General Investments and Outside Options: Some Recent Results Reconsidered*

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Abstract

Some recent contributions have studied equilibrium ex ante investments by buyers and sellers, followed by matching and bargaining over the match surplus, with diverse findings. This paper sets up a simple model with both matching frictions and costs of switching bargaining partners, which encompasses existing contributions. We show that when switching costs are zero, in the limit as matching frictions go to zero, there are a continuum of equilibria with investment between efficient and hold-up levels, but when switching costs are positive, the unique equilibrium investments are at the hold-up level. Under the same assumptions, the axiomatic approach of Cole, Mailath, and Postlewaite predicts that equilibrium investments will be efficient.

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1. Introduction

In many situations, buyers and sellers have to make complementary investments before knowing the identity of the agent with whom they will be matched. The classic example is that of workers and firms: workers and firms have to make decisions about human and physical capital investment, often before workers are matched with firms. Another example would be the marriage market. A key question is: what are the incentives to invest in this setting? Should we expect underinvestment or overinvestment?

This problem has received some attention in recent years, with different contributions modelling the post-investment matching process in different ways. Our focus[1] is on a literature in which investment is followed by matching or pairs of buyers and sellers, and then bilateral bargaining over match surplus. First, there two papers which assume that matching is stochastic i.e. agents on each side of the market are randomly assigned to those on the other, and that the quasi-rent from the match is divided by ex post bargaining (Acemoglu (1996), Acemoglu and Shimer (1999), Masters(1998)). The bargaining protocols studied in these papers allow for agents to have outside options: once matched, agents can switch[2], (possibly at a cost) to a new bargaining partner. Acemoglu (1996) considers costly switching and shows that even when switching costs are arbitrarily small, investments are at their hold-up levels i.e. where each agent invests up to the point where the marginal cost of investment equal to half the social benefit. Masters(1998) considers costless switching and claims that investments are at efficient levels.

Cole, Mailath and Postlewaite (2001a, b) henceforth CMP, take a different approach. In their set-up, matching and bargaining is not modelled explicitly, but the outcome of this second stage is constrained to be "stable" i.e. there is no pair of agents that by rematching and appropriately sharing resulting surplus can both be strictly better off than in the equilibrium. Their paper, although at first sight different to those of Acemoglu and Masters, is in fact related: CMP's motivation for using the stability concept is *"to capture the idea that the division of the surplus within any match should respect outside options"*. They are able to find examples of equilibria with underinvestment, efficient

[1] Burdett and Coles(2001) consider the same issue, but in a NTU framework i.e. match surplus is not transferable. Felli and Roberts(2000) assume allocation of buyers to sellers (or vice versa) is by auction.

[2] Masters also considers a protocol which only allows bargainers the outside option of returning to the search process. In this case, under the assumptions made by Masters (continuous investments, and homogenous agents) there is definitely underinvestment in equilibrium, as outside options never bind. However, in a companion paper, de Meza and Lockwood(2004) show that if either of these two assumptions are relaxed, overinvestment may occur.

investment, or overinvestment.

This paper attempts to reconcile these rather different results in a common framework, and also to correct some problems with the existing analysis. First, in his model, Acemoglu considers only a limiting case, where the matching process is frictionless i.e. where agents are immediately matched, and those wishing to opt out can instantaneously find another other agents to switch to. As he says himself, "*it is a convenient substitute for the more rigorous strategy of characterizing the steady state of a bargaining market with entry and exit*". Acemoglu(1996), p797. This is a limitation, as generally one would wish to model both switching costs and frictions in the matching process simultaneously: in particular, a good definition of a "competitive" market would be one where both frictions and switching costs are very low (but possibly not zero). Masters(1998) set-up does allow for matching frictions, but has no switching costs.

Our first contribution, prompted by Acemoglu's observation, is to set up and rigorously analyse the steady state of a simple model of entry and exit of agents, allowing for both matching frictions and switching costs, which encompasses both of these two contributions. We then consider equilibrium investments in this setting, both without and with switching costs. This reveals a mistake in Masters' analysis. On the bottom of p480 of Masters(1998), it is *assumed* that "an individual whose offer is rejected and who is contacted during negotiations leaves with a new suitor". This can only be rational if this individual - say the seller - is in a match with a buyer who has underinvested: otherwise, the seller will not switch. We show in this paper[3] that this creates a kink in payoffs at equilibrium investment which gives a *continuum of equilibria with between efficient and hold-up investment when switching costs are zero*. On the other hand, the insight of Acemoglu is confirmed: even with positive match frictions, a positive switching cost implies underinvestment in equilibrium. Therefore, the equilibrium correspondence mapping switching costs into equilibrium investments is upper-hemi-continuous at zero switching costs.

Finally, we turn to the axiomatic approach of CMP. We consider stable matching in our model, where all buyers, and all sellers, have the same costs of investment (although buyer and sellers may differ in these costs). Applying CMP's results to our model[4], we find that the equilibrium investments are the (unique) investments that maximize the

[3]Masters ignores this point - i.e. he assumes that the seller always takes the outside option - and so finds efficient investment in equilibrium.

[4]In CMP's model, all agents on a given side of the market are heterogenous, differing in costs of investment, whereas they are homogeneous in our model - although different side of the market (buyers and sellers) can differ in this way. Nevertheless, their results carry over.

net surplus from a match (the gains from trade minus the costs of investment). Under- and over-investment are not possible. So, there is consistency between the axiomatic and strategic approaches only when switching costs are assumed zero in the latter approach. This could be regarded as a strong assumption and shows that simply assuming stability is not without loss of generality.

2. The Model

2.1. Preliminaries

There are two types of agents: buyers and sellers, who interact over an infinite number of time periods $t = -1, 0, 1...$ All agents have a discount factor $\delta = e^{-r\Delta}$, where the period length is Δ.

The following events occur in each period t. First, unit measures of new buyers and sellers are born and simultaneously make investments β, $\sigma \in \Re_+$. Investments have a cost of $c(\beta)$, $\psi(\sigma)$, where $c(.)$, $\psi(.)$ are strictly increasing, differentiable, and convex. Then, a fraction $0 < \Delta a < 1$ of the measure of as yet unmatched[5] buyers and sellers are randomly matched with each other. That is[6], every seller is matched with a buyer(and vice versa) with probability Δa. Second, if both buyer and seller are matched, they decide simultaneously and independently whether to accept or reject the match. If one or both reject, they remain unmatched until the beginning of the next period. If they both accept, they then bargain over the division of the gains from trade, $y(\beta, \sigma)$. It is assumed that $y(\beta, \sigma)$ is strictly increasing, strictly concave and twice differentiable in its arguments, with first derivatives denoted y_β, y_σ. We also assume weakly complementary investments $y_{\beta\sigma} \geq 0$. Note that we do *not* require y to be symmetric in its arguments. This covers several cases[7]studied in the literature.

The bargaining procedure is as follows. Following Acemoglu and Masters, we allow

[5]As any buyer must exit matched with a seller, in any period, the measures of buyers and sellers in the unmatched state are the same.

[6]For concreteness, think of a two-stage matching process where measure Δa agents on either side of the market are randomly selected from the pool of the unmatched, and then these Δa workers and firms are randomly matched with each other. The existence of such procedure even with a continuum on each side of the market is guaranteed by the arguments of Alos-Ferrer(2002).

[7]One is is where y is separable, i.e. $y_{\beta\sigma} = 0$, so $y(\beta, \sigma) = u(\beta) - g(\sigma)$, where $u(\beta)$ is the utility or profit of the buyer from consumption of the good, and $g(\sigma)$ is the cost to the seller of supplying the good, and $g' < 0$, so σ can be interpreted as a cost-reducing investment. In labour market applications, it is more usual to assume $y_{\beta\sigma} > 0$ (see e.g. Acemoglu(1996), Lockwood and de Meza(2004)).

the bargainer to switch partners (possibly at a cost) while bargaining. First, in any time period, one of the agents is randomly selected to make an offer to the other. The responder can either accept or reject. If she accepts, agreement is reached, and the buyer and seller exit the market and trade. If she rejects, with probability $\Delta\lambda$ she receives an outside option which is a match with another agent on the opposite side of the market, which she can accept or reject. If she accepts the outside option, she must pay a switching cost $\varepsilon \geq 0$. With probability $1 - \Delta\lambda$, nothing further happens in that period. If the outside option is not available, or is rejected, the proposer becomes responder in the next period.

The switching cost can be interpreted as a mobility cost, or a flow loss because finding another partner takes time[8]. Note that if $\varepsilon = 0$, outside options are exactly as in Masters. The case $\varepsilon > 0$ can be thought of as the one analyzed by Acemoglu. Finally, to simplify the algebra, we assume $\lambda = a$, although nothing crucial hangs on this. Note that in contrast to the bilateral monopoly case (with just one buyer and seller), agents have two *outside options* in this model. First, an agent can reject a match, and continue searching. Second, once matched, exit to bargaining with a third agent, is possible.

2.2. Strategies and Equilibrium

Section 2.1 above describes a stochastic game played at periods $t = .. - 1.0, 1...$ by a continuum of players. We restrict attention to steady-state equilibria i.e. where flows into, and flow out of, the unmatched state are equal. Let the measure of unmatched agents at time t be μ_t : because, in equilibrium, no matches are refused, this evolves according to $\mu_t = 1 - \Delta a \mu_{t-1}$, so this requires simply $\mu_t = 1/(1 + \Delta a) = \mu^*$. We also assume symmetry i.e. all agents one side of the market invest the same, say β^* for buyers, and σ^* for sellers. So, in any continuation game, we can assume that all but a measure zero of as yet unmatched agents have invested β^*, σ^*. So, if a buyer b and a seller s are matched at the beginning of period t, the only[9] payoff-relevant aspects of the history of play for this pair are (i) their two investment levels β, σ : (ii) the equilibrium investment β^*, σ^* made by almost all as yet unmatched agents.

We will say that the match acceptance and bargaining strategies of an agent are *Markov* if they only depends on β^*, σ^*, his own investment, and, while bargaining, the investment

[8] In our set-up, this last possibility can be formalised as follows. Suppose that $\Delta\lambda = 1$, and the outside option is not an actual match with another agent, but entry into the search process with matches arriving at rate a.

[9] Note in particular, the measure of unmatched μ^* is not payoff-relavent as it does not affect the matching probabilities of the remaining agents (implicitly, there are constant returns to scale in the matching process).

of the agent he is matched with. Note that if (almost) all agents follow Markov strategies, the expected payoff to a buyer (resp. seller) being unmatched in at the beginning of any period $t = 1, 2, ...$ will only depend on that agent's own investment β (resp. σ) and β^*, σ^*, and not on any other aspect of the history of the game: let this payoff be $v_b(\beta; \beta^*, \sigma^*)$ for the buyer, and $v_s(\beta; \beta^*, \sigma^*)$ for the seller. So, $v_b(\beta; \beta^*, \sigma^*)$ measures the payoff to a buyer if he invests at some arbitrary level β (which we call a *deviant buyer* in what follows), while all other agents invest at the equilibrium level, and $v_b(\sigma; \beta^*, \sigma^*)$ has a similar interpretation for the seller.

Within this class of strategies, we will focus on *perfect* match acceptance and bargaining strategies of the agents. Such a match acceptance strategy is one where an agent accepts a match at any date iff doing so gives a higher payoff than continued search. Bargaining strategies are perfect if they are subgame-perfect in the bargaining game between the two partners, assuming that a match is accepted.

3. Analysis

3.1. Preliminary Results

With the assumption of Markov strategies, the outside options in the bargaining game are stationary. In particular, let w_b be the payoff to switching to bargaining with a seller who has invested σ^*, and where w_s be the payoff to switching to bargaining with a buyer who has invested β^*, including any switching cost. For the moment, we take w_b, w_s as exogenous. We can now draw on existing results for two-person bargaining games (Muthoo(1999)) to characterize the continuation equilibrium of the bargaining game between two matched agents.

Lemma 1. *Consider a buyer and seller who have invested β, σ respectively and thus have a gain from trade $y(\beta, \sigma) = y$ to be divided. Assume all other agents who are not yet matched have invested β^*, σ^*. In the limit as $\Delta \to 0$, there are three cases. First, neither buyer or seller have a binding outside option i.e. $w_b, w_s \leq 0.5y$, In this case, $\pi_b = \pi_s = 0.5y$. Second, the buyer has a binding outside option i.e. $w_b - \varepsilon > 0.5y \geq w_s - \varepsilon$. Then*

$$\pi_b = (1-\lambda)\frac{y}{2} + \lambda w_b, \ \pi_s = (1-\lambda)\frac{y}{2} + \lambda(y - w_b) \tag{3.1}$$

where $\lambda = \frac{0.5a}{r+0.5a}$. Third, the seller has a binding outside option i.e. $w_s > 0.5y$. Then

$$\pi_s = (1-\lambda)\frac{y}{2} + \lambda w_s, \ \pi_b = (1-\lambda)\frac{y}{2} + \lambda(y - w_s) \tag{3.2}$$

To summarize this result, it says first that when neither outside option binds, the two parties split the gains from trade equally. But, when the outside option of one agent $i = b, s$ binds, (3.1) or (3.2) says that i gets a weighted combination of his outside option and half the surplus, and j gets a weighted combination of the surplus and the residual payoff $y - w_i$. So, the agent who is facing a binding outside option is in a sense, the residual claimant. Moreover, the weight λ on the residual payoff goes to 1 as the matching process becomes frictionless i.e. as $a \to \infty$.

Now recall that $v_b(\beta; \beta^*, \sigma^*), v_s(\sigma; \beta^*, \sigma^*)$ are the payoffs to search to a buyer or seller who has invested β or σ given that all other agents have invested β^*, σ^*, and π_b, π_s to be the payoffs to a buyer and seller who are matched together and who have invested β, σ. In the limit as $\Delta \to 0$, and dropping the arguments, v_b, v_s must satisfy the usual recursive formulae that state that the return to search rv_i, must be equal to the expected capital gain in moving from the unmatched to matched state which is $a(\pi_i - v_i)$ if both agents agree to the match i.e. $\pi_i \geq v_i, \quad i = b, s$, and zero otherwise. So, solving for v_i :

$$v_i = \begin{cases} \phi \pi_i & \pi_i \geq v_i, \quad i = b, s \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

where $\phi = \frac{a}{a+r}$. So, assuming the match is accepted, v_i is π_i times a discount factor ϕ capturing match frictions.

Using Lemma 1 and (3.3), we can now give a complete characterization of $v_b(\beta; \beta^*, \sigma^*)$. Define a critical value of buyer investment $\tilde{\beta}$ as

$$0.5y(\tilde{\beta}, \sigma^*) = 0.5y(\beta^*, \sigma^*) - \varepsilon \tag{3.4}$$

This is the level of investment that will leave the seller indifferent between staying with the deviant buyer, and switching to an equilibrium buyer i.e. a buyer investing β^*.. Note that if $\varepsilon > 0$, $\tilde{\beta} < \beta^*$, and if $\varepsilon = 0$, $\tilde{\beta} = \beta^*$. Next, define the critical values

$$y(\beta_N, \sigma^*) = \phi y(\beta^*, \sigma^*), \; y(\beta_B, \sigma^*) = \phi 0.5y(\beta^*, \sigma^*) + \frac{a\varepsilon}{r} \tag{3.5}$$

As shown in detail in the proof of Lemma 2, β_N is the level of investment that leaves the seller indifferent between initially accepting a match with a deviant buyer and continuing to search, given that the seller's outside option *does not* bind in the subsequent bargaining. Also, β_B is the level of investment that leaves the seller indifferent between initially accepting a match with a deviant buyer, given that the seller's outside option *does* bind in the subsequent bargaining. We now have:

Lemma 2. *Take equilibrium investments β^*, σ^* as given. Then if $\varepsilon < \hat{\varepsilon} = \frac{r}{a+r} 0.5y(\beta^*, \sigma^*)$:*

$$v_b(\beta; \beta^*, \sigma^*) = \begin{cases} \phi 0.5y(\beta, \sigma^*) & \beta \geq \tilde{\beta} \\ \frac{\phi}{2r+a} \left[(r+a)y(\beta, \sigma^*) - a\left(0.5y(\beta^*, \sigma^*) - \varepsilon\right) \right], & \tilde{\beta} > \beta \geq \beta_B \\ 0, & \beta < \beta_B \end{cases} \tag{3.6}$$

and if $\varepsilon > \hat{\varepsilon}$:

$$v_b(\beta; \beta^*, \sigma^*) = \begin{cases} \phi 0.5y(\beta, \sigma^*) & \beta \geq \beta_N \\ 0, & \beta < \beta_N \end{cases} \tag{3.7}$$

Lemma 2 (and the analogous result for the deviant seller, Lemma 3 below) is crucial for our results on investment, and thus deserves some comment. If $\varepsilon < \hat{\varepsilon}$, then in a well-defined sense, the switching cost ε is small relative to the level of matching friction (inversely measured by a). In this case, a deviant buyer's choice of investments β can lead to three possible outcomes. First, if investment is high enough i.e. $\beta \geq \tilde{\beta}$, he will find a match with a seller that is acceptable to both, and in that match, neither partners' outside option will bind, so that they will share equally the surplus from the match, giving each a payoff of $0.5y(\beta, \sigma^*)$. Second, if investment is low enough i.e. $\beta < \beta_B$, no seller will accept a match with the deviant buyer, as the payoff to continuing to search for a match with an equilibrium buyer is greater. Finally, if β is intermediate, i.e. $\tilde{\beta} > \beta \geq \beta_B$, the match is acceptable to the seller, but both agents rationally anticipate that in the subsequent bargaining, the seller's outside option will bind, lowering the payoff of the buyer in the match.

If $\varepsilon > \hat{\varepsilon}$, then the switching cost ε is large relative to the level of matching friction (inversely measured by a). In this case, the intermediate outcome - where matching takes place, but where subsequently, the seller's outside option does not bind - cannot occur. The intuition is clear: a larger switching cost ε, other things equal, depresses the value of the outside option to the seller. So, for ε large enough, the seller's outside option will not bind.

Exactly the same characterization is possible of the seller's payoff to a unilateral deviation. Define the critical values

$$0.5y(\beta^*, \tilde{\sigma}) = 0.5y(\beta^*, \sigma^*) - \varepsilon, \;\; y(\beta^*, \sigma_N) = \phi y(\beta^*, \sigma^*), \;\; y(\beta^*, \sigma_B) = \phi 0.5y(\beta^*, \sigma^*) + \frac{a\varepsilon}{r} \tag{3.8}$$

These have analogous interpretations to the critical values in the buyer case. Then we have the analogous result:

Lemma 3. *Take equilibrium investments β^*, σ^* as given. Then if $\varepsilon < \hat{\varepsilon}$:*

$$v_s(\sigma; \beta^*, \sigma^*) = \begin{cases} \phi 0.5y(\beta^*, \sigma) & \sigma \geq \sigma \\ \frac{\phi}{2r+a}\left[(r+a)y(\beta^*, \sigma) - a\left(0.5y(\beta^*, \sigma^*) - \varepsilon\right)\right], & \tilde{\sigma} > \sigma \geq \sigma_B \\ 0, & \sigma < \sigma_B \end{cases} \tag{3.9}$$

and if $\varepsilon > \hat{\varepsilon}$:

$$v_s(\sigma; \beta^*, \sigma^*) = \begin{cases} \phi 0.5y(\beta^*, \sigma) & \sigma \geq \sigma_N \\ 0, & \sigma < \sigma_N \end{cases} \tag{3.10}$$

3.2. Equilibrium Investments

We have characterized payoffs to unilateral deviations from equilibrium investments for buyers and sellers. So, equilibrium investments must then satisfy the usual best-response conditions which say that unilateral deviations from equilibrium do not pay

$$v_b(\beta^*; \beta^*, \sigma^*) - c(\beta^*) \geq v_b(\beta'; \beta^*, \sigma^*) - c(\beta'), \text{ all } \beta' \in \Re_+, \tag{3.11}$$

$$v_s(\sigma^*; \beta^*, \sigma^*) - \psi(\sigma^*) \geq v_s(\sigma'; \beta^*, \sigma^*) - \psi(\sigma'), \text{ all } \sigma' \in \Re_+ \tag{3.12}$$

Next, define investment levels

$$\frac{a}{r+a} 0.5 y_\beta(\underline{\beta}, \underline{\sigma}) = c'(\underline{\beta}), \; \frac{a}{r+a} 0.5 y_\sigma(\underline{\beta}, \underline{\sigma}) = \psi'(\underline{\sigma}) \tag{3.13}$$

$$\frac{a}{2r+a} y_\beta(\overline{\beta}, \overline{\sigma}) = c'(\overline{\beta}), \; \frac{a}{2r+a} y_\sigma(\overline{\beta}, \overline{\sigma}) = \psi'(\overline{\sigma}) \tag{3.14}$$

Note that if we assume that investments are (weak) complements i.e. $y_{\beta\sigma} \geq 0$, that $\underline{\beta} < \overline{\beta}, \; \underline{\sigma} < \overline{\sigma}$. To interpret (3.13),(3.14), note first that $\underline{\beta}, \underline{\sigma}$ are the investment levels that maximize the net payoffs $v_b - c, \; v_s - \psi$ respectively, given that there are no binding options at the bargaining stage, and thus that the surplus from the match is equally divided. Also, note that $\overline{\beta}, \overline{\sigma}$ are the investment levels that maximize the net payoffs $v_b - c, \; v_s - \psi$ respectively, given that the investor is residual claimant at the bargaining stage, in the sense described above in Section 2.1. We then have:

Proposition 1. *If $\varepsilon > 0$, equilibrium investments are unique, and $\beta^* = \underline{\beta}, \; \sigma^* = \underline{\sigma}$. If $\varepsilon = 0$, assume that investments are (weak) complements i.e. $y_{\beta\sigma} \geq 0$. In equilibrium, there are a whole range of equilibrium investments $\underline{\beta} \leq \beta^* \leq \overline{\beta}$, and $\underline{\sigma} \leq \sigma^* \leq \overline{\sigma}$.*

This is our main result. The intuition is the following. Obviously, in equilibrium, as all agents on each side of the market are identical, outside options are not binding. If $\varepsilon > 0$, as can easily be checked from Lemmas 2 and 3, starting at equilibrium, a small enough change in (for example) a buyer's investment will not cause the seller's outside option to bind in any subsequent match for that buyer. Thus, any any change (upward or downward) in that buyer's investment will change the buyer's payoff by half of the marginal product of investment. Thus, each buyer is subject to hold-up on any change in investment and has a low incentive to invest.

Now consider the case $\varepsilon = 0$. Now, the outside option is simply the share of output in the alternative match, *not reduced by any switching cost.* This means that starting at the equilibrium investments, a small *cut* in investment by (say) the buyer will immediately cause the seller's outside option to bind in any future match that the buyer may have. In turn, this makes the buyer the "residual claimant" (in the sense defined in Section 2.1 above), so he bears almost 100% of the cost of a cut in investment when match frictions are small. This effect sustains an equilibrium with high investments $\overline{\beta}, \overline{\sigma}$. On the other hand, a small increase in investment is subject to hold-up exactly as in the case with strictly positive switching costs. Thus there is a kink (non-differentiability) in the buyer's payoff function at $\beta = \beta^*$, which generates multiple equilibria.

Now consider what happens as the matching process becomes frictionless, $(\alpha \to \infty)$. If $\varepsilon > 0$, from Proposition 1 and (3.13), equilibrium investments $\underline{\beta}, \underline{\sigma}$ tend to the "hold-up" levels[10], where half the marginal product from investment is equated to marginal cost. Hold-up investments have the interpretation of equilibrium investments made by a single buyer and seller when bargaining takes place without any outside options at the matching or bargaining stages - bilateral monopoly. What about the case with no switching costs? As $\alpha \to \infty$, from (3.14), $\overline{\beta}, \overline{\sigma}$ tend to the efficient investments that maximize the net surplus $S = y - c - \psi$. So, in the limit case, any investments between the holdup and efficient levels can be equilibrium investments.

3.3. Investments with Stable Matching

In CMP(2001), ex ante general investments are studied in a framework much as above, except where the matching process and bargaining is not modelled explicitly. Rather, conditional on investments, the assignment of buyers to sellers and the division of the surplus between them is simply assumed to be stable in the sense that "there are no pairs of agents who by matching and sharing the resulting surplus, can make themselves strictly better off". Their motivation for using the stability concept is "to capture the idea that the division of the surplus within any match should respect outside options".

Given that stable matching is one way of modelling the matching and bargaining

[10]Formally, the hold-up levels satisfy $0.5y_\beta(\beta_H, \sigma_H) = c'(\beta_H),\ 0.5y_\sigma(\beta_H, \sigma_H) = \psi'(\sigma_H)$.

outcome in the frictionless limit, it is natural to ask how the equilibrium investments with stable matching are related to the limiting investments with bargaining and outside options in our set-up. To answer this question, we note first that under our assumptions, already made on technology and costs, the *net surplus* $S = y - c - \psi$ from a match is strictly concave in investments β, σ.

Next, Proposition 5 of CMP says that equilibrium investments with stable matching are a local maximum of S.The intuition is that stability, plus a continuum of agents, guarantees that any agent who deviates from the equilibrium investment .is "residual claimant" in the sense that he bears the full value of any gain (or loss) in net surplus that results from his deviation. The conclusion[11] is that in our set-up, the unique equilibrium investments[12] with stable matching are the efficient investments $\hat{\beta}, \hat{\sigma}$ that maximize S.

This observation, combined with Proposition 1, indicates a discontinuity between one the one hand, the strategic approach in the limit as match frictions go to zero, and on the other hand, the axiomatic approach, *unless there are zero switching costs*. But if $\varepsilon = 0$, the limiting equilibrium investment in the model with frictions and an explicit matching and bargaining process converges - as frictions go to zero - to the equilibrium investment with frictionless (stable) matching.

4. Conclusions

Our conclusions are the following. First, there is consistency between the axiomatic and strategic approaches (i.e. they both predict the same equilibrium investments) only when in the second approach, switching is assumed costless. Second, there is continuity in the limit as switching costs go to zero: the equilibrium correspondence mapping the switching

[11] This reasoning does not constitute a formal proof, as the results of CMP(2001) are derived in a setting of strictly heterogenous agents i.e. where each agent has a strictly different cost of matching, and where each agent invests a different amount in equilibrium. However, the analog of their Proposition 5 for our model can easily be established, and is available on request.

[12] This result seems to be at variance with CMP(2001), who provide examples of overinvestment and underinvestment in equilibrium. But this is not really the case. Their examples rely on a non-concavity in the surplus function, which means that investments that generate a local (but not global) maximum of $S(\beta, \sigma)$ can be equilibrium investments, and this is also the case here, with homogenous agents. In particular, their examples are generated by assuming a revenue function of the form

$$y(\beta, \sigma) = \begin{cases} \beta\sigma, & \beta\sigma \leq 0.5 \\ 2(\beta\sigma)^2 & \beta\sigma < 0.5 \end{cases}$$

which is clearly not concave.

cost into equilibrium investments is upper hemi-continuous at this point.

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A. Proofs of Propositions

Proof of Lemma 1. If the buyer is proposer, he offers the seller

$$y - \pi_b = (1 - a\Delta)\delta\pi_s + a\Delta \max\{w_s, \pi_s\} \tag{A.1}$$

and if the seller is proposer, he offers

$$y - \pi_s = (1 - a\Delta)\delta\pi_s + a\Delta \max\{w_b, \pi_b\} \tag{A.2}$$

where π_b, π_b are the payoffs to buyer an seller respectively at the point where each is proposer. Note that (A.1) and (A.2) embody the restrictions that the responder will take the outside option if it is available and it is ex post rational to do so.

Now assume Case 1: i.e. $w_s, w_b \leq 0.5y$. Then standard arguments imply that (A.1) ,(A.2) have the unique solution $\pi_b = \pi_s = 0.5y$. Now assume Case 2: $w_b > 0.5y \geq w_s$. Assume provisionally that $w_s \leq \pi_s,\ w_b > \pi_b$. Then π_s, π_b solve

$$y - \pi_b = \delta\pi_s,\ y - \pi_s = (1 - a\Delta)\delta\pi_b + a\Delta w_b \tag{A.3}$$

Solving (A.3) for π_b, we get

$$\pi_b = \frac{(1-\delta)y + \delta\Delta a w_b}{1 - (1 - a\Delta)\delta^2} \tag{A.4}$$

Taking the limit in (A.4) as $\Delta \to 0$, using $\delta = e^{-r\Delta}$ and L'Hopital's rule gives

$$\pi_b = \frac{ry + aw_b}{2r + a}, \tag{A.5}$$

Finally, using (A.3),(A.5)

$$\pi_s = y - \pi_b = \frac{(r+a)y - aw_b}{2r + a} \tag{A.6}$$

Finally, from (A.5), we see that as $w_b - \varepsilon > 0.5y$, then $w_b - \varepsilon > \pi_b$, also. A similar argument shows that $w_s - \varepsilon < \pi_s$, confirming these assumptions. So, we have shown that in the limit, (A.1),(A.2) have a solution (A.5),(A.6). It is tedious but straightforward to show that this is the unique solution. Rearrangement of (A.5),(A.6) gives ((3.1).

Finally, Now assume Case 3: $w_s - \varepsilon > 0.5y \geq w_b - \varepsilon$. Then, an argument exactly like the one given implies that the limit solutions to (A.1),(A.2) are

$$\pi_s = \frac{ry + a(w_s - \varepsilon)}{2r + a},\ \pi_b = \frac{(r+a)y - a(w_s - \varepsilon)}{2r + a} \tag{A.7}$$

Rearranging (A.7) gives (3.2). This completes the proof. □

Proof of Lemma 2. Consider a match between a buyer with investment β - a $\beta-$buyer - and a seller with investment σ^* - an equilibrium seller. In this match, say that the buyer's outside option is binding if $w_b > 0.5y(\beta,\sigma^*)$, and that the seller's outside option is binding if $w_s > 0.5y(\beta,\sigma^*)$; the outside options are non-binding otherwise. Note that because all sellers are alike, in this match, the $\beta-$buyer's outside option is never binding. Also, the outside option of the equilibrium seller is to switch to a match with an equilibrium buyer.

Now, in matches between equilibrium buyers and sellers, outside options cannot be binding, so after switching, the seller can expect $0.5y(\beta^*,\sigma^*)$, which implies and outside option for the seller of $0.5y(\beta^*,\sigma^*)-\varepsilon$, net of the switching cost. So, by Lemma 1, the equilibrium seller's outside option is non-binding iff $0.5y(\beta,\sigma^*) \geq 0.5y(\beta^*,\sigma^*)-\varepsilon$. Thus, the seller's outside option is non-binding iff $\beta \geq \tilde{\beta}$, where $\tilde{\beta}$ is defined in (3.4). So, assuming that the match is initially accepted, from Lemma 1, if $\beta \geq \tilde{\beta}$, $\pi_b = \pi_s = 0.5y(\beta,\sigma^*)$, and if $\beta < \tilde{\beta}$,

$$\pi_s = \frac{ry(\beta,\sigma^*) + a\left(0.5y(\beta^*,\sigma^*)-\varepsilon\right)}{2r+a},\ \pi_b = \frac{(r+a)y(\beta,\sigma^*) - a\left(0.5y(\beta^*,\sigma^*)-\varepsilon\right)}{2r+a}$$

Now, consider the conditions under which the match is accepted by both partners. First, as all sellers are alike, the match will always be accepted by the buyer. So, the question is when the seller will accept. By rejecting the match, the seller can get $0.5y(\beta^*,\sigma^*)$, discounted by search frictions i.e. $\phi 0.5y(\beta^*,\sigma^*)$. Consider first the case $\beta \geq \tilde{\beta}$. In this case, the seller will accept the match if $\pi_s = 0.5y(\beta,\sigma^*) \geq \phi 0.5y(\beta^*,\sigma^*)$ from above, i.e. when $\beta \geq \beta_N$ where β_N is defined in (3.5). On the other hand, if $\beta < \tilde{\beta}$, then in this case, the seller will accept the match if

$$\begin{aligned}\pi_s &= \frac{ry(\beta,\sigma^*) + a\left(0.5y(\beta^*,\sigma^*)-\varepsilon\right)}{2r+a} \geq \phi 0.5y(\beta^*,\sigma^*) \\ &\iff y(\beta,\sigma^*) \geq \phi 0.5y(\beta^*,\sigma^*) + \frac{a\varepsilon}{r}\end{aligned}$$

So, in this case, the match will be accepted if $\beta \geq \beta_B$, where β_B is defined in (3.5) Next, note that

$$\begin{aligned}\beta_B &< \beta_N \Leftrightarrow \phi 0.5y(\beta^*,\sigma^*) + \frac{a\varepsilon}{r} < \phi y(\beta^*,\sigma^*) \Leftrightarrow \varepsilon < (1-\phi)0.5y(\beta^*,\sigma^*) = \hat{\varepsilon}. \\ \beta_N &< \tilde{\beta} \Leftrightarrow \phi y(\beta^*,\sigma^*) < y(\beta^*,\sigma^*) - 2\varepsilon \Leftrightarrow \varepsilon < (1-\phi)0.5y(\beta^*,\sigma^*) = \hat{\varepsilon}\end{aligned}$$

So, there are two cases.

(a) $\varepsilon < \hat{\varepsilon}$. Then $\beta_B < \beta_N < \tilde{\beta}$, and so either (i) $\beta \geq \tilde{\beta}$, in which case the match is accepted and the seller's outside option does not bind or (ii) $\beta_B \leq \beta < \tilde{\beta}$, in which case, the match is accepted and the seller's outside option binds,; (iii) $\beta_B > \beta$, in which case the match is rejected. So, formula (3.6) can now be derived as follows. In case (i), by Lemma 1, the deviant buyer gets

$0.5y(\beta, \sigma^*)$ if matched, and so the payoff to search is $\phi 0.5y(\beta, \sigma^*)$ from (3.3). In case (ii), by Lemma 1, the deviant buyer gets $\frac{1}{2r+a}\left[(r+a)y(\beta, \sigma^*) - a0.5y(\beta^*, \sigma^*)\right]$ if matched, and so the payoff to search is this term multiplied by ϕ. In case (iii), the deviant buyer is never matched and thus the payoff to search is zero.

(b) If $\hat{\varepsilon} < \varepsilon$. Then $\beta_B > \beta_N > \tilde{\beta}$. So, (i) if $\beta \geq \beta_N$, the match is accepted and the seller's outside option does not bind, and (ii) if $\beta < \beta_N$, the match is rejected. So, formula (3.7) can now be derived as follows. In case (i), by Lemma 1, the deviant buyer gets $0.5y(\beta, \sigma^*)$ if matched, and so the payoff to search is $\phi 0.5y(\beta, \sigma^*)$ from ( ). In case (ii), the deviant buyer is never matched and thus the payoff to search is zero. □

Proof. of Proposition 1. (i) Assume $\varepsilon > \hat{\varepsilon}$. A necessary condition for equilibrium is that a small deviation from equilibrium cannot make any agent better off. From (3.5), $\underline{\beta} < \beta^*$, so from Lemma 2, $v_b(\beta; \beta^*, \sigma^*)$ is differentiable in β a small enough neighborhood of β^*, with derivative $0.5\phi y_\beta(\beta^*, \sigma^*)$. In the same way, from Lemma 3, $v_s(\sigma; \beta^*, \sigma^*)$ is differentiable in the neighborhood of σ^*, with derivative $0.5\phi y_\sigma(\beta^*, \sigma^*)$. At equilibrium, these must be equal to $c'(\beta^*)$, $\psi'(\sigma^*)$ respectively, which given the strict concavity of y and convexity of c, ψ is only possible if $\beta^* = \underline{\beta}$, $\sigma^* = \underline{\sigma}$. To show that this is indeed an equilibrium, it suffices now to show that (i) no global deviation β' or σ' from equilibrium can make buyer or seller better off. But this follows immediately, as $v_b(\beta; \beta^*, \sigma^*)$ is bounded above by $\phi 0.5(\beta, \sigma^*)$ and $v_s(\sigma; \beta^*, \sigma^*)$ is bounded above by $\phi 0.5(\beta^*, \sigma)$.

(ii) Assume $0 < \varepsilon < \hat{\varepsilon}$. Then, $\tilde{\beta} < \beta^*$, so again from Lemma 2, Lemma 3, $v_b(\beta; \beta^*, \sigma^*)$ is differentiable in β a small enough neighborhood of β^*, with derivative $0.5\phi y_\beta(\beta^*, \sigma^*)$. In the same way $v_s(\sigma; \beta^*, \sigma^*)$ is differentiable in the neighborhood of σ^*, with derivative $0.5\phi y_\sigma(\beta^*, \sigma^*)\sigma^*)$. So, the argument of part (i) applies.

(iii) Assume $\varepsilon = 0$. So, equilibrium buyer investment β^* must maximize $v_b(\beta; \beta^*, \sigma^*) - c(\beta^*)$. Now, from Lemma 2, as $\tilde{\beta} = \beta^*$, $v_b(\beta; \beta^*, \sigma^*)$ is non-differentiable at $\beta = \beta^*$. But, it has left-hand and right-hand derivatives

$$\frac{a}{(2r+a)} y_\beta(\beta^*, \sigma^*),\ 0.5\phi y_\beta(\beta^*, \sigma^*)$$

respectively. So, in equilibrium, marginal cost for the buyer must be between these values i.e.

$$0.5\phi y_\beta(\beta^*, \sigma^*) \leq c'(\beta^*) \leq \frac{a}{(2r+a)} y_\beta(\beta^*, \sigma^*) \tag{A.8}$$

In the same way, in equilibrium

$$0.5\phi y_\sigma(\beta^*, \sigma^*) \leq \psi'(\sigma^*) \leq \frac{a}{(2r+a)} y_\sigma(\beta^*, \sigma^*) \tag{A.9}$$

Because investments are (weak) complements i.e. $y_{\beta\sigma} \geq 0$, any $(\underline{\beta}, \underline{\sigma}) \leq (\beta^*, \sigma^*) \leq (\overline{\beta}, \overline{\sigma})$ solves (A.8),(A.9). □