1. Proof of Sufficiency of the Equilibrium Conditions

(i) In this not-for-publication Appendix, we prove the sufficiency of the first-order conditions (3.8), (3.10) in the paper for equilibrium investment. To do this, we introduce the following notation. Let \( e \) be an arbitrary investment level, and let \( V_k(e) \) be the payoff in the unmatched state to a cost type \( k = h, l \) who has invested \( e \), net of the cost of investment. So, \( V_k(e) = v(e) - c_k e \), where \( v(e) \) denotes the payoff in the unmatched state (search) to an agent who has invested some arbitrary amount \( e \). Note therefore that \( v(e_h) = v_h, v(e_l) = v_l \), where \( v_h, v_l \) are the equilibrium payoffs to search defined in the paper. Of course, \( V(.) , v(.) \) also depend on equilibrium investments \( e_h, e_l \), but we suppress this dependence for clarity in what follows. So, it suffices to show that \( V_k(e) \) has a global maximum at \( e = e_k, k = h, l \). [Note that by the arguments in the paper, \( V_k(e) \) has a local maximum at \( e_k \).] We look at the cases of \( k = h, l \) separately, beginning with \( k = h \).

(ii) Consider first a deviation by an \( h \)-type to some \( e < e_h \). Depending on how low \( e \) is, there are a number of possibilities. If \( e \) is sufficiently close to \( e_h \), i.e. above some critical value \( e^* \), then (a) \( y(e_l, e) \geq v_l \), so \( l \)-types will accept a match with the deviant, and (ii) \( y(e_h, e)/2 \geq v_h \), so \( h \)-types will accept a match with the deviant, and divide the surplus equally. In this event, from (3.4) in the paper, the deviant’s payoff will be

\[
V_h(e) = \overline{V}_h(e) = \phi[\pi_h \left( \frac{y(e_h)}{2} \right) + \pi_l (y(e, e_l) - \phi_l \left( \frac{y(e_l, e_l)}{2} \right))] - c_h e
\]  

Note from (1.1) and the strict concavity of \( y \) in \( e \) that \( \overline{V}_h(e) \) is a strictly concave function of \( e \) with a global maximum at \( e = e_h \).

If \( e \) is such that either one of conditions (i) and (ii) is violated, the deviant will be at least weakly worse off relative to \( \overline{V}_h \), conditional on investment i.e. \( V_h(e) \leq \overline{V}(e) \), as shown in Figure 1(a). This is because either he is rejected by a matching partner, or because \( y(e_h, e)/2 < v_h \), so the deviant is now residual claimant in a match with another \( h \)-type, and thus receiving less than half the surplus, or some possible combination of these.

(ii) Now consider a deviation by an \( h \)-type to some \( e > e_h \). Depending on how high \( e \) is, there are a number of possibilities. Define \( y(e_l, e_h)/2 = v_l \). If \( e_h < e < e^* \), then the deviant \( h \)-type is still a residual claimant in a match with an \( l \)-type, and so from (1.1), \( V_h(e) = \overline{V}_h(e) \). If \( e > e^* \), then the deviant \( h \)-type is no longer a residual claimant in a match with an \( l \)-type, but rather the output is shared equally in the match, so then by calculations similar to those in the paper:

\[
V_h(e) = V^*_h(e) = \phi[\pi_h \left( \frac{y(e_h)}{2} \right) + \pi_l (y(e, e_l) - \frac{y(e_l, e_l)}{2})] - c_h e
\]  

It is then easily verified from (1.2) that \( V^*_h(e) \) is a strictly concave function of \( e \) with a global maximum at some \( e^*_h \). Moreover, comparing (1.1) and (1.2), \( e^*_h < e_h \) because \( y(e, e_l) \) is divided by two in (1.2).

Finally, at \( e^*_h \), by definition of \( e^*_h \), \( V^*_h(e^*_h) = \overline{V}_h(e^*_h) \).

(iii) Putting (i) and (ii) together, we see that \( V_h(e) \) must be as shown in Figure 1(a), i.e. a continuous and piecewise differentiable function of \( e \) with a global maximum at \( e = e_h \).

(iv) Consider a deviation by an \( l \)-type to some \( e \neq e_l \). Consider first \( e < e_l \). Depending on how low \( e \) is, there are a number of possibilities. If \( e \) is sufficiently close to \( e_l \), i.e. above some critical \( e^*_l \), then (i) \( y(e_l, e)/2 \geq v_l \), \( l \)-types will accept a match with the deviant and divide the surplus equally, and (ii)
\( v(e) \geq y(e_h, e)/2 \), so the deviant is residual claimant in a match with an \( h \)-type. In this event, from (3.3) in the paper, the payoff to deviating is

\[
V_l(e) = V_l(e) \equiv \phi_i \frac{y(e, e_l)}{2} - c_i e
\]  

(1.3)

If \( e \) is such that either one of conditions (i) and (ii) is violated, the deviant will be at least weakly worse off than at \( e = e_l \), i.e. \( V_l(e) \leq V_l(e_l) \), as shown in Figure 1(b). This is because either (a) he is no longer receiving his outside option, evaluated at \( e_l \) i.e. \( v_l \), but something less, or (b) so the deviant is now residual claimant in a match with another \( l \)-type, and thus receiving less than half the surplus, or (c) has a match rejected, or some possible combination of these.

(v) Now consider a deviation by an \( l \)-type to some \( e > e_l \). Then it is clear that no matter how high \( e \) is, the deviant’s continuation payoff must be less than \( y(e, e_l)/2 \), as all other agents have investment of at most \( e_l \). So, there are two possibilities. The first is that the deviant’s continuation payoff is less than \( y(e, e_h) \), in which case the deviant will accept a match with an \( h \)-type. In this case, \( V_l(e) = V_h(e) \) as defined in (1.3) above. The second is that the deviant rejects a match with an \( h \)-type. In this case, his continuation payoff satisfies

\[
rv(e) = \alpha \pi_t \left( \frac{y(e, e_l)}{2} - v(e) \right) \Rightarrow v(e) = \phi_i \frac{y(e, e_l)}{2}
\]

which, is the same as in (1.3), abSENT the cost of investment. So, again in this case, \( V_l(e) = V_l(e) \) as defined in (1.3) above.

(vi) Putting (iv) and (v) together, we see that \( V_l(e) \) must be as shown in Figure 1(b), i.e. a continuous and piecewise differentiable function of \( e \) with a global maximum at \( e_l \). This completes the proof. QED.
Figure 1(a)

Figure 1(b)