Some Remarks on the Ranking of Infinite Utility Streams*

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1 Introduction

A long tradition in welfare economics and moral philosophy, dating back at least to Sidgwick(1907) is the idea that all generations must be treated alike. Perhaps, the most forceful assertion of this idea comes from Ramsey (1928) who declared that any argument for preferring one generation over another must come "merely from the weakness of the imagination". The "equal treatment of all generations" or the intergenerational equity principle has been formalised in the subsequent literature as the axiom of *Anonymity*, which requires that two infinite utility streams be judged indifferent to one another if one can be obtained from the other through a permutation of utilities of a *finite* number of generations. Since it also seems "natural" to require that any social evaluation of infinite utility streams respond positively to an increase in the utility of any generation, the Pareto Axiom is also desirable. Unfortunately, Diamond(1965) showed that there is no social welfare function satisfying these axioms along with a continuity axiom. In a more recent paper, Basu and Mitra(2003) prove a more general result by showing that the continuity axiom is superfluous.

These impossibility results are for social welfare *functions*. For many purposes, it is sufficient to have a *social welfare ordering* which allows for comparisons of all infinite utility streams. In an important paper, Svensson (1980) showed that such an ordering satisfying Anonymity and the Pareto axioms does exist. However, Svensson's proof is non-constructive. He constructs a *pre-order*[1] satisfying Anonymity and the Pareto axioms, and then appeals to Szpilrajn's Lemma which guarantees the existence of an ordering extension of any pre-order. Since it is possible to show the existence of an ordering satisfying the two basic axioms, a natural step forward is to explore the existence issue of orderings satisfying additional properties. Several recent papers have taken this route. For instance, Asheim and Tungodden (2004) and Bossert et al (2007) provide characterizations of different infinite-horizon versions of the leximin principle. An infinite-horizon version of utilitarianism is characterised by Basu and Mitra (2007).[2]

Like Svensson (1980), these papers also construct pre-orderings satisfying desirable properties and then invoke Szpilrajn's lemma to assert the existence of an ordering. Of course, it is one thing to *know* that different infinite utility streams can be compared consistently, and quite another thing to know *how*

[1] A pre-order is a binary relation satisfying reflexivity and transitivity. A pre-order which is also *complete* is an ordering.

[2] See also Fleurbaey and Michel (2003), Hara et al (2006), Asheim et al (2007) for other possibility theorems.

to make such comparisons. The latter requires explicit knowledge of the form of the ordering extension. Unfortunately, there is no constructive proof of these existence theorems. Indeed, Zame (2007) demonstrates that an ordering satisfying the two basic axioms cannot be explicitly described!

In this note, I describe some of the recent pre-orders which have been proposed in the literature. Analogous to the literature on the rankings of social states for finite societies, these pre-orders are the infinite horizon versions of classic utilitarianism and the leximin principles. There are basically two different ways in which leximin and utilitarianism have been extended to the infinite horizon context. I use some simple examples to illustrate how these different approaches compare with one another, and argue informally that one method is perhaps better than the other.

I then go on to explore a consequence of imposing *Separability*[3] axioms on social welfare orderings. It is known that no ordering can satisfy the Pareto axiom and a *Strong Anonymity* axiom which requires that two utility streams be judged indifferent if one is obtained from another by means of a permutation of an infinite number of generations.But, how severe is this violation? Is there any *systematic* bias in how generations are treated? Since we do not know the exact form of the various ordering extensions, these questions have remained unanswered. I show that over a limited class of comparisons, Separability along with a weaker version of the Pareto axiom implies that the social welfare ordering must exhibit *time preference*.[4] In other words, there is indeed a systematic bias in the way in which different generations are treated.

2 The Framework

Let $\mathbb{N}$ be the set of natural numbers. Let $\mathbb{R}_+$ be the set of non-negative real numbers, and denote $X \equiv \mathbb{R}_+^{\mathbb{N}}$. Then, X is the set of infinite utility sequences. A typical element of X is an infinite-dimensional vector $x = (x_1, x_2, \ldots, x_n, \ldots)$, and the interpretation is that x_n represents the utility experienced by generation n.

Let s be a *finite* sequence with elements from J. Then, s_∞ denotes the infinite sequence in which s is repeated infinitely often, while $(s)_k$ will denote

[3]Separability means that the ranking of two utility streams should not depend on the utility levels of generations who are indifferent between the two utility streams.

[4]Banerjee and Mitra (2007) show that a Paretian social welfare function on the domain of infinite utility streams must also exhibit a preference for the present over the future.

the sequence in which s is repeated k times. For any $n \in \mathbb{N}$, and $k \in \mathbb{R}_+$, $(k)_n$ denotes the finite sequence in which k is repeated n times.

For any $x \in X$, let $x^{-n} = (x_1, \ldots, x_n)$, be the first n terms of the sequence x, and x^{+n} be the sequence $(x_{n+1}, \ldots,)$. Hence $x = (x^{-n}, x^{+n})$.

Given any $x \in X$, and $n \in \mathbb{N}$, let $\widetilde{x}_n$ be the permutation of x^{-n} which ensures that $\widetilde{x}^{-n}$ is a *non-decreasing* sequence; let $I(x^{-n}) = x_1 + x_2 + \ldots x_n$.

I will use the following notation for vector inequalities on X. For any $x, y \in X$, (i) $x \geq y$ if $x_n \geq y_n$ for all $n \in \mathbb{N}$; (ii) $x > y$ if $x \geq y$ and $x \neq y$.

A *social welfare relation* (henceforth SWR) is a binary relation on X, which is *reflexive* and *transitive*. For any SWR R, the interpretation is that if xRy, then x is considered to be at least as good as y. The symmetric and asymmetric components of R are denoted by the binary relations I and P. Of course, for any $x, y \in X$, xIy if xRy and yRx, while xPy if xRy and not yRx. A *social welfare ordering* (henceforth SWO) is a SWR which is also *complete*.

A SWR R is a *subrelation* to another SWR R' if for all $x, y \in X$, (i) xIy implies $xR'y$ and (ii) xPy implies $xP'y$. If R is a subrelation to R', then R' is an *extension* of R.

A finite permutation of $\mathbb{N}$ is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that there exists $m \in \mathbb{N}$ with $\sigma(n) = n$ for all $n \in \mathbb{N} \setminus \{1, \ldots, m\}$. Let $\sigma(x)$ denote the utility sequence which results from a finite permutation σ of x. I will use Σ to define the set of all finite permutations of $\mathbb{N}$.

Two fundamental axioms in the recent literature on intergenerational social welfare rankings are the Strong Pareto principle and Finite Anonymity. These are defined below.

Strong Pareto : For all $x, y \in X$, if $x > y$ then xPy.

Finite Anonymity: For all $x \in X$, and for all $\sigma \in \Sigma$, $xI\sigma(x)$.

The Suppes-Sen *grading principle* is the SWR R_S defined on X as follows.

$$\text{For all } x, y \in X, xR_Sy \text{ iff } \exists\, \sigma \in \Sigma \text{ such that } \sigma(x) \geq y$$

The grading principle R_S is transitive, but not complete and hence is not a SWO. However, it is a subrelation of any SWR satisfying the Strong Pareto and Finite Anonymity axioms. Svensson (1980) established the existence of a SWO satisfying these two axioms by invoking Szpilrajn's lemma to conclude that some extension of R_S must be a SWO.

3 Utilitarian and Leximin Relations

Since it is possible to show the existence of SWO's satisfying the two basic assumptions, a natural step forward has been to explore the possibilities of having social welfare orderings satisfying additional assumptions. In the context of a finite society, much of the literature has focussed on axiomatic characterizations of utilitarian and leximin social welfare orderings.[5] Several recent papers have taken this route in the ranking of infinite utility streams. For instance, Asheim and Tungodden (2004) and Bossert et al (2007) provide characterizations of different infinite-horizon versions of the leximin principle. Infinite-horizon versions of utilitarianism are characterised by both Basu and Mitra (2007) and Asheim and Tungodden(2004). However, as I have mentioned earlier, none of these papers actually *construct* a SWO satisfying these axioms, as they all use some version of Szpilrajn's Lemma to demonstrate the existence of SWOs. This section contains an informal discussion of the various rules, focusing partly on the difficulty which may arise if comparisons are based solely on social welfare relations.

Consider, for example, different formulations of infinite-horizon utilitarianism.

Basu and Mitra(2007) define what they call the *utilitarian* SWR R_U as follows.

$$\forall x, y \in X, xR_Uy \text{ iff } \exists n \in \mathbb{N} \text{ such that } (I(x^{-n}), x^{+n}) \geq (I(y^{-n}), y^{+n})$$

A different and more traditional method of comparing infinite utility streams (without using discounting) is by employing the *overtaking* principle. There are two versions of the overtaking criterion - the *catching up* criterion and the *overtaking* criterion. Denote the corresponding SWRs as R_C and R_O. The formal definitions follow.

$$\forall x, y \in X, xR_Cy \text{ iff } \exists \bar{n} \in \mathbb{N} \text{ such that } I(x^{-n}) \geq I(y^{-n}) \forall n \geq \bar{n}$$

$$\begin{aligned} \forall x, y \in X, xR_Oy \text{ iff either } & (i) \exists \bar{n} \in \mathbb{N} \text{ such that } I(x^{-n}) > I(y^{-n}) \forall n \geq \bar{n} \\ \text{or } & (ii) \exists \bar{n} \in \mathbb{N} \text{ such that } I(x^{-n}) = I(y^{-n}) \forall n \geq \bar{n} \end{aligned}$$

There are different extensions of the Rawlsian leximin criterion to the infinite case. In order to define these extensions, it is useful to define the leximin criterion on *finite* utility streams. So, let s and r be two finite sequences both of length k, with $\widetilde{s}$ and $\widetilde{r}$ being the permutations of s and t which ensure that they are non-decreasing sequences. Then,

$$sR_l^k r \text{ iff } \widetilde{s} = \widetilde{r} \text{ or } \exists j < k \text{ such that } \widetilde{s}_i = \widetilde{r}_i \forall i < j \text{ and } \widetilde{s}_j > \widetilde{r}_j$$

[5] d'Aspremont and Gevers(2002) is an elegant survey of this literature.

One version of the infinite horizon leximin rule, due to Asheim and Tungodden(2004) is the following.

$$\forall x, y \in X, xR_L^1 y \text{ iff } \exists \bar{n} \in \mathbb{N} \text{ such that } \forall n \geq \bar{n}, x^{-n} R_l^n y^{-n}$$

An alternative formulation of infinite horizon leximin is due to Bossert et al (2007). They define the following SWR.

$$\forall x, y \in X, xR_L^2 y \text{ iff } \exists n \in \mathbb{N} \text{ such that } x^{-n} R_l^n y^n \text{ and } x^{+n} \geq y^{+n}$$

These definitions demonstrate that there have been essentially two ways of extending utilitarianism and leximin to the infinite horizon. First, one can define one utility stream x to be at least as good as another utility stream y according to the utilitarian (respectively leximin) principle if there is some period $\bar{n}$, such that *all* finite truncations of x of length greater than $\bar{n}$ are deemed to be at least as good as the corresponding finite truncations of y according to the finite version of utilitarianism (respectively leximin). The catching up and overtaking criteria in the case of utilitarianism, and R_L^1 in the case of leximin fall in this category. Alternatively, one can declare x to be better than y if there is some finite truncation of x of length n which is better than that of y according to the utilitarian (or leximin) criterion *and* x^{+n} is at least weakly Pareto preferred to y^{+n}. The utilitarian SWR R_U and the leximin SWR R_L^2 belong to this category.

The relationship between these two categories is clear. Consider, for instance, the two leximin criteria. If $xR_L^2 y$, then from some period onwards, no term in the sequence x is smaller than the corresponding term in y. So, if the truncation of x is better than the truncation of y prior to this period according to the leximin criterion, then x must come out better than y in *all* subsequent comparisons. Hence, $xR_L^1 y$ must be true. A similar argument holds for R_U on the one hand and R_O, R_C on the other hand. These "facts" are summarised below.

Proposition 1 *(i) R_U is a subrelation of R_O, which in turn is a subrelation of R_C.*

(ii) R_L^2 is a subrelation of R_L^1.

Leximin and utilitarianism are of course very different criteria, and much has been written about the two classes in the finite context. Clearly, a different kind of comparison can also be made in the ranking of infinite utility streams.

For instance, is R_U a "better" representative of utilitarianism than R_O? Or what about R_L^1 versus R_L^2?

Consider, first the comparison between different representatives of utilitarianism. Clearly, R_U is more conservative in declaring one social state to be preferred to another. So, at first sight, it may seem that R_U is less likely to jump to erroneous conclusions in so far as strict preference is concerned. Indeed, this is precisely the point made by Basu and Mitra (2007) in the context of an example which is almost identical to Example 1 below.

However, comparisons between social welfare relations can sometimes be slightly misleading. For suppose R_U is unable to compare between two alternative utility streams, while R_O can compare the two. Do we "complete" R_U by declaring the two states to be indifferent? This can be problematic as I demonstrate in Example 1. The problem arises because what matters is not the SWR R_U itself, but its *ordering extension(s).* In principle, a SWR may have more than one ordering extension. Also, if R is a subrelation of R', then the ordering relation of R' must also be an ordering extension of R, although the converse may not be true.[6]

Given any SWR R, let $\bar{R}$ denote some ordering extension of R. Of course, $R = \bar{R}$ if R is a SWO.

Consider the following example.

Example 1 *Let* $x = (2, (1, 0)_\infty)$*,* $y = ((1, 0)_\infty)$*, and* $z = (1, (1, 0)_\infty)$*.*

Then, xP_Oy but neither xR_Uy nor yR_Ux. Basu and Mitra(2007) actually argue that this is a virtue of R_U because "there are an infinite number of future generations who rank x below y". But, now let us consider $\bar{R}_U$. Suppose $x\bar{I}_Uy$. Then, the same intuitive reasoning must force us to conclude that $y\bar{I}_Uz$. But, now we are in trouble because x Pareto dominates z, and so xP_Uz leading to $x\bar{P}_Uz$. Transitivity decrees that $x\bar{P}_Uy$. Hence, the ordering extension of R_U must express a strict preference between x and y even if R_U itself prefers to remain silent!

In fact, this example demonstrates the following impossibility theorem.

Veto Power of Infinitely Many Generations(VPIMG) : For all $x, y \in X$, if $|\{n \in \mathbb{N} | x_n > y_n\}| = \infty$, then $x\bar{R}y$.

[6] In fact, if the ordering extension of R' is the *only* ordering extension of R, then there is nothing to choose between R and R'.

Theorem 1 *There is no SWO satisfying VPIMG and the Strong Pareto Principle.*

Proof. Consider x, y, z defined in Example 1. If $\bar{R}$ satisfies Strong Pareto and VPIMG, then $x\bar{P}y$ from Strong Pareto, and $y\bar{I}z$ and $y\bar{I}x$. This is a violation of transitivity. ■

Remark 3.1 *Notice that the impossibility is precipitated even without any appeal to the Finite Anonymity condition.*

Basu and Mitra (2007) also argue that a "robustness" check on a SWR is to check whether the ranking between pairs of infinite utility streams provided by the SWR is preserved for discount factors close to one in the discounted present value social welfare function. They then construct an ingenious example of a pair of alternatives x and y such that xP_Oy but not xR_Uy, and the discounted present value of y is strictly higher than that of x *for every* $\delta \in (0, 1)$. In their example, the limiting values as $\delta \to 1$ exist and are equal. Basu and Mitra argue that this implies that x and y should be deemed indifferent.

However, apart from the fact that discounting even as a robustness check goes against the spirit of anonymity, it is not clear that the two utility streams would be deemed indifferent according to $\bar{R}_U$. Moreover, as the next example demonstrates, R_U may fail to pass judgement on a pair of utility streams even when the discounted present value of one is strictly higher than that of another for all discount factors.

Example 2 *Choose numbers* $b > a > 0$*, and let* $x = (2b, 0)_\infty$ *while* $y = (b, a)_\infty$*.*

In this case, every finite truncation of x has a higher sum than that of y and so x is preferred to y according to the overtaking criterion and hence the catching up criterion. But, R_U cannot compare between x and y since an infinite number of generations prefer y to x, although for all $\delta \in (0, 1)$, $\sum_{n=1}^{\infty} \delta^{n-1}(x_n - y_n) > 0$.

Of course, being able to pass judgement is not necessarily a virtue. This is illustrated in the next example.

Example 3 *Let* $x = (1, 0)_\infty$*, while* $y = (0, 1)_\infty$*.*

In this example, x and y are noncomparable according to the overtaking criterion, but the catching up criterion declares x to be strictly preferred to y. In the first stream, generations in odd periods get a utility of 1. The role of the "rich" and "poor" generations is reversed in the second utility stream, with generations in even periods getting the higher utility. Clearly, a social welfare ordering satisfying any intuitive notion of intergenerational equity should declare the two utility sequences to be indifferent. Notice that R_C does satisfy Finite Anonymity, but not the stronger version which requires that two utility streams be judged indifferent if one is obtained from another by means of a permutation of an infinite number of generations.[7]

I come now to a comparison of the two versions of the leximin principle. Consider again Example 1. Suppose $n > 2$ and n is even. Then, $\widetilde{x}^{-n}$ will have $n/2 - 1$ 0s, followed by $n/2$ 1s and one 2. If n is odd, then $\widetilde{x}^{-n}$ will have $(n-1)/2$ 0s followed by the same number of 1s and finally 2. On the other hand, for n even, $\widetilde{y}^{-n}$ will have an equal number of 0s and 1s. For n odd, $\widetilde{y}^{-n}$ will have $(n+1)/2$ 0s and $(n-1)/2$ 1s. So, for all finite truncations, $x^{-n} P_l^n y^{-n}$ and hence $x P_L^1 y$. On the other hand, for no n will x^{+n} Pareto dominate y^{+n} and so R_L^2 cannot compare the two. However, R_L^2 implicitly gives veto power to coalitions of infinitely many generations. Theorem 1 demonstrates the problem with this principle.

However, the following example suggests that perhaps R_L^1 declares strict preference even when it should not.

Example 4 *Let* $x = (0, 2_\infty)$ *and* $y = 1_\infty$.

The SWR R_L^2 cannot compare the two utility streams. But, for all n, $\widetilde{x}^{-n} = (0, 2, \ldots, 2)$ while $\widetilde{y}^{-n} = (1, \ldots, 1)$. Hence, $y P_L^1 x$. Notice however that except for $t = 1$, $\widetilde{x}_t^{-n} > \widetilde{y}_t^{-n}$. Nevertheless, y is declared to be better than x. It is known that while the finite horizon leximin rule has no individual as a dictator, it is characterized by "positional dictatorship" of the worst-off rank. The extreme importance given to one rank is obviously less defensible when there are an infinity of positions.

4 Separability

In the context of a *finite* society, the axiomatic literature on the characterization of interpersonally comparable social welfare rankings such as leximin and classic

[7]However, no SWR satisfies Strict Pareto and Strong Anonymity. See also the next section where a Separability axiom along with a weaker version of the Pareto principle also results in x being declared better than y.

utilitarianism uses a *separability* condition which requires that the ranking of two utility or welfare vectors should be independent of the utility levels of "unconcerned" individual, that is individuals who have the same levels of utility in the two vectors. In this section, I explore consequences of using the Separability axiom in the ranking of infinite utiltiy streams.

There are different ways of formalising separability when there are an infinite number of generations. A particularly weak version of separability is to require that if the "first" generation has the same utility level in two utility streams, then the ranking of these two utility streams can only depend on the utility sequences from period two onwards.

Separability : $\forall\ a \in \mathbb{R}_+,\ \forall x, y \in X,\ (a, x)R(a, y) \leftrightarrow xRy$

This condition, under the name of *Stationarity*, has been used extensively in the literature on the utility ranking of infinite consumption streams.[8]

A stronger form of separability extends the definition to infinite sets of unconcerned individuals. That is, suppose the comparison is between $x = (3, 0, 1)_\infty$ and $y = (2, 0, 2)_\infty$. Then, all generations $t = 2+3r$ where $r = 0, 1, \ldots$ have the same utility in x and y. So, the stronger form of separability requires that the ranking between x and y coincide with that between $(3, 1)_\infty$ and 2_∞.

I need some more notation to define this stronger form of separability. Given any $x \in X$ and $T \subset \mathbb{N}$, let x_{-T} denote the subsequence of x which takes values only in $\mathbb{N} - T$. For any $x, y \in X$, let $T(x, y) = \{n \in \mathbb{N} | x_n = y_n\}$.

Strong Separability: For all $x, y \in X$, $xRy \leftrightarrow x_{-T(x,y)}Ry_{-T(x,y)}$

A possible reason for objecting to the Strict Pareto condition is that when there are an infinite number of generations, one utility stream should not be judged strictly superior to another if just a single individual is better off and all others are indifferent. A much weaker requirement is a "non-perversity" condition which states that the social ranking must not respond negatively to an increase in individual utilities. This condition is formalized below.

Monotonicity: $\forall x, y \in X$, if $x > y$, then xRy.

Even if a single individual's strict preference (along with other individuals' indifference) should not translate into social strict preference, it may be argued that if a sufficiently large but finite number of individuals strictly prefer x to y and all others are indifferent, then x should be strictly preferred to y. This is the *Finite Pareto* principle.

[8] See for instance Koopmans (1960¡ Fishburn and Rubinstein)(1982). A slightly stronger condition has been labeled the "Independent Futures" condition by Fleurbaey and Michel (2003).

Finite Pareto Principle: There is $\bar{n} \in \mathbb{N}$ such that $\forall x, y \in X$, if $|\{i \in \mathbb{N}|x_i > y_i\}| \geq \bar{n}$ and $x > y$, then xPy.

The three versions of utilitarian pre-orderings as well as the leximin pre-order R_L all satisfy Separability. Since they satisfy the Strict Pareto condition, they obviously satisfy Monotonicity and the Finite Pareto principle. Notice, however, that Szpilrajn's lemma cannot be used to assert the existence of a SWO satisfying these three conditions. Consider, for instance, $\bar{R}_U$, the ordering extension of R_U. The fact that R_U satisfies Separability does not rule out the existence of a pair x, y in X and a in $\mathbb{R}_+$ such that the pairs (x, y) and $(a+x, a+y)$ are noncomparable according to R_U, but $x\bar{R}_U y$ and $(a, y)\bar{P}_U(a, x)$! Of course, it is not easy to check whether such a pair exists since we do not know the functional nature of $\bar{R}_U$.

In the following, I describe a characteristic of any SWO satisfying Separability, Monotonicity and the Finite Pareto principle *if* such an ordering exists. I show that separability and strong separability imply *time preference* within a limited class of comparisons, with the stronger form of separability leading to time preference for a larger class of comparisons. I will first describe the type of comparisons which result in time preference.

Throughout this section, I restrict attention to utility sequences which are infinite repetitions of finite sequences. That is, comparisons are between sequences of the form s_∞ and s'_∞.

Now, consider two finite sequences $s = (p, (0)_k)$ and $s' = ((0)_k, p)$ where $p > (0)_k$ for some integer k. So, s and s' are both sequences of length $2k$, where x is the sequence in which the terms $p_1, \ldots, p_k$ occur first followed by a sequence of k zeroes. In y, the first k terms are 0, followed by the sequence p. Also, note that each p_i is non-negative with at least term being strictly positive. Of course, any social welfare relation in the finite context satisfying Anonymity must declare s and s' to be socially indifferent, while discounting will imply that s is better than s'. One can also say that in the comparison of infinite utility streams, the social rule exhibits time preference if it declares that s_∞ is strictly preferred to s'_∞. The next proposition shows that any SWO satisfying Monotonicity, Separability and the Finite Pareto principle must exhibit this kind of time preference.

Proposition 2 : *Suppose R is a SWO satisfying Monotonicity, the Finite Pareto Principle and Separability. Then, xPy if $x = (p, (0)_k)_\infty$, $y = ((0)_k, p)_\infty$ where $p > (0)_k$ for some integer k.*

Proof. Consider any SWO R satisfying Monotonicity and Separability. Choose any x and y satisfying the conditions in the proposition.

I show that R cannot satisfy the Finite Pareto principle when yRx.

Define $z = (p, x)$.

Step 1: I first prove that yRx implies zIy.

Notice that $x \equiv (p, y)$ and $y = ((0)_k, x)$.

Since yRx, the repeated application of Separability ensures that $(p, y)R(p, x)$. Hence, xRz.

Since $p > (0)_k$, Monotonicity ensures that $(p, x)R((0)_k, x)$. So, zRy. Hence, from transitivity, xRy. So, xIy. This is turn ensures that xIz and so zIy.

Now, define $z^n = ((p)_n, x)$ so that $z^1 \equiv z$. Also, define $y^n = ((0)_{kn}, p)$, so that $y^1 \equiv y$.

Step 2: I now show that $z^n I z^{n+1}$ and $y^n I y^{n+1}$.

Now,

$$\begin{aligned} z^n &= ((p)_n, x) \\ &= ((p)_n, p, y) \end{aligned}$$

From repeated use of Separability

$$((p)_{n+1}, x)I((p)_{n+1}, y)$$

Using Separability again,

$$z^{n+1} \equiv ((p)_{n+1}, x)I(p_{n+1}, y) \equiv z^n$$

Similarly,

$$\begin{aligned} y^{n+1} &= ((0)_{k(n+1)}, x) \\ &= ((0)_{kn}, y) \end{aligned}$$

Making repeated use of Separability,

$$y^{n+1} \equiv ((0)_{kn}, x)I((0)_{kn}, x) \equiv y^n$$

Steps 1 and 2 establish the theorem. Step 1 showed that $z^1 I y^1$. Step 2 shows that for all $n \in \mathbb{N}$, $z^1 I z^n$ and $y^1 I y^n$. Hence, for all $n \in \mathbb{N}$, $z^n I y^n$. This shows that the Finite Pareto principle is violated since $(p)_n > (0)_{kn}$. ■

The following proposition follows easily from the earlier one.

Proposition 3 : *Suppose R is a SWO satisfying Monotonicity, the Finite Pareto Principle and Strong Separability. Then, xPy if $x = (p, (0)_k)_\infty$, $y = ((0)_k, p)_\infty$ where $p > (0)_m$ for some integers m, k. with $m \leq k$.*

Proof. Of course, Strong Separability implies Separability and so the case of $m = k$ has been proved already. So, take any m, k with $m < k$ and $x = (p, (0)_k)_\infty$ and $y = ((0)_k, p)_\infty$ where $p > (0)_m$.

Since $m < k$, $x_n = y_n = 0$ for all $n = m+1, \ldots, k-1, 2(m+1), \ldots, 2(k-1), \ldots, t(m+1), \ldots, t(k-1), \ldots$. Using Strong Separability, xRy iff $x'Ry$ where $x' = (p, (0)_m)_\infty$ and $y' = ((0)_m, p)_\infty$. Since p is also of length m, we know from Proposition 2 that $x'Py'$. Hence, xPy. ■

Corollary 1 *If $x = (1, 0)_\infty$ and $y = (0, 1)_\infty$, then xPy.*

This is obviously a gross violation of intergenerational equity! It shows that any SWO satisfying the stipulated properties must exhibit some form of time preference. Notice that R_C does rank $(1, 0)_\infty$ over $(0, 1)_\infty$. But, as I have remarked in the previous section, this is an unappealing feature of the catching up criterion. This section shows however that this is also an inevitable cost associated with the Separability axiom.

5 References

G.B. Asheim, W. Buchholz, B. Tungodden (2001) "Justifying sustainability," *J. Environ. Econ. Manage.* **41**, 252-268.
G.B. Asheim, W. Buchholz, B. Tungodden(2004) "Resolving distributional conflicts between generations," *Rev. Econ. Stud.***24**, 221-230.
K. Banerjee, T. Mitra(2006) "On the impatience implications of Paretian social welfare functions", **43**, 236-248.
K. Basu, T. Mitra(2003) "Aggregating infinite utility streams with intergenerational equity: the impossibility of being Paretian," *Econometrica* **71**, 1557-1563.
K. Basu, T. Mitra(2007)"Utilitarianism for infinite utility streams: A new welfare criterion and its axiomatic characterization," *Journal of Economic Theory* **133**, 350-373.
d'Aspremont, C. and L.Gevers(2002) "Interpersonal comparability of welfare and social choice theory", in K.J.Arow, A.K. Sen and K.Suzumura eds., *Handbook of Social Choice Theory*, Elsevier, Amsterdam.
P. Diamond (1965) "The evaluation of infinite utility streams", *Econometrica*, **33**, 170-177.
M. Fleurbaey, P. Michel (2003) " Intertemporal Equity and extension of the Ramsey Principle", *J. Math. Econ.* **39**, 777-802.
Fishburn, P., and A. Rubinstein (1982) "Time Preference", *International Economic Review*, **23**, 677–694.
T.C. Koopmans (1960) " Stationary ordinal utility and impatience", *Econometrica* **28**, 287-309.
A.K. Sen (1971) *Collective Choice and Social Welfare*, Oliver and Boyd, Edinburgh.
A.K. Sen (1977) "On weights and measures: informational constraints in social welfare analysis", *Econometrica***45**, 1539-1572.
H.Sidgwick (1907) *The Method of Ethics*, Macmillan, London.
P. Suppes (1966) "Some formal models of grading principles", *Synthese***6**, 284-306.
L.-G. Svensson (1980) "Equity among generations", *Econometrica***48**, 1251-1256.
W.R. Zame (2007) "Can Utilitarianism be Operationalized?", *Theoretical Economics*, **2**, 187-202.