Markets with Bilateral Bargaining and Incomplete Information[1]

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Abstract

We study the relationship between bargaining and competition with incomplete information. We consider a model with two uninformed and identical buyers and two sellers. One of the sellers has a privately-known reservation price, which can either be *Low* or *High*. The other seller's reservation price is commonly known to be *in between* the Low and High values of the privately-informed seller. Buyers move in sequence, and make offers with the second buyer observing the offer made by the first buyer. The sellers respond simultaneously. We show that there are two types of (perfect Bayes) equilibrium. In one equilibrium, the buyer who moves second does better. In the second equilibrium, buyers' expected payoffs are equalised, and the price received by the seller with the known reservation value is determined entirely by the equuilibrium of the two-player game between a single buyer and an informed seller. We also discuss extensions of the model to multiple buyers and sellers, and to the case where both sellers are privately informed.

1 Introduction

One fruitful way of modelling the microstructure of markets has been to conceive of them as the results of pairwise meetings between economic agents, with the market outcome being determined by the various agreements concluded by those pairs who agree to trade.This approach goes back a long way (see, for example, the housing market example in Shubik's book [14]); the modern interest in it dates back to the papers of Rubinstein and Wolinsky [11], Gale [9] and Binmore and Herrero [2] and the ensuing debate on the nature and properties of the equilibria generated.

These papers were concerned with random matching in large markets. Rubinstein and Wolinsky [12] discussed markets with small numbers of buyers and sellers and their work was followed up by Hendon and Tranaes [10] and Chatterjee and Dutta [3] amongst others. Chatterjee and Dutta [3] consider a model of a market in which sellers compete for heterogeneous buyers and, in a setting that has some features of auction-like competition and of bilateral bargaining. They show that in general one cannot obtain uniform prices across pairs or efficient (immediate) trade in this setting.

All the models mentioned above have assumed *complete information.* As is well-known, a literature on bilateral bargaining under incomplete information also developed around the same time.[1] However, possibly because of the general perception of the difficulty in obtaining determinate results in this literature without using equilibrium refinements, there has been no work that we know of that addresses small markets with some incomplete information and with the features of competition for bargaining partners that occur in some of the complete information papers.

[1]See the illuminating survey by Ausubel, Cramton and Deneckere[1], and the references cited there.

This paper attempts to make a start in studying the relationship between bargaining and competition with incomplete information, using as our basis a simplified version of a model of bilateral bargaining with two types that appears as a sub-model in Chatterjee and Samuelson [4]. Our purpose here, of course, is not just to fill a perceived gap in the literature. The interaction of competition and incomplete information has potentially interesting implications for the value of outside options and how this changes with incomplete information, a problem studied in a different setting by Fudenberg, Levine and Tirole[7] and Samuelson [13]. In the first model only a single seller has the ability to switch among buyers and would do so in the event of a rejection from a buyer signalling that the buyer is of a recalcitrant type. We discuss the incentive to switch in this way, but like Chatterjee and Dutta [3], add competition among sellers as well as a finite number of players on both sides of the market.

Our basic setup is as follows (a more formal description appears in Section 2): There are two buyers and two sellers.[2] One of the sellers has a privately-known reservation price, which can either be *Low* or *High* with commonly-known probabilities. The other seller has no private information, and his reservation price is commonly known to be *in between* the Low and High values of the privately-informed seller. The two buyers have the same commonly-known value, which is greater than the High seller reservation price. The buyers move in sequence and make offers with the second buyer observing the offer made by the first buyer. The sellers respond simultaneously[3] and accept or reject the offers made. Any acceptance leads to the trading pair leaving the market. In the next period, buyers again make offers and sellers accept or reject. Future payoffs are discounted with the common discount factor being δ.

[2]We discuss extensions to more buyers and sellers in the last section.

[3]We also consider what would happen if the sellers move in the order in which they are named by the buyers (if only one seller receives offers only that seller moves).

What would intuition suggest about a market of this nature? One might expect, first, competition among buyers to equalise equilibrium expected payoffs for the buyers (in which case the order in which they move would not matter in equilibrium). One might also expect that if the probability is high that the privately-informed seller is of *Low* type, that seller will reap the benefits of buyer competition with the opposite being true if the informed seller is more likely to be a *High* type, so that weakness could be strength. One might also surmise that the reservation price of the known seller would play a crucial role in determining prices in the first case and the reservation price of the *High* type in the second. [4]

It turns out there are two types of (perfect Bayes) equilibrium, one in which the intuition about equal expected payoffs of the buyers is satisfied and the other in which the second buyer to move does better. More surprisingly, if we consider the first kind of equilibrium, the price received by the known seller is entirely driven by the payoffs in the two-player incomplete information game, so that no switch occurs as described in the previous paragraph.

Moreover, we demonstrate through an example that when *both* the sellers are privately informed, even though their reservation prices are independent draws, the first kind of equilibrium with payoffs to buyers being order-independent need not exist.

It seems natural to compare our results to Shubik's discussion of the housing market, especially the attainment of the core allocation. The incomplete information of course leads to potential inefficiency through delay, so there is no hope of achieving the complete-information core. However, the equality in expected payoffs between the buyers seems a good proxy for the core, as in some loose sense we have equality in expectations of prices. However, this is not true

[4] These were our own initial intuitions about this problem.

in general if there is “too much” private information.

The outline of the rest of the paper is as follows: The next section introduces the notation and the explicit description of the model. Section 3 considers the complete information benchmark, in there are no privately informed sellers. Section 4 describes the two-player bargaining game with incomplete information and is based on Chatterjee and Samuelson [4]. Section 5 contains the basic analysis of the four-player game. Section 6 has the example with two privately-informed sellers and Section 7 discusses markets with more sellers and buyers in addition to providing concluding remarks.

2 The Model and Notation

There are *two* identical buyers B_1 and B_2. Each buyer has one unit demand for an indivisible good. The buyers' common and commonly-known valuation for the good is $v > 0$. There are also two sellers. Each seller owns one unit of the good. The first seller, to be denoted S_M, has a reservation value of M for the good, and this is common knowledge. The second seller's reservation value is private information to the seller. However, it is common knowledge that her reservation value is either H with probability π or L with probability $1 - \pi$, where $v > H > M \geq L$. In what follows, we simplify notation by setting $L = 0$. We will sometimes refer to the second seller as the *informed* seller, and denote her as S_I.

We consider the following infinite horizon bargaining game. in which only buyers make offers. In each period, the two buyers make offers to the sellers *sequentially*, the order of offers being random. An offer is simply a price p at which the buyer is willing to buy one unit of the good. The offer is targeted to a particular seller, since they are not identical. After both offers are on the table, the sellers decide whether to accept at most one of the offers.[5]

Matched pairs, if any, leave the market. If some pair is left unmatched, then the bargaining proceeds to the next period, in which the unmatched buyer(s) again make price offers to the unmatched seller(s). All players have the same discount factor $\delta \in (0, 1)$. All players are risk neutral.

We adopt the terminology of Fudenberg and Tirole [8] and denote each period as a "stage" in this game, to avoid the use of "subgames" in a game of incomplete information. We will also use their equilibrium concept of "Perfect Bayes' Equilibrium", namely sequential rationality at every stage given beliefs at that stage and beliefs being compatible with Bayes' theorem on and, wherever possible, off the equilibrium path.

Note that a stage in which a buyer and S_I have left the market and the other players remain begins a complete-information subgame (with a trivial solution). If a buyer and seller S_M have traded and left the market, the ensuing game is a two-player bagaining game of one-sided incomplete information with two types. This too has a determinate sequential equilibrium, to be discussed in the next section. We essentially adopt part of the Chatterjee-Samuelson [4] paper for this part. In that paper, there is a one-sided incomplete information "subgame" with two-sided offers. However the informed player's offers are always rejected in the equilibrium constructed there except possibly in the last stage. The game with the uninformed player being the sole proposer therefore has an easily derived equilibrium. [6]

The specification in which the buyers move in sequence might need some comment. We specify the model in this way rather than having buyers make

[5]Our results do not depend qualitatively on whether sellers move simultaneously or sequentially, though some details differ as pointed out later.

[6]See, for instance, Deneckere and Liang [5]. The game with a continuum of types was solved by Sobel and Takahashi [15] and Fudenberg, Levine and Tirole ([6] and there is no substantive difference in the results. So, we do not claim any novelty for our reformulation of the relevant part of Chatterjee-Samuelson ..

simultaneous targeted offers, as in Chatterjee and Dutta [3], mainly for analytical tractability. However, one can think of buyers moving in continuous time and extraneous irrelevant factors determining who moves first in a particular stage. This rules out strategically choosing whether to move first or second; such a restriction does not matter if the order of moves is payoff-irrelevant in equilibrium.

3 The Complete Information Game

In this section, we briefly describe the nature of equilibrium payoffs when seller valuations are also commonly known. The main purpose of this section is to act as a benchmark for the case when one of the sellers is privately informed about his reservation value - the case that is of principal interest in this paper.

We consider the case where seller reservation values are publicly known to be M and L. What will be the nature of equilibrium payoffs in this case? Intuition suggests that there should be competition for S_L, and this competition "should" drive up the price offered to the Low seller to M, which is also offered to S_M. Hence, in this equilibrium, buyer payoffs will be equalised at $v - M$.

Indeed, this will be one set of equilibrium payoffs. However, there is also another set of equilibrium payoffs. Suppose buyer B_1 is the first to make offers. Then, B_1 "knows" that if she offers a price $p < M$ to seller S_L, then B_2 will win over S_L with a slightly higher price p'. Hence, B_1 knows that her payoff canno exceed $v - M$. On the other hand, she can always ensure herself a payoff of $v - M$ by offering a price M to S_M. Notice, however, that if B_1 does make this offer to S_M, then B_2 can trade with S_L at the low price of L.

Hence, this suggests that there will be a second set of equilibrium payoffs where buyer payoffs are not equalised because B_1 essentially drops out of a contest she cannot win.[7]

The proposition below summarises the previous discussion.

Proposition 1 *The following constitute the only sets of equilibrium payoffs in the bargaining game when seller valuations are commonly known to be M and L.*

(i) Both buyers buy at the common price of $p = M$ giving rise to buyer payoffs of $v - M$. Seller S_M has zero payoff while seller S_L derives a payoff of $M - L$.

(ii) Buyer B_1 (the first buyer to make an offer in the initial period), has a payoff of $v - M$, while B_2 has a payoff of $v - L$. Both sellers get zero payoff.

Proof. We first describe equilibrium strategies which give rise to these payoffs.[8]

The following strategies support the first set of payoffs.

(a) Buyer B_1 offers a price of M to S_L in the initial period.

(b.1) If B_1 has offered a price of at least M to S_L, then B_2 offers M to S_M.

(b.2) If B_1 has offered $p < M$ to S_L, then B_2 offers $p' = \max(p, L)$ to S_L.

(b.3) If B_1 has made an offer to M, then B_2 offers L to S_L.

(c) If S_L receives only one offer p, then she accepts this offer iff $p \geq L$. If she receives two offers, then she accepts the higher of the two offers if this is at least as high as L. If both buyers offer the same price $p \geq L$, then she accepts the offer from B_2.

(d) If S_M receives only one offer p, then she accepts this offer iff $p \geq M$. If she receives two offers, then she accepts the higher of the two offers if this is

[7] Of course, this equilibrium arises due to the fact that buyers make offers sequentially.

[8] We do not claim that there are only two sets of equilibrium *strategies*.

at least as high as M. She uses any tie-breaking rule if both buyers offer the same $p \geq M$.

In subsequent periods, if only one pair is unmatched, then the players play the unique equilibrium of the two-player game, where the buyer offers a price exactly equal to the reservation value of the remaining seller. If both pairs are unmatched, then all players play the equilibrium strategies corresponding to the second set of equilibrium payoff which are described below.

In the second equilibrium buyer B_1 offers M to S_M, instead of to S_L. All other strategies are as described earlier.

We leave it to the reader to check that these indeed constitute equilibrium strategy profiles.

To verify that these are the only equilibrium payoffs possible, simply note that B_1 cannot obtain a payoff higher than $v - M$. For if she did, then she must be trading with S_L at a price $p < M$. Since B_2 makes her offer *after* B_1, she can make a slightly higher payoff and win over S_L. ■

4 The Two-Player Game with Incomplete Information

Play of the four-player game may lead to a two-player "subgame" involving the informed seller and one of the buyers. In fact, as we show in the next section, this continuation game will be reached with positive probability along the equilibrium path when S_M accepts the targeted offer made to her while S_I rejects the offer made to her with some probability. In this section, we briefly review the results on the equilibrium of this two-player game.

Since the subgame has only one buyer and one seller, we simplify notation by denoting the buyer as B and the (informed) seller as S. Suppose the subgame starts in period t', and let $\pi_{t'}$ be the initial probability that the seller's reservation value is L. We will describe the *unique* equilibrium which is essentially the one described in Chatterjee and Samuelson [4] and Deneckere and Liang [5]. [9]

It is convenient to count time "backwards". That is, period t means that the game will end t periods from now. Of course, this assumes that the game ends in finite time. Fortunately, it turns out that for any $\delta < 1$, the game ends in a finite number of periods $N(\delta)$. Moreover, as δ tends to one, $N(\delta)$ is *uniformly bounded.*[10]

Construct an *increasing* sequence of probabilities $\{0, q_1, \ldots, q_t, \ldots\}$. Recall that π^0 is the initial probability that S_I is of the Low type, and define $p_t \equiv \delta^t H$ for all $t = 1, \ldots, N(\delta)$. The nature of the equilibrium path is the following. Suppose that in period t, the play of the game so far and Bayes Rule implies that $\pi_t \in (q_t, q_{t+1}]$ is the updated probability that the seller is the Low type. Then, B offers p_t. The High seller rejects this offer with probability one, while the Low seller accepts this with a probability which implies through Bayes Rule that the updated probability π_{t-1} equals q_{t-1}. If $\pi_t < q_1$, then B offers H. This offer is accepted by both types of S.

The cut-off points q_t are chosen such that the buyer is indifferent between making the offer $\delta^t H$ and ending the game in t periods or offering $\delta^{t-1}H$ and ending the game one period earlier. So, at q_1, B is indifferent between offering H and δH. The latter offer is accepted with probability one by L. Hence, B's expected payoff from the offer δH is $q_1(v - \delta H) + (1 - q_1)\delta(v - H)$. Equating this to $v - H$, we get $q_1 = \frac{v-H}{v}$.

It is trivial to check that the Low type seller's behavior is optimal given B's specified strategy. For suppose, he receives the offer p_t. If he rejects this

[9] There is a small difference in our description of the equilibrium from that of Chatterjee and Samuelson. They specify an alternating offers extensive form so that buyers make offers every two periods. Since B makes an offer in every period in our model, there is a difference in the rate of discounting.

[10] This is shown in Deneckere and Liang [5]

offer, his payoff next period is p_{t-1}. Since $\delta p_{t-1} = p_t$, he is indifferent between rejecting and accepting this offer.

It is also easy to show that the 2-player game has a unique equilibrium. Clearly, after every history of the game, equilibrium must be unique if $\pi_t < q_1$ as this essentially becomes a 2-player complete information game. A form of "backward induction" argument can be used to establish uniqueness.

5 The Four-Player Game with Incomplete Information

In this section, we consider the four-player game described in Section 2.

We use the cutoffs q_t derived in the previous section. Recall that if π is below q_1, the two-player game essentially becomes a complete information game with the high offer made to the seller and if π is between q_1 and q_2, the two-player game would last at most for two periods. We first consider the four-player game in this ranges of values of π as an example of what happens in equilibrium in this game. We then extend the analysis to all values of π.

Let $\pi^0 \in [q_1, q_2)$ be the initial probability of type L. Define $\bar{p}_1^M$ and p_t^M as follows: (i) $v - \bar{p}_1^M = \pi^0(v - \delta H) + (1 - \pi^0)\delta(v - H)$ and (ii) $p_1^M = M + (1 - \pi^0)\delta(H - M)$.

Example 1 *W.l.o.g. let B_1 move first as the outcome of the random draw. The following is an equilibrium of the game for sufficiently high δ.*

B_1 offers $\bar{p}_1^M$ to S_M. The offers of B_2 depend on the offers made by B_1, and are described below.

(i) If B_1 offers a price $p \geq \bar{p}_1^M$ to S_M, then B_2 offers $p_1 = \delta H$ to S_I.

(ii) If B_1 offers a price $p < \bar{p}_1^M$ to S_M, then B_2 offers $p + \epsilon$ also to S_M.

(iii) If B_1 offers H or higher to S_I, B_2 offers M to S_M.

(iv) If B_1 makes an offer $p \in (\delta H, H)$ to S_I, B_2 makes an offer p_1^M to S_M.

(v) If B_1 makes an offer $p \in (\delta^2 H, \delta H)$ to S_I, then B_2 offers $\tilde{p}_1^M = M + (1 - \pi^0\alpha)\delta(H - M)$ to S_M where α is as defined in the response for type L below.

(vi) Finally, if B_1 makes an offer $p \leq \delta^2 H$ to S_I, B_2 makes an offer $M + \delta(\bar{p}_1^M - M)$ to M.

Seller S_I, type L accepts all offers $p \geq \delta H$, rejects all offers $p \leq \delta^2 H$ and accepts offers in $(\delta^2 H, \delta H)$ with probability α such that $q_1 = \frac{\pi^0(1-\alpha)}{\pi^0(1-\alpha)+1-\pi^0}$,. Type H accepts all offers p $\geq$H and rejects all offers below H; and S_M accepts any offer greater than his expected continuation payoff, which could be either p_1^M or $\tilde{p}_1^M$, depending on the offer made to S_I.[11]

If the initial offers are rejected, the game goes into the following period with all four players and with π either unchanged (=π^0), $\pi = q_1$ or $\pi = 0$. If $\pi = \pi^0$, the strategies above are played. If $\pi = q_1$, the offer to S_I randomises between δH and H. An analogue of (i) above then determines the offer made to S_M. If $\pi = 0$, the complete information strategies described in Section 3 are used, that is H is offered to both sellers. Thus, the equilibrium outcome path is: B_1 offers $\bar{p}_1^M$ to S_M and B_2 offers δH to S_I, S_M and type L accept and in the next period, B_2 offers H to S_I who accepts.

If S_I accepts and S_M rejects, the buyer remaining offers S_M a price of M in the following period. If S_M accepts and S_I rejects, the ensuing game is a two-player game of incomplete information and the strategies are as described in the previous section.

Proof. The argument constructs two prices for seller S_M, her continuation payoff, given in (ii), and the price obtained by competition among the buyers

[11] We have not set down details of possible deviations by B_2. They do not affect the sellers' response strategies.

as given in (i). We shall show that, in fact, the second is strictly higher than the first, so S_M always finds it optimal to accept $\bar{p}_1^M$. Seller S_I here plays the two-player game of the previous section with one of the buyers, so the two-player analysis carries over. The buyers follow strategies that equalise their expected payoffs.

We check deviations. If B_1 deviates and offers $p > \bar{p}_1^M$ to S_M, S_M accepts and B_1 is worse off. If B_1 offers $p < \bar{p}_1^M$, B_2 offers a higher price which S_M accepts, thus giving B_1 the two-player expected payoff with S_I but one period later. This makes him strictly worse off. If B_1 deviates and offers to S_I, the best resulting expected payoff is exactly equal to that obtained by offering $\bar{p}_1^M$ to S_M and therefore no gain is realised. If B_1 does not make a serious offer or makes a rejected offer, B_2 induces acceptance by making an offer to S_M of $M + \delta(\bar{p}_1^M - M)$, thus making B_1 worse off. Deviations by B_2 can be shown similarly to be unprofitable.

For the sellers, S_M will accept $\bar{p}_1^M$, since this is strictly greater than his continuation payoff, p_1^M. To see this, we explicitly calculate

$$\bar{p}_1^M - p_1^M = (v - M)(1-\delta) + \delta\pi^0 H - \pi^0 v + \delta\pi^0(v - M) \quad (1)$$

$$= v(1-\delta)(1-\pi^0) + \delta\pi^0(H - M) - M(1-\delta) \quad (2)$$

As $\delta \to 1$, the first and the third terms go to 0 and the second term is positive. [12]

We can similarly check that the rest of his strategy is optimal for S_M, namely to accept anything at least as high as his continuation payoff. The response strategy of S_I is the same as in the corresponding two-player game with one buyer. This is optimal because S_M finds it optimal to accept the equilibrium offer, and so S_I faces a two-player continuation game. If the offers are such that

[12] If $\delta = 0$ and π_0 is close to q_1, then the expression is positive. However when π_0 is close to q_2, the expression is negative. (With $\delta = 0$, $p_1^M = M$.)

S_M will reject but S_I type L is supposed to accept, a rejection by L signals he is a H type. But he can only obtain the H offer in the following period. The offer is such that L is indifferent between accepting and rejecting and getting the high offer in the next period. ■

Remark 1 *Note that if the sellers were to respond in the order they were named rather than simultaneously, there would be no change as long as S_M moves first. If S_I moves first, S_M's continuation payoff would depend on whether S_I accepted or rejected. This would not make a difference on the equilibrium path because B_1 would still be indifferent between making an offer of $\bar{p}_1^M$ to S_M or p_1 to S_I and thus would not gain by deviating. So in fact S_M would move first. But if S_I were chosen by B_1, the offer from B_2 to S_M could be either M or $M+\delta(H-M)$ depending on B_2's belief about S_I's probability of accepting.*

One would expect the (high) price needed to obtain a trade with type L of S_I when the probability of L is small to drive buyer competition for S_M. What happens when this probability is high? Suppose for instance that S_I is "almost certainly" of the Low type. Surely, the buyers should be competing to trade with S_I? The next lemma shows, surprisingly, that for sufficiently high δ, the competition is always over S_M.

Define the following sequences of prices for all $t = 1, \ldots,$ with $a_t = \pi_t \alpha_t$ the equilibrium acceptance probability for such an offer in the two-player incomplete information game.

(i) $p_t^I = \delta^t H$.

(ii) $p_t^M = M + \delta(1 - a_t)(\bar{p}_{t-1}^M - M)$.

(ii) $\bar{p}_t^M = v - \left[(v - p_t^I)a_t + (1 - a_t)\delta(v - \bar{p}_{t-1}^M)\right]$

(iii) $\hat{p}_t^M = \max(p_t^M, \bar{p}_t^M)$.

We now prove a lemma, which we shall refer to as the "competition lemma".

Lemma 1 *For all* $t = 1, \ldots,$ *there exists* $\tilde{\delta}(t)$ *such that for all* $\delta \geq \tilde{\delta}(t)$, $\hat{p}_t^M = \bar{p}_t^M$.

Proof. We show that for all $t \geq 1$ and for sufficiently high δ, $\bar{p}_t^M \geq p_t^M$.

$$
\begin{aligned}
\bar{p}_t^M - p_t^M &= v - \left[(v - p_t^I)a_t + (1 - a_t)\delta(v - \bar{p}_{t-1}^M)\right] - M - \delta(1 - a_t)(\bar{p}_{t-1}^M - M) \\
&= (v - M)(1 - \delta + \delta a_t) - a_t(v - p_t^I) \\
&= (1 - \delta)(v - M) + a_t(\delta v - \delta M - v + \delta^t H) \\
&= (1 - \delta)(v - M) + a_t(\delta^t H - \delta M - (1 - \delta)v).
\end{aligned}
$$

We have remarked earlier that for all $\delta < 1$, the equilibrium duration of the two-player incomplete information game is uniformly bounded by say T^*. Fix $t = T^*$. It is sufficient to show that the second term (the co-efficient of a_t) is non-negative for some $\tilde{\delta}$. Note that this is increasing in δ; at $\delta = 1$, it is clearly positive. Therefore there exists a $\tilde{\delta} < 1$ such that for $\delta > \tilde{\delta}$, the second term is positive. If this is true for $t = T^*$, it is clearly true for smaller values of t. Therefore, $\bar{p}_t^M > p_t^M$ whenever $\delta \geq \tilde{\delta}$. ■

We now construct the equilibrium for $\delta > \tilde{\delta}$ such that the expected payoff to the buyer does not depend upon the order in which the offers are made. We utilise four sequences, one of probabilities and three of prices, $\{q_t\}, \{p_t^I\}, \{\bar{p}_t^M\}, \{p_t^M\}$. The interesting feature here is that competition results in S_M getting more than his continuation game expected payoff. This is because S_M's continuation payoff is the combination of two terms. If S_I accepts, S_M is at the mercy of the other buyer who gives him M in the next period. If S_I rejects, she is more likely to be a H type and gets a higher equilibrium payoff. This drives up S_M's payoff in the following period as buyers potentially compete for his good. The driving force in the competition is the incomplete information in the game.

Proposition 2 *Define sequences* $p_t^I, \bar{p}_t^M, p_t^M$ *from conditions (i)-(iii) in the preceding lemma. Let* q_t *be defined as in the two-player game with incomplete information. Let* p_{ikt}[13] *represent the offer made by buyer* i *to seller* S_k *when* $\pi \epsilon [q_t, q_{t+1})$*; let* B_1 *, without loss of generality, be the first mover in each period. Let* $\delta \geq \tilde{\delta}$*, where* $\tilde{\delta}$ *has been defined in the competition lemma. There is one equilibrium in which the buyers obtain the same expected payoffs* u_i*. The common expected payoff* $u = u_1 = u_2$ *is the expected buyer payoff in the two-player incomplete information game with the given value of* π^0*, which we denote by* $v_B(\pi^0)$.

The stationary[14] *strategies that sustain these equilibrium payoffs are as follows:*

(i) B_1 *chooses* $p_{1Mt} = \bar{p}_t^M$ *and does not make an offer to* S_I.

(ii) If $p_{1Mt} \geq \bar{p}_t^M$, B_2 *chooses* $p_{2It} = p_t^I$; S_M *accepts,* S_I *of type L accepts with a probability sufficient to make* $\pi = q_{t-1}$ *in the next period,* S_I *of type H rejects any offer less than H.*

(iii) If $p_{1Mt} < \bar{p}_t^M$, B_2 *chooses* $p_{2Mt} = p_{1Mt} + \epsilon$ *such that* $p_{2Mt} \leq \bar{p}_t^M$ *and* $p_{2Mt} \geq M + \delta(\bar{p}_t^M - M)$, S_M *accepts* p_{2Mt}, S_I *has no move.*

(iv) If $p_{1It} \geq p_t^I$ *and* B_1 *does not make an offer to* S_M, B_2 *offers* p_t^M *to* S_M, S_M *accepts,* S_I *of type L uses the same acceptance strategy* a_t *as in the two-player incomplete information game.*

(v) If $p_{1It} \in [p_{t-1}^I, p_t^I)$, $p_{2Mt} = p_t^M(\bar{a}_t)$, *where* $\bar{a}_t$ *is the equilibrium acceptance probability of the corresponding two-player incomplete information game, player* S_M *accepts. Player* S_I*'s (L type) acceptance decision implies that* $\bar{a}_t$ *is the acceptance probability.*

[13] Each buyer can choose only a single value of p_{ikt} in this game.

[14] By "stationary" we mean independent of history apart from the updated value of π and of the set of players remaining in the game.

(vi) If B_2 deviates from (ii), S_I of type L responds according to the two-player game equilibrium strategy, S_M accepts if $\bar{p}_t^M$ is at least as high as his continuation payoff given the acceptance probability for S_I.

(vii) If $p_{2Mt} > 0$, S_M accepts any $p_{2Mt} \geq p_t^M(a_t)$, S_M's continuation payoff given an acceptance probability of a_t by the L type of S_I. The response behaviour of S_I (L type) follows that of the seller in the two-person incomplete information game with an uninformed buyer.

Proof. Consider deviations by B_1. If he chooses $p_{1Mt} > \bar{p}_t^M$, he is worse off because (a) S_M accepts any offer greater than her expected continuation payoff, p_t^M and, by the competition lemma, $\bar{p}_t^M > p_t^M$, and (b) B_2 is better off making an offer to S_I than choosing $p_{2Mt} > \bar{p}_t^M$, so B_2 will not offer such a price to S_M. If $p_{1Mt} < \bar{p}_t^M$, B_2 raises the offer by (iii) above, S_M accepts p_{2Mt}, and B_1 gets an expected payoff equal to the discounted buyer payoff in the incomplete information game with S_I. From the definition of $\bar{p}_t^M$, this is strictly less than $v - \bar{p}_t^M$. If B_1 chooses to make an offer to S_I, S_I will respond as in the two-player incomplete information game and, again by the definition of $\bar{p}_t^M$, B_1 will not be strictly better off with the optimal p_{1It}. B_2 moves second. If she deviates (1) by not following (ii), she is worse off since p_t^I is the equilibrium offer in the ensuing two-player incomplete information continuation game (since S_M will accept); (2) by not following (iii), she is clearly worse off by the definition of $\bar{p}_t^M$; (3) by not following (iv), she is worse off because S_M accepts any offer at least as high as his continuation payoff for which p_t^M is an upper bound and $v - p_t^M > v - \bar{p}_t^M$, by definition. The responses of the sellers are clearly optimal from the two-player continuation games and the four-player game with the updated value of π. ■

Remark 2 *Out-of-equilibrium beliefs do not play a significant role here because buyers make offers. Their deviations (and deviations by S_M) cannot signal* *anything about S_I's type by the requirement of "no signalling what you don't know". Player S_I always has a positive probability of accepting or rejecting and deviations in these probabilities are not observable. The sole exception is if the offer to S_I is $p \geq H$. In this case, a rejection does not change beliefs.*

Remark 3 *The comment after the first example in this section about the order of responses holds more generally.*

As the complete information analysis of Section 3 would suggest, this is not the only equilibrium in stationary strategies. There is another equilibrium in which the first mover in each period makes an offer to S_I and the second proposer offers S_M that seller's continuation payoff. We write this as a proposition. (We are again restricting our attention to sufficiently high values of δ.)

Proposition 3 *There exists an equilibrium in stationary strategies where the first buyer to move, B_1, obtains an expected payoff $u_1' = v_B(\pi^0)$, B_2 obtains $u_2' = v - M$ and $u_2' > u_1'$.*

Proof. The strategies that sustain these as equilibrium payoffs are obtained from (iv) to (vii) of the previous proposition. B_1 chooses $p_{1It} > 0$, making the equilibrium offer in the two-player incomplete information bargaining game for the given value of π. A deviation to making an offer to S_M will not increase this payoff, from the previous proposition. If B_1 makes an offer to S_I, B_2 offers M to S_M, who accepts any offer $p \geq M$. The continuation payoff for S_M is 0. If $\pi < q_1$, B_1 makes an offer of H to S_I, who accepts with probability 1. S_M will then accept any offer $p \geq M$. Since, in each period, B_2 makes an offer p_{2Mt} equal to the continuation payoff of S_M, backward induction shows that the continuation payoff must be 0 in each period. S_I responds as in her equilibrium strategy in the two-player, incomplete information game.

If B_1 makes an offer to S_M, the response from B_2 follows (ii) and (iii) from the previous proposition. This ensures B_1 does not gain by deviating. It is clear that B_2, S_I will not gain by deviating. ■

These two are equilibria in stationary strategies. One can think of the second as essentially a decomposition into two separate two-player games, one with incomplete information and one with complete information. The first equilibrium shows that putting the four players together can give rise to competition and to outcomes different from the two-player games for some of the players.

We can clearly combine the two equilibria to obtain others. For example, take the second equilibrium discussed above. Suppose that, if there is no agreement in the first period, the players switch to the first equilibrium (in which the first proposer makes an offer to S_M). In this case, the first-period offer by B_2 to S_M would be $p_t^M > M$, since S_M has a continuation payoff greater than 0. However, we can identify the following properties of all equilibria.

Proposition 4 *In all equilibria of the* 4*-player game, after every history, the following hold.*

- *The offer to the informed player S_I as well as her response is identical to that of the two-player game with a single buyer.*
- *The first proposer B_1 obtains an expected payoff $v_B(\pi)$.*
- *The payoff to S_M varies between* 0 *and $\bar{p}_t^M - M$.*

Proof. To prove the first point, consider the first period t where $\pi_t \leq q_1$. An offer of H is optimal for a buyer in the two-player game and is accepted by S_I with probability 1. Clearly a higher offer is not optimal in the four-player game since even the type H seller will accept an offer of H with probability one. A lower offer is not optimal because the type H seller will reject this, and the definition of q_1 implies that it is better to offer H instead. So, in the four-player game S_I will get the same offer for $\pi_t \leq q_1$. Now consider type L of S_I playing a pure strategy in equilibrium at some period τ in the four-player game. In equilibrium, the pure strategy cannot be to reject with probability 1, because no updating takes place and the buyer will increase her offer. Suppose the pure strategy is to accept with probability 1. Then, in period $\tau - 1$, $\pi_{\tau-1} = 0$, and the buyer must offer H. But, incentive compatibility for the low type implies the offer that is accepted is δH and optimality for the buyer implies $\pi_\tau \leq q_2$.

For other values of π_t, type L of S_I must be playing a non-degenerate mixed strategy. Let t' be the first period (counting backwards) in which S_I in the four-player game gets an offer $p^I_{t'}$ strictly greater than the equilibrium offer in the two-player game for $\pi_{t'}$. (A strictly lower offer will clearly not occur in equilibrium.) If S_I, type L, plays a randomised behavioural strategy, he must be indifferent between accepting $p^I_{t'}$ or rejecting and accepting the two-player equilibrium offer in period $t' - 1$. Therefore $p^I_{t'} = \delta p^I_{t'+1}$. But this is exactly the equilibrium offer in the two-player game, contradicting our hypothesis.

For the second and third parts, note that the first buyer to make a proposal can choose either S_I or S_M. If she chooses S_I, she has to offer the two-player game offer and gets a payoff of $v_B(\pi_t)$. If she chooses S_M she has to offer a price that cannot be bid up by the buyer following, i.e. $\bar{p}^M_t$. This shows point 2 of the proposition. If B_1, being indifferent between S_I and S_M randomises in period $t - 1$, the continuation payoff for S_M in period t will depend on the sequence of such randomisations employed by the first proposer in periods $t - 1$ onwards. The minimum continuation payoff for S_M will be obtained if the first proposer always makes an offer to S_I- a payoff of 0. The highest payoff will be obtained if B_1 always chooses S_M, a payoff of $\bar{p}^M_t - M$. ■

Remark 4 *It is not possible to rule out rejection with probability 1 by S_M.*

This could happen, for example, if the randomisation chosen by B_1 in periods $t-1$ onwards depended on the offer made by B_2 in period t.

The preceding discussion has been based on the protocol where the order of proposers is chosen randomly at the beginning of the game. Suppose, alternatively, that each buyer is chosen as first proposer with equal probability in each period. Clearly, there is no difference in the first equilibrium in which the buyers have the same expected payoff. The second equilibrium also survives. Suppose B_1 and B_2 have been chosen in that order in a particular period. B_1 might consider making a non-serious offer so as to wait for the chance to make an offer to S_M in the following period. However, a non-serious offer to S_I will (a) not result in any updating of π and (b) S_M will accept the equilibrium offer from B_2, so that B_1 will not have S_M available in the next period. If B_1 makes an offer to S_M, the optimal offer does not increase his payoff beyond $v_B(\pi)$. Therefore, a change in the protocol does not affect the equilibrium.

6 Extensions

In this section, we consider some extensions of the basic model considered earlier.

6.1 Many Buyers and Sellers

The results of the basic model extends easily to the case when there are "many" buyers and sellers, provided only one seller has private information. Suppose there are $n > 2$ buyers and sellers, with each buyer's valuation being v, while sellers $1, \ldots, n-1$ have known reservation values $M_1 \geq \ldots \geq M_{n-1} \geq 0$. Seller n is the informed seller, and her valuation is either H with probability $1 - \pi_0$ or $L = 0$ with probability π_0, where

$$v > H > M_1$$

Suppose δ is sufficiently high. Then, there is an equilibrium in which all buyers get the same expected payoff $u(\pi_0)$, where $u(\pi_0)$ is the expected buyer payoff in the 2-person game where π_0 is the initial probability that the informed seller is of the low type.

We describe informally the strategies which sustain this equilibrium. Without loss of generality, let $B_1, \ldots, B_n$ be the order in which buyers make offers. Then, each buyer B_i , $i < n$ offers $\bar{p}_t^M$[15] to some seller S_i, $i < n$ so that each seller receives only one offer. Seller B_n makes the equilibrium offer of the 2-person bargaining game with an informed seller. Sellers $1, \ldots, n-1$ accept their offers, while S_n's response mimics that of the informed seller in the 2-person game. B_n has no incentive to deviate becasue she is essentially playing the 2-person gane with an informed seller. If some other buyer B_i offers a lower price $p < \bar{p}_t^M$ to seller $i < n$, then this does not help because buyer B_n then offers $p + \epsilon$ to the same seller, who obviously accepts the higher offer. Thus, deviation results in B_i tarding with B_n one period later.

As before, there is also an equilibrium in which buyers who make offers later in the sequence get higher payoffs.[16]

6.2 Two Privately-Informed Sellers

Suppose now that both sellers are privately informed. If both sellers are ex ante identical -that is, both sellers have an identical probability of bewing the low type- then the 4-person market essentialy splits up into two 2-person markets. The interesting case is when the two sellers are not ex ante identical. In particular, will there still be an equilibrium in which both buyers obtain the same expected payoff? We construct an example in which there is no equilibrium with both buyers obtaining the same expected payoff.

[15] As before, the price $\bar{p}_t^M$ is such that $v - \bar{p}_t^M = u(\pi_0)$.

[16] The inequality in buyer payoffs will be strict if the reservation values $M_1, \ldots, M_{n=1}$ are all distinct.

Let $v = 5, H = 4, \delta = \frac{3}{4}, \pi_0^1 = \frac{1}{2}, \pi_0^2 = \frac{4}{7}$, where π_0^1, π_0^2 are the initial probabilities that sellers 1 and 2 are of the low type.

We first calculate the cut-offs q_1, q_2, q_3.

If the probability of the low type is q_1, the buyer (in the 2-player game) is indifferent between offering H and δH, the latter being accepted with probability one by the low type. This immediately yields

$$q_1 = \frac{v - H}{v} = \frac{1}{5}$$

Similarly, the buyer is indifferent between offering δH and $\delta^2 H$ when $\pi_0 = q_2$. An offer of δH is accepted with probability one by the low type. Let the probability of acceptance of $\delta^2 H$ be α_{21}. So,

$$(v - \delta H)q_2 + (1 - q_2)\delta(v_H) = (v - \delta^2 H)q_2\alpha_{21} + (1 - q_2\alpha_{21})\delta(v - H)$$

Hence,

$$\alpha_{21} = \frac{5}{8}$$

Also, from Bayes Rule,

$$q_2 = \frac{q_1}{1 - \alpha_{21}(1 - q_1)} = \frac{2}{5}$$

When $\pi_0 = q_3$, the buyer is indifferent between offering $\delta^2 H$ and $\delta^3 H$. Let $V_B(\delta^3 H)$ and $V_B(q_2)$ denote the buyer's expected payoff from the offer $\delta^3 H$ and the equilibrium payoff when $\pi_0 = q_2$. Then,

$$V_B(\delta^3 H) = (v - \delta^3 H)q_3\alpha_{32} + (1 - q_3\alpha_{32})\delta V_B(q_2) \tag{3}$$

where α_{32} is the probability of acceptance which along with Bayes Rule implies that the updated probability of the seller being the low type is q_2. Now,

$$V_B(q_2) = (v - \delta H)q_2 + (1 - q_2)\delta(v - H) = \frac{5}{4}$$

Substituting in equation 3, we get

$$V_B(\delta^3 H) = \frac{53}{16} q_3\alpha_{32} + (1 - q_3\alpha_{32})\frac{15}{16}$$

Also,

$$\begin{aligned} V_B(\delta^2 H) &= (v - \delta^2 H) q_3\alpha_{31} + (1 - q_3\alpha_{31})\delta(v - H) \\ &= \frac{11}{4} q_3\alpha_{31} + (1 - q_3\alpha_{31})\frac{3}{4} \end{aligned}$$

where α_{31} is the probabbility of acceptance by the low type which results in an updated probability of $\pi = q_1$.

Equating $V_B(\delta^2 H)$ and $V_B(\delta^2 H)$ yields

$$16a_{31} - 19a_{32} = \frac{3}{2} \tag{4}$$

where $a_{ik} = q_i\alpha_{ik}$.

Since $(1-a_{31}) = (1-a_{32})(1-a_{21})$, substitution in equation 4 yields $a_{32} = \frac{5}{14}$. Finally, since $q_3 = q_2(1 - a_{32}) + a_{32}$, we have

$$q_3 = \frac{43}{70}$$

So, $q_1 = \frac{1}{5}, q_2 = \frac{2}{5}, q_3 = \frac{43}{70}$.

Let B_1 make the offer to S_1. We first calculate the expected payoff of B_1.

The offer to S_1 must be $\delta^2 H = \frac{9}{4}$. If a denotes the probability of acceptance by the low type, then the updated probability, after rejection, is q_1. Hence,

$$q_1 = \frac{\pi_0^1 - a}{1 - a}$$

This yields

$$a = \frac{3}{8}$$

When the updated probability that S_1 is the Low type is q_1, the buyer is indifferent between offering H and δH. So, the expected payoff of B_1 is

$$E(B_1) = (v - \delta^2 H)a + (1 - a)\delta(v - H) = \frac{3}{2}$$

So, we need to check whether there is an equilibrium in the 4-player game where $E(B_2) = \frac{3}{2}$. First, there cannot be such an equilibrium where S_2 accepts the price offer with probability one. For suppose there is indeed such an equilibrium. Then, since rejection would imply that the seller is of type H, the price offer p must be at least $\delta H = 3$. But if $p \geq 3$, then

$$E(B_2) \leq (v-3)\pi_0^2 + (1-\pi_0^2)\delta(v-H) = \frac{41}{28} < \frac{3}{2}$$

Suppose that an offer of p brings forth a mixed response from the low type of S_2. Since S_2 is indifferent between accepting and rejecting p, p must equal the discounted value of the seller's expected payoff if he rejects p. The latter is calculated as follows. With probability $a = \frac{3}{8}$, the other seller has accepted the offer, and so this is the probability with which S_2 will be involved in a 2-player game in the next period. The next period is a 4-player game with residual probability. In this game, the equilibrium offer (to S_1) is $\delta H = 3$. Letting $\hat{\pi}$ denote the updated probability that S_2 is of the low type, we get

$$p = \delta(aV_S(\hat{\pi}) + (1-a)\delta H) \tag{5}$$

where $V_S(\hat{\pi})$ is the equilibrium offer to S in the 2-player game when the initial probability of the low type is $\hat{\pi}$.

Case 1: $\hat{\pi} > q_2$. Then, $V_S(\hat{\pi}) = \delta^2 H = \frac{9}{4}$. Substituting in equation 5, we get

$$p = \frac{261}{128}$$

We now calculate the expected payoff to B_2. Let $\hat{a}$ denote the probability with which p is accepted by B_2. Since $\hat{a}$ results in the updated probability of $\hat{\pi}$ (from π_0^2),

$$\hat{a} = \frac{\pi_0^2 - \hat{\pi}}{1-\hat{\pi}}$$

Also, let $V_B(\hat{\pi})$ denote the expected payoff to the buyer in the 2-person game when the initial probability that the seller is of the low type is $\hat{\pi}$. Then,

$$\begin{aligned} V_B(\hat{\pi}) &= (v-\delta^2 H)\frac{\hat{\pi}-\frac{1}{5}}{1-\frac{1}{5}} + \frac{1-\hat{\pi}}{1-\frac{1}{5}}\delta(v-H) \\ &= \frac{1}{4}(10\hat{\pi}+1) \end{aligned}$$

So,

$$\begin{aligned} E(B_2) &= (v-p)\hat{a} + (1-\hat{a})\delta(aV_B(\hat{\pi}) + (1-a)(v-H) \\ &= \left(5-\frac{261}{128}\right)\frac{(\frac{4}{7}-\hat{\pi})}{(1-\hat{\pi})} + \frac{\frac{3}{7}}{(1-\hat{\pi})}\frac{3}{4}\left(\frac{3}{8}\frac{1}{4}(10\hat{\pi}+1)+\frac{5}{8}\right) \\ &= \frac{379}{128}\left(\frac{4-7\hat{\pi}}{7(1-\hat{\pi})}\right) + \frac{9}{28(1-\hat{\pi})}\left(\frac{15\hat{\pi}}{16}+\frac{23}{32}\right) \\ &= \frac{1723-2383\hat{\pi}}{896(1-\hat{\pi})} \end{aligned}$$

Equating this to $E(B_1) = \frac{3}{2}$ yields

$$\hat{\pi} = \frac{379}{1039} < 0.4 = q_2$$

Hence, Case 1 cannot occur.

Case 2 : Suppose $\hat{\pi} \in (q_1, q_2)$.

Then, $V_S(\hat{\pi}) = \delta H = 3$.

Substituting in equation 5, we get

$$p = \frac{9}{4}$$

Then, the expected payoff to B_2 is

$$\begin{aligned} E(B_2) &= (5-\frac{9}{4})\frac{\frac{4}{7}-\hat{\pi}}{1-\hat{\pi}} + \frac{\frac{3}{7}}{(1-\hat{\pi})}\frac{3}{4}\left(\frac{3}{8}(\frac{5}{4}\hat{\pi}+\frac{3}{4})+\frac{5}{8}\right) \\ &= \frac{44-77\hat{\pi}}{28(1-\hat{\pi})} + \frac{9}{28(1-\hat{\pi}_2)}\left(\frac{15}{32}\hat{\pi}+\frac{29}{32}\right) \end{aligned}$$

Equating this to $E(B_1) = \frac{3}{2}$ yields $\hat{\pi} > 1$, which is clearly not possible.

This shows that there cannot be an equilibrium in which both buyers get equal expected payoffs.

Remark 5 *However, there will be an equilibrium in which B_1 (the first buyer to make an offer)and B_2 both offer $\delta^2 H$ to S_1 and S_2 respectively. The seller responses are identical to that in the equilibria of the 2-person games. In this equilibrium, $E(B_2) > E(B_1)$.*

7 Conclusions

This paper attempts to model competition among small numbers of market participants with incomplete information. The small numbers makes random matching less desirable as a model and we consider players making targeted offers to particular individuals on the other side. All offers are made by buyers, so as to keep the bargaining-theoretic complexity to a minimum. We find that there are equilibria in which buyers' expected payoffs are equalised in equilibrium if only one of the sellers has private information. (Adding more buyers and sellers with complete information does not matter.) However, if an additional privately informed seller is present, such an equilibrium need not exist and the second buyer to move has an advantage. Surprisingly the competition is always driven by the incomplete information and not by the values of the complete information sellers, in contrast to the complete information model.

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