{-(0,0),(0,1)},z_{0,0}+1,z_{0,1}-1) &= \frac{Q}{1-Q}\frac{z_{0,1}}{z_{0,0}+1}U(\boldsymbol{z}).
\end{aligned}
\tag{78}
$$

Hence the objective function can be further rewritten as

$$\begin{aligned}
&\frac{n}{2}\Bigg(\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\left(\zeta^1(0,0)\varrho(0,0,\boldsymbol{z})+\zeta^1(0,1)\varrho(0,1,\boldsymbol{z})\right)\\
&\quad-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\zeta^1(1,1)\varrho(0,0,\boldsymbol{z})\\
&\quad-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} \left(\frac{1-Q}{Q}\right)U(\boldsymbol{z})\zeta^1(1,1)\varrho(0,1,\boldsymbol{z})\\
&\quad-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}>0, z_{1,1}=0}} \left(\frac{Q}{1-Q}\right)\zeta^1(1,0)U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z})\\
&\quad-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}>0, z_{1,1}=0}} U(\boldsymbol{z})\zeta^1(1,0)\varrho(0,1,\boldsymbol{z})\Bigg)+C=\\
&\frac{n}{2}\Bigg(\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\left(\zeta^1(0,0)-\zeta^1(1,1)\right)\varrho(0,0,\boldsymbol{z})\\
&\quad+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\left(\zeta^1(0,1)-\left(\frac{1-Q}{Q}\right)\zeta^1(1,1)\right)\varrho(0,1,\boldsymbol{z})\\
&\quad+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}>0, z_{1,1}=0}} U(\boldsymbol{z})\left(\zeta^1(0,0)-\left(\frac{Q}{1-Q}\right)\zeta^1(1,0)\right)\varrho(0,0,\boldsymbol{z})\\
&\quad+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}>0, z_{1,1}=0}} U(\boldsymbol{z})\left(\zeta^1(0,1)-\zeta^1(1,0)\right)\varrho(0,1,\boldsymbol{z})\\
&\quad+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=0, z_{1,1}=0}} U(\boldsymbol{z})\left(\zeta^1(0,0)\varrho(0,0,\boldsymbol{z})+\zeta^1(0,1)\varrho(0,1,\boldsymbol{z})\right)\Bigg)+C,
\end{aligned}$$

where

$$C=\frac{n}{2}\left(\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\frac{\zeta^1(1,1)}{z_{1,1}+1}+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0}} U(\boldsymbol{z})\frac{\zeta^1(1,0)}{z_{1,0}+1}\right).$$

Using (14), we can rewrite the objective function as

$$
\begin{aligned}
\frac{n}{2}\Bigg(& \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\,(1-p-r)\,\varrho(0,0,\boldsymbol{z}) \\
& + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z}) \left(\frac{r(1-r)(1-2p)}{Q}\right) \varrho(0,1,\boldsymbol{z}) \\
& + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}>0, z_{1,1}=0}} U(\boldsymbol{z}) \left(\frac{r(1-r)(1-2p)}{1-Q}\right) \varrho(0,0,\boldsymbol{z}) \\
& + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}>0, z_{1,1}=0}} U(\boldsymbol{z})\,(r-p)\,\varrho(0,1,\boldsymbol{z}) \\
& + (1-p) \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=0, z_{1,1}=0}} U(\boldsymbol{z})\left((1-r)\varrho(0,0,\boldsymbol{z}) + r\varrho(0,1,\boldsymbol{z})\right)\Bigg) + C
\end{aligned}
$$

and constant C as

$$
C = \frac{n}{2}\left(\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z}) \frac{pr}{z_{1,1}+1} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0}} U(\boldsymbol{z}) \frac{p(1-r)}{z_{1,0}+1} \right).
$$

By the probability constraints (11), if $z_{0,1} = z_{1,0} = z_{1,1} = 0$ then $z_{0,0} = n-1$ and

$$
\varrho(0,0,\boldsymbol{z}) = \frac{1}{n}.
$$

Hence, for any $A \in \mathbb{R}$ and $B \in \mathbb{R}$,

$$
\begin{aligned}
& \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0}} U(\boldsymbol{z})\left(A\varrho(0,0,\boldsymbol{z}) + B\varrho(0,1,\boldsymbol{z})\right) = \\
& \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} AU(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,0}=z_{1,1}=0}} U(\boldsymbol{z})\frac{A}{n} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0}} U(\boldsymbol{z})B\varrho(0,1,\boldsymbol{z}) = \\
& \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} AU(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \frac{Q^{n-1}}{2^{n-1}}\frac{A}{n} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0}} U(\boldsymbol{z})B\varrho(0,1,\boldsymbol{z}).
\end{aligned}
\tag{79}
$$

In addition, by the probability constraints (11), if $z_{1,0} = z_{1,1} = 0$ and $z_{0,1} > 0$ then

$$
\varrho(0,0,\boldsymbol{z}) = \frac{1 - z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,0),(0,1)}, z_{0,0}+1, z_{0,1}-1)}{z_{0,0}+1}.
$$

Hence

$$
\begin{aligned}
& \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} AU(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) = \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} U(\boldsymbol{z})A\left(\frac{1 - z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,0),(0,1)}, z_{0,0}+1, z_{0,1}-1)}{z_{0,0}+1}\right) = \\
& \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} U(\boldsymbol{z})\left(\frac{A}{z_{0,0}+1}\right) - \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} AU(\boldsymbol{z})\left(\frac{z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,0),(0,1)}, z_{0,0}+1, z_{0,1}-1)}{z_{0,0}+1}\right).
\end{aligned}
$$

Using (78), this can be further rewritten as

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} U(\boldsymbol{z})\left(\frac{A}{z_{0,0}+1}\right)$$
$$-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} \frac{1-Q}{Q}\frac{z_{0,0}+1}{z_{0,1}}U(\boldsymbol{z}_{-(0,0),(0,1)},z_{0,0}+1,z_{0,1}-1))A\left(\frac{z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,0),(0,1)},z_{0,0}+1,z_{0,1}-1)}{z_{0,0}+1}\right)=$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} U(\boldsymbol{z})\left(\frac{A}{z_{0,0}+1}\right)-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,0}>0}} \frac{(1-Q)A}{Q}U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}).$$

Inserting this in (79) we obtain

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0}} U(\boldsymbol{z})\left(A\varrho(0,0,\boldsymbol{z})+B\varrho(0,1,\boldsymbol{z})\right)=$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} U(\boldsymbol{z})\left(\frac{A}{z_{0,0}+1}\right)-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,0}>0}} \frac{(1-Q)A}{Q}U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})+\frac{Q^{n-1}}{2^{n-1}}\frac{A}{n}+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0}} U(\boldsymbol{z})B\varrho(0,1,\boldsymbol{z})=$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} U(\boldsymbol{z})\left(\frac{A}{z_{0,0}+1}\right)+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,0}>0}} \left(B-\frac{(1-Q)A}{Q}\right)U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})+$$
$$\frac{Q^{n-1}}{2^{n-1}}\frac{A}{n}+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,0}=z_{1,0}=z_{1,1}=0}} U(\boldsymbol{z})B\varrho(0,1,\boldsymbol{z})=$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} U(\boldsymbol{z})\left(\frac{A}{z_{0,0}+1}\right)+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,0}>0}} \frac{(A+B)Q-A}{Q}U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})+\frac{Q^{n-1}}{2^{n-1}}\frac{A}{n}+\frac{(1-Q)^{n-1}}{2^{n-1}}\frac{B}{n}. \tag{80}$$

Since

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} U(\boldsymbol{z})\frac{1}{z_{0,0}+1}=\frac{1}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{Q}\right)$$

so (80) can further be rewritten as

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0}} U(\boldsymbol{z})\left(A\varrho(0,0,\boldsymbol{z})+B\varrho(0,1,\boldsymbol{z})\right)$$
$$=\frac{A}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{Q}\right)+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,0}>0}} \frac{(A+B)Q-A}{Q}U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})+\frac{A}{n2^{n-1}}Q^{n-1}+\frac{B}{n2^{n-1}}(1-Q)^{n-1}$$
$$=\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,0}>0}} \frac{(A+B)Q-A}{Q}U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})+\frac{1}{n2^{n-1}}\left(\frac{A(1-(1-Q)^n)+BQ(1-Q)^{n-1}}{Q}\right). \tag{81}$$

Using (81) (with $A = 1 - r$ and $B = r$), we can rewrite the objective function as

$$\frac{n}{2}\Bigg(\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\,(1-p-r)\,\varrho(0,0,\boldsymbol{z})$$

$$+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\left(\frac{r(1-r)(1-2p)}{Q}\right)\varrho(0,1,\boldsymbol{z})$$

$$+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}>0, z_{1,1}=0}} U(\boldsymbol{z})\left(\frac{r(1-r)(1-2p)}{1-Q}\right)\varrho(0,0,\boldsymbol{z})$$

$$+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}>0, z_{1,1}=0}} U(\boldsymbol{z})\,(r-p)\,\varrho(0,1,\boldsymbol{z})$$

$$+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,0}>0}} \frac{(1-p)(Q+r-1)}{Q}U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})\Bigg)+C,$$

which is the formulation given in Equation (17). Constant C is

$$C = \frac{n}{2}\left(\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\frac{pr}{z_{1,1}+1} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0}} U(\boldsymbol{z})\frac{p(1-r)}{z_{1,0}+1}\right) - \frac{1}{2^n}\left(\frac{(1-r)(1-(1-Q)^n)+rQ(1-Q)^{n-1}}{Q}\right).$$

Since

$$\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\frac{1}{z_{1,1}+1} = \frac{1}{n2^{n-1}Q}\left(2^n-(2-Q)^n\right)$$

and

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0}} U(\boldsymbol{z})\frac{1}{z_{1,0}+1} = \frac{1}{n2^{n-1}(1-Q)}\left((2-Q)^n-1\right)$$

so constant C can be rewritten as

$$C = \frac{1}{2^n}\left(\frac{2^n-(2-Q)^n}{Q}pr + \frac{(2-Q)^n-1}{1-Q}p(1-r) - \frac{(1-r)(1-(1-Q)^n)+rQ(1-Q)^{n-1}}{Q}\right),$$

which is the value given in Equation (18).

D.4 Derivation of Equations (19) and (20)

In this section we derive formulations of the BIC constraints given in Equations (19) and (20).

By Lemma 1, either $\varrho(1,0,\boldsymbol{z}) = 0$ or $z_{1,1} = 0$. Using that and the resulting derivations of $\varrho(1,1,\boldsymbol{z})$ given in (15) and $\varrho(1,0,\boldsymbol{z})$ given in (16) in the case of $x_0 = 0$, we can rewrite the LHS of the BIC constraint as follows:

$$
\begin{aligned}
&\sum_{y_0\in\{0,1\}}\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\zeta(0,y_0)\left(\varrho(0,y_0,\boldsymbol{z})-\varrho(1,y_0,\boldsymbol{z})\right)\\
&\quad=\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\left(\zeta(0,0)\left(\varrho(0,0,\boldsymbol{z})-\varrho(1,0,\boldsymbol{z})\right)+\zeta(0,1)\left(\varrho(0,1,\boldsymbol{z})-\varrho(1,1,\boldsymbol{z})\right)\right)\\
&\quad=\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\Bigg(\zeta(0,0)\varrho(0,0,\boldsymbol{z})+\\
&\qquad\zeta(0,1)\Bigg(\varrho(0,1,\boldsymbol{z})\\
&\qquad\qquad-\frac{1-z_{0,0}\varrho(0,0,\boldsymbol{z}_{-(0,0),(1,1)},z_{0,0}-1,z_{1,1}+1)-z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,1),(1,1)},z_{0,1}-1,z_{1,1}+1)}{z_{1,1}+1}\Bigg)\Bigg)+\\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0}} U(\boldsymbol{z})\Bigg(\zeta(0,0)\Bigg(\varrho(0,0,\boldsymbol{z})\\
&\qquad\qquad-\frac{1-z_{0,0}\varrho(0,0,\boldsymbol{z}_{-(0,0),(1,0)},z_{0,0}-1,z_{1,0}+1)-z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,1),(1,0)},z_{0,1}-1,z_{1,0}+1)}{z_{1,0}+1}\Bigg)\\
&\qquad+\zeta(0,1)\Bigg(\varrho(0,1,\boldsymbol{z})\\
&\qquad\qquad-\frac{1-z_{0,0}\varrho(0,0,\boldsymbol{z}_{-(0,0),(1,1)},z_{0,0}-1,z_{1,1}+1)-z_{0,1}\varrho(0,1,\boldsymbol{z}_{-(0,1),(1,1)},z_{0,1}-1,z_{1,1}+1)}{z_{1,1}+1}\Bigg)\Bigg).
\end{aligned}
$$

This can be further rewritten as

$$
\begin{aligned}
&\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\left(\zeta(0,0)\varrho(0,0,\boldsymbol{z})+\zeta(0,1)\varrho(0,1,\boldsymbol{z})\right)+\\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z}_{-(0,0),(1,1)},z_{0,0}+1,z_{1,1}-1)\zeta(0,1)\frac{(z_{0,0}+1)\varrho(0,0,\boldsymbol{z})}{z_{1,1}}+\\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z}_{-(0,1),(1,1)},z_{0,1}+1,z_{1,1}-1)\zeta(0,1)\frac{(z_{0,1}+1)\varrho(0,1,\boldsymbol{z})}{z_{1,1}}+\\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0,z_{1,0}>0}} U(\boldsymbol{z}_{-(0,0),(1,0)},z_{0,0}+1,z_{1,0}-1)\zeta(0,0)\frac{(z_{0,0}+1)\varrho(0,0,\boldsymbol{z})}{z_{1,0}}+\\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0,z_{1,0}>0}} U(\boldsymbol{z}_{-(0,1),(1,0)},z_{0,1}+1,z_{1,0}-1)\zeta(0,0)\frac{(z_{0,1}+1)\varrho(0,1,\boldsymbol{z})}{z_{1,0}}\\
&-\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\frac{\zeta(0,1)}{z_{1,1}+1}-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0}} U(\boldsymbol{z})\frac{\zeta(0,0)}{z_{1,0}+1}.
\end{aligned}
$$

Using (78) this can be rewritten as

$$\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\left(\zeta(0,0)\varrho(0,0,\boldsymbol{z})+\zeta(0,1)\varrho(0,1,\boldsymbol{z})\right)+$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\zeta(0,1)\varrho(0,0,\boldsymbol{z})+$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} \left(\frac{1-Q}{Q}\right) U(\boldsymbol{z})\zeta(0,1)\varrho(0,1,\boldsymbol{z})+$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} \left(\frac{Q}{1-Q}\right) U(\boldsymbol{z})\zeta(0,0)\varrho(0,0,\boldsymbol{z})+$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\zeta(0,0)\varrho(0,1,\boldsymbol{z})$$
$$-\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\frac{\zeta(0,1)}{z_{1,1}+1}-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0}} U(\boldsymbol{z})\frac{\zeta(0,0)}{z_{1,0}+1}.$$

Reorganizing the summands we obtain

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z})+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} \left(\frac{1-Q}{Q}\right) U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})+$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} \left(\frac{Q}{1-Q}\right) U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z})+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})+$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=z_{1,0}=0}} U(\boldsymbol{z})\left(Q\varrho(0,0,\boldsymbol{z})+(1-Q)\varrho(0,1,\boldsymbol{z})\right)$$
$$-\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\frac{1-Q}{z_{1,1}+1}-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0}} U(\boldsymbol{z})\frac{Q}{z_{1,0}+1}.$$

Similarly, in the case of $x_0 = 1$, we can rewrite the LHS of the BIC constraint as

$$
\begin{aligned}
&\sum_{y_0\in\{0,1\}}\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\zeta(1,y_0)\left(\varrho(1,y_0,\boldsymbol{z})-\varrho(0,y_0,\boldsymbol{z})\right)\\
&=\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\left(\zeta(1,0)\left(\varrho(1,0,\boldsymbol{z})-\varrho(0,0,\boldsymbol{z})\right)+\zeta(1,1)\left(\varrho(1,1,\boldsymbol{z})-\varrho(0,1,\boldsymbol{z})\right)\right)\\
&=-\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\left(\zeta(1,0)\varrho(0,0,\boldsymbol{z})+\zeta(1,1)\varrho(0,1,\boldsymbol{z})\right)\\
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\zeta(1,1)\varrho(0,0,\boldsymbol{z})\\
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} \left(\frac{1-Q}{Q}\right)U(\boldsymbol{z})\zeta(1,1)\varrho(0,1,\boldsymbol{z})\\
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} \left(\frac{Q}{1-Q}\right)U(\boldsymbol{z})\zeta(1,0)\varrho(0,0,\boldsymbol{z})\\
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\zeta(1,0)\varrho(0,1,\boldsymbol{z})\\
&+\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\frac{\zeta(1,1)}{z_{1,1}+1}+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0}} U(\boldsymbol{z})\frac{\zeta(1,0)}{z_{1,0}+1}.
\end{aligned}
$$

Reorganizing the summands and using (14) we obtain

$$
\begin{aligned}
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z})-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})\\
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z})-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})\\
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=z_{1,0}=0}} U(\boldsymbol{z})\left((1-Q)\varrho(0,0,\boldsymbol{z})+Q\varrho(0,1,\boldsymbol{z})\right)+\\
&\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\frac{Q}{z_{1,1}+1}+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0}} U(\boldsymbol{z})\frac{1-Q}{z_{1,0}+1}.
\end{aligned}
$$

Thus the BIC constraints can be written as, for $x_0 = 0$:

$$
\begin{aligned}
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z})+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} \left(\frac{1-Q}{Q}\right)U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})+\\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} \left(\frac{Q}{1-Q}\right)U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z})+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z})+\\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=z_{1,0}=0}} U(\boldsymbol{z})\left(Q\varrho(0,0,\boldsymbol{z})+(1-Q)\varrho(0,1,\boldsymbol{z})\right)\geq\\
&\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\frac{1-Q}{z_{1,1}+1}+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0}} U(\boldsymbol{z})\frac{Q}{z_{1,0}+1},
\end{aligned}
$$

and, for $x_0 = 1$:

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=z_{1,0}=0}} U(\boldsymbol{z})\left((1-Q)\varrho(0,0,\boldsymbol{z}) + Q\varrho(0,1,\boldsymbol{z})\right) \leq$$
$$\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\frac{Q}{z_{1,1}+1} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0}} U(\boldsymbol{z})\frac{1-Q}{z_{1,0}+1}.$$

Using

$$\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\frac{1}{z_{1,1}+1} = \frac{1}{n2^{n-1}Q}\left(2^n - (2-Q)^n\right),$$

and

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0}} U(\boldsymbol{z})\frac{1}{z_{1,0}+1} = \frac{1}{n2^{n-1}(1-Q)}\left((2-Q)^n - 1\right),$$

we can rewrite the BIC constraint for $x_0 = 0$ as

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} \left(\frac{1-Q}{Q}\right) U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} \left(\frac{Q}{1-Q}\right) U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=z_{1,0}=0}} U(\boldsymbol{z})\left(Q\varrho(0,0,\boldsymbol{z}) + (1-Q)\varrho(0,1,\boldsymbol{z})\right) \geq$$
$$\frac{1}{n2^{n-1}}\left(\frac{2^n(1-Q)^2 + (2Q-1)(2-Q)^n - Q^2}{Q(1-Q)}\right).$$

and further, using (by (81) with $A = Q$ and $B = 1-Q$)

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=z_{1,0}=0}} U(\boldsymbol{z})\left(Q\varrho(0,0,\boldsymbol{z}) + (1-Q)\varrho(0,1,\boldsymbol{z})\right)$$
$$= \frac{1}{n2^{n-1}}\left(\frac{Q(1-(1-Q)^n) + (1-Q)Q(1-Q)^{n-1}}{Q}\right) = \frac{1}{n2^{n-1}},$$

as

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}>0}} \left(\frac{1-Q}{Q}\right) U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} \left(\frac{Q}{1-Q}\right) U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) \geq$$
$$\frac{1}{n2^{n-1}}\left(\frac{2^n(1-Q)^2 + (2Q-1)(2-Q)^n - Q}{Q(1-Q)}\right),$$

which is the formulation of the BIC constraint foro $x_0 = 0$ given in Equation (19).

Similarly, the BIC constraint for $x_0 = 1$ can be rewritten as

$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{1,1}=z_{1,0}=0}} U(\boldsymbol{z})\left((1-Q)\varrho(0,0,\boldsymbol{z}) + Q\varrho(0,1,\boldsymbol{z})\right) \leq \frac{2^n-1}{n2^{n-1}}$$

and further, using (by (81) with $A = 1 - Q$ and $B = Q$)

$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{1,1}=z_{1,0}=0}} U(\boldsymbol{z})\left((1-Q)\varrho(0,0,\boldsymbol{z}) + Q\varrho(0,1,\boldsymbol{z})\right)$$
$$= \frac{1}{n2^{n-1}}\left(\frac{(1-Q)(1-(1-Q)^n) + Q^2(1-Q)^{n-1}}{Q}\right) = \frac{1}{n2^{n-1}}\left(\frac{1-Q+(2Q-1)(1-Q)^{n-1}}{Q}\right),$$

as

$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{1,1}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{1,1}=z_{1,0}=0 \\ z_{0,0}>0}} \left(\frac{2Q-1}{Q}\right) U(\boldsymbol{z})\varrho(0,1,\boldsymbol{z}) \leq \frac{1}{n2^{n-1}}\left(\frac{2^nQ-1-(2Q-1)(1-Q)^{n-1}}{Q}\right),$$

which is the formulation of the BIC constraint for $x_0 = 1$ given in Equation (20).

D.5 Derivation of Equation (42)

$$\Pi^D(p,r) = \frac{n}{2} \sum_{(x_0,y_0)\in\{0,1\}^2} \sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\zeta^1(x_0,y_0)\varrho^{\mathsf{D}}(x_0,y_0,\boldsymbol{z}) =$$

$$\frac{n}{4}\left((1-r)\sum_{x_0\in\{0,1\}}\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\varrho^{\mathsf{D}}(x_0,0,\boldsymbol{z}) + r\sum_{x_0\in\{0,1\}}\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\varrho^{\mathsf{D}}(x_0,1,\boldsymbol{z})\right) =$$

$$\frac{n}{2}\left((1-r)\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,0}+z_{1,0}=n-1}} U(\boldsymbol{z})\frac{1}{n} + r\sum_{m=0}^{n-1}\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}+z_{1,1}=m}} U(\boldsymbol{z})\frac{1}{m+1}\right) =$$

$$\frac{1}{2^n}\Bigg((1-r)\sum_{(z_{0,0},z_{1,0})\in S_2(n-1)}\binom{n-1}{z_{0,0},z_{1,0}}Q^{z_{0,0}}(1-Q)^{z_{1,0}}$$

$$+ rn\sum_{m=0}^{n-1}\frac{1}{m+1}\binom{n-1}{m}\sum_{(z_{0,1},z_{1,1})\in S_2(m)}\binom{m}{z_{0,1},z_{1,1}}Q^{z_{1,1}}(1-Q)^{z_{0,1}}$$

$$\sum_{(z_{0,0},z_{1,0})\in S_2(n-m-1)}\binom{n-m-1}{z_{0,0},z_{1,0}}Q^{z_{0,0}}(1-Q)^{z_{1,0}}\Bigg) =$$

$$\frac{1}{2^n}\left(1-r+rn\sum_{m=0}^{n-1}\frac{1}{m+1}\binom{n-1}{m}\right) = \frac{1}{2^n}\left(1-r+r(2^n-1)\right) = r - \frac{2r-1}{2^n}.$$

D.6 Derivation of Equation (44)

$$\Pi^B(p,r) = \frac{1}{2^n}\left(\frac{pr(1-Q)2^n - r(1-r)(2p-1)((2-Q)^n+Q^n) + Q(p-r)}{Q(1-Q)}\right)$$

$$-\frac{1}{2^n}\left(\frac{r(1-r)(2p-1)(2^n(1-Q)^2 + Q(1-Q^n) - (2-Q)^n(1-Q))}{Q^2(1-Q)}\right)$$

$$-\frac{1}{2^n}\left(\frac{r(1-r)(2p-1)(2Q-1)\max\left(2^n(1-Q)+Q(1-Q)^{n-1}-(2-Q)^n,0\right)}{Q^2(1-Q)}\right)$$

$$= \frac{1}{2^n}\left(\frac{pr(1-Q)2^n + Q(p-r)}{Q(1-Q)}\right) - \frac{1}{2^n}\left(\frac{r(1-r)(2p-1)(2^nQ(1-Q)+Q))}{Q^2(1-Q)}\right)$$

$$-\frac{1}{2^n}\left(\frac{r(1-r)(2p-1)(2Q-1)\max\left(Q(1-Q)^{n-1},(2-Q)^n-2^n(1-Q)\right)}{Q^2(1-Q)}\right)$$

$$= r - \frac{2r-1}{2^n}\left(\frac{1}{Q(1-Q)}\right)\Bigg(p(1-p)$$

$$+\left(\frac{r(1-r)(2p-1)^2}{Q}\right)\max\left(Q(1-Q)^{n-1},(2-Q)^n-2^n(1-Q)\right)\Bigg).$$

D.7 Derivation of Equation (57)

In this section we derive formulation of the objective function given in Equation (57).

Using Equations (56) in the Appendix we can rewrite the objective function as follows:

$$
\begin{aligned}
\frac{n}{2}\Bigg(&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0, z_{1,1}>0}} U(\boldsymbol{z})\left(\zeta^1(1,1)\varrho(1,1,\boldsymbol{z}) + \zeta^1(0,1)\left(\frac{1 - z_{1,1}\varrho(1,1,\boldsymbol{z}_{-(0,1),(1,1)}, z_{0,1}+1, z_{1,1}-1)}{z_{0,1}+1}\right)\right) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{1,1}>0}} U(\boldsymbol{z})\Bigg(\zeta^1(1,1)\varrho(1,1,\boldsymbol{z}) + \zeta^1(0,1)\left(\frac{1 - z_{1,1}\varrho(1,1,\boldsymbol{z}_{-(0,1),(1,1)}, z_{0,1}+1, z_{1,1}-1)}{z_{0,1}+1}\right) + \\
&\qquad + \zeta^1(0,0)\left(\frac{1 - z_{1,1}\varrho(1,1,\boldsymbol{z}_{-(0,0),(1,1)}, z_{0,0}+1, z_{1,1}-1)}{z_{0,0}+1}\right)\Bigg) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0, z_{1,0}>0, z_{1,1}=0}} U(\boldsymbol{z})\left(\zeta^1(1,1)\varrho(1,1,\boldsymbol{z}) + \zeta^1(1,0)\varrho(1,0,\boldsymbol{z}) + \zeta^1(0,1)\left(\frac{1 - z_{1,0}\varrho(1,0,\boldsymbol{z}_{-(0,1),(1,0)}, z_{0,1}+1, z_{1,0}-1)}{z_{0,1}+1}\right)\right) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0, z_{1,0}=0, z_{1,1}=0}} U(\boldsymbol{z})\left(\zeta^1(1,1)\varrho(1,1,\boldsymbol{z}) + \zeta^1(1,0)\varrho(1,0,\boldsymbol{z}) + \frac{\zeta^1(0,1)}{z_{0,1}+1}\right) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{1,0}>0, z_{1,1}=0}} U(\boldsymbol{z})\Bigg(\zeta^1(1,1)\varrho(1,1,\boldsymbol{z}) + \zeta^1(1,0)\varrho(1,0,\boldsymbol{z}) + \zeta^1(0,1)\left(\frac{1 - z_{1,0}\varrho(1,0,\boldsymbol{z}_{-(0,1),(1,0)}, z_{0,1}+1, z_{1,0}-1)}{z_{0,1}+1}\right) + \\
&\qquad + \zeta^1(0,0)\left(\frac{1 - z_{1,0}\varrho(1,0,\boldsymbol{z}_{-(0,0),(1,0)}, z_{0,0}+1, z_{1,0}-1)}{z_{0,0}+1}\right)\Bigg) \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,0}=z_{1,1}=0}} U(\boldsymbol{z})\left(\zeta^1(1,1)\varrho(1,1,\boldsymbol{z}) + \zeta^1(1,0)\varrho(1,0,\boldsymbol{z}) + \frac{\zeta^1(0,1)}{z_{0,1}+1} + \frac{\zeta^1(0,0)}{z_{0,0}+1}\right)\Bigg)
\end{aligned}
$$

and, further, as follows:

$$
\begin{aligned}
\frac{n}{2}\Bigg(&\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\zeta^1(1,1)\varrho(1,1,\boldsymbol{z}) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}>0}} U(\boldsymbol{z}_{-(0,1),(1,1)}, z_{0,1}-1, z_{1,1}+1)\zeta^1(0,1)\left(\frac{1-(z_{1,1}+1)\varrho(1,1,\boldsymbol{z})}{z_{0,1}}\right) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0, z_{0,0}>0}} U(\boldsymbol{z}_{-(0,0),(1,1)}, z_{0,0}-1, z_{1,1}+1)\zeta^1(0,0)\left(\frac{1-(z_{1,1}+1)\varrho(1,1,\boldsymbol{z})}{z_{0,0}}\right) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0}} U(\boldsymbol{z})\zeta^1(1,0)\varrho(1,0,\boldsymbol{z}) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0, z_{0,1}>0}} U(\boldsymbol{z}_{-(0,1),(1,0)}, z_{0,1}-1, z_{1,0}+1)\zeta^1(0,1)\left(\frac{1-(z_{1,0}+1)\varrho(1,0,\boldsymbol{z})}{z_{0,1}}\right) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,1}=0, z_{0,0}>0}} U(\boldsymbol{z}_{-(0,0),(1,0)}, z_{0,0}-1, z_{1,0}+1)\zeta^1(0,0)\left(\frac{1-(z_{1,0}+1)\varrho(1,0,\boldsymbol{z})}{z_{0,0}}\right) + \\
&\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,0}=z_{1,1}=0}} U(\boldsymbol{z})\frac{\zeta^1(0,1)}{z_{0,1}+1} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,0}=z_{1,1}=0}} U(\boldsymbol{z})\frac{\zeta^1(0,0)}{z_{0,0}+1}\Bigg).
\end{aligned}
$$

Reorganizing the summands this can be rewritten as

$$\frac{n}{2}\Bigg(\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}>0}} \frac{z_{0,1}U(\boldsymbol{z})\zeta^1(1,1)-(z_{1,1}+1)U(\boldsymbol{z}_{-(0,1),(1,1)},z_{0,1}-1,z_{1,1}+1)\zeta^1(0,1)}{z_{0,1}}\varrho(1,1,\boldsymbol{z})+$$

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0,z_{0,0}>0}} \frac{z_{0,0}U(\boldsymbol{z})\zeta^1(1,1)-(z_{1,1}+1)U(\boldsymbol{z}_{-(0,0),(1,1)},z_{0,0}-1,z_{1,1}+1)\zeta^1(0,0)}{z_{0,0}}\varrho(1,1,\boldsymbol{z})+$$

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0,z_{0,0}=0}} U(\boldsymbol{z})\zeta^1(1,1)\varrho(1,1,\boldsymbol{z})+$$

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0,z_{0,1}>0}} \frac{z_{0,1}U(\boldsymbol{z})\zeta^1(1,0)-(z_{1,0}+1)U(\boldsymbol{z}_{-(0,1),(1,0)},z_{0,1}-1,z_{1,0}+1)\zeta^1(0,1)}{z_{0,1}}\varrho(1,0,\boldsymbol{z})+$$

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,1}=0,z_{0,0}>0}} \frac{z_{0,0}U(\boldsymbol{z})\zeta^1(1,0)-(z_{1,0}+1)U(\boldsymbol{z}_{-(0,0),(1,0)},z_{0,0}-1,z_{1,0}+1)\zeta^1(0,0)}{z_{0,0}}\varrho(1,0,\boldsymbol{z})+$$

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,0}=z_{0,1}=z_{0,1}=0}} U(\boldsymbol{z})\zeta^1(1,0)\varrho(1,0,\boldsymbol{z})\Bigg)+C,$$

where C is a constant equal to

$$\frac{n}{2}\Bigg(\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}>0}} \frac{U(\boldsymbol{z}_{-(0,1),(1,1)},z_{0,1}-1,z_{1,1}+1)\zeta^1(0,1)}{z_{0,1}}+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0,z_{0,0}>0}} \frac{U(\boldsymbol{z}_{-(0,0),(1,1)},z_{0,0}-1,z_{1,1}+1)\zeta^1(0,0)}{z_{0,0}}+$$

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0,z_{0,1}>0}} \frac{U(\boldsymbol{z}_{-(0,1),(1,0)},z_{0,1}-1,z_{1,0}+1)\zeta^1(0,1)}{z_{0,1}}+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,1}=0,z_{0,0}>0}} \frac{U(\boldsymbol{z}_{-(0,0),(1,0)},z_{0,0}-1,z_{1,0}+1)\zeta^1(0,0)}{z_{0,0}}+$$

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,0}=z_{1,1}=0}} U(\boldsymbol{z})\frac{\zeta^1(0,1)}{z_{0,1}+1}+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,0}=z_{1,1}=0}} U(\boldsymbol{z})\frac{\zeta^1(0,0)}{z_{0,0}+1}\Bigg).$$

By (9),

$$\begin{aligned}
U(\boldsymbol{z}_{-(0,1),(1,1)},z_{0,1}-1,z_{1,1}+1) &= \frac{Q}{1-Q}\frac{z_{0,1}}{z_{1,1}+1}U(\boldsymbol{z})\\
U(\boldsymbol{z}_{-(0,0),(1,1)},z_{0,0}-1,z_{1,1}+1) &= \frac{z_{0,0}}{z_{1,1}+1}U(\boldsymbol{z})\\
U(\boldsymbol{z}_{-(0,1),(1,0)},z_{0,1}-1,z_{1,0}+1) &= \frac{z_{0,1}}{z_{1,0}+1}U(\boldsymbol{z})\\
U(\boldsymbol{z}_{-(0,0),(1,0)},z_{0,0}-1,z_{1,0}+1) &= \frac{1-Q}{Q}\frac{z_{0,0}}{z_{1,0}+1}U(\boldsymbol{z})
\end{aligned}\tag{82}$$

and by (10)

$$\begin{aligned}
\zeta^1(1,1) &= qr\\
\zeta^1(1,0) &= q(1-r)\\
\zeta^1(0,1) &= (1-q)r\\
\zeta^1(0,0) &= (1-q)(1-r).
\end{aligned}$$

Using that, we can rewrite the objective function as

$$\frac{n}{2}\Bigg(\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}>0}}\frac{rU(\boldsymbol{z})}{1-Q}((1-Q)q-Q(1-q))\varrho(1,1,\boldsymbol{z})+$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0,z_{0,0}>0}}U(\boldsymbol{z})(qr-(1-q)(1-r))\varrho(1,1,\boldsymbol{z})+$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,0}=z_{0,1}=0}}qrU(\boldsymbol{z})\varrho(1,1,\boldsymbol{z})+$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0,z_{0,1}>0}}U(\boldsymbol{z})(q(1-r)-(1-q)r)\varrho(1,0,\boldsymbol{z})+$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,1}=0,z_{0,0}>0}}\frac{(1-r)U(\boldsymbol{z})}{Q}(Qq-(1-Q)(1-q))\varrho(1,0,\boldsymbol{z})+$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,0}=z_{0,1}=z_{1,1}=0}}U(\boldsymbol{z})q(1-r)\varrho(1,0,\boldsymbol{z})\Bigg)+C$$

and constant C as

$$\frac{n}{2}\Bigg(\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}>0}}\frac{Q}{1-Q}\frac{U(\boldsymbol{z})}{z_{1,1}+1}(1-q)r+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0,z_{0,0}>0}}\frac{U(\boldsymbol{z})}{z_{1,1}+1}(1-q)(1-r)+$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0,z_{0,1}>0}}\frac{U(\boldsymbol{z})}{z_{1,0}+1}(1-q)r+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,1}=0,z_{0,0}>0}}\frac{1-Q}{Q}\frac{U(\boldsymbol{z})}{z_{1,0}+1}(1-q)(1-r)+\tag{83}$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,0}=z_{1,1}=0}}U(\boldsymbol{z})\frac{\zeta^1(0,1)}{z_{0,1}+1}+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,0}=z_{1,1}=0}}U(\boldsymbol{z})\frac{\zeta^1(0,0)}{z_{0,0}+1}\Bigg).$$

By (56) in the case of $z_{0,0}=z_{0,1}=0$, $\varrho(1,1,\boldsymbol{z})=1/(z_{1,1}+1)$, and in the case of $z_{0,0}=z_{0,1}=z_{1,1}=0$, $\varrho(1,0,\boldsymbol{z})=1/n$. In addition, in the case of $z_{0,1}=z_{1,0}=z_{1,1}=0$, $z_{0,0}=n$. Using that and rewriting the coefficients at the summands, the objective function can be rewritten as

$$\frac{n}{2}\Bigg(\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0,z_{0,1}>0}}(q-r)U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z})+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}>0}}\frac{r(1-r)(2q-1)}{1-Q}U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z})+$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,1}=0,z_{0,0}>0}}\frac{r(1-r)(2q-1)}{Q}U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z})+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0,z_{0,0}>0}}(q+r-1)U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z})\Bigg)+C',$$

where

$$C' = \frac{n}{2}\Bigg(\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}>0}} \frac{Q}{1-Q}\frac{U(\boldsymbol{z})}{z_{1,1}+1}(1-q)r + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0, z_{0,0}>0}} \frac{U(\boldsymbol{z})}{z_{1,1}+1}(1-q)(1-r) +$$

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0, z_{0,1}>0}} \frac{U(\boldsymbol{z})}{z_{1,0}+1}(1-q)r + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,1}=0, z_{0,0}>0}} \frac{1-Q}{Q}\frac{U(\boldsymbol{z})}{z_{1,0}+1}(1-q)(1-r) +$$

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,0}=z_{0,1}=0}} \frac{U(\boldsymbol{z})}{z_{1,1}+1}qr + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,0}=z_{0,1}=z_{1,1}=0}} \frac{U(\boldsymbol{z})}{n}q(1-r) +$$

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,0}=z_{1,1}=0}} \frac{U(\boldsymbol{z})}{z_{0,1}+1}(1-q)r + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=z_{1,0}=z_{1,1}=0}} \frac{U(\boldsymbol{z})}{n}(1-q)(1-r)\Bigg),$$

which is the formulation given in (57).

D.8 Derivation of Equations (59) and (60)

In this section we derive formulations of the BIC constraints given in Equations (59) and (60).

Using (56), the LHS of the BIC constraints can be rewritten as follows:

$$\sum_{(x_0,y_0)\in\{0,1\}^2}\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\zeta(x_0,y_0)\beta(x_1,x_1',\boldsymbol{z})\varrho(x_0,y_0,\boldsymbol{z})$$

$$= \sum_{\boldsymbol{z}\in S_4(n-1)} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z}) \sum_{(x_0,y_0)\in\{0,1\}^2} \zeta(x_0,y_0)\varrho(x_0,y_0,\boldsymbol{z})$$

$$= \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0}} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\Bigg(\zeta(1,1)\varrho(1,1,\boldsymbol{z}) + \zeta(1,0)\varrho(1,0,\boldsymbol{z}) +$$

$$\zeta(0,1)\left(\frac{1 - z_{1,0}\varrho(1,0,\boldsymbol{z}_{-(0,1),(1,0)}, z_{0,1}+1, z_{1,0}-1) - z_{1,1}\varrho(1,1,\boldsymbol{z}_{-(0,1),(1,1)}, z_{0,1}+1, z_{1,1}-1)}{z_{0,1}+1}\right)\Bigg) +$$

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0}} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\Bigg(\zeta(1,1)\varrho(1,1,\boldsymbol{z}) + \zeta(1,0)\varrho(1,0,\boldsymbol{z}) +$$

$$\zeta(0,1)\left(\frac{1 - z_{1,0}\varrho(1,0,\boldsymbol{z}_{-(0,1),(1,0)}, z_{0,1}+1, z_{1,0}-1) - z_{1,1}\varrho(1,1,\boldsymbol{z}_{-(0,1),(1,1)}, z_{0,1}+1, z_{1,1}-1)}{z_{0,1}+1}\right) +$$

$$\zeta(0,0)\left(\frac{1 - z_{1,0}\varrho(1,0,\boldsymbol{z}_{-(0,0),(1,0)}, z_{0,0}+1, z_{1,0}-1) - z_{1,1}\varrho(1,1,\boldsymbol{z}_{-(0,0),(1,1)}, z_{0,0}+1, z_{1,1}-1)}{z_{0,0}+1}\right)\Bigg).$$

This can be further rewritten as

$$\begin{aligned}
&\sum_{\boldsymbol{z}\in S_4(n-1)} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\left(\zeta(1,1)\varrho(1,1,\boldsymbol{z})+\zeta(1,0)\varrho(1,0,\boldsymbol{z})\right)\\
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0}} \beta(x_1,x_1',\boldsymbol{z}_{-(0,1),(1,0)},z_{0,1}-1,z_{1,0}+1)U(\boldsymbol{z}_{-(0,1),(1,0)},z_{0,1}-1,z_{1,0}+1)\zeta(0,1)\frac{(z_{1,0}+1)\varrho(1,0,\boldsymbol{z})}{z_{0,1}}\\
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0}} \beta(x_1,x_1',\boldsymbol{z}_{-(0,1),(1,1)},z_{0,1}-1,z_{1,1}+1)U(\boldsymbol{z}_{-(0,1),(1,1)},z_{0,1}-1,z_{1,1}+1)\zeta(0,1)\frac{(z_{1,1}+1)\varrho(1,1,\boldsymbol{z})}{z_{0,1}}\\
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} \beta(x_1,x_1',\boldsymbol{z}_{-(0,0),(1,0)},z_{0,0}-1,z_{1,0}+1)U(\boldsymbol{z}_{-(0,0),(1,0)},z_{0,0}-1,z_{1,0}+1)\zeta(0,0)\frac{(z_{1,0}+1)\varrho(1,0,\boldsymbol{z})}{z_{0,0}}\\
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} \beta(x_1,x_1',\boldsymbol{z}_{-(0,0),(1,1)},z_{0,0}-1,z_{1,1}+1)U(\boldsymbol{z}_{-(0,0),(1,1)},z_{0,0}-1,z_{1,1}+1)\zeta(0,0)\frac{(z_{1,1}+1)\varrho(1,1,\boldsymbol{z})}{z_{0,0}}\\
&+\sum_{\boldsymbol{z}\in S_4(n-1)} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\frac{\zeta(0,1)}{z_{0,1}+1}+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0}} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\frac{\zeta(0,0)}{z_{0,0}+1}.
\end{aligned}$$

Notice that, by (54), for any $d,d'\in\{0,1\}$ and for any $x,x_1'\in\{0,1\}$ such that $x_1\neq x_1'$,

$$\begin{aligned}
&\beta(x_1,x_1',\boldsymbol{z}_{-(0,1),(1,0)},z_{0,1}-1,z_{1,0}+1)=\beta(x_1,x_1',\boldsymbol{z})+Q([x_1'=0]-[x_1'=1]),\\
&\beta(x_1,x_1',\boldsymbol{z}_{-(0,1),(1,1)},z_{0,1}-1,z_{1,1}+1)=\beta(x_1,x_1',\boldsymbol{z})+Q-[x_1'=1],\\
&\beta(x_1,x_1',\boldsymbol{z}_{-(0,0),(1,0)},z_{0,0}-1,z_{1,0}+1)=\beta(x_1,x_1',\boldsymbol{z})+1-Q-[x_1'=1],\\
&\beta(x_1,x_1',\boldsymbol{z}_{-(0,0),(1,1)},z_{0,0}-1,z_{1,1}+1)=\beta(x_1,x_1',\boldsymbol{z})+(1-Q)([x_1'=0]-[x_1'=1]).
\end{aligned} \tag{84}$$

Using (82) and (84) the LHS of the BIC constraints can be rewritten as

$$\begin{aligned}
&\sum_{\boldsymbol{z}\in S_4(n-1)} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\left(\zeta(1,1)\varrho(1,1,\boldsymbol{z})+\zeta(1,0)\varrho(1,0,\boldsymbol{z})\right)\\
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0}} \left(\beta(x_1,x_1',\boldsymbol{z})+Q([x_1'=0]-[x_1'=1])\right)U(\boldsymbol{z})\zeta(0,1)\varrho(1,0,\boldsymbol{z})\\
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0}} \left(\frac{Q}{1-Q}\right)\left(\beta(x_1,x_1',\boldsymbol{z})+Q-[x_1'=1]\right)U(\boldsymbol{z})\zeta(0,1)\varrho(1,1,\boldsymbol{z})\\
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} \left(\frac{1-Q}{Q}\right)\left(\beta(x_1,x_1',\boldsymbol{z})+1-Q-[x_1'=1]\right)U(\boldsymbol{z})\zeta(0,0)\varrho(1,0,\boldsymbol{z})\\
&-\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} \left(\beta(x_1,x_1',\boldsymbol{z})+(1-Q)([x_1'=0]-[x_1'=1])\right)U(\boldsymbol{z})\zeta(0,0)\varrho(1,1,\boldsymbol{z})\\
&+\sum_{\boldsymbol{z}\in S_4(n-1)} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\frac{\zeta(0,1)}{z_{0,1}+1}+\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0}} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\frac{\zeta(0,0)}{z_{0,0}+1}.
\end{aligned}$$

Reorganizing the summands and using

$$\zeta(x_0,y_0)=\begin{cases} Q, & \text{if } x_0=y_0\\ 1-Q, & \text{otherwise.}\end{cases} \tag{85}$$

we obtain

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0}} Q(1-Q)([x_1'=1]-[x_1'=0])U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0}} Q([x_1'=1]-Q)U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} (1-Q)([x_1'=1]-(1-Q))U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} Q(1-Q)([x_1'=1]-[x_1'=0])U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,0}=z_{0,1}=0}} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\left(Q\varrho(1,1,\boldsymbol{z})+(1-Q)\varrho(1,0,\boldsymbol{z})\right)$$
$$+\sum_{\boldsymbol{z}\in S_4(n-1)} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\frac{1-Q}{z_{0,1}+1} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0}} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\frac{Q}{z_{0,0}+1}.$$

By Lemma (8), either $\varrho(1,0,\boldsymbol{z})=0$ or $z_{1,1}=0$. Therefore, the LHS of the BIC constraint can be rewritten as

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0, z_{1,1}=0}} Q(1-Q)([x_1'=1]-[x_1'=0])U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0}} Q([x_1'=1]-Q)U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{0,0}>0}} (1-Q)([x_1'=1]-(1-Q))U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} Q(1-Q)([x_1'=1]-[x_1'=0])U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,0}=z_{0,1}=z_{1,1}=0}} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})(1-Q)\varrho(1,0,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,0}=z_{0,1}=0}} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})Q\varrho(1,1,\boldsymbol{z}) +$$
$$\sum_{\boldsymbol{z}\in S_4(n-1)} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\frac{1-Q}{z_{0,1}+1} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0}} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\frac{Q}{z_{0,0}+1}.$$

By the probability constraints (56), if $z_{0,0}=z_{0,1}=z_{1,1}=0$ then $\varrho(1,0,\boldsymbol{z})=1/n$ and, if $z_{0,0}=z_{0,1}=0$ then $\varrho(1,1,\boldsymbol{z})=1/(z_{1,1}+1)$. Hence the LHS of the BIC constraint can be further rewritten as

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0, z_{1,1}=0}} Q(1-Q)([x_1'=1]-[x_1'=0])U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0}} Q([x_1'=1]-Q)U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{0,0}>0}} (1-Q)([x_1'=1]-(1-Q))U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} Q(1-Q)([x_1'=1]-[x_1'=0])U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,0}=z_{0,1}=z_{1,1}=0}} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\frac{1-Q}{n} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,0}=z_{0,1}=0}} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\frac{Q}{z_{1,1}+1} +$$
$$\sum_{\boldsymbol{z}\in S_4(n-1)} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\frac{1-Q}{z_{0,1}+1} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0}} \beta(x_1,x_1',\boldsymbol{z})U(\boldsymbol{z})\frac{Q}{z_{0,0}+1}.$$

Using that, the BIC constraint for $x_1 = 1$ and $x_1' = 0$ is

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0, z_{1,1}=0}} Q(1-Q)U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}>0}} Q^2U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{0,0}>0}} (1-Q)^2U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} Q(1-Q)U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z})$$
$$\leq \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,0}=z_{0,1}=z_{1,1}=0}} \beta(1,0,\boldsymbol{z})U(\boldsymbol{z})\frac{1-Q}{n} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,0}=z_{0,1}=0}} \beta(1,0,\boldsymbol{z})U(\boldsymbol{z})\frac{Q}{z_{1,1}+1} +$$
$$\sum_{\boldsymbol{z}\in S_4(n-1)} \beta(1,0,\boldsymbol{z})U(\boldsymbol{z})\frac{1-Q}{z_{0,1}+1} + \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0}} \beta(1,0,\boldsymbol{z})U(\boldsymbol{z})\frac{Q}{z_{0,0}+1}.$$

Using

$$\sum_{\boldsymbol{z}\in S_4(n-1)} \beta(1,0,\boldsymbol{z})U(\boldsymbol{z})\frac{1}{z_{0,1}+1} = \frac{1}{n2^{n-1}}\left(\frac{Q^2(2^n-(1+Q)^{n-1}((n-1)(1-Q)+2))}{1-Q}\right),$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,1}=0}} \beta(1,0,\boldsymbol{z})U(\boldsymbol{z})\frac{1}{z_{0,0}+1} = \frac{1}{n2^{n-1}}\left(\frac{(1-Q)((Q+1)^{n-1}((n-1)Q^2+1)-(n-1)Q-1)}{Q}\right),$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,0}=z_{0,1}=0}} \beta(1,0,\boldsymbol{z})U(\boldsymbol{z})\frac{1}{z_{1,1}+1} = \frac{(n-1)(1-Q)}{n2^{n-1}}\left(1-(1-Q)^n\right),$$
$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{0,0}=z_{0,1}=z_{1,1}=0}} \beta(1,0,\boldsymbol{z})U(\boldsymbol{z})\frac{1}{n} = \frac{(n-1)Q(1-Q)^n}{n2^{n-1}}$$

and dividing both sides by $Q(1-Q)$, we can rewrite the RHS of the BIC constraint for

$x_1 = 1$ and $x_1' = 0$ to obtain

$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}>0, z_{1,1}=0}} U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}>0}} \left(\frac{Q}{1-Q}\right) U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{0,0}>0}} \left(\frac{1-Q}{Q}\right) U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z})$$
$$\leq \frac{1}{n2^{n-1}} \left(\frac{2^n Q^2 - (2Q-1)(1+Q)^n + Q - 1}{Q(1-Q)}\right),$$

which is the formulation given in (59).

Similarly, the BIC constraint for $x_1 = 0$ and $x_1' = 1$ is

$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}>0, z_{1,1}=0}} Q(1-Q)U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}>0}} Q(1-Q)U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{0,0}>0}} Q(1-Q)U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} Q(1-Q)U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z})$$
$$\geq - \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,0}=z_{0,1}=z_{1,1}=0}} \beta(0,1,\boldsymbol{z})U(\boldsymbol{z})\frac{1-Q}{n} - \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,0}=z_{0,1}=0}} \beta(0,1,\boldsymbol{z})U(\boldsymbol{z})\frac{Q}{z_{1,1}+1} -$$
$$\sum_{\boldsymbol{z} \in S_4(n-1)} \beta(0,1,\boldsymbol{z})U(\boldsymbol{z})\frac{1-Q}{z_{0,1}+1} - \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ z_{0,1}=0}} \beta(0,1,\boldsymbol{z})U(\boldsymbol{z})\frac{Q}{z_{0,0}+1}.$$

Using

$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ z_{0,0}=z_{0,1}=z_{1,1}=0}} \beta(0,1,\boldsymbol{z})U(\boldsymbol{z})\frac{1}{n} = \frac{1}{n2^{n-1}}(n-1)(-Q^2)(1-Q)^{n-1}$$
$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ z_{0,0}=z_{0,1}=0}} \beta(0,1,\boldsymbol{z})U(\boldsymbol{z})\frac{1}{z_{1,1}+1} = \frac{1}{n2^{n-1}} \left(\frac{(1-Q)^n(nQ^2-(1-Q)^2) - (1-Q)((n+1)Q-1)}{Q}\right)$$
$$\sum_{\boldsymbol{z} \in S_4(n-1)} \beta(0,1,\boldsymbol{z})U(\boldsymbol{z})\frac{1}{z_{0,1}+1} = \frac{1}{n2^{n-1}} \left(Q((1+Q)^{n-1}((n-1)(1-Q)+2) - 2^n)\right)$$
$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ z_{0,1}=0}} \beta(0,1,\boldsymbol{z})U(\boldsymbol{z})\frac{1}{z_{0,0}+1} = \frac{1}{n2^{n-1}}(1-Q)\left(n+1-(1+Q)^{n-1}(n(1-Q)+Q+1)\right)$$

and dividing both sides by $Q(1-Q)$, we can rewrite the RHS of the BIC constraint for $x_1 = 0$ and $x_1' = 1$ to obtain

$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}>0, z_{1,1}=0}} U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}>0}} U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z}) +$$
$$\sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}=z_{1,1}=0, z_{0,0}>0}} U(\boldsymbol{z})\varrho(1,0,\boldsymbol{z}) + \sum_{\substack{\boldsymbol{z} \in S_4(n-1) \\ \text{s.t. } z_{0,1}=0, z_{0,0}>0}} U(\boldsymbol{z})\varrho(1,1,\boldsymbol{z})$$
$$\geq \frac{1}{n2^{n-1}} \left(\frac{(1+Q)^n(2Q-1) - Q^2 2^n - Q + 1}{Q(1-Q)}\right),$$

which is the formulation given in (60).

D.9 Calculations and auxiliary results

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} U(\boldsymbol{z})\frac{1}{z_{0,0}+1} = \frac{1}{Q}\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} \binom{n-1}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} \frac{Q^{z_{1,1}+z_{0,0}+1}(1-Q)^{z_{0,1}+z_{1,0}}}{2^{n-1}} \frac{1}{z_{0,0}+1} =$$
$$\frac{1}{n2^{n-1}Q}\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0,\\ z_{0,1}>0}} \binom{n}{z_{0,0}+1, z_{0,1}, z_{1,0}, z_{1,1}} Q^{z_{1,1}+z_{0,0}+1}(1-Q)^{z_{0,1}+z_{1,0}} =$$
$$\frac{1}{n2^{n-1}Q}\sum_{\substack{\boldsymbol{z}_{-(1,0),(1,1)}\in S_2(n)\\ \text{s.t. } z_{0,0}>0, z_{0,1}>0}} \binom{n}{z_{0,0}, z_{0,1}} Q^{z_{0,0}}(1-Q)^{z_{0,1}} =$$
$$\frac{1}{n2^{n-1}Q}\left(\sum_{\boldsymbol{z}_{-(1,0),(1,1)}\in S_2(n)} \binom{n}{z_{0,0}, z_{0,1}} Q^{z_{0,0}}(1-Q)^{z_{0,1}} - Q^n - (1-Q)^n\right)$$
$$= \frac{1}{n2^{n-1}}\left(\frac{1-Q^n-(1-Q)^n}{Q}\right). \tag{86}$$

$$\sum_{\boldsymbol{z}\in S_4(n-1)} U(\boldsymbol{z})\frac{1}{z_{1,1}+1} = \frac{1}{Q}\sum_{\boldsymbol{z}\in S_4(n-1)} \binom{n-1}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} \frac{Q^{z_{1,1}+z_{0,0}+1}(1-Q)^{z_{0,1}+z_{1,0}}}{2^{n-1}} \frac{1}{z_{1,1}+1} =$$
$$\frac{1}{n2^{n-1}Q}\sum_{\boldsymbol{z}\in S_4(n-1)} \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}+1} Q^{z_{1,1}+z_{0,0}+1}(1-Q)^{z_{0,1}+z_{1,0}} =$$
$$\frac{1}{n2^{n-1}Q}\sum_{\substack{\boldsymbol{z}\in S_4(n)\\ z_{1,1}>0}} \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}} =$$
$$\frac{1}{n2^{n-1}Q}\left(\sum_{\boldsymbol{z}\in S_4(n)} \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}} - \sum_{\substack{\boldsymbol{z}\in S_4(n)\\ z_{1,1}=0}} \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}}\right) =$$
$$\frac{1}{n2^{n-1}Q}\left(2^n - \sum_{\boldsymbol{z}_{-(1,1)}\in S_3(n)} \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}} Q^{z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}}\right)$$
$$= \frac{1}{n2^{n-1}Q}\left(2^n - (2-Q)^n\right). \tag{87}$$

$$
\begin{aligned}
\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0}} U(\boldsymbol{z})\frac{1}{z_{1,0}+1} &= \frac{1}{1-Q}\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0}} \binom{n-1}{z_{0,0},z_{0,1},z_{1,0},z_{1,1}} \frac{Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}+1}}{2^{n-1}} \frac{1}{z_{1,0}+1} = \\
& \frac{1}{n2^{n-1}(1-Q)} \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0}} \binom{n}{z_{0,0},z_{0,1},z_{1,0}+1,z_{1,1}} Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}+1} = \\
& \frac{1}{n2^{n-1}(1-Q)} \sum_{\substack{\boldsymbol{z}_{-(1,1)}\in S_3(n)\\ z_{1,0}>0}} \binom{n}{z_{0,0},z_{0,1},z_{1,0}} Q^{z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}} = \\
& \frac{1}{n2^{n-1}(1-Q)} \Bigg(\sum_{\boldsymbol{z}_{-(1,1)}\in S_3(n)} \binom{n}{z_{0,0},z_{0,1},z_{1,0}} Q^{z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}} \\
& \qquad - \sum_{\substack{\boldsymbol{z}_{-(1,1)}\in S_3(n)\\ z_{1,0}=0}} \binom{n}{z_{0,0},z_{0,1},z_{1,0}} Q^{z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}} \Bigg) = \\
& \frac{1}{n2^{n-1}(1-Q)} \Bigg((2-Q)^n - \sum_{\boldsymbol{z}_{-(1,0),(1,1)}\in S_2(n)} \binom{n}{z_{0,0},z_{0,1}} Q^{z_{0,0}}(1-Q)^{z_{0,1}} \Bigg) \\
&= \frac{1}{n2^{n-1}(1-Q)} \left((2-Q)^n - 1 \right).
\end{aligned}
\tag{88}
$$

$$
\begin{aligned}
\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0\\ z_{0,0}>0}} U(\boldsymbol{z})\frac{1}{z_{0,1}+1} &= \frac{1}{1-Q}\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0\\ z_{0,0}>0}} \binom{n-1}{z_{0,0},z_{0,1},z_{1,0},z_{1,1}} \frac{Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}+1}}{2^{n-1}} \frac{1}{z_{0,1}+1} = \\
& \frac{1}{n2^{n-1}(1-Q)} \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0\\ z_{0,0}>0}} \binom{n}{z_{0,0},z_{0,1}+1,z_{1,0},z_{1,1}} Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}+1} = \\
& \frac{1}{n2^{n-1}(1-Q)} \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0\\ z_{0,0}>0, z_{0,1}>0}} \binom{n}{z_{0,0},z_{0,1},z_{1,0},z_{1,1}} Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}} = \\
& \frac{1}{n2^{n-1}(1-Q)} \Bigg(\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{1,0}=z_{1,1}=0}} \binom{n}{z_{0,0},z_{0,1},z_{1,0},z_{1,1}} Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}} \\
& \qquad - \sum_{\substack{\boldsymbol{z}\in S_4(n)\\ z_{0,1}=z_{1,0}=z_{1,1}=0\\ z_{0,0}>0}} \binom{n}{z_{0,0},z_{0,1},z_{1,0},z_{1,1}} Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}} \\
& \qquad - \sum_{\substack{\boldsymbol{z}\in S_4(n)\\ z_{0,0}=z_{1,0}=z_{1,1}=0\\ z_{0,1}>0}} \binom{n}{z_{0,0},z_{0,1},z_{1,0},z_{1,1}} Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}} \Bigg) \\
&= \frac{1}{n2^{n-1}} \left(\frac{1-Q^n-(1-Q)^n}{1-Q} \right).
\end{aligned}
\tag{89}
$$

$$\sum_{\substack{\boldsymbol{z}\in S_4(n)\\ z_{0,1}>0,z_{1,1}>0}} \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} Q^{z_{0,0}+z_{1,1}}(1-Q)^{z_{0,1}+z_{1,0}} =$$

$$\Bigg(\sum_{\boldsymbol{z}\in S_4(n)} \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} Q^{z_{0,0}+z_{1,1}}(1-Q)^{z_{0,1}+z_{1,0}}$$

$$- \sum_{\boldsymbol{z}_{-(0,1)}\in S_3(n)} \binom{n}{z_{0,0}, z_{1,0}, z_{1,1}} Q^{z_{0,0}+z_{1,1}}(1-Q)^{z_{1,0}} \tag{90}$$

$$- \sum_{\boldsymbol{z}_{-(1,1)}\in S_3(n)} \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}} Q^{z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}}$$

$$+ \sum_{\boldsymbol{z}_{-(0,1),(1,1)}\in S_2(n)} \binom{n}{z_{0,0}, z_{1,0}} Q^{z_{0,0}}(1-Q)^{z_{1,0}} \Bigg) =$$

$$2^n - (1+Q)^n - (2-Q)^n + 1.$$

By Equation (90),

$$\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}>0}} U(\boldsymbol{z}) \frac{1}{z_{0,1}+1} = \frac{1}{1-Q} \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}>0}} \binom{n-1}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} \frac{Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}+1}}{2^{n-1}} \frac{1}{z_{0,1}+1} =$$

$$\frac{1}{n2^{n-1}(1-Q)} \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}>0}} \binom{n}{z_{0,0}, z_{0,1}+1, z_{1,0}, z_{1,1}} Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}+1} =$$

$$\frac{1}{n2^{n-1}(1-Q)} \sum_{\substack{\boldsymbol{z}\in S_4(n)\\ z_{0,1}>0,z_{1,1}>0}} \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} Q^{z_{0,0}+z_{1,1}}(1-Q)^{z_{0,1}+z_{1,0}}$$

$$= \frac{1}{n2^{n-1}} \left(\frac{2^n - (1+Q)^n - (2-Q)^n + 1}{1-Q} \right). \tag{91}$$

$$
\begin{aligned}
\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0, z_{1,0}>0}} U(\boldsymbol{z})\frac{1}{z_{0,1}+1} &= \frac{1}{1-Q} \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0, z_{1,0}>0}} \binom{n-1}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} \frac{Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}+1}}{2^{n-1}} \frac{1}{z_{0,1}+1} = \\
& \frac{1}{n2^{n-1}(1-Q)} \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ z_{1,1}=0, z_{1,0}>0}} \binom{n}{z_{0,0}, z_{0,1}+1, z_{1,0}, z_{1,1}} Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}+1} = \\
& \frac{1}{n2^{n-1}(1-Q)} \sum_{\substack{\boldsymbol{z}_{-(1,1)}\in S_3(n)\\ z_{0,1}>0, z_{1,0}>0}} \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}} Q^{z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}} = \\
& \frac{1}{n2^{n-1}(1-Q)} \Bigg(\sum_{\boldsymbol{z}_{-(1,1)}\in S_3(n)} \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}} Q^{z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}} \\
& \qquad - \sum_{\boldsymbol{z}_{-(0,1),(1,1)}\in S_2(n)} \binom{n}{z_{0,0}, z_{1,0}} Q^{z_{0,0}}(1-Q)^{z_{1,0}} \\
& \qquad - \sum_{\boldsymbol{z}_{-(1,0),(1,1)}\in S_2(n)} \binom{n}{z_{0,0}, z_{0,1}} Q^{z_{0,0}}(1-Q)^{z_{0,1}} \\
& \qquad + Q^n \Bigg) \\
&= \frac{1}{n2^{n-1}} \left(\frac{(2-Q)^n + Q^n - 2}{1-Q} \right).
\end{aligned} \tag{92}
$$

$$
\begin{aligned}
& \sum_{\substack{\boldsymbol{z}\in S_4(n)\\ z_{0,1}=z_{1,1}=0,\\ z_{0,0}>0, z_{1,0}>0}} \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}} = \\
& \left(\sum_{(z_{0,0}, z_{1,0})\in S_2(n)} \binom{n}{z_{0,0}, z_{1,0}} Q^{z_{0,0}}(1-Q)^{z_{1,0}} - Q^n - (1-Q)^n \right) = \\
& 1 - Q^n - (1-Q)^n.
\end{aligned} \tag{93}
$$

By Equation (93),

$$
\begin{aligned}
\sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0,\\ z_{1,0}>0}} U(\boldsymbol{z})\frac{1}{z_{0,0}+1} &= \frac{1}{Q} \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0,\\ z_{1,0}>0}} \binom{n-1}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} \frac{Q^{z_{1,1}+z_{0,0}+1}(1-Q)^{z_{0,1}+z_{1,0}}}{2^{n-1}} \frac{1}{z_{0,0}+1} = \\
& \frac{1}{n2^{n-1}Q} \sum_{\substack{\boldsymbol{z}\in S_4(n-1)\\ \text{s.t. } z_{0,1}=z_{1,1}=0,\\ z_{1,0}>0}} \binom{n}{z_{0,0}+1, z_{0,1}, z_{1,0}, z_{1,1}} Q^{z_{1,1}+z_{0,0}+1}(1-Q)^{z_{0,1}+z_{1,0}} = \\
& \frac{1}{n2^{n-1}Q} \sum_{\substack{\boldsymbol{z}\in S_4(n)\\ z_{0,1}=z_{1,1}=0,\\ z_{0,0}>0, z_{1,0}>0}} \binom{n}{z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}} Q^{z_{1,1}+z_{0,0}}(1-Q)^{z_{0,1}+z_{1,0}} \\
&= \frac{1}{n2^{n-1}} \left(\frac{1 - Q^n - (1-Q)^n}{Q} \right).
\end{aligned} \tag{94}
$$