Too Much Investment? A Problem of Endogenous Outside Options*

David de Meza
LSE

Ben Lockwood
University of Warwick

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Abstract

This paper shows that when agents on both sides of the market are heterogenous, varying in their costs of investment, ex ante investments by firms and workers (or buyers and sellers more generally) may be too high when followed by stochastic matching and bargaining over quasi-rents. The overinvestment is caused by the fact that low-cost agents, by investing more, can increase the value of their outside option and thus shift rent away from high-cost investors. Overinvestment occurs, however, only when the outside option principle applies. When match break-up is exogenous and random, rent-shifting incentives can be present, but are always dominated by the hold-up effect. Numerical simulations show that overinvestment can occur given parameter values calibrated to OECD labour markets.

Keywords: hold-up, coordination failure, matching, over-investment.
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1. Introduction

In many situations, economic agents have to make complementary investments before knowing the identity of the agent with whom they will be matched. The classic example is that of workers and firms: workers and firms have to make decisions about human and physical capital investment, often before workers are matched with firms. Another example is the marriage market. A key question is: what are the incentives to invest in this setting? Should we expect underinvestment or overinvestment?

This problem has received some attention in recent years, with different contributions modelling the post-investment matching process in various ways. Our focus is on stochastic matching with the quasi-rent from the match divided by ex-post bargaining as analyzed by Acemoglu (1996), Acemoglu (1997), Acemoglu and Shimer (1999) and Masters (1998). The bargaining protocols studied in these papers allow for agents to have outside options, either in the form of a return to the search process or switching between partners. In all these papers, equilibrium investments are too low. The reason is essentially the hold-up problem. That is, outside options are ineffective in "protecting" investors from the hold-up problem, because they do not bind in equilibrium, or indeed for small deviations from equilibrium investments. So, any agent increasing their investment by a small enough increment only receives half the gross economic return, implying under-investment.

1 For an earlier literature (not involving matching) on strategic complementarity and market failure are Scitovsky (1954), Murphy, Shleifer and Vishny (1989), Redding (1996).
2 Cole, Malaith and Postlethwaite (2001a,b) - henceforth CMP - show that if the matching process is modeled axiomatically rather than explicitly, so that the outcome of the matching process is assumed to be stable, overinvestment equilibria are possible. A detailed discussion of the relationship of our findings to the CMP papers can be found in Section 7.
3 Masters (1998) claims that when the outside option takes the form of switching between partners, the outcome is efficient. De Meza and Lockwood (2006) show that there is an error in this argument: if switching is costless, as in Masters, there are a continuum of investment equilibria between the hold-up and efficient levels, and if there is any cost $\varepsilon > 0$ of switching, only the hold-up level of investment can be an equilibrium.
4 A related partial-equilibrium literature initiated by Grossman and Hart (1986) and Hart and Moore (1990) studies investments that are to some degree relationship specific and undertaken post matching. In Hart (1995, p37), the marginal return to investment is plausibly assumed higher in the relationship for which it is designed than outside it. The result is underinvestment for familiar hold up reasons. A further modification is to assume the nature of the investment is discretionary. For an agent with a binding outside option, partner-specific investment has no payoff whereas investment that enhances productivity with outsiders yields a private return but no social benefit. So if firms can choose the type of investment, as, for example, in Watson (2002, p178 Q2), they underinvest in partner-specific capital but overinvest in capital that will never be productively used. It seems reasonable to characterize the
In all these contributions, agents are assumed homogenous. In this paper, we show that homogeneity is important for the under-investment result. Specifically, if agents are heterogenous, differing in investment costs, outside options may become binding in equilibrium. Then, it turns out that outside options may do more than "protect" investors against hold-up: they may provide incentives for over-investment.

This works as follows. We assume that a fraction of firms and workers have a low cost of investment (low-cost agents) and the complementary fraction have a high cost of investment (high-cost agents). Suppose further for simplicity that high-cost agents do not invest in equilibrium, because the cost of investment is too high. With sufficient search frictions, a low-cost agent will match with a high-cost agent (non-assortative matching). Assume further, for simplicity, that investments are perfectly complementary: so, an investment by the low-cost agent in a non-assortative match is completely unproductive. Nevertheless, investment by the low-cost agent enhances his bargaining power in such a match by creating - or increasing the value of - a binding outside option, because the investment increases the value of a match with another low-cost agent who has also invested. This rent-transfer opportunity is privately profitable but not socially beneficial.

Note that overinvestment requires "intermediate" match frictions. For rent transfer to be relevant, match frictions must be high enough to ensure that non-assortative matches occur in equilibrium, but not so high that the outside option of a low-cost agent does not bind in these matches. For the case of a CES revenue function, we provide a complete characterization of the set of parameter values for which overinvestment can occur. Interestingly, overinvestment is possible for a range of match friction values consistent with average unemployment durations in OECD countries. Numerical simulations also indicate that this set tends to be larger, the bigger the proportion of low-cost agents, and the lower the returns to scale in the revenue function.

One question is whether overinvestment occurs in our model because of the precise way that the outside option appears in the bargaining solution i.e. the payoff is equal to the maximum of equal division of the revenue from the match and the return to search

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5 Friction must be intermediate only in the precise sense that for fixed values of the other parameters, and normalizing match frictions between zero and one, overinvestment occurs only for frictions in an open interval strictly in [0, 1]. Thus, overinvestment can occur when match friction is quite close to 1, as is the case for OECD labour markets.

6 Specifically, we calibrate the match arrival rate over a month as in the inverse of unemployment duration for a range of OECD countries. Combined with an annual discount rate of 5%, this gives a range of values of the match friction parameter between 0.9 and 1.
(the outside option). To check this, we consider an alternative bargaining protocol where the return to the unmatched state occurs randomly via stochastic match break-up. In this case, it is well-known (e.g. Muthoo(1999)) that the division of the revenue from the match follows Nash’s axiomatic formula, where the threat points from the match are proportional to the returns to search i.e. the returns to search are effectively "inside options". In this case, we will show that while there may be a rent-shifting motive for investment (to increase the value of the threat point), this is always dominated by the hold-up effect, so that in equilibrium, there is always underinvestment.

Last but not least, are there any observable implications of our theory? A well-known empirical puzzle is that higher education yields substantial private returns yet in cross-country studies, the effect of higher education investment on GNP growth is weak. Signalling theory provides the best known explanation of this. But, If we interpret the ex ante investment in our model as the acquisition of higher education, our rent-transfer effect provides an alternative explanation of the facts that does not rely on asymmetric information.

In our model, private return to education can exceed the social return. Moreover, an increase in the number of low-cost investors (those willing and able to invest in higher education) will increase the outside option of these investors and thus reduce the wages of non-investors (non-graduates). So the absolute wage of non graduates will be lower in an equilibrium with high numbers of graduates, as the evidence suggests.

How might we try to distinguish our explanation from signalling? If our explanation is important, there must be extensive “over-qualification”. That is, many people are employed in jobs for which their educational qualifications are unnecessary. Evidence that this is indeed the case is provided by Sicherman (1991) and Goos and Manning (2003).

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 derives the overinvestment results when the outside option principle applies. Section 4 changes the bargaining assumptions by introducing an exogenous probability of match breakup and shows this rules out overinvestment. Section 5 discusses related literature, and Section 6 concludes.

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7 For example, Ashenfelter and Rouse (1998) find the private return to higher education in the US is of the order of 10% whereas Barro and Lee (1994) find that, unlike secondary education, higher education is not significant in explaining the variation in international growth rates.

8 For example, Acemoglu (1999) documents that in the US and elsewhere that an upsurge in the number of graduates has been associated with falls in the absolute wage of non graduates.
2. The Model

2.1. Preliminaries

There are two types of agents: firms and workers. Both are infinitely lived. Time is discrete, with a period length of $\Delta$, and runs infinitely forward and back, and all agents have a discount factor $\delta = e^{-r\Delta}$. The following events occur in each period $t$. First, a measures $\pi_i$ of both workers and firms of type $i = h, l$ enter the pool of unmatched, with $\pi_h + \pi_l = 1$. A type $i$ agent has investment cost $c_i(e) = c_i e$, $i = h, l$ and $c_h > c_l$, all $e$. So, $h$-types have a higher cost of investment than $l$-types.

Then, a fraction $0 < \Delta a < 1$ of the measure of as yet unmatched firms and workers, are randomly matched with each other. That is, every worker is matched with a firm (and vice versa) with probability $\Delta a$. If both firm and worker are matched, they decide simultaneously and independently whether to accept or reject the match. Should one or both reject, then nothing further happens to these agents until the next period.

If they both accept, events are as follows. First, one of them is then randomly selected to be proposer in bargaining over revenue. A firm which has invested $e_w$ and a worker which has invested $e_f$ can produce present value of revenue of $y(e_w, e_f)$: the properties of this revenue function are discussed below. Then, all such proposers, plus all proposers in matches formed in previous periods that have not yet reached agreement over the division of revenue, can propose a division of revenue. The responder can accept or reject the division, or terminate the match. If the proposal is accepted, the matched firm and worker start producing in the following period. If the responder rejects, then he is proposer at $t + \Delta$. If the responder terminates the match, both parties return to the unmatched state at the beginning of the next period, $t + \Delta$.

Note that in contrast to the bilateral monopoly case (with just one firm and one worker), agents have two outside options in this model. First, an agent can reject a match, and continue searching. Second, a responder can exit back to the searching state. It is these outside options that drive our results.

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9 As any firm must exit matched with a worker, in any period, the measures of firms and workers in the unmatched state are the same.

10 For concreteness, think of a two-stage matching process where measure $\Delta a$ agents on either side of the market are randomly selected from the pool of the unmatched, and then these $\Delta a$ workers and firms are randomly matched with each other. The existence of such procedure even with a continuum on each side of the market is guaranteed by the arguments of Alos-Ferrer(2002).
2.2. The Revenue Function

We assume that $y(e_w, e_f)$ is non-negative, symmetric, and strictly increasing and strictly concave and twice differentiable for all $(e_w, e_f) \in \mathbb{R}_+^2$, with $y_i$ denoting the derivative with respect to the $i$th argument. Moreover, we will assume that $y$ is supermodular: given differentiability, this is just the condition $y_{12} > 0$. That is, the inputs are strict complements. One class of functions that satisfies all of these assumptions is the symmetric CES revenue function

$$y(e_w, e_f) = (0.5e_w^\rho + 0.5e_f^\rho)^{\alpha/\rho} + y_0, \quad \rho < 1, \quad 1 > \alpha > \rho$$

(2.1)

Note that if $\rho \leq 0$, inputs are essential i.e. $y(0,e) = y(e,0) = y_0$, all $e$. Specifically, $\rho = 0$ is the Cobb-Douglas case $y(e_w, e_f) = e_w^{\alpha/2}e_f^{\alpha/2} + y_0$.

2.3. Strategies and Equilibrium

Section 2.2 above describes a stochastic game played at periods $t = -2, 1, 0, 1, 2...$ by a continuum of players. We focus on symmetric steady-state Markov equilibrium. This imposes several restrictions.

First, symmetry and the steady-state requires that agents on both side of the market of a given cost type choose the same investments at all dates: that is, if a firm (or worker) of cost type $i = l, h$ enters the market at $t$, he invests $e_i^*$. Second, the steady-state requires that for each cost type, inflows to, and outflows from, the unmatched state are equal. Thus, we must distinguish the share of type $i = l, h$ agents in the stock of the unmatched $(\lambda_h, \lambda_l)$ from the shares of new agents of type $i = l, h$ who enter the unmatched state at any date $(\pi_h, \pi_l)$. It turns out that in equilibrium, there is always a simple relationship between $(\lambda_h, \lambda_l)$ and $(\pi_h, \pi_l)$, although this relationship depends on whether matching is assortative or not.

With non-assortative matching (i.e. all agents accept all matches) $\lambda_i$ is clearly equal to $\pi_i$, as both types are exiting at the same rate - every agent exists with probability $\Delta a$ over a time period. As explained in the Appendix, if matching is assortative, unless $\pi_h = \pi_l = 0.5$, different types exit at different rates. This implies that in the steady state, $\lambda_h, \lambda_l$ are non-linear functions of $\pi_h, \pi_l$. But it is shown in the Appendix that any pair $\lambda_h, \lambda_l$ can be generated by appropriate choice of $\pi_h, \pi_l$, so we can work directly with $\lambda_h, \lambda_l$ in what follows.

Given the previous discussion, if a firm $f$ and a worker $w$ are matched at the beginning of period $t$, the only payoff-relevant aspects of the history of play for this pair are (i) their
two investment levels $e_w, e_f$ : (ii) the distribution of equilibrium investments across all as yet unmatched agents, which is characterized by $(e^*_h, e^*_l)$.

We will say that the match acceptance and bargaining strategies of an agent are Markov if his decisions only depend on $e_w, e_f$, and $(e^*_h, e^*_l)$. In what follows, we assume Markov strategies. Within this class of strategies, we will focus on perfect match acceptance and bargaining strategies of the agents. Such a match acceptance strategy is one where an agent accepts a match at any date if doing so gives a higher payoff than continued search. Bargaining strategies are perfect if they are subgame-perfect in the alternating-offers bargaining game between the two partners $f$ and $w$.

3. Overinvestment Results

In any (symmetric steady state Markov) equilibrium, the outside option principle must apply: that is, the revenue from a match must be divided equally, unless one of the parties has an outside option worth more than half the revenue, in which case his outside option is said to bind and he receives the value of this outside option. To generate conditions under which overinvestment occurs, we construct a (symmetric steady state Markov) equilibrium where (i) matching is non-assortative (NAM) and (ii) where, when an $h$–type matches with an $l$–type, the outside option of the $l$–type binds. Call such an equilibrium a N-B equilibrium.

3.1. The N-B Equilibrium

In an N-B equilibrium, the present value expected payoffs to continued search for the two types, denoted $v_h, v_l$, satisfy the following dynamic programming equations in the limit as $\Delta \to 0$:

$$rv_l = a\lambda_h(v_l - v_l) + a\lambda_l(y(e_l, e_l) - v_l)$$

(3.1)

$$rv_h = a\lambda_h(y(e_h, e_h) - v_h) + a\lambda_l(y(e_h, e_l) - v_l - v_h)$$

(3.2)

The first equation follows because when matched with an $h$–type (which occurs with probability $\Delta a\lambda$) the $l$–type gets no surplus from the match. The second follows because when matched with an $l$–type (which occurs with probability $\Delta a\lambda_l$) the $h$–type is residual claimant. Solving (3.1), (3.2), we get

$$v_l = \phi_l \frac{y(e_l, e_l)}{2}$$

(3.3)

$$v_h = \phi [\lambda_h \frac{y(e_h, e_h)}{2} + \lambda_l(y(e_h, e_l) - \phi_l \frac{y(e_l, e_l)}{2})]$$

(3.4)
Finally, NAM and a binding outside option respectively require:

\[ v_l > \frac{y(e_h, e_l)}{2} \]  

\[ y(e_h, e_l) \geq v_h + v_l \] (3.6)

So, given investments, \( e_h, e_l \), (3.3)-(3.6) fully characterize the N-B equilibrium.

It remains to find the equilibrium investments. Suppose that an individual \( l \)-agent deviates by a small amount from equilibrium investment \( e_l^* \) to \( e' \). Then, as his outside option continues to bind in a match with an \( h \)-type (for a small enough deviation), his payoff net of investment costs is

\[ \phi_l \frac{y(e', e_l^*)}{2} - c_l e' \] (3.7)

So, the equilibrium investment must maximize this expression i.e.

\[ \frac{\phi_l}{2} y_1(e_l^*, e_l^*) = c_l \] (3.8)

where \( y_l \) denotes the first derivative of \( y \). By the same argument, if an individual \( h \)-agent deviates by a small amount from equilibrium investment \( e_h^* \) to \( e' \). Then, as he is still residual claimant in a match with an \( h \)-type (for a small enough deviation), his payoff net of investment costs is

\[ \phi_h \frac{y(e', e_h^*)}{2} + \phi_l (y(e', e_l^*) - \frac{y(e_l^*, e_l^*)}{2}) - c_h e' \] (3.9)

So, the equilibrium investment must maximize this expression i.e.

\[ \frac{\phi_h}{2} y_1(e_h^*, e_h^*) + \phi_l y_1(e_h^*, e_l^*) = c_h \] (3.10)

So, equations (3.8),(3.10) are the first-order necessary conditions for equilibrium investments.

However, some discussion of sufficient conditions is required. By assumption, \( y \) is strictly concave in investments, so this might appear to ensure that (3.8),(3.10) are also sufficient. But, there is the additional complication that large deviations in \( e \) away from the equilibrium level can cause the "regime" facing the deviant to change e.g. whether or not he faces a binding outside option in a given kind of match. For example, if the \( l \)-type chooses an \( e' \) sufficiently below \( e_l^* \), he will face first a binding outside option in a match with another \( l \), then as \( e' \) falls further, he will face a binding outside option in a match with an \( h \)-type, \( l \)-types will reject a match with the deviant, etc. But, as all these changes make the deviant worse off, his payoff to (downward) deviation must be
bounded above by (3.7). So, if \( e_t^* \) maximizes (3.7), it must certainly be a global maximum for the \( l \)-type. A similar argument implies that an \( h \)-type’s payoff to (upward) deviation must be bounded above by (3.9). So, if \( e_h^* \) maximizes (3.9), it must certainly be a global maximum for the \( h \)-type. So, to conclude, the N-B equilibrium is fully characterized by (3.3),(3.4),(3.5),(3.6), (3.8),(3.10).

3.2. Overinvestment

As payoffs are linear in consumption, the natural efficiency criterion is the sum of the payoffs to search net of investment costs at some levels of investment for each type \( e_l, e_h \) (aggregate surplus). In N-B equilibrium, aggregate surplus can be written as:

\[
W(e_h, e_l) = \lambda_h v_h + \lambda_l v_l - \lambda_l c_l e_t^* - \lambda_h c_h e_h^* \tag{3.11}
\]

where in the second line, we have used (3.3),(3.4). Differentiating (3.11), and collecting terms:

\[
\frac{1}{\lambda_l} \frac{\partial W(e_h^*, e_l^*)}{\partial e_l} = \lambda_h \phi_2(e_h^*, e_l^*) + (1 - \lambda_h \phi) y_1(e_l^*, e_l^*) - c_l \tag{3.12}
\]

where in the second line, we have used (3.8). So, investment of the \( l \)-types is locally too high if the term on the RHS of the second line of (3.12) is negative. Inspection of this term, using \( y_1(e_l^*, e_l^*) = y_2(e_h^*, e_l^*) \) gives:

**Proposition 1.** In N-B equilibrium, \( e_l^* \) is locally too high i.e. \( \frac{\partial W(e_l^*, e_l^*)}{\partial e_l} < 0 \), iff (i) \( \lambda_h \phi > 0.5 \); (ii) investments are sufficiently complementary i.e.

\[
\frac{y_1(e_l^*, e_l^*)}{y_1(e_l^*, e_h^*)} > \frac{\lambda_h \phi}{\phi_1[\lambda_h \phi - 0.5]} \tag{3.13}
\]

Condition \( \lambda_h \phi > 0.5 \) is intuitive. First, it is more likely to hold, the higher \( \lambda_h \), as the higher \( \lambda_h \), the higher the negative fiscal externality imposed on the \( h \)-types. Second, it is more likely to hold the higher \( \phi \), as the higher \( \phi \), the more likely are \( hl \) matches.

Condition (3.13) can be explained and interpreted as follows. First, the RHS of the inequality in (3.13) is by definition, strictly greater than 1 so (3.13) requires \( y_1(e_l^*, e_l^*) > y_1(e_l^*, e_h^*) \). Second, it is easy to show that \( e_l^* > e_h^* \) in equilibrium\(^{11}\). These two facts imply

\(^{11}\)For suppose not i.e. \( e_l^* \leq e_h^* \), but continue to assume that \( l \)-types have a binding outside option i.e. (3.5) holds. Then \( y(e_l^*, e_l^*) < y(e_h^*, e_l^*) \), and moreover, \( v_l \) must be less than (due to match frictions) a weighted average of \( 0.5y(e_l^*, e_l^*) \) and \( v_l \) itself. Thus, \( v_l < 0.5y(e_h^*, e_l^*) \), contradicting (3.5).
that some complementarity in investments is necessary for overinvestment. The reason
is that we know that for the standard holdup reason there is too little investment in an
ll match so overinvestment requires that the marginal product of $e_l$ is lower in an hl
match. As the only difference between the two types of match is that $e_h$ is lower, the
complementarity requirement follows.

So, generally, an overinvestment equilibrium will exist if (i) all the conditions for N-B
equilibrium exist, and (ii) the conditions in Lemma 2 hold. We now wish to obtain
conditions in terms of underlying model parameters for which these conditions hold si-
multaneously. To do this, some simplifying assumptions are required. We make two such
assumptions.

A1. First, we assume that $h$–types face a cost of investment that is prohibitively
costly i.e. $c_l = \infty$, so that they do not invest in equilibrium, or alternatively, they may
not be able to invest because e.g. they are credit-constrained.

A2. Second, we assume a CES production function with essential investments i.e.
(2.1) holds with $\rho \leq 0$.

The main simplification with A1, A2 is the following. First, A1 implies $e_h^* = 0$, and A2
then implies that $y_l(e_l^*, e_h^*) = y_l(e_l^*, 0) = 0$, as $y(e, 0) \equiv y_0$, all $e$. Thus, (3.13) in Lemma 2
automatically holds as long as $\lambda_h \phi > 0.5$. Moreover, conditions (3.5)-(3.6) reduce to the
condition that $y_0$ in (2.1) lie in a certain interval. We can thus prove (see Appendix):

**Proposition 2.** Assume A1, A2. Then, if $\lambda_h \phi > 0.5$, and

$$b^+ = \frac{\kappa \phi_l^{1/(1-\alpha)} y_l^{\alpha/(1-\alpha)}}{(1 - \phi_l)} > y_0 c_l^{\alpha/(1-\alpha)} > \frac{\kappa 0.5 \phi_l^{1/(1-\alpha)}}{1 - 0.5 \theta \phi_l} = b^-$$

where $\theta = \frac{1 - \phi + \phi \lambda_h}{1 - \phi + 0.5 \phi \lambda_h}$, $\kappa = \frac{\alpha}{2}^{\alpha/(1-\alpha)} > 0$, then there exists an overinvestment equilib-
rium. There is always a non-empty set of parameters for which there exists an overin-
vestment equilibrium. For any fixed values of the other parameters, an overinvestment
equilibrium exists if match frictions are "intermediate" i.e. $\phi$ lies in an interval strictly in
[0, 1].

Note that the conditions for existence of an overinvestment equilibrium are given
entirely in terms of parameters $\phi$, $\lambda_h$ (or $\lambda_l$), $\alpha$, $c_l$; in particular, the precise elasticity
of substitution between inputs, $\rho$, does not matter, as long as it is non-positive. The
concavity of the production function, as measured by $\alpha$, does matter, however.

To prove that there are always parameters for which an overinvestment equilibrium
exists, we turn to numerical simulations, which also give us a feel for how big this set of
parameters is. Our numerical simulations, reported in Figure 1 below, proceed as follows.
First, we wish to choose ranges of parameter values which are "realistic" for our main application, the labour market. We begin with the match friction parameter, \( \phi = \frac{a}{a+r} \). In the model, in steady-state equilibrium, \( a \) is by definition equal to the flow (over time period \( \Delta \)) of new entrants to the pool of unmatched, divided by the stock of unmatched. So, \( 1/a \) is equal to the average time to find a match. Empirically, this corresponds to average unemployment duration. Typical unemployment durations measured in months for OECD countries range between 14 months (France) and 3 months (the US), with an average for Europe of about 6 months (Pissarides(2007)). This suggests a range of values for \( a \) of \( 1/3 \) to \( 1/14 \).

Next, following Pissarides (2007), we choose an annual discount rate of 5%, implying a monthly discount rate of \( \ln(1.05)/12 = 0.0041 \). Overall, this gives a range of values of \( \phi \), 0.934-0.987, i.e. indicating that according to this measure, the labour market is close to frictionless. So, we shall let \( \phi \) range between 0.9 and 1. Nevertheless, as we shall see, for reasonable values of the other parameters, it is possible to find overinvestment.

The other parameters here are \( \alpha \), the returns to scale of the production functions, and \( \lambda_h \). These are much more difficult to calibrate from labour market data. First, on \( \alpha \), even if there are constant returns to all inputs, the investments considered here may be only a small subset of inputs (e.g. investment in IT training by workers, and capital investments complementary to IT training, for example, computer hardware, by firms) so \( \alpha \) could be quite small. We let \( \alpha \) take on the values 0.1, 0.5, 0.9. Finally, for \( \lambda_h \), we need \( \lambda_h > 0.5/\phi \) for \( \phi \geq 0.9 \), i.e. \( \lambda_h > 0.55 \). So, we let \( \lambda_l \) range between 0.04 and 0.44.

Figure 1 in here

These Figures graph \( b^+, b^- \) as functions of the match friction parameter \( \phi = \frac{a}{a+r} \), which must lie between 0.9 and 1. Specifically, Figures 1(a) to 1(f) show \( \phi \) along the horizontal axis, and \( b^+, b^- \) on the vertical axis. The set of parameter values satisfying (3.14) is shown by the shaded area in each case. So, from Proposition 2, the set of parameter values for which overinvestment equilibrium exists is just the shaded area. Generally, we see that for every configuration of parameter values illustrated, this set is non-empty. In several cases, this region is quite large, verifying our claim that overinvestment is a realistic possibility in the labour market.

Now consider the effects of parameter changes. First Figures (a)-(c) show what happens as the fraction of investors \( \lambda_l \) increases. This clearly increases the size of the shaded area. This is because the higher \( \lambda_l \), the higher the value of the outside option of an \( l-\)type when matched with an \( h-\)type, and so the more likely it is that the outside option will bind. Figures (d)-(f) show what happens as the returns to scale of the production function
changes. Increases in $\alpha$ tend to reduce the size of the interval between $b^+, b^-$ at any given value of $\phi$.

Finally, we turn to the important fact, stated in Proposition 2, that for any fixed values of the other parameters, an overinvestment equilibrium exists if match frictions are intermediate i.e. $\phi$ is in an interval strictly in $[0, 1]$. First, how can that be reconciled with Figure 1, where $\phi$ can take on very high values? Simply fix a point on the vertical axis and there is always an interval on the horizontal axis that generates coordinates in the shaded area. So, "intermediate" in Proposition 2 has a very precise meaning; it does not mean, for example, that values of $\phi$ in the middle of the feasible range $[0, 1]$ i.e. around 0.5 always generate overinvestment.

Second, why are "intermediate" frictions required for overinvestment? The reason is that an N-B equilibrium only occurs with intermediate match frictions. Match frictions must be low enough to ensure that the $l$-type’s outside option is binding, but must be high enough to ensure that matching is non-assortative. Moreover, the only way in which overinvestment can arise is in N-B equilibrium, and thus a necessary condition for overinvestment is that match frictions are intermediate.12

4. Underinvestment with Random Match Break-Up

4.1. The Bargaining Outcome

The model is as above, except that we now modify the bargaining in the following way. As before, bargaining over revenue $y$ is alternating offers, with random selection of proposer, but with the modification (following Muthoo(1999)) that if an agent rejects the offer, the match breaks up with probability $p = \Delta \beta$ before he can make a counter-offer. Moreover, the responder cannot actively opt for the outside option if the match does not break up. If the match breaks up, the worker and firm return to the unmatched state, where they get $v_w, v_f$ respectively. Assume $v_w + v_f < y$.

By Corollary 4.3 of Muthoo(1999), in the limit as $\Delta \rightarrow 0$, the shares of $y$ satisfy the

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12 A formal proof that underinvestment occurs with any other kind of equilibrium is provided in de Meza and Lockwood (2004). The idea is that there are only two other possibilities; (i) assortative matching, where by definition, the outside option cannot bind, and (ii) non-assortative matching, with non-binding outside options. In either case, there is no rent-shifting incentive for investment.
well-known "split the difference rules" i.e. payoffs are:

\[ u_w = \frac{\beta}{r+\beta}v_w + \frac{1}{2}(y - \frac{\beta}{r+\beta}v_w - \frac{\beta}{r+\beta}v_f) \]
\[ u_f = \frac{\beta}{r+\beta}v_f + \frac{1}{2}(y - \frac{\beta}{r+\beta}v_w - \frac{\beta}{r+\beta}v_f) \]

where \( \frac{\beta}{r+\beta}v_w, \frac{\beta}{r+\beta}v_f \) can be interpreted as "threat points". This is intuitive; the lower \( \beta \), the less the exogenous match break-up matters for the outcome via the threat point. It helpful for what follows to rewrite these formulae as

\[ u_w = \frac{1}{2}y + \frac{1}{2}\beta(v_w - v_f) \]  \hspace{1cm} (4.1)
\[ u_f = \frac{1}{2}y + \frac{1}{2}\beta(v_f - v_w) \]  \hspace{1cm} (4.2)

i.e. the possibility of match break-up only matters for the shares if the firm and worker in the bargain have different returns to search.

4.2. Equilibrium with Assortative Matching

Assume that a symmetric equilibrium has assortative matching. Then, as firms and workers behave symmetrically, let \( v_h, v_l \) denote the value of being in the search state for either a firm or worker of type \( h, l \), and \( u_h, u_l \) denote the expected payoff in the matched state (both conditional on equilibrium investments). Then, \( v_h, v_l \) satisfy the following dynamic programming equations in the limit as \( \Delta \to 0 \);

\[ rv_h = a\lambda_h(u_h - v_h), \quad rv_l = a\lambda_l(u_l - v_l) \]  \hspace{1cm} (4.3)

Moreover, from (4.1),(4.2), as both firm and worker have the same returns to search:

\[ u_h = 0.5y(e_h,e_h), \quad u_l = 0.5y(e_l,e_l) \]  \hspace{1cm} (4.4)

So, solving (4.3),(4.4) for \( v_h, v_l \):

\[ v_h = \phi_h0.5y(e_h,e_h), \quad v_l = \phi_l0.5y(e_l,e_l) \]  \hspace{1cm} (4.5)

Suppose that an individual \( k \)-type agent deviates by a small amount from equilibrium investment \( e_k^* \) to \( e' \). Then, from (4.5), his payoff net of investment costs will be \( \phi_k0.5y(e',e_k) - e' c_k, \quad k = h, l \). Thus, at the investment stage, investments by agents of type \( k = h, l \) maximise these expressions, and thus satisfy the first-order conditions:

\[ c_k = \phi_k0.5y(e_k^*,e_k^*), \quad k = h, l. \]  \hspace{1cm} (4.6)
4.3. Equilibrium with Non-Assortative Matching

Now assume that equilibrium has non-assortative matching. As before, \( v_h, v_l \), denote the value of being in the search state for either a firm or worker of type \( h, l \). But now let \( u_{hh}, u_{hl} \) denote the expected payoff in the matched state for an \( h \)–type matched with an \( h \) or \( l \)–type, and \( u_{lh}, u_{ll} \) denote the expected payoff matched state for an \( l \)–type matched with an \( h \) or \( l \)–type. From (4.1),(4.2) above, these satisfy

\[
\begin{align*}
  u_{hh} &= 0.5y(e_h, e_h), \quad u_{hl} = 0.5y(e_l, e_l), \\
  u_{lh} &= 0.5y(e_h, e_l) + 0.5\frac{\beta}{r + \beta}(v_h - v_l) \\
  u_{ll} &= 0.5y(e_l, e_l) + 0.5\frac{\beta}{r + \beta}(v_l - v_h)
\end{align*}
\]

Moreover, \( v_h, v_l \) satisfy the following dynamic programming equations in the limit as \( \Delta \to 0; \)

\[
\begin{align*}
  rv_h &= a\lambda_h(u_{hh} - v_h) + a\lambda_l(u_{hl} - v_h) \\
  rv_l &= a\lambda_l(u_{ll} - v_l) + a\lambda_h(u_{lh} - v_l)
\end{align*}
\]

Equations (4.7)-(4.11) reduce to two simultaneous equations in \( v_h, v_l \) which can be written conveniently as :

\[
\begin{align*}
  v_h &= \frac{a\lambda_h 0.5y(e_h, e_h) + a\lambda_l 0.5y(e_l, e_l) - \frac{a\lambda_l}{2} \frac{\beta}{r + \beta} v_l}{r + a - \frac{a\lambda_l}{2} \frac{\beta}{r + \beta}} \\
  v_l &= \frac{a\lambda_l 0.5y(e_l, e_l) + a\lambda_h 0.5y(e_h, e_h) - \frac{a\lambda_h}{2} \frac{\beta}{r + \beta} v_h}{r + a - \frac{a\lambda_h}{2} \frac{\beta}{r + \beta}}
\end{align*}
\]

Suppose that an individual \( h \) agent deviates by a small amount from equilibrium investment \( e_h^* \) to \( e' \). Then, from (4.12), his payoff net of investment costs will be

\[
\frac{a\lambda_h 0.5y(e', e_h^*) + a\lambda_l 0.5y(e', e_l^*) - \frac{a\lambda_l}{2} \frac{\beta}{r + \beta} v_l}{r + a - \frac{a\lambda_l}{2} \frac{\beta}{r + \beta}} - c_h
\]

noting that a type \( h \) agent perceives \( v_l \) as exogenous when choosing \( e' \). So, the equilibrium \( e_h \) maximises (4.14) and thus solves the first-order condition:

\[
\frac{a\lambda_h 0.5y_1(e_h^*, e_h^*) + a\lambda_l 0.5y_1(e_l^*, e_l^*)}{r + a - \frac{a\lambda_l}{2} \frac{\beta}{r + \beta}} = c_h
\]

Note that given \( e_h^* \), and using \( y_{11} < 0 \) from the strict concavity of the production function, \( e_h^* \) is increasing in the match break-up parameter \( \beta \), due to a rent-shifting effect: a higher
increases the payoff to match break-up for the \( h \)-type and thus his share of the cake. By the same argument, the equilibrium \( e_l \) solves:

\[
\frac{a\lambda_t 0.5y_t(e^*_l, e^*_h) + a\lambda_h 0.5y_t(e^*_l, e^*_h)}{r + a - \frac{a\lambda_h \beta}{2}} = c_l
\] (4.16)

4.4. The Underinvestment Result

The key question is of course, whether there is over- or under-investment in equilibrium. It turns out that whether matching is assortative or not, there is always underinvestment. In the assortative case, this is straightforward; because "mixed" matches never occur, there is no rent-shifting incentive to counteract the under-incentive to invest because of hold-up. With non-assortative matching, there is, as established above, a rent-shifting incentive. Less obviously, it turns out that the rent-shifting effect, although present, is never enough to fully offset the hold-up effect.

To see this formally, define welfare as

\[
W(e_h, e_l) = \lambda_h v_h + \lambda_l v_l - \lambda_l c_l e^*_l - \lambda_h c_h e^*_h
\]

where \( e^*_h, e^*_l \) solve (4.6) in the case of assortative matching, and (4.12),(4.13) in the case of non-assortative matching. Then we can prove (see Appendix):

**Proposition 3.** In the model with random match break-up, equilibrium investments of both types are always locally too low, whether matching is assortative or not i.e. \( \frac{\partial W(e^*_h, e^*_l)}{\partial e_h} > 0, \frac{\partial W(e^*_h, e^*_l)}{\partial e_l} > 0. \)

Finally, one intuition for why the rent-shifting incentive is always dominated by the hold-up effect is the following. With assortative matching the private benefit to an \( l \)-type’s investment includes the rent-shifting effect in \( hl \) matches. However, with random match break up the extra payoff is half the discounted rise in the payoff in an \( ll \) match whereas in the outside option case it is the full discounted rise. It is this difference that allows overinvestment in the latter but not in the former case.

5. Non-Markov Equilibrium

Our equilibrium concept is "supported" by the assumption of a particular continuation equilibrium in the sub-game once investments have been chosen i.e. a symmetric, steady-state, Markov equilibrium. We have five defenses. First, our main objective in this paper is to demonstrate the possibility of overinvestment equilibria, not to characterize
all the equilibria in the model. Second, the restrictions on strategies seem reasonable, and are widely used in the matching and bargaining literature (e.g. Osborne and Rubinstein (1990), Coles and Muthoo (1998)). Thirdly, this particular equilibrium is "focal": it is stationary, and the relationship between the outside options (at the match and bargaining stages) and the incentive to invest is particularly clear. Fourthly, in a large market, an agent’s bargaining history in other matches may not be known.

Finally, it is not clear that there are any non-Markov perfect equilibria. In a related model of a frictionless exchange economy Rubinstein and Wolinsky (1990) do construct non-Markov equilibria. A finite number of homogeneous buyers and sellers are randomly matched in pairs and bargain over the price of an indivisible good. All perfect equilibria in Markovian strategies yield the competitive price but in the absence of discounting there is a continuum of equilibrium prices once non-Markov strategies (actually strategies that depend on the identity of an agent even though this is not payoff relevant) are allowed.

This result is not robust however, for as soon as positive discounting is introduced there is a unique perfect equilibrium. Gale and Sabourian (2006) extend the model to allow heterogeneous types and show there is a unique Markov-perfect equilibria but do not have results for other equilibria. We conjecture that given investments as in Rubinstein and Wolinsky (1990) our model has no non-Markov equilibria. If then there is to be a non-Markov equilibrium it must be that, relative to the Markov case, exercising an outside option is punished or rewarded in the subsequent matches. However, no partner has an incentive to reward or punish as they are then out of the game and so cannot themselves be touched.

6. Related Literature

Our work is related to a number of recent papers which consider the efficiency of ex ante investments. These can be divided into two, depending on how the post-investment process for allocating agents to one another is modelled. Acemoglu (1996), Acemoglu (1997), Acemoglu and Shimer (1999) Burdett and Coles (2001), and Masters (1998) all assume that post investment, agents are brought together via stochastic matching with frictions, much as in this paper. Of these papers, only Burdett and Coles (2001) find that overinvestment can occur in equilibrium. Acemoglu (1996) was the first paper to consider the implications of stochastic matching for ex ante investments. Moreover, in the Appendix of his paper, an explicit extensive-form bargaining game with outside options is considered. Unlike our paper, agents who are already matched can switch immediately at some cost \( \varepsilon > 0 \) to alternative partners. He finds there will be hold-up i.e. both firms
and workers will not capture the full returns on their investments, and thus both will underinvest.

Masters (1998) considers a framework very similar to our own in both the assumptions about the matching process and the bargaining rule. Specifically, matching is stochastic with potential match arrivals following a Poisson process, and outside options are exactly as in our model i.e. a return to the unmatched state.\textsuperscript{13} He also finds underinvestment in equilibrium. Acemoglu (1997) and Acemoglu and Shimer (1999) consider related models, and the findings are similar: when the surplus is split by bilateral bargaining between firm and worker, there is generally underinvestment.\textsuperscript{14} What is common to all these models is that agents on each side of the market are homogenous. So, our paper shows that with agent heterogeneity, overinvestment is possible.

Burdett and Coles (2001) have a matching model of the marriage market where both men and women can make investments in their “pzazz” (sex appeal) prior to matching. Unlike the other papers just discussed, utility is non-transferable, so relationship surplus is not bargained over. More pzazz means it is possible to match with a higher pzazz partner and an overinvestment equilibrium may emerge. It is fairly clear, however, that their overinvestment result does depend on the impossibility of bargaining over relationship surplus (non-transferable utility). In their paper, investments are additive, so our results (Proposition 1) show that were bargaining possible, so that (say) a low pzazz male could "buy" a high pzazz female by a cash transfer, overinvestment could not occur.

We are aware of two papers that consider ex ante investments followed by competitive, or frictionless, mechanisms for pairing or matching agents (e.g. firms and workers): Cole, Mailath and Postlewaite (2001a) and Felli and Roberts (2002).\textsuperscript{15} Cole, Mailath and Postlewaite (2001a), henceforth CMP, consider a matching model in which buyers and sellers make investment decisions noncooperatively prior to entering a frictionless matching and bargaining process that is modelled as a cooperative game. The outcome of this second stage is constrained to be "stable" i.e. there is no pair of agents that by rematching and appropriately sharing the resulting surplus can both be strictly better off than in

\textsuperscript{13}He also considers another bargaining rule, where agents can search while bargaining, and claims that in this case investments are efficient. However, there is a mistake in this argument, discussed in more detail by de Meza and Lockwood (2006): there is generally underinvestment, even with this bargaining protocol.

\textsuperscript{14}In these papers, alternative forms of wage determination, such as wage posting, are also discussed, and sometimes this restores efficiency.

\textsuperscript{15}Cole, Mailath and Postlewaite (2001b) is less closely related to our work, as it assumes a finite number of agents. In this case, the decision of any two agents to match has an external effect on the opportunities available to other agents, so the set-up is a bit different.
the equilibrium. Their paper is particularly relevant to ours for two reasons. First, their motivation for using the stability concept is "to capture the idea that the division of the surplus within any match should respect outside options". Second, they are able to find examples of equilibria with overinvestment.

However, there are several limitations to their overinvestment results. Most importantly, Proposition 5 of CMP says that equilibrium investments with stable matching are at a local maximum of net surplus (the revenue from a match minus investment costs, \( S = y(e_w, e_f) - c(e_w) - c(e_f) \) in our notation). The intuition is that stability, plus a continuum of agents, guarantees that any agent who deviates from the equilibrium investment is "residual claimant" in the sense that he bears the full value of any gain (or loss) in net surplus that results from his deviation. The implication is that if \( S \) is concave, the unique equilibrium investments with stable matching are the efficient investments that maximize \( S \). So, their example of overinvestment relies on \( S \) being non-concave.\(^{16}\) In particular, CMPs over-investment example has a non-concave revenue function \( y \).\(^{17}\) In contrast, our model has a well-behaved concave revenue function; overinvestment is due to a different mechanism.\(^{18}\)

Moreover, in their model, typically there are multiple stable equilibria to the post-investment game, and thus multiple equilibria overall, which may generally involve underinvestment, efficient investment, or overinvestment. In their cooperative framework, without an explicit bargaining model, they have no criterion for selecting among these equilibria. Our paper differs in that the standard dynamic matching and bargaining model utilized is fully noncooperative and so follows the Nash program.

Finally, Felli and Roberts (2002) analyze a model with a fixed number of heterogeneous buyers and sellers, and investment only by one side of the market. There are no search frictions. Following investment, a Bertrand-style game is assumed where firms bid for

\(^{16}\)CMP also have an example of overinvestment with a finite number of agents and discrete (binary) investment levels. However, they are at pains to explain that overinvestment in the example is due to the finiteness of the set of agents, and in particular, the implication of finiteness that the equilibrium payoff of an agent changes when his partner changes his investment. They regard this property of equilibrium as undesirable and in the main part of the paper, move to a model with a continuum of agents where this phenomenon is ruled out by assumption.

\(^{17}\)In our notation, their example is

\[
y(e_w, e_f) = \begin{cases} 
eq \frac{1}{2} & e_w e_f \leq 0.5 \\
2(e_w e_f)^2 & e_w e_f < 0.5 
\end{cases}
\]

which is clearly not concave.

\(^{18}\)As implied by the fact that our framework requires search frictions which are absent from CMP.
workers (or vice versa). Efficiency requires assortative matching with investment increasing in intrinsic quality. The Bertrand assumption implies agents capture their differential productivities, so an equilibrium with fully efficient matching and investment always exists. Other equilibria are possible though, where there is not an everywhere monotonic relationship between intrinsic quality and investment levels. In such equilibria, unlike in our model, the agents invest efficiently, conditional on the match that they anticipate. But, relative to the first-best, overinvestment by an agent by - for example - a relatively low quality worker is possible, because he anticipates being hired by a very high-quality firm. There is no general tendency to over- or under-investment.

7. Conclusions

A recent literature examines agents’ incentives to make investments prior to entering a stochastic matching process, followed by bargain over the surplus from the match. This literature argues that because outside options are not binding, investors are held up and will under-invest in equilibrium. We have shown that this finding is not robust. If agents are heterogenous, outside options will in general help to determine investment incentives, and overinvestment, rather than underinvestment, may result. But, if match break-up is random, rather than an option that can be excercised, the hold-up effect dominates any rent-shifting effects, and there is underinvestment.

8. References


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A. Proofs of Propositions

Proof of Proposition 2. Note that as \( y(e_h, e_l) = y(0, e_l) = y_0 \), conditions (3.5), (3.6) reduce to

\[
v_h + v_l \leq y_0 < 2v_l \tag{A.1}\]

Moreover, using (3.3), (3.4), (3.6) can be written

\[
y_0 \geq \left( \frac{1 - \phi + \phi \lambda_h}{1 - \phi + 0.5\phi \lambda_h} \right) v_l \tag{A.2}\]

So, from (A.1),(A.2), the condition for the binding outside option and non-assortative matching conditions to be satisfied together become

\[
2v_l > y_0 \geq \left( \frac{1 - \phi + \phi \lambda_h}{1 - \phi + 0.5\phi \lambda_h} \right) v_l \equiv \theta v_l \tag{A.3}\]

Now from the CES assumption (2.1), we have

\[
y(e_l, e_l) = e_l^\alpha + y_0 = \Delta + y_0 \tag{A.4}\]

Combining this with (3.3) gives \( v_l = 0.5\phi_l(y_0 + \Delta) \). But then, assuming \( y_0 > 0 \), (A.3) reduces to

\[
\frac{\phi_l \Delta}{1 - \phi_l} > y_0 > \frac{0.5\theta \phi \Delta}{1 - 0.5\theta \phi_l} \tag{A.5}\]

Finally, \( e_l \) satisfies the FOC (3.8):

\[
\frac{\alpha \phi_l}{2} (e_l^\rho)^{\rho-1} e_l^{\rho-1} = c_l \implies e_l^* = \left( \frac{\alpha \phi_l}{2 c_l} \right)^{1/(1-\alpha)}
\]

So,

\[
\Delta = (e_l^*)^\alpha = \kappa \left( \frac{\phi_l}{c_l} \right)^{\alpha/(1-\alpha)}, \quad \kappa = \left( \frac{\alpha}{2} \right)^{\alpha/(1-\alpha)} > 0 \tag{A.6}\]

and thus, combining (A.5), (A.6), we need

\[
\frac{\kappa \phi_l^{1/(1-\alpha)}}{(1 - \phi_l)c_l^{\alpha/(1-\alpha)}} > y_0 > \frac{\kappa 0.5\theta \phi \phi_l^{1/(1-\alpha)}}{(1 - 0.5\theta \phi_l)c_l^{\alpha/(1-\alpha)}}
\]

which gives (3.14). Finally, we need to prove that for any fixed values of the other parameters, an overinvestment equilibrium exists if match frictions are intermediate i.e. \( \phi \) is in an interval strictly in \([0, 1]\). First, as \( \phi \to 1, 0.5\theta, \phi_l \to 1 \) also, so that \( b^+, b^- \to \infty \), so that for \( \phi \) close enough to 1, any fixed \( y_0 c_l^{\alpha/(1-\alpha)} < b^- \), so no overinvestment equilibrium can exist for \( \phi \) close enough to 1. Second, as \( \phi \to 0, \phi_l \to 0 \), so \( b^+, b^- \to 0 \), so that for \( \phi \)
close enough to 1, any fixed \( y_0 c_i^{\alpha/(1-\alpha)} > b^+ \), so no over-investment equilibrium can exist for \( \phi \) close enough to 0. \( \square \)

**Proof of Proposition 3.** (i) **Assortative Matching.** In this case,

\[
W(e_h, e_l) = \lambda_h \phi_h 0.5y(e_h, e_h) + \lambda_l \phi_l 0.5y(e_l, e_l) - \lambda_l c_l e_l - \lambda_h c_h e_h
\]

So,

\[
\frac{1}{\lambda_h} \frac{\partial W}{\partial e_h} = \phi_h y_1(e_h, e_h) - c_h
\]

\[
= 0.5\phi_h y_1(e_h, e_h) > 0
\]

where in the second line we have used the equilibrium condition for \( e_h \) (4.6) to substitute out \( c_h \). A similar argument applies in the case of \( e_l \).

(ii) **Non-Assortative Matching.** In this case, to write down the welfare function, we have to first solve explicitly for \( v_h, v_l \). First, from (4.12),(4.13), we have;

\[
(v_h - v_l) = \frac{C}{r + a - \frac{a \beta}{2 r + \beta}}, \quad C = a\lambda_h 0.5(y_h - y_m) + a\lambda_l 0.5(y_m - y_l)
\]

So,

\[
v_h = \frac{a\lambda_h 0.5y_h + a\lambda_l 0.5y_m}{r + a} + \frac{a\lambda_l \beta}{2} \frac{C}{r + \beta} \frac{1}{r + a - \frac{a \beta}{2 r+\beta}}
\]

\[
v_l = \frac{a\lambda_l 0.5y_l + a\lambda_h 0.5y_m}{r + a} - \frac{a\lambda_h \beta}{2} \frac{C}{r + \beta} \frac{1}{r + a - \frac{a \beta}{2 r + \beta}}
\]

But from (A.7),(A.8):

\[
\lambda_h v_h + \lambda_l v_l = \frac{a\lambda_h^2 0.5y_h + a\lambda_h \lambda_l y_m + a\lambda_l^2 0.5y_l}{r + a}
\]

i.e. as expected, the rent-shifting effects (the terms in \( C \)) net out. So, social welfare is

\[
W(e_h, e_l) = \frac{a\lambda_h^2 0.5y(e_h, e_h) + a\lambda_h \lambda_l (e_h, e_l) + a\lambda_l^2 0.5y(e_l, e_l)}{r + a} - \lambda_h c_h e_h - \lambda_l c_l e_l
\]

(A.9)

So, differentiating (A.9) and using this with the equilibrium condition for \( e_h \), (4.15), to
substitute out \( c_h \), we get:

\[
\frac{1}{\lambda_h} \frac{\partial W}{\partial e_h} = \frac{a\lambda_h y_1(e_h, e_h) + a\lambda y_1(e_h, e_l)}{r + a} - c_h
\]

\[
= \frac{a\lambda_h y_1(e_h, e_h) + a\lambda y_1(e_h, e_l) - a\lambda h 0.5y_1(e, e_h) - a\lambda 0.5y_2(e, e_l)}{r + a - \frac{a\lambda l}{2} \beta}{(r + a)}
\]

\[
> 0 \text{ as } 1 > \lambda_l \frac{\beta}{r + \beta}
\]

An exactly similar argument applies to the \( l \)-type. \( \square \)

**Formulae for \( \lambda_l, \lambda_h \) with Assortative Matching.** Let \( \nu_i \) be the measure of those unmatched at time \( t \) of type \( i \): by the assumption of the steady state, this is independent of \( t \). Then, \( \nu_h, \nu_l \) solve the two non-linear equations

\[
\nu_h = \pi_h + \nu_h (1 - a \frac{\nu_h}{\nu_h + \nu_l}), \quad \nu_l = \pi_l + \nu_l (1 - a \frac{\nu_l}{\nu_h + \nu_l}) \quad (A.10)
\]

The explanation is the following. In either case, a measure \( \pi_i \) of new agents enters the unmatched state in any period. For an existing agent of type \( i \), the probability of exiting the process is \( a \) times the probability of finding a type \( i \) to match with, which is \( \frac{\nu_i}{\nu_h + \nu_l} \).

Equations (A.10) solve to give

\[
\lambda_i = \frac{\nu_i}{\nu_h + \nu_l} = \frac{\pi_i + \sqrt{\pi_h \pi_l}}{1 + 2 \sqrt{\pi_h \pi_l}} \quad (A.11)
\]

It is clear from (A.11) that any \( \lambda_i \in [0, 1] \) can be generated by appropriate choice of \( \pi_h, \pi_l \).
Figure 1: Parameter Values for which Overinvestment Equilibrium Exists

(a): $\lambda_l=0.04, \alpha=0.5$

(b): $\lambda_l=0.24, \alpha=0.5$

(c): $\lambda_l=0.44, \alpha=0.5$
(d): $\lambda_{ij}=0.24$, $\alpha=0.1$

(e): $\lambda_{ij}=0.24$, $\alpha=0.5$

(f): $\lambda_{ij}=0.24$, $\alpha=0.9$