Multiple Equilibria in the Citizen-Candidate Model of Representative Democracy.

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Abstract

De Sinopoli and Turrini (1999) present an example to show that the notion of equilibrium proposed in the Besley-Coate model of representative democracy (*political equilibria* (Besley-Coate, 1997)) may fail to capture all aspects of the behaviour of rational agents. Thus, Political Equilibria may not conform to the principle of iterated elimination of dominated strategies. In this note, we show that requiring political equilibria to be iteratively undominated rules out unreasonable equilibria only for those cases where at least four candidates stand for election.

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1 Introduction

The citizen-candidate model (Besley-Coate(1997), Osborne-Slivinski(1996)) is being increasingly used to model decision-making in environments where Condorcet winners may not exist. The model has three stages of activity; (i) citizens decide on whether to stand for office, and incur a small cost if they do so; (ii) citizens vote for the candidates who stand, and the winner is elected by plurality rule; (iii) the candidate who is elected implements her most preferred policy from a fixed set of alternatives. Although the citizen-candidate model is a major advance over existing models, a major problem is that there are typically multiple equilibria at the voting stage, due to plurality rule (Dhillon and Lockwood(2000)). These multiple equilibria at stage (ii) generate multiple equilibria to the game as a whole, which Besley and Coate call political equilibria (PE). Osborne-Slivinski(1996) resolve this problem by assuming that voters vote sincerely. Sincere voting, however is an arbitrary rule for selecting strategies. In contrast, Besley-Coate (1997) impose the requirement that, conditional on any set of candidates, the voting equilibrium must be weakly undominated. Not surprisingly, this weak refinement at the voting stage still leaves many equilibria, some of them not very credible[1].

In this paper, we investigate whether imposing a stronger refinement on the (Nash) equilibrium at the voting stage, conditional on any set of candidates, eliminates any PE. Our refinement is that voting strategies be *iteratively* weakly undominated. We call PE with this refinement imposed at the voting stage *iteratively weakly undominated political equilibria* (IWUPE). Our justification for this refinement is twofold. First, that if it is common knowledge than agents will not play weakly dominated strategies, then it is "reasonable" that rational voters will not play their "second round" weakly dominated strategies, and so on. Some formal justification of this is in Rajan(1998). Second, it has been shown by Sinopoli(2000) that more standard refinements (perfection, properness) do not have much bite in plurality vot-

[1] Besley and Coate declare "for those who would like a clean empirical prediction, our multiple equilibria will raise a sense of dissatisfaction."

ing games; in particular, requiring subgame-perfection only rules out Nash equilibria where Condorcet losers are elected. [This does not eliminate any PE, except possibly one-candidate ???]

De Sinopoli and Turrini(1999) initiated this approach to refining PE in a paper where they present an example with four candidates and one winner where the voting strategies are not iteratively weakly dominated, and moreover, requiring them to be so eliminates this equilibrium. This result raises the question of whether iterated deletion of weakly dominated strategies also refines political equilibrium outcomes in the case of one, two, and three candidate equilibria. This paper answers this question, fully and negatively. We show[2] that if there exists a PE with fewer than four candidates, and a given set of winner(s), then there also exists a IWUPE with the same candidate set and the same winner(s). So, this paper complements De Sinopoli and Turrini(1999); together, they how that iterated deletion of weakly dominated strategies also refines political equilibrium outcomes only when the number of candidates is at least four.

We describe the Besley-Coate model in greater detail in Section 2. Section 3 then discusses the main results. Section 4 concludes.

2 The Citizen-Candidate Model of Representative Democracy

Besley-Coate (1997) consider a community of n citizens, who may select a representative to implement a policy alternative. Each citizen $i \in N = \{1, ...n\}$ has a finite action set X_i representing the policy alternatives available to him if elected. It is possible that citizens may be of different competencies i.e. $X_i \neq X_j$. If no-one is elected, a default policy $x_0 \in \cap_{i \in N} X_i$

[2]We do not show that every PE is also a IWUPE. This is generally not true; as shown by Dhillon and Lockwood(2000), in any voting game, the set of iteratively weakly undominated strategy profiles may be much smaller than the set of weakly undominated ones.

is selected. Voters have preferences over who represents them, as well the alternatives they choose, so utility functions are defined on $X \times N \cup \{0\}$, $X = \cup_{i \in N} X_i$, i.e. $\pi_i(x, j)$ is the utility for i if j is elected and chooses action x. If no-one is elected, utilities are $\pi_i(x_0, 0)$.

The political process has three stages. At stage 1, citizens face a binary decision: to stand for election (enter) or not. At stage 2, voting takes place, and in stage 3, the elected representatives choose policy. We discuss each stage in turn.

At the final stage, once elected, a citizen i will therefore choose their own most preferred policy (assumed to be unique):

$$x_i^* = \arg\max_{x \in X_i} \pi_i(x, i)$$

Since for every citizen's most preferred point $x_i^* \in X_i$ is known, the induced preferences of citizens over *candidates* are given by $u_i(j) = \pi_i(x_i^*, j),\ i, j \in N$. We assume that these induced preferences over candidates are strict: i.e. $u_i(j) \neq u_i(k)$, all $i, j, k \in N,\ j \neq k$. Also, $u_i(0) \equiv \pi_i(x_0, 0)$.

At the second stage, voting is by plurality rule: each voter has one vote, which she can cast for any one of the set $C \subset N$ of candidates who stand, and the candidate with the greatest number of votes wins. If a set of two or more candidates have the greatest number of votes, every candidate in this set is selected with equal probability. Let $W \subset C$ be the set of candidates with the most votes, which we call the *winset.* Then voter payoffs over some W are:

$$u_i(W) = \frac{1}{\#W} \sum_{j \in W} u_i(j),\ i \in N$$

Formally, let $\alpha_i = j$ if voter i votes for candidate $j \in C$. Then $\alpha = (\alpha_1, ..., \alpha_n)$ denotes a *vote profile.* Let $W(\alpha, C) \subset C$ denote the winset, given the vote profile α and candidate set C. The utility to voter i from α (given C) is then $u_i(\alpha, C) \equiv u_i(W(\alpha, C))$. A *Nash equilibrium* profile α^* is defined in usual way as a profile where α_i^* is a best response to α_{-i}^*, all $i \in N$.

Let $S \subset C^n$ be any set of voting profiles, with $S = \times_{i \in N} S_i$. Say that α_i *is undominated relative to* S if (i) $\alpha_i \in S_i$; (ii) there does not exist $\alpha_i' \in S_i$ such that $u_i(\alpha_i', \alpha_{-i}, C) \geq u_i(\alpha_i, \alpha_{-i}, C)$, all $\alpha_{-i} \in S_{-i}$, and $u_i(\alpha_i', \alpha_{-i}, C) > u_i(\alpha_i, \alpha_{-i}, C)$, some $\alpha_{-i} \in S_{-i}$. Now define the sequence of sets of vote profiles $\{A^0, A^1, A^2...\}$ for i as follows: $A^0 = C^n$, and $A^n = \times_{i \in N} A_i^n$, where A_i^n is the set of voting actions for i that are undominated relative to A^{n-1}, all $i \in N$. As the set of voting actions is finite, A^n converges after a finite number of steps to some A^∞, which is the set of vote profiles that are *iteratively* weakly undominated. It is always non-empty. Also, the A^n are understood to be conditional on C.

Besley and Coate define a *voting equilibrium* to be a α^* which is (i) Nash equilibrium; and (ii) weakly undominated i.e. $\alpha^* \in A^1$. We will focus on a stronger refinement of Nash equilibrium i.e. where $\alpha^* \in A^\infty$. Formally, an *iteratively weakly undominated voting equilibrium is* a α^* which is (i) Nash equilibrium; and (ii) *iteratively* weakly undominated i.e. $\alpha^* \in A^\infty$.

Finally, we turn to the entry stage. Any citizen can run for office, but if they run, they incur a small cost δ. If no-one runs for office, the default policy x_0 is implemented. In the first stage, citizens decide non-cooperatively on their entry: $\gamma_i \in \{0, 1\}$ denotes the entry[3] decision for i. When deciding upon candidacy, citizens all anticipate[4] the same voting equilibrium $\alpha^*(C)$ among the multiple equilibria at the voting stage, given any possible set of candidates C. Denote the strategy profile at the entry stage by $\gamma = \{\gamma_1, ..., \gamma_n\}$.

We can now state our equilibrium concepts. A *weakly undominated political equilibrium* (WUPE) of this game is a $(\gamma^*, \alpha^*(.))$ if (i) γ^* is an equilibrium of the entry stage, given $\alpha^*(.)$ and (ii) $\alpha(C) \in C^n$ is a weakly undominated Nash equilibrium in the voting subgame, for every $C \subset N$. Our WUPE is Besley and Coate's political equilibrium: we add the qualifier to make explicit the refinement assumed at the voting stage. An *iteratively weakly undominated political equilibrium* (IWUPE) of this game is a $(\gamma^*, \alpha^*(.))$ if (i)

[3] We do not allow citizens to randomise.

[4] This is represented as in De Sinopoli and Turrini (1999) by the function $\alpha(.) : 2^N \rightarrow (N \cup \{0\})^N$.

γ^* is an equilibrium of the entry stage, given $\alpha^*(.)$ and (ii) $\alpha(C) \in C^n$ is an iteratively weakly undominated Nash equilibrium in the voting subgame, for every $C \subset N$.

It is helpful for future reference to state the equilibrium entry conditions in either case, which are first, that $i \in C$ must prefer to enter, given α^* i.e.

$$u_i(\alpha^*(C^*/\{i\}), C^*/\{i\}) \leq u_i(\alpha^*(C^*), C^*) - \delta,\ i \in C^* \tag{1}$$

and second, that any $j \notin C$ must prefer not to enter, given α^* i.e.

$$u_j(\alpha^*(C^* \cup \{k\}), C^* \cup \{k\}) - \delta \leq u_j(\alpha^*(C^*), C^*),\ j \notin C^* \tag{2}$$

Finally, we state the assumptions we need (in addition to those made by Besley and Coate(1998)) for our analysis. First, we assume a "no indifference over lotteries" condition i.e.

NI. $u_i(W) \neq u_i(W')$, for all i and all $W \neq W',\ W, W' \subset N$.

This condition ensures that the order of deletion of weakly dominated strategies does not matter[5], and thus implies that is important to ensure that the solution concept we use is well defined.

Our second assumption, already made above, and purely for convenience, is that voters cannot abstain. When voting is costless and, as in our version of the Besley-Coate model preferences over candidates are strict, abstention is always weakly dominated so that ruling out abstention is without loss of generality.

3 Analysis

Let $(\gamma^*, \alpha^*(.))$ be some WUPE, and let $C(\gamma^*) = C^*$ be the equilibrium set of candidates given entry decisions γ^*. We will show that as long as $\#C^* \leq 3$, for any WUPE with equilibrium candidate set C^*, and winset

[5] That is, the calculation of A^∞ does not depend on the order in which weakly dominated strategies are deleted. See Marx and Swinkels (1997).

$W(C^*, \alpha^*(C^*))$, there is a IWUPE $(\gamma^{**}, \alpha^{**}(.))$ with the *same* equilibrium set of candidates and the same winset - and therefore the same outcome in terms of policy chosen and political representation.

We proceed as follows. First, $\alpha^{**}(.)$ *must* generate the same winset as $\alpha^*(.)$ when the candidate set is the equilibrium one:

$$W(C, \alpha^{**}(C^*)) = W(C, \alpha^*(C^*)) \tag{3}$$

Second, the incentives to enter must be the same in the original WUPE and the constructed IWUPE. That is, the entry conditions (1),(2) must continue to hold when α^* is replaced *by* α^{**} i.e.

$$u_i(\alpha^{**}(C^*/\{i\}), C^*/\{i\}) \leq u_i(\alpha^{**}(C^*), C^*) - \delta,\ i \in C^* \tag{4}$$

$$u_k(\alpha^{**}(C^* \cup \{k\}), C^* \cup \{k\}) - \delta \leq u_k(\alpha^{**}(C^*), C^*),\ j \notin C^* \tag{5}$$

So, for any C^* with $\#C^* \leq 3$, we must show that we can construct some $\alpha^{**} \in A^\infty$ such that (3)-(5) hold.

Now let Ψ be the set of candidate sets comprising C^* and those sets arising from unilateral deviations from equilibrium entry decisions[6]. Note that conditions (3)-(5) impose conditions on $\alpha^{**}(C)$ when $C \in \Psi$. For candidate sets *not* in Ψ, $\alpha^{**}(.)$ can be defined arbitrarily, subject to the requirement that it is an iteratively undominated profile. That is, we can set

$$\alpha^{**}(C) \in A^\infty(C), \text{ all } C \in \mathcal{N}/\Psi^* \tag{6}$$

where $\mathcal{N}$ is the power set of N. Note that (6) is always possible as $A^\infty(C)$ is always non-empty for all non-empty C.

It is helpful to break our complex task into steps by classifying political equilibria by the *number* of candidates. Following Besley and Coate, 1997,

[6]Formally,

$$\Psi = \{C \subset N \,|C = C^*,\ C = C^*/\{i\},\ i \in C^*,\ C = C^* \cup \{j\},\ j \notin C^*\}$$

say that a political equilibrium is a $m-$*candidate* political equilibrium if m candidates stand for election in the equilibrium. We first have:

Proposition 1. *For any 1-candidate WUPE with equilibrium candidate set $C^* = \{i\}$, and winset $W(C^*, \alpha^*(C^*)) = \{i\}$, there is a IWUPE $(\gamma^{**}, \alpha^{**}(.))$ with the same equilibrium set of candidates and the same winset.*

Proof. First, $\alpha^{**}(.)$ is defined[7] on Ψ as follows. For $C = \{i\}$, or $C = \{i, j\},\ j \in N$, set $\alpha^{**}(.) = \alpha^*(.)$. As we have set $\alpha^{**}(.) = \alpha^*(.)$, (3)-(5) hold from the fact that $\alpha^*(.)$ is part of a WUPE. To conclude, we must verify that $\alpha^*(C)$ is iteratively undominated for all $C \in \Psi$. The case $C = \{i\}$ is trivial, as every voter has only one strategy, so we must have $A^1(i) = A^\infty(i) = \{i\}^n$. In the case $C = \{i, j\},\ j \in N$, the only undominated strategy for any voter is to vote sincerely, so $A^1(C)$ is a singleton, so again iterated deletion does not reduce it i.e. $A^1(C) = A^\infty(C)$. □

To deal with two-candidate equilibria, we first need the following Lemmas. Let $\Gamma(C)$ be the voting (subgame) with candidate set C. A *strict Nash equilibrium* (Harsanyi, 1973) of $\Gamma(C)$ is a vector of voting decisions α^* where $u_i(\alpha_i^*, \alpha_{-i}^*\ ) > u_i(\alpha_i, \alpha_{-i}^*)$ all $\alpha_i \in C,\ \alpha_i \neq \alpha_i^*$, all $i \in N$. We then have:

Lemma 1. *Any strict Nash equilibrium α^* is iteratively undominated i.e. $\alpha^* \in A^\infty$.*

Proof. A strict Nash equilibrium is a profile of pure strategies $(\alpha_1^*, \alpha_2^*, ..., \alpha_n^*)$, such that each α_i^* is a *unique* best response to the profile α_{-i}^*. Thus, none of these strategies can be deleted in the first round. Moreover if this profile survives for all players at any round k of iterated deletion, they must survive in round $k+1$. This is because iterated deletion means that any player has the same or fewer strategies at every round, so if a strategy was a unique best response to a profile which survived round k, it will continue to be a unique best response to this profile in round $k+1$. □

For the proof of the next Lemma, the following notation will be useful. Let $\omega_{-i}(\alpha_{-i})$ be a vector recording the total votes for each candidate; given a

[7] Obviously, $C/\{i\} = \emptyset$, so $\alpha(\emptyset)$ is not defined.

strategy profile α_{-i} i.e. when individual i is not included. We suppress the dependence of ω_{-i} on α_{-i} except when needed and refer to ω_{-i} as a *vote distribution.* Clearly i's best response to ω_{-i} depends only on the information in ω_{-i}.

Lemma 2. *Any weakly undominated Nash equilibrium $\alpha^*(C)$ of a voting game $\Gamma(C)$, where $\#W(\alpha^*,\ C) > 1$ is a iteratively weakly undominated Nash equilibrium, and remains so even in $\Gamma(C \cup \{k\})$ for any $k \notin C$.*

Proof. Since all candidates in $W(\alpha^*, C)$ are tied, i.e. $\omega_i = \omega_j, \quad \forall i, j \in W(\alpha^*, C)$, it follows that every voter is pivotal between all elements of $W(\alpha, C)$. Clearly, voting for his best alternative in $W(\alpha^*, C)$ is a unique best response for any voter[8]. Hence any weakly undominated Nash equilibrium α^* with $\#W(\alpha^*, C) > 1$ must be a strict Nash equilibrium. But then by Lemma 1, $\alpha^*(C)$ is also an iteratively weakly undominated Nash equilibrium.

Now consider the game $\Gamma(C \cup \{k\})$. Assume that j is voter $i's$ most preferred candidate in $W(\alpha^*,\ C)$. The vector of votes i faces given $\alpha^*(C)$ is such that (w.l.o.g.) $\omega_j = \omega_l - 1$, $\forall l \neq j$, and $j, l \in W(C)$. It is sufficient to show that voting for candidate k is not a best response for in $\Gamma(C \cup k)$. Note that $n \geq 3$ since there $\#W(C) > 1$, so $\#C > 1$. Therefore, $n \geq 4$ (otherwise we cannot have a tie between two candidates in the two candidate game). Thus, if voter i deviates to k, he would ensure that $W(C \cup \{k\}) = W(C \backslash \{j\})$. Thus, voting for j remains a unique best response. Thus, $\alpha^*(C)$ remains a strict Nash equilibrium of the game $\Gamma(C \cup \{k\})$. Again, by Lemma 1, $\alpha^*(C)$ is also an iteratively weakly undominated Nash equilibrium in $\Gamma(C \cup \{k\})$.$\square$

We now turn to two-candidate WUPE. Note that the entry condition (1) requires that if there are two candidates, both must be in the winset - otherwise, the one that does not win would withdraw (Besley and Coate(1998). So, our second result is:

Proposition 2. *For any 2-candidate WUPE with equilibrium candidate set $C^* = \{i, j\}$, and winset $W(C^*, \alpha^*(C^*)) = \{i, j\}$, there is an IWUPE*

[8]This also implies, given our NI condition, that *all* voters will vote for their best alternative in $W(C)$

$(\gamma^{**}, \alpha^{**}(.))$ *with the same equilibrium set of candidates and the same win-set.*

Proof. First, $\alpha^{**}(.)$ is defined[9] on Ψ as follows. For $C = C^*$, or $C = C^*/\{i\}$, $i \in C^*$, set $\alpha^{**}(.) = \alpha^*(.)$. For $C = C^* \cup \{k\}$, $k \notin C^*$, set $\alpha^{**}(C^* \cup \{k\}) = \alpha^*(C^*)$. Note that by construction, (3),(4) are satisfied. Also, note that (5) is satisfied. First, note that

$$W(C^* \cup \{k\}, \alpha^{**}(C^* \cup \{k\})) = W(C^* \cup \{k\}, \alpha^*(C^*)) = W(C^*, \alpha^*(C^*)) \quad (7)$$

i.e. given α^{**}, the winner is unchanged if k enters. So, from (7),

$$u_k(\alpha^{**}(C^* \cup \{k\}), C^* \cup \{k\}) = u_k(\alpha^*(C^*), C^*)$$

and consequently (5) holds as $\delta > 0$.

Again, to conclude, we must verify that $\alpha^*(C)$ is iteratively undominated for all $C \in \Psi$. For $C = C^*$, or $C = C^*/\{i\}$, $i \in C^*$, an argument identical to the proof of Proposition 1 shows this. For $C = C^* \cup \{k\}$, Lemma 2 implies that $\alpha^*(C)$ is an iteratively undominated voting profile in the game $\Gamma(C^* \cup k)$, as required. $\square$

We now turn to the most complex case, that of 3-candidate WUPE. First, with three candidates, there may in principle, be one, two or three winners. It turns out that the case of two winners is impossible under our assumption of strict preferences. The case of three winners can be dealt with using Lemma 2, following the proof of Proposition 2. However, in the case of one winner, Lemma 2 no longer applies, and so we must find some other argument to construct an IWUPE with one winner. To illustrate our argument, we first present an example of a 3-candidate WUPE with one winner where we can find an IWUPE with the same outcome.

We need the following notation and lemma before this example. Fix some candidate set C with $\#C = 3$. Let N_i be the set of voters who rank candidate $i \in C$ as worst, with $n_i = \#N_i$. Let $q = \max_{l \in C} \{n_l/n\}$, and let w_i denote

[9] Obviously, $C/\{i\} = \emptyset$, so $\alpha(\emptyset)$ is not defined.

citizen $i's$ worst candidate in C, all $i \in N$. Now define a critical value of q as:

$$q_n = \begin{cases} 1 - \frac{1}{n} - \frac{1}{n}\lceil\frac{n+1}{3}\rceil, & n \text{ odd} \\ 1 - \frac{1}{n}\lceil\frac{n+2}{3}\rceil, & n \text{ even} \end{cases} \tag{8}$$

where $\lceil x \rceil$ denotes the smallest integer larger than x, and $\lfloor x \rfloor$ denotes the largest integer smaller than x. Finally, in this section, we assume w.l.o.g that $n \geq 4$. We then have the following useful result, constructed from various results of Dhillon and Lockwood, 2000:

Lemma 3. *Assume $\#C = 3$. If $q \leq q_n$, then any weakly undominated strategy profile in $\Gamma(C)$ is also iteratively weakly undominated i.e. $A^1(C) = A^\infty(C)$. Moreover, $A^1(C)$ is a subset of the set of iteratively undominated strategy profiles in $\Gamma(C \cup l)$ i.e. $A^1(C) \subset A^\infty(C \cup l)$.*

Proof. Every $\alpha_i \in C$ except $\alpha_i = w_i$ is weakly undominated in C, so $A^1(C) = \times_i C/w_i$ (Dhillon and Lockwood, Lemma 1). Moreover, by Theorem 2 of Dhillon and Lockwood, as $q \leq q_n$, $A^\infty(C) = \times_i C/w_i$. So,$A^1(C) = A^\infty(C)$ as required. Finally, consider $\Gamma(C \cup l)$. Define the *full reduction* of $(C \cup l)^n$, $V = \times_i V_i$, to be the set of strategy profiles where every $\alpha_i \in V_i$ is undominated relative to V (Marx and Swinkels(1997)). Then, by definition, $V = A^\infty(C \cup l)$. We will show that $A^1(C) \subset V$ for all $\alpha_i' \in C \cup l$. To do this, it is sufficient to show that every $\alpha_i \in C/w_i$ is undominated relative to $(C \cup l)^n$. In turn, it is sufficient to show that every is a unique best response in $C \cup l$ to some α_{-i} in $(C \cup l)^{n-1}$.

To prove this, let $\tilde{\alpha}_i \in C/w_i$. As $\tilde{\alpha}_i \in A^\infty(C)$, there exists $\tilde{\alpha}_{-i} \in A^\infty_{-i}(C)$ such that $\tilde{\alpha}_i$ is the unique best response in $A^\infty_i(C)$ to $\tilde{\alpha}_{-i}$. It is now sufficient to show that $\tilde{\alpha}_i$ continues to be the unique best response in $A^\infty_i(C)$ to $\tilde{\alpha}_{-i}$, when voter i can also choose l. To see this, note that the vote distribution $\omega_{-i}(\tilde{\alpha}_{-i})$ must have some alternative in C getting two or more votes when $n \geq 5$, as $\#C = 3$. So, voting $\alpha_i = l$ in response to $\tilde{\alpha}_{-i}$ cannot affect the outcome, and so $\tilde{\alpha}_i$ remains a unique best response in $C \cup \{l\}$ to $\tilde{\alpha}_{-i}$. CASE WhEN N=4?? $\square$

This is a powerful result which allows treatment of the 3-candidate case.

Example

There are eight citizens with preferences over N as follows:

$$\begin{aligned}
1 &: 1 \succ 8 \succ 5 \succ 3 \succ 2 \succ 4 \succ 6 \succ 7 \\
2 &: 2 \succ 8 \succ 1 \succ 3 \succ 5 \succ 4 \succ 6 \succ 7 \\
3 &: 3 \succ 8 \succ 5 \succ 1 \succ 2 \succ 4 \succ 6 \succ 7 \\
4 &: 4 \succ 8 \succ 5 \succ 3 \succ 1 \succ 2 \succ 6 \succ 7 \\
5 &: 5 \succ 8 \succ 1 \succ 3 \succ 2 \succ 4 \succ 6 \succ 7 \\
6 &: 6 \succ 8 \succ 1 \succ 3 \succ 5 \succ 2 \succ 4 \succ 7 \\
7 &: 7 \succ 8 \succ 1 \succ 3 \succ 5 \succ 2 \succ 4 \succ 6 \\
8 &: 8 \succ 5 \succ 3 \succ 1 \succ 2 \succ 4 \succ 6 \succ 7
\end{aligned}$$

Let $(\gamma^*, \alpha^*(.))$ represent a WUPE in this game, with an equilibrium set of 3 candidates $C^* = \{1, 3, 5\}$, and one winner, $W(C^*, \alpha^*(C^*)) = \{5\}$. We will first describe $\alpha^*(.)$ and verify that it does induce the equilibrium entry decisions. Then, we will show that there is an IWUPE with the same set of candidates and winset.

Description of $\alpha^*(.)$

First, $\alpha(C^*) = (5, 1, 5, 5, 5, 1, 3, 5)$, thus candidate 5 wins. This is a Nash equilibrium, since no voter is pivotal, and moreover, the profile is undominated ($\alpha(C^*) \in A^1(C^*)$) as no-one votes for their worst candidate.

Voting profiles and winsets in all the two candidate games generated by withdrawal of one of the equilibrium candidates are as follows:

$$\begin{aligned}
\alpha(C^*/\{1\}) &= (5, 3, 3, 5, 5, 3, 3, 5),\ W(C^*/\{1\}) = \{3, 5\} \\
\alpha(C^*/\{3\}) &= (1, 1, 5, 5, 5, 1, 1, 5),\ W(C^*/\{3\}) = \{1, 5\} \\
\alpha(C^*/\{5\}) &= (1, 1, 3, 3, 1, 1, 1, 3),\ W(C^*/\{5\}) = \{1\}
\end{aligned}$$

It is clear that withdrawal is suboptimal for all candidates. For example, if candidate 1 withdraws, he gets a lottery over 3 and 5 which is worse for