1. Proof of Sufficiency of the Equilibrium Conditions

(i) In this not-for-publication Appendix, we prove the sufficiency of the first-order conditions (3.8), (3.10) in the paper for equilibrium investment. To do this, we introduce the following notation. Let e be an arbitrary investment level, and let $V_k(e)$ be the payoff in the unmatched state to a cost type $k = h, l$ who has invested e, net of the cost of investment. So, $V_k(e) \equiv v(e) - c_k e$, where $v(e)$ denotes the payoff in the unmatched state (search) to an agent who has invested some arbitrary amount e. Note therefore that $v(e_h) = v_h$, $v(e_l) = v_l$, where v_h, v_l are the equilibrium payoffs to search defined in the paper. Of course, $V(.), v(.)$ also depend on equilibrium investments e_h, e_l, but we suppress this dependence for clarity in what follows. So, it suffices to show that $V_k(e)$ has a global maximum at $e = e_k,\ k = h, l$. [Note that by the arguments in the paper, $V_k(e)$ has a *local* maximum at e_k.] We look at the cases of $k = h, l$ separately, beginning with $k = h$.

(ii) Consider first a deviation by an $h-$type to some $e < e_h$. Depending on how low e is, there are a number of possibilities. If e is sufficiently close to e_h, i.e. above some critical value $\underline{e}_h$, then (i) $y(e_l, e) \geq v_l$, so $l-$types will accept a match with the deviant, and (ii) $y(e_h, e)/2 \geq v_h$, so $h-$types will accept a match with the deviant, and divide the surplus equally. In this event, from (3.4) in the paper, the deviant's payoff will be

$$V_h(e) = \overline{V}_h(e) \equiv \phi[\pi_h \frac{y(e, e_h)}{2} + \pi_l(y(e, e_l) - \phi_l \frac{y(e_l, e_l)}{2})] - c_h e \tag{1.1}$$

Note from (1.1) and the strict concavity of y in e that $\overline{V}_h(e)$ is a strictly concave function of e with a global maximum at $e = e_h$.

If e is such that either one of conditions (i) and (ii) is violated, the deviant will be at least weakly worse off relative to $\overline{V}_h$, conditional on investment i.e. $V_h(e) \leq \overline{V}(e)$, as shown in Figure 1(a). This is because either he is rejected by a matching partner, or because $y(e, e_h)/2 < v_h$, so the deviant is now residual claimant in a match with another $h-$type, and thus receiving less than half the surplus, or some possible combination of these.

(ii) Now consider a deviation by an $h-$type to some $e > e_h$. Depending on how high e is, there are a number of possibilities. Define $y(e_l, \overline{e}_h)/2 = v_l$. If $e_h < e < \overline{e}_h$, then the deviant $h-$type is still a residual claimant in a match with an $l-$type, and so from (1.1), $V_h(e) = \overline{V}_h(e)$. If $e > \overline{e}_h$, then the deviant $h-$type is no longer a residual claimant in a match with an $l-$type, but rather the output is shared equally in the match, so then by calculations similar to those in the paper:

$$V_h(e) = V_h^*(e) \equiv \phi[\pi_h \frac{y(e, e_h)}{2} + \pi_l \frac{y(e, e_l)}{2}] - c_h e \tag{1.2}$$

It is then easily verified from (1.2) that $V_h^*(e)$ is a strictly concave function of e with a global maximum at some e_h^*. Moreover, comparing (1.1) and (1.2), $e_h^* < e_h$ because $y(e, e_l)$ is divided by two in (1.2). Finally, at $\overline{e}_h$, by definition of $\overline{e}_h$, $V_h^*(\overline{e}_h) = \overline{V}_h(\overline{e}_h)$.

(iii) Putting (i) and (ii) together, we see that $V_h(e)$ must be as shown in Figure 1(a), i.e. a continuous and piecewise differentiable function of e with a global maximum at $e = e_h$.

(iv) Consider a deviation by an $l-$type to some $e \neq e_l$. Consider first $e < e_l$. Depending on how low e is, there are a number of possibilities. If e is sufficiently close to e_l, i.e. above some critical $\underline{e}_l$, then (i) $y(e_l, e)/2 \geq v_l$, $l-$types will accept a match with the deviant and divide the surplus equally, and (ii)

$v(e) \geq y(e_h, e)/2$, so the deviant is residual claimant in a match with an $h-$type. In this event, from (3.3) in the paper, the payoff to deviating is

$$V_l(e) = \overline{V}_l(e) \equiv \phi_l \frac{y(e, e_l)}{2} - c_l e \tag{1.3}$$

If e is such that either one of conditions (i) and (ii) is violated, the deviant will be at least weakly worse off than at $e = e_l$, i.e. $V_l(e) \leq \overline{V}(e_l)$, as shown in Figure 1(b). This is because either (a) he is no longer receiving his outside option, evaluated at e_l i.e. v_l, but something less, or (b) so the deviant is now residual claimant in a match with another $l-$type, and thus receiving less than half the surplus, or (c) has a match rejected, or some possible combination of these.

(v) Now consider a deviation by an $l-$type to some $e > e_l$. Then it is clear that no matter how high e is, the deviant's continuation payoff must be less than $y(e, e_l)/2$, as all other agents have investment of at most e_l . So, there are two possibilities. The first is that the deviant's continuation payoff is less than $y(e, e_h)$, in which case the deviant will accept a match with an $h-$type. In this case, $V_l(e) = \overline{V}_h(e)$ as defined in (1.3) above. The second is that the deviant rejects a match with an $h-$type. In this case, his continuation payoff satisfies

$$rv(e) = a\pi_l(\frac{y(e, e_l)}{2} - v(e)) \Longrightarrow v(e) = \phi_l \frac{y(e, e_l)}{2}$$

which, is the same as in (1.3), absent the cost of investment. So, again in this case, $V_l(e) = \overline{V}_l(e)$ as defined in (1.3) above.

(vi) Putting (iv) and (v) together, we see that $V_l(e)$ must be as shown in Figure 1(b), i.e. a continuous and piecewise differentiable function of e with a global maximum at e_l. This completes the proof. QED.

Figure 1(a)

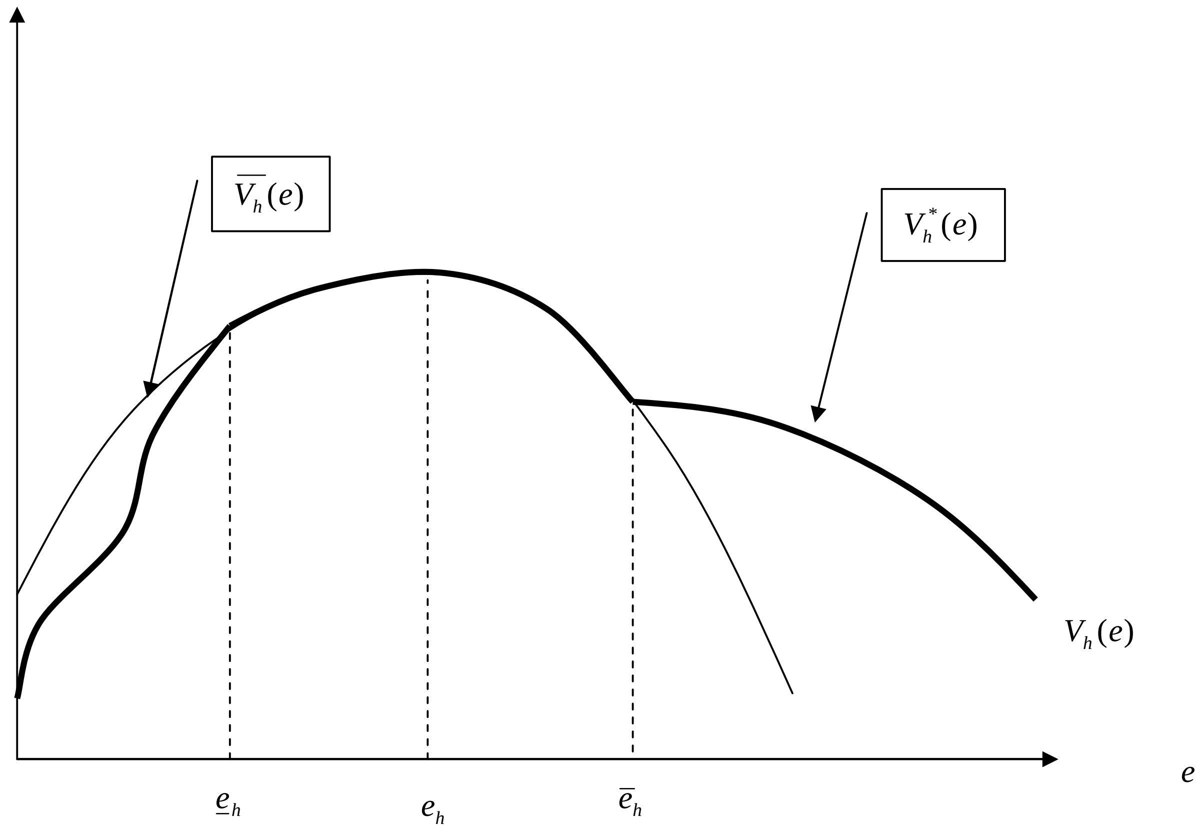

Figure 1(b)