Costly Voting and Inefficient Participation

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Abstract

We study the efficiency of participation equilibria in a model of costly voting where voters' preferences combine both private values and heterogeneous beliefs generated by noisy signals about a common, payoff-relevant state of the world. Initially, we assume that voters make their participation decision before acquiring any private information. Conditional on participation, we show that there is a unique symmetric Bayesian equilibrium in weakly dominated pure strategies where voting is either according to private values or according their beliefs. When voting takes place according to private values, there is a unique participation equilibrium with excessive participation. In contrast, when voting takes place according to signals, (a) too little information is aggregated at a voting equilibrium and in the vicinity of equilibrium, more participation is *always* Pareto-improving, (b) multiple Pareto-ranked voting equilibria may exist and in particular, compulsory voting may Pareto dominate voluntary voting. Finally, we show that our results are robust when participation is conditioned on partial private information. However, we obtain strikingly different results when voters condition participation on full private information.

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1. Introduction

Many decisions are made by majority voting. In most cases, participation in the voting process is both voluntary and costly. The question then arises whether the level of participation is efficient i.e. is there too much or too little voting? Borgers (2004) addresses this issue in a model with costly voting and private values. He identifies a negative externality from voting: the decision of one voter to vote lowers the probability that any other voter is pivotal, and thus reduces the benefit to voting of all other agents. A striking result of the paper is that the negative externality implies that compulsory voting is *never* desirable: all voters are strictly better off at the (unique) voluntary voting equilibrium. An implication of this global result is a local one: in the vicinity of an equilibrium, lower participation is *always* Pareto-improving.

In this paper, we re-examine the nature of inefficiency of majority voting in a model with costly participation, but where voters differ not only also in innate private values *but also beliefs* about which alternative is best. We have two motivations for this line of inquiry. First, in the experimental psychological literature, it is found that beliefs about the relative likelihood of different candidates or policies being effective (cognitive influences) influence voter behavior just as much as likes and dislikes (affective influences). For example, Lavine et. al. (1998), using the National Election Studies conducted by the University of Michigan, show that voting behavior in four US presidential election was significantly determined not just by emotional attitudes towards the candidates, but also by voters' assessments of certain personal characteristics of the candidates (moral, intelligent, dishonest, weak, etc).

Second, in economics, there is already a substantial literature, focussed on voting by juries, that assumes that voters have the same innate preferences, but differ in their beliefs about which alternative is best (the Condorcet Jury literature: see e.g. Austen-Smith and Banks(1996), McClellan(1998), Wit(1998), Feddersen and Pesendorfer(1997)). Specifically, this literature works with a Bayesian set-up, where all voters agree on which of two alternatives is best, given a state of the world, but voters all receive different private signals about the state of the world, which generates heterogeneity in beliefs. The focus of this literature is how well various voting rules aggregate the information in the signals, given that voters behave strategically.

In this paper, we adopt the modelling of heterogeneity of beliefs used in the Condorcet Jury literature, and also allow agents to differ in innate preferences over the two alternatives. In this setting, in addition to the negative "pivot" externality identified by Borgers

(2004), there is[1] a potential positive informational externality from voting: an individual voter, by basing his voting decision on his informative signal, improves the quality of the collective decision for all voters. On the face of it, is not at all obvious, in general, which of these two externalities dominates, and therefore, in principle, equilibrium participation may be too high or too low.

In addition to modelling both beliefs and private values, we also examine the key issue of timing. Our base case is when voters observe their signals and their private values after the participation has been made. However, we show that the results of the base case (described more fully below) carry over to cases where voters observe partial information (*either* the signal *or* the private value) before participation. Scenarios that fit this timing of events include situations where (a) voters have to be able to commit to participate before acquiring information (like an appointments committee or jury service) or (b) voters have to participate to acquire information as much of the relevant information is obtained by participation at the meeting itself (like a town hall meeting or a departmental staff meeting). On the other hand, allowing voters to condition their participation decision on their private information and signal captures better the order of events in general (parliamentary or presidential) elections. In this case, the results are very different.

In more detail, a finite set of potential voters (agents) must choose between two alternatives. The preference of any agent over these alternatives depends on both on a binary state of the world and a private preference parameter. For the most part, we assume purely for convenience[2] that payoff to any agent from either alternative is additively separable in the state of the world and the private preference parameter (private value), with a weight on the latter of λ.

Initially, we consider the following timing of events. First, the state of the world is realized. Then, agents privately observe their cost of voting and decide whether to participate in the election. If they decide to participate, they observe a noisy signal about the state of the world, and the random variable that determines their private value. All participating agents then vote over the two alternatives.

We begin by characterizing the voting decision, conditional on participation. We show that *independently of the numbers of agents who have decided to participate*, there is a (generically) unique symmetric weakly undominated Bayes-Nash equilibrium of the

[1] As Borgers remarks *" In a common value model of voting...there will be positive externalities to voting which can mitigate or outweigh the negative externality which we identify. In such a model one cannot expect as clear-cut results as we obtain here".*

[2] All the results extend to the case where the payoff is multiplicatively separable, so nothing crucial hinges on additivity.

voting sub-game. If $\lambda < \hat{\lambda}$, all participants will vote according to their signal i.e. vote for the alternative that is best in the state of the world that is forecast to be more likely, given their signal. If[3] $\lambda > \hat{\lambda}$, all participants will vote according to their private value. Moreover, $\hat{\lambda}$ is increasing in the accuracy of the signal and in the limit $\hat{\lambda}$ tends to a half. So, conditional on participation, if $\lambda < \hat{\lambda}$, the private information of voters is aggregated efficiently: if $\lambda > \hat{\lambda}$, this information is not used at all.

We then study the participation decision. We focus on symmetric equilibria where agents participate iff their cost of doing so is below some critical value. We show that if in the subsequent voting subgame, voting is according to signals i.e. when $\lambda < \hat{\lambda}$, participation equilibrium may not be unique: typically, there will be several equilibria. This is contrast to the case of $\lambda > \hat{\lambda}$, where participation equilibrium is unique. Moreover, we find that other things equal, the incentive to participate is lower when voters rationally anticipate voting according to their signal than when they anticipate voting according to their private value.

The key issue is the efficiency of the participation decision. As remarked above, in our model, there are two opposing externalities at work, the pivot externality, and the informational externality. Our finding is that which externality dominates depends *entirely* on which voting equilibrium prevails. Specifically, if $\lambda < \hat{\lambda}$, so that voting is according to signals, participation is inefficiently low: a small coordinated increase in the participation externality makes all agents better off ex ante. If $\lambda > \hat{\lambda}$, so that voting is according to private values, participation is inefficiently high: a small coordinated decrease in the participation externality makes all agents better off ex ante.

Some additional results follow in the case where voting is according to the signal. Under some conditions, voting equilibria can be Pareto-ranked, with an equilibrium with more voters Pareto-dominating the equilibrium with fewer voters, on the average. Moreover, we also show that there are conditions under which compulsory voting Pareto-dominates voluntary majority voting. We also demonstrate that the additivity of preferences in the state of the world and the private preference parameter is not crucial: our results extend straightforwardly when preferences are multiplicative in these two components.

Finally, we examine the robustness of our results to changes in the timing of events. We begin by pointing out that for extreme values of λ i.e. $\lambda = 0$ and $\lambda = 1$, it makes no difference to our analysis whether or not voters can condition their participation decision

[3] At $\hat{\lambda}$, conditional on participation, each voter is indifferent about voting either according to their signal or PV.

on their private information. For an arbitrary value of λ strictly between 0 and 1, we show that all the preceding results extend to the case where agents can observe *either* their signal *or* their private value before making their participation decision. This is important, because in many actual voting situations, it is more realistic to think of heterogenous beliefs as being generated by observation of the mass media, etc. *before* any irrevocable participation decision is made.

However, if agents can observe *both* their signals and their private values before participating, then an additional complication arises: even though, conditional on participation, our existing results on voting behavior go through, in general, the expected benefit to voting when pivotal will depend on whether these two pieces of information "agree" or not, with benefit being higher in the former case. Therefore, a participation strategy will have two cutoffs, where the lower (higher) cutoff corresponds to a situation where these two pieces of information "disagree" ("agree"). When voting is according to signals, (a) small increase in the cutoff when signals and private values "agree" is ex-ante Pareto improving while (b) a small decrease in the cutoff when signals and private values "disagree" is ex-ante Pareto improving. When voting is according to private values, a small decrease in the cutoff when signals and private values "agree" is ex-ante Pareto improving. However, in this case, although a small change in the cutoff when signals and private values "disagree" is, in general, ex-ante Pareto improving, the direction of the Pareto improving change is unclear.

Related literature is as follows. As our model combines both private values and heterogeneous beliefs, some of our results generalize existing results in Borgers (2004) while other results we obtain here are novel. Our result that there is a unique equilibrium with excessive participation when $\lambda > \hat{\lambda}$, generalizes Borgers' main result for the special case where $\lambda = 1$. In contrast, our results when $\lambda < \hat{\lambda}$, that locally participation is inefficiently high, that there are multiple Pareto ranked equilibria, that compulsory participation may dominate voluntary participation are new as are our results for the case when voters are able to condition participation on full private information.

Apart from Borgers' paper, and the Condorcet Jury literature discussed above, this paper is related to some other recent papers on voting with incomplete information[4]. First, our paper is one of the very few (to our knowledge) to study information aggregation

[4]We should also mention Osborne, Rosenthal and Turner (2000) who study a model of costly participation. However, the focus of our paper and the formal model differs from their paper. They do not explicitly model voting and agents have complete information. Moreover, they do not consider the efficiency of participation equilibria.

in a setting where agents have heterogenous preferences: others include Feddersen and Pesendorfer (1996), (1997) and Coughlan (2000). Our paper is distinctive because we allow for participation costs, so that the number of agents actually voting is variable. In this respect, it is closest[5] to Feddersen and Pesendorfer (1996), which explains why abstention (which is a form of non-participation) might be observed in equilibrium. However, they are key differences. First, we focus on the efficiency of equilibrium i.e. whether participation is too high or too low, whereas this is not directly addressed[6] by Feddersen and Pesendorfer. Second, in Feddersen and Pesendorfer, abstention is due to the "swing voter's curse", whereas in our model, it is generated by an explicit cost of participation.

Our model is also related[7] to that of Persico(2001), who considers the design of voting rules and committee size when committee members have to pay for informative signals. However, we are addressing a rather different issue; Persico studies the optimal design of a committee subject to the constraint that members are given the correct incentives to acquire information: we are looking at how information acquisition is sub-optimal, given a voting particular rule (majority voting) and fixed size of the electorate.

Finally, our inefficiency result when $\lambda < \hat{\lambda}$ is similar to Goeree and Grosser (2004) who also study a model of costly majority voting similar to Borgers(2004), with the generalisation that voters' preference parameters can be positively correlated. They find that when this correlation is sufficiently high, participation is inefficiently low. Although the results are similar, the explanations are different: in our paper, under-participation is due to the public good nature of signals (if i votes according to his signal, that is beneficial for j) whereas in Goeree and Grosser, under-participation is due to the corelation of private values. Also, in their framework, the question of *why* private values may be correlated is not addressed. This is not a problem in our framework, because signals are positively correlated as long as they are informative.

[5] Coughlan (2000) demonstrates the existence of an equilibrium in the Condorcet Jury model with a mistrial outcome. This is a version of the CJ model where the defendant is only acquitted (convicted) if there is a unanimuous vote for doing so: otherwise, there is a mistrial. Moreover, jurors have heterogenous preferences over type I and type II errors.In this equilibrium, voters always vote informatively i.e. with their signals.

[6] They establish the asymptotic result that if the number of potential voters is large, the election fully aggregates information (i.e. the winning alternative is almost surely the same a in the case where the electorate is fully informed): in this sense, participation is asymptotically efficient.,

[7] Our model is more general in what it assumes about the cost of observing signals (costs may be heterogenous, and are privately observed, whereas in Persico's paper, there is a homogenous cost which is common knowledge), but otherwise, more special (it does not allow for asymmetry in priors or over the cost of different types of mistaken decisions).

In the next section, we set out the model. Section 3 characterizes participation equilibria. Section 4 contains the main results on the sign of the externalities and efficiency of equilibrium. Sections 5 and 6 discuss possible extensions, and the last section concludes. Unless stated otherwise, all proofs are gathered together in the appendix.

2. The Model

2.1. Alternatives, Preferences and Costs

There is a set $N = \{1, ..n\}$ of agents, who can collectively choose between two alternatives, A and B. Voters have payoffs over alternatives that depend on both their own private value and the state of nature. There are two states of nature $s \in \{s_A, s_B\}$, and two possible private values for any voter. Specifically, voter $i's$ private value is a random variable a^i with support $\{A, B\} = T$, and with $\Pr(a^i = A) = 0.5$.

As in Feddersen and Pesendorfer(1997), we allow the payoff w^i of agent i from alternative $L \in \{A, B\}$ to depend on both s and a^i. We begin by focussing on the special case where w is additive[8] in (L, s) and (L, a^i) i.e.

$$w(L : a^i, s) = \lambda u(L, a^i) + (1 - \lambda)v(L, s),\ \lambda \in [0, 1].$$

Moreover, we assume that $u(L, a^i) = 1$ if $a^i = L$, and 0 otherwise, and $v(A, s_A) = v(B, s_B) = 1$ and 0 otherwise. That is, any voter values alternative L more if either his private value or the state of the world, or both, "agree" with L.

Here λ parameterizes the importance of the private value relative to the signal. One way of interpreting λ is to note that $\lambda > 0$ *biases* a given voter either in favour or against a given alternative, once a^i has been determined. For example, suppose that $a^i = A$. Then, up to a constant $(1 - \lambda)$, payoffs over pairs (s, L) are of the form

	A	B
s_A	$1 + \beta$	0
s_B	β	1

where $\beta = \lambda/(1 - \lambda)$ is the bias in favour of alternative A.

Agents have identical priors over $\{s_A, s_B\}$: all believe that each s in $\{s_A, s_B\}$ is equally likely. Prior to the decision to vote, voter $i \in N$ privately observes signal $\sigma^i \in \{A, B\}$. The probability of signal L, conditional on state s_L is $q > 0.5$, $L = A, B$. Conditional[9]

[8] The case where w is multiplicative is studied in section 6 below.

[9] Of course, the signals are unconditionally correlated (indeed, affiliated).

on s, the $(\sigma^1, ..\sigma^n)$ are i.i.d. Moreover, variables $(a^1, a^2, ..a^n)$ are also assumed i.i.d., and any a^i is also independent of $s, \sigma^1, ..\sigma^n$.

We assume that participation in the election is costly: i.e. it is costly to attend a meeting, or go to a polling station. Specifically, each voter $i \in N$ incurs a privately observed cost of participation, c^i: if he wishes to vote, he must pay this cost[10]. We assume that the c^i are i.i.d. across individuals: c^i is distributed on support $[\underline{c}, \bar{c}] \subset \Re_{++}$ with the probability distribution $F(c)$. Moreover, any c^i is also independent of $s, a^1, a^2, ..a^n,\ \sigma^1, ..\sigma^n$.

2.2. Order of Events and Solution Concept

Initially, we assume the sequence of events is as follows.

Step 0. The state of the world is realized, and each $i \in N$ privately observes c^i and decides whether to participate or not.

Step 1. All i who have decided to participate, observe a^i, σ^i, the total number of participants, say m, and vote either for A or for B.

Step 2. The alternative with the most votes is selected. If both A, B get equal numbers of votes, each is selected with probability 0.5.

Finally, our equilibrium concept is subgame perfect Bayesian equilibrium, with three additional relatively weak assumptions[11]. First, we suppose all agents, who are identical ex-ante, behave alike in equilibrium (symmetry). Second, we rule out randomization[12]. Third, we assume that player's equilibrium strategy at the voting stage is admissible (weakly undominated). Call any subgame perfect Bayesian equilibrium satisfying these three conditions a *participation equilibrium.*

2.3. Discussion

Two comments are in order on the timing described in Section 2.2 above. First, at Step 1, we assume that voter who have chosen to participate observe the total number of participants. In this case, $i's$ information set is (a^i, c^i, σ^i, m). This is the natural assumption to make if voting takes place at a meeting of some kind. An alternative is that

[10] Below, we argue that under very weak assumptions, this can also be interpreted as the cost of purchasing, or observing, the signal, σ^i.

[11] We are following Borgers(2004) in making these three assumptions.

[12] In Condorcet Jury models, this is generally a too strong assumption, as a Bayes-Nash equilibrium in pure strategies generally does not exist (Austen-Smith and Banks(1996), Wit(1998)). Howoever, due to the symmetry of our model and the voting rule of majority voting, a pure-strategy equilibrium always exists.

they do not i.e. at Step 1, $i's$ information set is (a^i, c^i, σ^i). However, as we demonstrate in Section 5, under the assumptions made below on strategies in the voting subgame, it makes no difference which of these assumptions hold: there is always a unique equilibrium at the voting stage.

Second, note that the "preference" information (a^i, σ^i) arrives after the participation decision has been made. This is an important assumption. As shown in Section 6.2, if (a^i, σ^i) arrives simultaneously with c^i, we would have to allow for the additional possibility that the participation decision is conditioned on whether σ^i, a^i "agree" or not, and in this case, the characterization of the equilibrium changes. On the other hand, if *only one piece* of preference information (i.e. *either* a^i *or* σ^i) arrives before the participation decision, our main results are robust to this change in timing (Section 6.1). In both cases, the assumption that signals are conditionally independent, models the idea that different individuals interpret a public signal (like a debate) differently.

It is also worth noting that our model nests some existing models as special cases. Most obviously, when $\lambda = 1$, our model reduces to that of Borgers (2004). Conditional on a given number, m, of voters participating, and $\lambda = 0$, the voting subgame is a special case of the Condorcet Jury model developed by Austen-Smith and Banks (1996) and several subsequent papers (see e.g. the references in Duggan and Martinelli (2001)). In particular, we have made a number of symmetry assumptions that are sufficient to ensure that in this subgame, when $\lambda = 0$, informative voting i.e. voting according to one's signal is also equilibrium behavior[13]. This is deliberate: otherwise, the analysis becomes intractable, and more importantly, we wish to focus on the interplay between preferences and beliefs in the simplest possible setting.

3. Participation Equilibrium

In the above environment, the n voters play a two-stage game of incomplete information. We solve the game backwards in the usual way, so we begin with the voting subgame when potential voters have made their participation decisions.

3.1. Voting

At stage 1, a strategy for i is of the form $\gamma^i : \{\sigma_A, \sigma_B\} \times \{A, B\} \times \{1, ..., N-1\} \rightarrow \{A, B\}$. We study the Bayes-Nash equilibria of this subgame, conditional on the (common

[13] Specifically, (in the notation of Austen-Smith and Banks) we assume that $q_a = q_b = q$, so that thier Theorem 1 applies.

knowledge) event that $l+1 \leq n$ voters have decided to participate. By assumption, we restrict our attention to symmetric equilibria i.e. where $\gamma^i = \gamma$, all i. Say that a voter votes *according to her private value (PV)* if $\gamma(a^i, \sigma^i, m) = a^i$, whatever σ^i. Say that a voter votes *according to her signal* if $\gamma(a^i, \sigma^i, m) = K$ iff $\sigma^i = K$, whatever a^i. Then, we have:

Proposition 1. *Conditional on a fixed number of participants $l+1$, there is a critical value $0 < \hat{\lambda} < 1$, where $\hat{\lambda} = (q-0.5)/q$, such that (i) if $\lambda \geq \hat{\lambda}$, there is always a symmetric equilibrium of the voting sub-game where those participating vote to according to their PV; (ii) if $\lambda \leq \hat{\lambda}$, there is always a symmetric equilibrium of the voting sub-game where those participating vote according to their signal. Moreover, these are the only symmetric equilibria in admissible, pure strategies when $\lambda \neq \hat{\lambda}$.*

Note that in our equilibrium, when voters vote according to their signals, they do so non-strategically i.e. ignoring any information inferred from the fact that they are pivotal. As remarked above, this is due to the fact that all the hypotheses of Theorem 1 of Austen-Smith and Banks(1996) are assumed satisfied.

There is full information aggregation in the sense of Feddersen and Pesendorfer(1997) - i.e. the choice of A, B would not change if all private information were common knowledge if and only if $\lambda < \hat{\lambda}$. Otherwise, there is no information aggregation: the outcome is the same as if no signals were observed by every voter.

3.2. Participation

We will study the symmetric participation equilibrium where any citizen votes iff c^i is below a cutoff $\tilde{c}$. Let $p = F(\tilde{c})$. In choosing $\tilde{c}$, every voter is assumed to anticipate the relevant (unique) continuation symmetric voting equilibrium, as described in Proposition 1, and so evaluates the expected benefit of participation on that basis. We first calculate the expected benefit to participation to some i, conditional on exactly l other agents participating, and agent i being pivotal.

Let $B(l)$ be the expected benefit to i from voting before information (σ^i, a^i) is known, given that exactly l other agents decide to participate. Also, let

$$f(m : p, n) = \binom{n}{m} p^m (1-p)^{n-m}$$

be the probability of m successes in n trials with success probability p. Then we have:

Lemma 1. *The expected benefit to voting given that exactly l other agents decide* *to participate is*

$$B(l) = \begin{cases} B_p(l) = \lambda\pi(l; 0.5)0.5 & \lambda > \hat{\lambda} \\ B_s(l) = (1-\lambda)\pi(l; q)I(l)(q-0.5) & \lambda < \hat{\lambda} \end{cases} \tag{3.1}$$

and $I(l) = 1$ *if* l *is even, and* 0 *otherwise. where*

$$\pi(l; x) = \begin{cases} f(\frac{l}{2} : l, x) & l \text{ even} \\ f(\frac{l-1}{2} : l, x) & l \text{ odd} \end{cases} \tag{3.2}$$

At $\lambda = \hat{\lambda}$, $B(l)$ *can equal* $B_p(l)$ *or* $B_s(l)$, *depending on which continuation equilibrium occurs.*

Now let $B(p)$ be the expected benefit to voting before information (σ^i, a^i) is known, given that all other agents decide to participate with probability p. This is simply the expected value of $B(l)$, given that l agents participate with probability $f(l : p, n-1)$;

$$B(p) = \begin{cases} B_p(p) = \lambda \sum_{l=0}^{n-1} f(l : p, n-1)\pi(l; 0.5)0.5 & \lambda > \hat{\lambda} \\ B_s(p) = (1-\lambda) \sum_{l=0}^{n-1} f(l : p, n-1)\pi(l; q)I(l)(q-0.5) & \lambda < \hat{\lambda} \end{cases} \tag{3.3}$$

At $\lambda = \hat{\lambda}$, $B(l)$ can equal $B_p(l)$ or $B_s(l)$, depending on which continuation equilibrium occurs.

It is now clear that if all other voters play a participation strategy with voting probability p, then $i's$ (strict) best response is to participate if $c^i < B(p)$ and not if $c^i > B(p)$. Following Borgers, we call this a *cutoff strategy,* and we denote the cutoff generally by $\hat{c}$. Generally, c^* is an *equilibrium cutoff strategy* if $c \leq B(F(c^*))$, all $c \leq c^*$, and $c \geq B(F(c^*))$, all $c \geq c^*$. A symmetric Bayesian equilibrium in cutoff strategies (for now on, just a *participation equilibrium*) is a c^* where every voter participates if $c \leq c^*$ and abstains otherwise.

Say that a participation equilibrium is *interior* $c^* = B(F(c^*))$, $\underline{c} < c^* < \overline{c}$ or at a *corner* if $\underline{c} > B(0)$ or $\overline{c} < B(1)$. can now show that there is at least one symmetric Bayesian equilibrium in cutoff strategies.

Proposition 2. *If* $\lambda > \hat{\lambda}$, *there is a unique symmetric Bayesian equilibrium in cutoff strategies. If* $\lambda \leq \hat{\lambda}$, *there is at least one symmetric Bayesian equilibrium in cutoff strategies.*

Proof. For fixed λ, existence of at least one equilibrium follows from the continuity of $B(F(.))$ on $[\underline{c}, \overline{c}]$, plus the possibility of corner equilibria. If $\lambda \geq \hat{\lambda}$ the continuation voting equilibrium is PV voting, so uniqueness follows directly from Proposition 1 of Borgers(2001). This in turn follows from the fact that $B_p(p)$ is decreasing in p. □

This result leaves open the possibility that multiple equilibria exist when $\lambda \leq \hat{\lambda}$, and the following example confirms this.

Example 1 (**Multiple Equilibria**).
Assume $n = 3$, $\lambda < \hat{\lambda}$ and that c is uniform on $[0, \overline{c}]$. In this case, from (3.3), we have:

$$B(p) = (1 - \lambda)(q - 0.5)[2p^2 q(1 - q) + (1 - p)^2] \tag{3.4}$$

Note that $p^* = F(c^*) = c^*/\overline{c}$, so assuming an interior equilibrium, the equilibrium condition $B(F(c)) = c$ can be rewritten in terms of p as $B(p) = p\overline{c}$, or explicitly as

$$(q - 0.5)[2p^2 q(1 - q) + (1 - p)^2] = \frac{p\overline{c}}{1 - \lambda} \tag{3.5}$$

This is a quadratic in p, with two roots:

$$p = \frac{(2 + \alpha) \pm \sqrt{(2 + \alpha)^2 - 8q(1 - q) - 4}}{2[2q(1 - q) + 1]} \tag{3.6}$$

where $\alpha = \overline{c}/(1 - \lambda)(q - 0.5)$. If we take $q = 0.75$, and $\overline{c}/(1 - \lambda) = 0.09$, then it is easy to check that the two roots are

$$p^* = \frac{1.3119}{1.375}, \; p^{**} = \frac{1.0481}{1.375}$$

i.e. the voting game has two interior equilibria. Note also for these numbers and $\lambda = 0$ that $B(1) = 0.09375 > \overline{c}$, so there may also be also a corner equilibrium where $p^{***} = 1$. $\|$

This is illustrated below. It is clear from the Figure that multiple equilibria are due to the non-monotonicity of the benefit function $B(p)$. when voting is based on signals.

Figure 1 in here

What about the relative incentives to participate when voting is based on private common values or signals? Note three differences between $B_s(l)$ and $B_p(l)$. First, in the private values case, there is a benefit to voting even when the number of voters is odd. Second q is replaced by 0.5 in $\pi(l : .)$ as any voter cannot predict how any other will vote, given that he decides to vote at all. As $q(1 - q)$ is maximized at $q = 0.5$, we can assert that $\pi(l : 0.5) > \pi(l : q)$, all $q \neq 0.5$. Finally, the benefit from one's most preferred alternative relative to random selection rises from $q - 0.5$ to 0.5 as in the private values case, voters are *sure* which alternative is best. Now write $B_s(p, \lambda), B_p(p, \lambda)$ to emphasize the dependence of the two benefits on λ. Then the above discussion implies that as all these three differences raise the benefit to voting in the private values case, so that it is always true that

$$B_p(p,\lambda) > B_s(p,1-\lambda),\ 0 \le p \le 1 \tag{3.7}$$

for all values of λ for which a pure-strategy voting equilibrium exists i.e. $\lambda \in [0, \underline{\lambda}]$. Let $c_p(\lambda)$ be the unique equilibrium cost cutoff in the private values case, and let $c_s(\lambda)$ be the *highest* equilibrium cutoff in the common values case (this is well-defined by Proposition 1).

A second important fact is the following. By inspection of (3.1), $B_s(l) = B_p(l)$ when $\lambda = \hat{\lambda}$. But then from (3.2),(3.4), $B_s(p,\hat{\lambda}) < B_p(p,\hat{\lambda})$. This gives us a virtually complete characterization of how the equilibrium cutoff, and thus the equilibrium participation probability p, varies are the weight λ varies. This is most easily illustrated by a diagram, rather than a formal proposition.

Figure 2 in here

As this figure shows, starting at $\lambda = 1$, the participation probability declines continuously as the weight on PV component of preferences decreases, until $\hat{\lambda} < 0.5$ is reached. At $\hat{\lambda}$, there are two kinds of voting equilibrium and thus two possible participation probabilities: as $B_s(p,\hat{\lambda}) < B_p(p,\hat{\lambda})$, the participation probability associated with voting according to signals (say, p_s) is discretely below that associated with voting according to PV (say, p_p). Then, as λ falls to zero, the participation probability increases continuously as the weight on common values component of preferences increases.

4. The Inefficiency of Participation Equilibria

4.1. Participation Externalities

We begin by defining the ex ante payoff to any citizen (i.e. prior to observing a^i, σ^i, c^i), conditional on a given cutoff strategy. First, note that if $\lambda \ge \hat{\lambda}$, PV voting occurs. In this case, if i does not participate, his payoff is just 0.5, whatever λ, as a^i is uncorrelated with a^j and with s. If he does participate, his payoff is $0.5 + B_p(F(\hat{c})) - c$. So, this ex ante payoff is

$$w(\hat{c}) = 0.5 + \int_{\underline{c}}^{\hat{c}} (B_p(F(\hat{c})) - c)f(c)dc$$

So,

$$\frac{dw}{d\hat{c}} = (B_p(F(\hat{c})) - \hat{c}) + \int_{\underline{c}}^{\hat{c}} B_p'(F(\hat{c}))f(\hat{c})f(c)dc$$

Now at an interior equilibrium, $B_p(F(c^*)) = c^*$, so

$$\frac{dw}{d\hat{c}}\,|_{\hat{c}=c^*} = \int_{\underline{c}}^{c^*} B_p'(F(c^*))f(c^*)f(c)dc < 0$$

where in the inequality, we have used the fact, proved by Borgers (2004), Proposition 1, that $B_p(p)$ is decreasing in p. This says that in equilibrium, the cutoff is locally too high: all agents could be made better off if they could coordinate to vote with a slightly lower probability. The cause is the negative pivot externality: an increase in $\hat{c}$ raises p, and thus decreases the probability that any agent is pivotal, and thus their utility.

Now we consider the case if $\lambda \leq \hat{\lambda}$. Here, the expected payoff is rather different. The key difference is that the payoff to any agent i if he does not participate depends (positively) on the participation probabilities of others. Consider the expectation of $v(L, s)$ if exactly m citizens vote according to their signals and denote this $v(m)$. This is simply equal to the probability of making the correct decision i.e. :

$$v(m) = \begin{cases} \sum_{k=(m+1)/2}^{m} f(k : m, q) & \text{if } m \text{ is odd} \\ \sum_{k=\frac{m}{2}+1}^{m} f(k : m, q) + 0.5f(\frac{m}{2} : m, q) & \text{if } m \text{ is even} \end{cases}$$

It is well-known that $v(m+1) > v(m)$ i.e. more signals there are, the lower the probability of error. Now, let $v_1(p), v_0(p)$ be the expected values of $v(m)$ given participation and non-participation for a given citizen i respectively, given also that all $j \neq i$ participate with probability p. These are:

$$v_0(p) = \sum_{m=0}^{n-1} f(m : p, n-1)u(m), \;\; v_1(p) = \sum_{m=0}^{n-1} f(m : p, n-1)u(m+1)$$

A very useful property of $v_1(p), v_0(p)$, proved in the Appendix, is that they are both increasing in p.

Lemma 2. *Both $v_1(p), v_0(p)$ are increasing in p for all $p \in [0, 1)$.*

Now define the ex ante payoff to any citizen (i.e. prior to observing a^i, σ^i, c^i), conditional on a given cutoff strategy when $\lambda \leq \underline{\lambda}$, so that signal voting occurs. In this case, if i does not participate, his payoff is just $\lambda 0.5 + (1-\lambda)v_0(p)$, as a^i is uncorrelated any signal. If he does participate, , his payoff is $\lambda 0.5 + (1-\lambda)(v_0(p)) + B_s(p)$. So, his ex ante payoff conditional on cutoff $\tilde{c}$ is

$$w(\hat{c}) = 0.5\lambda + (1-\lambda)v_0(p) + \int_{\underline{c}}^{\hat{c}} (B_s(F(\hat{c})) - c)fdc \tag{4.1}$$

Now differentiating,

$$\frac{dw}{d\hat{c}} = (B_s(F(\hat{c})) - \hat{c})f(\hat{c}) + (1-\lambda)v_0'(F(\hat{c}))f(\hat{c}) + \int_{\underline{c}}^{\hat{c}} B_s'(F(\hat{c}))f(\hat{c})f(c)dc$$

Now at an interior equilibrium, $B_p(F(c^*)) = c^*$, so

$$\begin{aligned} \frac{dw}{d\hat{c}}\,|_{\hat{c}=c^*} &= (1-\lambda)v_0'(F(c^*))f(c^*) + \int_{\underline{c}}^{\hat{c}} B_s'(F(c^*))f(c^*)f(c)dc \\ &= \left[(1-\lambda)v_0'(F(c^*)) + B_s'(F(c^*))F(c^*)\right] f(c^*) \end{aligned} \tag{4.2}$$

So, there is an externality in the choice of cutoff that can be analytically decomposed into two parts. The first, measured by $(1-\lambda)v_0'(F(c^*))$ is the effect on any agent's expected utility of he decides not to participate of an increase in the cutoff (and thus participation probability) of others, and is positive from Lemma 1. This captured the information-pooling externality referred to above.

The second, measured by $B_s'(F(c^*))$ is the is the effect on any agent's benefit from participation of an increase in the cutoff (and thus participation probability) of others. Following the case of private values voting, we might call thus the pivot externality. The example above indicates that generally, B_s' can be positive or negative as p and therefore c varies. However, we are able to show that the information-pooling externality dominates the pivot externality, and so overall, the externality is positive.

The proof is simple. By definition, $B_s(p) \equiv (1-\lambda)(v_1(p) - v_0(p))$, so, differentiating this expression and rearranging,

$$(1-\lambda)v_0'(p) + B_s'(p)p \equiv (1-\lambda)(\ (1-p)v_0'(p) + pv_1'(p)) > 0 \tag{4.3}$$

Combining (4.2) and (4.3), noting that $p = F(c^*)$ at equilibrium, we see that $\frac{dU}{d\hat{c}}\,|_{\hat{c}=c^*} > 0$.. We can summarize our results as follows.

Proposition 3. *If $\lambda > \hat{\lambda}$, and the unique equilibrium is interior, the equilibrium cutoff is locally too high i.e. $\frac{dU}{d\hat{c}}\,|_{\hat{c}=c^*} < 0$, and so a small decrease in the cutoff $\hat{c}$ from c^* is always ex ante Pareto-improving..If $\lambda < \hat{\lambda}$, at any interior equilibrium, the cutoff is too low i.e. $\frac{dU}{d\hat{c}}\,|_{\hat{c}=c^*} > 0$ and so a small increase in the cutoff $\hat{c}$ from c^* is always ex ante Pareto-improving. If $\lambda = \hat{\lambda}$, the equilibrium cutoff may be too high or too low.*

The second part of Proposition 3 contrasts sharply with Borgers' results. His global result with private values establishes that it is never optimal to force agents to vote i.e. to raise $\hat{c}$ to $\overline{c}$. However, the proof of this result also establishes the local result that a small *decrease* in the cutoff $\hat{c}$ from c^* is always ex ante Pareto-improving. In this sense, Proposition 3 shows how a move from private values to common values reverses the nature of the inefficiency of voting equilibria.

4.2. Compulsory vs Voluntary Voting

The striking result of Borgers' paper is that in the case of private values, voluntary voting always dominates compulsory voting. That result is the global analog of the local inefficiency result stated in Proposition 3 above. By an application of the arguments in Borgers' paper, in particular the proof of Proposition 2, it is possible to show that when $\lambda > \hat{\lambda}$, compulsory voting is always desirable. That is, Borgers' result generalizes to the case when there are also a common element to payoffs, as long as these are sufficiently unimportant so that voters vote according to their private values.

What about the case where $\lambda < \hat{\lambda}$, so agents according to their signals? Consider two symmetric voting rules with cutoffs c^* and c^{**} such that $c^* < c^{**}$. Then, using (4.1), and integrating by parts, the difference between the expected payoffs at the two equilibria can be written as

$$w(c^{**})-w(c^*) = \int_{c^*}^{c^{**}} \left((1-\lambda)v_0'(F(c)) + F(c)B_s'(F(c))\right) f(c)dc + \int_{c^*}^{c^{**}} \left(B_s(F(c)) - c\right) f(c)dc \tag{4.4}$$

By Proposition 3, we know that the first integral is positive. However, the sign of the second integral is ambiguous as $B_s(p) - c$ is, in general, non-monotonic. This makes it impossible to obtain a general Pareto-ranking of equilibria. In particular, we cannot show that, in general, a Bayesian equilibrium with a higher cutoff value Pareto dominates a Bayesian equilibrium with a lower cutoff value. In general, it is also not possible to show that compulsory majority voting Pareto dominates Bayesian equilibrium outcomes with voluntary majority voting. However, the following results can be stated.

Proposition 4. *Assume that $\lambda < \hat{\lambda}$. Suppose that there are multiple voting equilibria as represented by cutoffs: $c_1 < .. < c_k < ... < c_m$. If either (a) $m \geq 2$, and $B_s(1) < \bar{c}$ or (b) $m \geq 3$, there is some k, $1 \leq k \leq m-1$, such that the the voting equilibrium c_{k+1} Pareto dominates the voting equilibrium c_k. If $B_s(1) \geq \bar{c}$, then $c_m = \bar{c}$, and this equilibrium Pareto-dominates equilibrium c_{m-1}* i.e. *starting at c_{m-1}, imposing compulsory voting is Pareto-improving.*

Proof. As $B_s(1) < \bar{c}$ remark that at $p = F(c_m)$, $B_s'(p) < 0$. As $m \geq 2$, it follows that there is at least one Bayesian equilibrium with cutoff c_k, for some k, $1 \leq k \leq m-1$ so that $B_s'(p) > 0$, $p = F(c_k)$ for some $k < m$. As $B_s'(p) > 0$, $p = F(c_k)$, for some $k < m$, $B_s(F(c)) > c$, $c \in (c_k, c_{k+1})$. Alternatively, suppose there exist at least three voting equilibria. Then, there is at least one voting equilibrium with cutoff c_k so that $B_s'(p) \geq 0$, $p = F(c_k)$ for some $k < m$. As $B_s'(p) \geq 0$, $p = F(c_k)$, for some $k < m$, $B_s(F(c)) \geq c$, $c \in (c_k, c_{k+1})$.So, in both cases, from (4.4), $U(c_{k+1}) > U(c_k)$ i.e. the voting equilibrium with the cutoff c_{k+1} Pareto dominates the voting equilibrium with cutoff c_k. Next, given that $B_s(1) \geq \bar{c}$, $c_m = \bar{c}$ follows directly from Proposition 1. By definition of c_m, c_{m-1}, $B_s(F(c)) \geq c$, $c \in (c_{m-1}, c_m)$. So, from (4.4), $U(\bar{c}) = U(c_m) > U(c_{m-1})$ i.e. compulsory voting Pareto-dominates voluntary voting equilibrium c_m. □

Can compulsory voting lead to a Pareto-improvement when $\bar{c}$ is not a voting equilibrium threshold? The following example shows that this a robust possibility. In this example, there is a unique equilibrium with $\hat{c} < \bar{c}$, and starting at this equilibrium, imposing compulsory voting leads to a strict Pareto-improvement.

Example 2 (Compulsory Voting May be Desirable).
The Example is the same as Example 1 i.e. $n = 3$, $\lambda = 0$, and uniform distribution of costs. Ex ante payoffs in this example can be computed from formula (4.1), which in this case simplifies to

$$U(p) = v_0(p) + pB(p) - \frac{1}{\bar{c}} \int_0^{p\bar{c}} cdc = u_0(p) + pB(p) - \bar{c}p^2/2$$

for any voting probability p. We already have computed a formula for $B(p)$ i.e. (3.4) in Example 1. Also, note that

$$v_0(p) = 0.5(1-p)^2 + 2p(1-p)q + p^2 q$$

So, using (3.4), in the above formula, we conclude that

$$U(p) = 0.5(1-p)^2 + 2p(1-p)q + p^2 q + p(q-0.5)[2p^2 q(1-q) + (1-p)^2] - \bar{c}p^2/2 \quad (4.5)$$

Now let $q = 0.75$, and ψ be the value of c for which the larger root of (3.5) is equal to 1. This will be the value for which $B(1) = \psi$, and $B(1) = (q-0.5)2q(1-q) = 0.09375$. Then from Figure 1, it is clear that for $\bar{c} > \psi$, there will be a unique equilibrium given by the smaller root to (3.5): the larger root is greater than 1 and so cannot be an equilibrium probability. So, take $\bar{c} = 0.0938$. Then $\alpha = \bar{c}/(q-0.5) = 0.3752$. In this case, there is a unique interior equilibrium with voting probability given by the smaller root to (3.6) i.e.

$$p^* = \frac{0.99947}{1.375} = 0.72689 \quad (4.6)$$

Now substituting $\bar{c} = 0.0938$ and $q = 0.75$ in (4.5), after some simplification, we get:

$$U(p) = 0.5 + 0.75p - 0.7969p^2 + 0.34375p^3 \quad (4.7)$$

So, $U(1) = 0.79685 > 0.75613 = U(p^*)$ i.e. compulsory voting leads to a strict Pareto-improvement. Indeed, from (4.7), it can be shown that $U(p)$ is everywhere increasing in $p \in [0, 1]$. ‖

5. Two Robustness Results

5.1. Multiplicative Preferences

In this section, we demonstrate that the results derived in the preceding sections under the assumption that w is additively separable, extends to the case when w is multiplicative i.e.

$$w(L : a^i, s) = [u(L, a^i)]^\lambda [v(L, s)]^{(1-\lambda)}, \ \lambda \in [0, 1].$$

As before, λ parameterizes the importance of private preferences relative to the state of world in determining overall preference over the two alternatives. Moreover, we assume that $u(L, a^i) = \overline{u}$ if $a^i = L$, and $\underline{u} > 0$ otherwise, and $v(A, s_A) = v(B, s_B) = \overline{v}$ and $\underline{v} > 0$ otherwise.

Again, one way of interpreting λ is to note that $\lambda > 0$ biases a given voter either in favour or against a given alternative, once a^i has been determined. For example, suppose that $a^i = A$. Then, up to a constant, payoffs over pairs (s, L) are of the form

	A	B
s_A	β	δ
s_B	$\beta\delta$	1

where $\beta = (\overline{u}/\underline{u})^\lambda > 1$, $\delta = (\underline{v}/\overline{v})^{1-\lambda} < 1$. The structure of information and the order of moves is identical to that described in section 2. We begin with the voting subgame when potential voters have made their participation decisions. As before a strategy for i is of the form $\gamma^i : \{\sigma_A, \sigma_B\} \times \{A, B\} \times \{1, ..., N-1\} \to \{A, B\}$. Then, we have:

Proposition 5. *Let $0 < \hat{\lambda} < 1$ be the unique root on $[0, 1]$ of*

$$f(\lambda) = \frac{\overline{u}^\lambda}{0.5\overline{u}^\lambda + 0.5\underline{u}^\lambda} - \frac{q\overline{v}^{1-\lambda} + (1-q)\underline{v}^{1-\lambda}}{0.5\overline{v}^{1-\lambda} + 0.5\underline{v}^{1-\lambda}}$$

Conditional on a fixed number $l + 1$ of participants , (i) if $\lambda \geq \tilde{\lambda}$, there is always a symmetric equilibrium of the voting sub-game where those participating vote to according to their PV; (ii) if $\lambda \leq \tilde{\lambda}$, there is always a symmetric equilibrium of the voting sub-game where those participating vote according to their signal. Moreover, these are the only symmetric equilibria in weakly dominant, pure strategies when $\lambda \neq \tilde{\lambda}$.

The reason for assuming that $\underline{u}, \underline{v} > 0$ should now be clear. If $\underline{u} = \underline{v} = 0$, for example, $f = 2[\overline{u} - q\overline{v}]$, which is either positive or negative independently of λ.This means that (for example, if $\overline{u} > q\overline{v}$) the only voting equilibrium is a private values one unless $\lambda = 0$, so there is a discontinuity at $\lambda = 0$.

As before, we study the symmetric participation equilibrium where any citizen votes iff c^i is below a cutoff $\tilde{c}$ where $p = F(\tilde{c})$ and every every voter is assumed to anticipate the relevant (unique) continuation symmetric voting equilibrium, as described in Proposition 6 and we compute the expected benefit to participation to some i, conditional on exactly l other agents participating, and agent i being pivotal. When l is even using an argument along the lines of the additive case, by computation, the gain from voting in the Common values signal voting equilibrium is

$$s(\lambda) = [0.5\overline{u}^{\lambda} + 0.5\overline{u}^{\lambda}][(q - 0.5)\, 0.5\overline{v}^{1-\lambda} + (1 - q - 0.5)\, 0.5\underline{v}^{1-\lambda}]$$

while the gain from voting in the PV equilibrium is

$$p(\lambda) = [0.5\overline{v}^{1-\lambda} + 0.5\underline{v}^{1-\lambda}][0.5\overline{u}^{\lambda} - 0.5\underline{u}^{\lambda}]$$

When l is odd, as in the additive case there is a zero gain from voting in the Common values signal equilibrium while in the gain from voting in the PV equilibrium remains $p(\lambda)$. Therefore, the expected benefit to i of voting, given a probability p of participation by all other voters, is

$$B(p) = \begin{cases} B_p(p) = \lambda \sum_{l=0}^{n-1} f(l : p, n-1)\pi(l; 0.5)p(\lambda) & \lambda > \hat{\lambda} \\ B_s(p) = (1-\lambda) \sum_{l=0}^{n-1} f(l : p, n-1)\pi(l; q)I(l)s(\lambda) & \lambda < \hat{\lambda} \end{cases} \tag{5.1}$$

Given a fixed λ, $s(\lambda)$, $p(\lambda)$ are just constants. So, using the same arguments as in the additive case, the main results of the paper, namely Propositions 2 and 3, carry over directly to the multiplicative case.

5.2. An Unknown Number of Voters

So far, for clarity of exposition, we have assumed that at the voting stage, every participant can observe the participation outcome i.e. the number of agents who have decided to vote. As remarked above, this restricts the applicability of the model. Here, we consider a different information structure where at Step 1, no agent observes m, the number of voters. This assumption obviously describes decisions made via the ballot box, or in large public meetings, rather than small committees. So, we replace Steps 0 and 1 by:

Step 1. All i who have decided to participate, observe a^i, σ^i, and vote either for A or for B.

It is easy to see that this different information structure makes no difference to the equilibrium outcome. This is because by the arguments of Proposition 1, if $\lambda \leq \hat{\lambda}$, there exists a symmetric voting equilibrium where agents vote according to their signals, *whatever* m, and if $\lambda \geq \hat{\lambda}$, there exists a symmetric voting equilibrium where agents vote according to their private preference, *whatever* m.

To be more precise about this, note that a voting strategy for i is now a map γ^i : $\{\sigma_A, \sigma_B\} \times \{A, B\} \rightarrow \{A, B\}$. Symmetric strategies are $\gamma^i = \gamma$, all $i \in N$ as before. So, Proposition 1 carries over to this case i.e. if $\lambda \leq \hat{\lambda}$, there is an equilibrium γ^* where all participants vote according to their signals. Then, if $\lambda \geq \hat{\lambda}$, there is an equilibrium γ^* where all participants vote according to their private preference. Moreover, these are the only symmetric equilibria in admissible, pure strategies when $\lambda \neq \hat{\lambda}$. Therefore, participation equilibrium is identical to the case when m is observed at the voting stage, and consequently, Propositions 2-4 apply directly.

6. The Role of Timing

6.1. Conditioning on Partial Preference Information

So far, we have assumed that agents do not observe either their signal or private value before they take the participation decision. This is restrictive; for example, as pointed out above, a^i could be interpreted as being observed costlessly though introspection, and σ^i could be interpreted as being observed costlessly though observation of mass media (television, newspapers, etc.), without having to commit to a costly participation decision. Here, we argue that if agents can observe *either* a^i or σ^i (but not both) before making the participation decision, there is still a voting and participation equilibrium exactly as described in Propositions 1 and 2. Thus, if this equilibrium occurs, it will still be inefficient as described in Proposition 3.

Suppose for example that agents observe c^i, a^i before the participation decision. If the voting subgame outcome is as described in Proposition 1, *and* all other agents $j \neq i$ follow a cutoff rule of participating if $c^j \leq \hat{c}$, where $\hat{c}$ is independent of a^j, then the gain to participation for any i is precisely (3.3), whatever a^i. So, it is a best response for i to also choose a cutoff rule independent of i for participation. This argument applies also if c^i, σ^i is observed before the participation decision.

6.2. Conditioning on All Preference Information

Now, we assume the sequence of events is as follows.

Step 0. The state of the world is realized, and each $i \in N$ privately observes c^i, a^i,σ^i and decides whether to participate or not.

Step 1. All i who have decided to participate vote either for A or for B.

Step 2. The alternative with the most votes is selected. If both A, B get equal numbers of votes, each is selected with probability 0.5.

We also continue to assume additive preferences. First, note that conditional on participation, the symmetric voting equilibrium is just as described in Proposition 1. But now, the participation decision can be conditioned on c^i, a^i,σ^i rather than just c^i. This potentially complex participation decision can, however, be simplified by noting (i) there is a gain to participation only in the event that an agent is pivotal; (ii) in this event, and the gain to participation depends on whether the PV and the signal of the agent *agree* ($\sigma^i = S_K,\ a^i = K$) or *disagree* ($\sigma^i = S_K,\ a^i = L$), with the gain being larger in the first case. So, it seems likely that there will be an equilibrium where the cutoff cost below which participation takes place will depend on whether signals agree (cutoff c_A) or disagree (cutoff c_D), with $c_A > c_D$.

However, the characterization of c_A, c_D is difficult, because at the voting stage, every participating agent rationally anticipates that any other may have her private preference and the signal of the agent agree (in which case, that other agent is voting with her signal, whatever the nature of the voting equilibrium), or disagree. We provide a complete characterization in the case of just two agents. For simplicity, we also assume $\underline{c} = 0$, and $\overline{c}$ large enough so that there is always an interior solution for c_A. Then we have:

Proposition 6. *Assume $n = 2$. If $\lambda < \hat{\lambda}$, then $c_A > c_D \geq 0$* solve

$$\begin{aligned} -F(c_A)(1-\eta)\lambda 0.5 + (1-F(c_A))((1-\lambda)q - 0.5) &= c_D \\ F(c_A)(1-\eta)\lambda 0.5 + (1-F(c_A))((1-\lambda)(q-0.5) + \lambda 0.5) &= c_A \end{aligned}$$

If $\lambda > \hat{\lambda}$, then, $c_A > c_D \geq 0$ solve

$$\begin{aligned} &0.5F(c_D)(1-\eta)\lambda 0.5 + (F(c_A) - 0.5F(c_D))\eta((1-\lambda)(0.5-\chi) + \lambda 0.5) \\ &+(1-F(c_A))((1-\lambda)(1-q) + \lambda - 0.5) = c_D \end{aligned}$$

$$\begin{aligned} &0.5F(c_D)\eta((1-\lambda)(\chi - 0.5) + \lambda 0.5) - (F(c_A) - 0.5F(c_D))(1-\eta)\lambda 0.5 \\ &+(1-F(c_A))((1-\lambda)q + \lambda - 0.5) = c_A \end{aligned}$$

Next, we extend the characterization of the efficiency of participation equilibria in the case of two agents. Let the ex ante payoff to any citizen (i.e. prior to observing

$a^i, \sigma^i, c^i)$, conditional on a given cutoff strategy $(\hat{c}_D, \hat{c}_A)$ be denoted as $w(\hat{c}_D, \hat{c}_A)$. The following proposition characterizes the efficiency of participation equilibria.

Proposition 7. *If $\lambda > \hat{\lambda}$, and $1 > c_A > c_D > 0$, (a) the equilibrium cutoff c_A is locally too high and a small decrease in $\hat{c}_A$ from c_A increases $w(\hat{c}_D, \hat{c}_A)$ and (b) the equilibrium cutoff c_D is locally inefficient and a small change in $\hat{c}_D$ from c_D increases $w(\hat{c}_D, \hat{c}_A)$. If $\lambda < \hat{\lambda}$, and $1 > c_A > c_D > 0$, (a) the equilibrium cutoff c_A is locally too low and a small increase in in $\hat{c}_A$ from c_A increases $w(\hat{c}_D, \hat{c}_A)$ and (b) the equilibrium cutoff c_D is locally too high and a small decrease in $\hat{c}_D$ from c_D increases $w(\hat{c}_D, \hat{c}_A)$.*

7. Conclusion

In this paper, we have shown that in a model of costly voting where preferences are a convex combination of a private values component and a common values component, when the weight on the Common values component is sufficiently high, the nature of the inefficiency of voting equilibrium identified in Borgers (2004) is reversed: in the vicinity of a Bayesian equilibrium, higher participation is *always* Pareto-improving. In addition, we have also shown that Pareto ranked participation equilibria may exist and moreover, compulsory majority voting can Pareto dominate voluntary majority voting. The key behind all the results in this paper lies in the finding that there are two different externalities at work: the negative "pivot" externality identified by Borgers (2004) and the positive information externality. In the vicinity of a Bayesian equilibrium, the positive informational externality may outweigh the negative "pivot" externality implying that there is too little participation in the voting process. We, then, examine the role of timing in obtaining our results and show that when voters condition participation on information, the nature of inefficiency of voting equilibria change quite dramatically.

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A. Appendix

Proof of Proposition 1. (i) We start with the additive case. The key to the proof is that (given l other voters), voter i is pivotal only in two possible events:
Case 1: l even and $\frac{l}{2}$ voters vote for A while the other $\frac{l}{2}$ voters vote for B.
Case 2: l odd and (i) $\frac{l+1}{2}$ voters have voted for A, and $\frac{l-1}{2}$ for B, or (ii) vice versa.

So, in general, to demonstrate an equilibrium, we only need to show that the relevant voting strategy played by i is a best response to the same voting strategy played by the other l voters in each of these two cases.

(ii) The private value equilibrium is constructed as follows. W.l.o.g, assume that Case 1 applies (the argument is the same in case 2), and assume all other $j \neq i$ vote with their private values. Voter $i's$ payoff to voting with her private value is lowest when $a^i \neq \sigma^i$. In this case, as she is pivotal , if she votes with her private value, the alternative chosen is a^i. We now calculate the expected payoffs to these two choices, conditional on the available relevant information at the time of voting, namely σ^i, a^i and the event that i is pivotal, which we denote by "piv".

$$\begin{aligned} E_s[w(a^i &: a^i, s) \left|\text{piv}, \sigma^i, a^i\right] = \lambda u(a^i, a^i) + (1-\lambda).E_s[v(a^i, s) \left|\text{piv}, \sigma^i, a^i\right] \quad \text{(A.1)} \\ &= \lambda + (1-\lambda).[q0. + (1-q)1] \\ &= \lambda + (1-\lambda).(1-q) \end{aligned}$$

where "piv" refers to the event that i is pivotal, given the strategies of the other voters. Note that $E_s[v(a^i, s)\,|\text{piv}, \sigma^i] = 1-q$ as i is voting contrary to his signal. Alternatively, if she votes with her signal, her expected payoff is

$$\begin{aligned} E_s[w(\sigma^i &: a^i, s) \left|\text{piv}, \sigma^i, a^i\right] = \lambda u(\sigma^i, a^i) + (1-\lambda).E_s[v(\sigma^i, s) \left|\text{piv}, \sigma^i, a^i\right] \quad \text{(A.2)} \\ &= \lambda.0 + (1-\lambda).[q1. + (1-q)0] \\ &= (1-\lambda).q \end{aligned}$$

So, she prefers to vote with her private value if $\lambda \geq (q-0.5)/q = \hat{\lambda}$.

(iii) The common value signal equilibrium is constructed as follows.

Case 1: : l even and $\frac{l}{2}$ voters vote for A while the other $\frac{l}{2}$ voters vote for B. Conditional on this event, $E_s[v(\sigma^i, s)\,|\text{piv}, \sigma^i] = q$, $E_s[v(a^i, s)\,|\text{piv}, \sigma^i] = 1-q$, so the condition for voting according to her common value signal is the reverse of $\lambda \geq \hat{\lambda}$ i.e.$\lambda \leq \hat{\lambda}$.

Case 2(i) Assume w.l.o.g. that $\sigma^i = A$. So, $\frac{l+1}{2}$ voters have voted for A, and $\frac{l-1}{2}$ for B. In this case, i infers if he is pivotal, that there are two more signals in favor of q than against[14]. So,

$$E_s[v(\sigma^i, s) \left|\text{piv}, \sigma^i, a^i\right.] = \chi = \frac{q^2}{q^2 + (1-q)^2}, \; E_s[v(a^i, s) \left|\text{piv}, \sigma^i, a^i\right.] = 1 - \chi$$

Then, by formula (A.2), replacing q by χ, the expected payoff to voting according to the common value signal is thus $(1-\lambda)\chi$. From formula (A.1), replacing q by χ, the expected payoff to voting according to private value is thus $\lambda + (1-\lambda).(1-\chi)$.

Case 2(ii). Assume w.l.o.g. that $\sigma^i = A$. So, $\frac{l+1}{2}$ voters have voted for B, and $\frac{l-1}{2}$ for A. In this case, i infers that if he is pivotal, there are equal numbers of signals in favor A and B, so:

$$E_s[v(\sigma^i, s) \left|\text{piv}, \sigma^i, a^i\right.] = E_s[v(a^i, s) \left|\text{piv}, \sigma^i, a^i\right.] = 0.5$$

Then, by formula (A.2), replacing q by 0.5, the expected payoff to voting according to the common value signal is thus $(1-\lambda)0.5$. From formula (A.1), replacing q by 0.5, the expected payoff to voting according to private value is thus $\lambda + (1-\lambda).(1-0.5)$.

Conditional on l odd, i does not know which of these 2(i) and 2(ii) has occurred when he decides whether to vote with his signal or his private value. But, as the signals are (unconditionally) correlated, the first sub-case is more likely than the second: in fact, it is simple to calculate that the probability of 2(i) is $\eta = q^2 + (1-q)^2 > 0.5$, and the probability of 2(ii) is $1-\eta$.

The argument is the following. Given that $\sigma^i = A$, the probabilities of 2(i) and 2(ii) are

$$\Pr(i) = qC + (1-q)C', \; \Pr(ii) = qC' + (1-q)C$$

respectively, where $C = f(\frac{l+1}{2} : l, q)$, $C' = f(\frac{l-1}{2} : l, q)$. So, the relative probability of sub-case (i) is

$$\frac{\Pr(i)}{\Pr(i) + \Pr(ii)} = \frac{qC + (1-q)C'}{C + C'} = q^2 + (1-q)^2 = \eta$$

So, the overall expected gain to i from voting according to his signal, rather than for his personal preference, is

$$\begin{aligned} \Delta &= \eta[(1-\lambda)\chi - \lambda - (1-\lambda)\chi] + (1-\eta)[0.5(1-\lambda) - \lambda - (1-\lambda)0.5] \\ &= (1-\lambda)(2q-1) - \lambda \end{aligned}$$

[14] Formally, χ is equal to the posterior probability that the state is (say) A, given that there are $l+1$ (resp. $l-1$) signals in favour of A (resp. B). Using Bayes' rule, after some simplification, we get the formula in the text.

So, $\Delta \geq 0$ if $\lambda \leq \hat{\lambda}$.

(iv) Fix a value of l, the number of other voters have chosen to participate. For any voter i and both the PV component of his preferences and his signal agree, he is indifferent about voting according to either his PV or signal and moreover, either of the two preceding choices weakly dominates all other pure strategies. When the PV component and the signal disagree, the computations reported in (i) and (ii), taken together, imply that when $\lambda \leq \hat{\lambda}$, voting according to the Common values signal weakly dominates all other pure strategies while if $\lambda \geq \hat{\lambda}$, voting according to PV weakly dominates all other pure strategies.□

Proof of Lemma 1. (i) $\lambda > \hat{\lambda}$. In this case all agents rationally anticipate that voting according to private values will take place in the voting subgame. The expected gain to voting is the probability that i is pivotal, times the expected benefit to voting, given that he is pivotal. There are then two subcases.

(a) l *even.* In this case, for i to be pivotal, $l/2$ other voters vote for each of A and B according to their private preference, so this event occurs with probability $f(\frac{l}{2} : l, 0.5)$. So, the expected payoff to not voting for i is 0.5. If i votes, he gets alternative a^i with probability 1, which gives him an expected payoff of $(1-\lambda)0.5+\lambda$. The difference is $\lambda 0.5$.

(b) l *odd.* In this case, for i to be pivotal, $(l+1)/2$ other voters vote for L, and $(l-1)/2$ other voters vote for K. If $a^i = L$, the gain to voting is zero as the outcome is not affected. If $a^i = K$, then the payoff to voting is 0.5 (as the two alternatives tie), and the payoff to not voting is $(1-\lambda)0.5$ (as L is chosen with probability 1). Again, the gain to voting is $\lambda 0.5$, and the probability of this event is $f(\frac{l-1}{2} : l, 0.5)$.

(ii) $\lambda < \hat{\lambda}$. In this case all agents rationally anticipate that voting according to signals will take place in the voting subgame. The expected gain to voting is the probability that i is pivotal, times the expected benefit to voting, given that he is pivotal. There are then two subcases.

(a) l *even.* In this case, for i to be pivotal, $l/2$ other voters vote for each of A and B according to their signal. As the signals are unconditionally correlated, this event occurs with probability $f(\frac{l}{2} : l, q)$. So, the expected payoff to not voting for i is 0.5. If i votes, he gets alternative σ^i with probability 1, which gives him an expected payoff of $(1-\lambda)q + \lambda 0.5$. The difference is $(1-\lambda)(q-0.5)$.

(b) l *odd.* In this case, for i to be pivotal, $(l+1)/2$ other voters vote for L, and $(l-1)/2$ other voters vote for K. If $\sigma^i = L$, the gain to voting is zero as the outcome is not affected. If $\sigma^i = K$, then voter i (conditioning on his own signal and the event that he is pivotal) assesses both states of the world as equally likely. So, expected payoff, conditioning only on the fact that he is pivotal, is the same whichever alternative is chosen i.e. his gain to voting is zero. □

Proof of Lemma 2. First,

$$v_1(p') - v_1(p) = \sum_{m=0}^{n-1}(f(m : p', n-1) - f(m : p, n-1))u(m+1).$$

Now, for $p' > p$, $\{f(m : p', n-1)\}_{m=0}^{n-1}$first-order stochastically dominates $\{f(m : p, n-1)\}_{m=0}^{n-1}$. Moreover, $v(m{+}1)$ is monotonically increasing in m. So, from Rothschild and Stiglitz(1970), we have $v_1(p') - v_1(p) \geq 0$. The proof is the same for $v_0(p)$. □

Proof of Proposition 5. This follows closely the proof of Proposition 1. The PV equilibrium is constructed as follows. W.l.o.g, assume that Case 1 applies (the argument is the same in case 2). Voter $i's$ payoff to voting with her private preference is lowest when $a^i \neq \sigma^i$. In this case, as she is pivotal , if she votes with her private value, the alternative chosen is a^i. Note that in this case, her expected payoff is

$$\begin{aligned} E[w(a^i \quad &: \quad a^i, s) \left|\text{piv}, \sigma^i, a^i\right] = [u(a^i, a^i)]^\lambda E[(v(a^i, s))^{(1-\lambda)} \left|\text{piv}, \sigma^i, a^i\right] \\ &= \quad \overline{u}^\lambda [0.5\overline{v}^{1-\lambda} + 0.5\underline{v}^{1-\lambda}] \end{aligned}$$

Alternatively, if she votes with her common value signal, her expected payoff is

$$\begin{aligned} E[w(\sigma^i \quad &: \quad a^i, s) \left|\text{piv}, \sigma^i, a^i\right] = E[u(\sigma^i, a^i)]^\lambda E[(v(\sigma^i, s))^{(1-\lambda)} \left|\text{piv}, \sigma^i, a^i\right] \\ &= \quad [0.5\overline{u}^\lambda + 0.5\underline{u}^\lambda][q\overline{v}^{1-\lambda} + (1-q)\underline{v}^{1-\lambda}] \end{aligned}$$

Then, it is easily checked that the gain to voting with preferences is

$$\begin{aligned} E[w(a^i \quad : \quad a^i, s) \left|\text{piv}, \sigma^i, a^i\right] - E[w(\sigma^i : a^i, s) \left|\text{piv}, \sigma, a^i\right] = \\ [0.5\overline{v}^{1-\lambda} + 0.5\underline{v}^{1-\lambda}][0.5\overline{u}^\lambda + 0.5\underline{u}^\lambda]f(\lambda) \end{aligned}$$

So, an equilibrium exists with voting with preferences iff $f(\lambda) \geq 0$. A similar argument, following closely the additive case, establishes that an equilibrium exists with voting with signals iff $f(\lambda) \leq 0$. Finally, it is easily checked that $f(0) < 0$, $f(1) > 0$, and that f is strictly increasing in λ. So, $f(\lambda)$ has a unique root, as claimed. □

Proof of Proposition 6. (a) $\lambda < \hat{\lambda}$. In this case, the unique voting equilibrium in the subgame is where voting is according to signals. W.l.o.g., we calculate the expected benefit to participation for agent 1. Also, note that the (unconditional) probability that voter 2 participates and $\sigma^2 \neq a^2$ is $p_D = 0.5F(c_D)$, and that the (unconditional) probability that voter 2 participates and $\sigma^2 = a^2$ is $p_A = F(c_A) - 0.5F(c_D)$.

Case 1: $a^1 = \sigma^1$.

Let $\overline{p} = p_D + p_A = F(c_A)$. Here, the payoffs to not voting and voting are

$$\begin{aligned} u_n &= \overline{p}\,(\eta((1-\lambda)\chi+\lambda)+(1-\eta)(1-\lambda)0.5)+(1-\overline{p})0.5 \\ u_v &= \overline{p}\,(\eta((1-\lambda)\chi+\lambda)+(1-\eta)0.5)+(1-\overline{p})((1-\lambda)q+\lambda) \end{aligned}$$

The explanation is as follows. If 1 does not vote, with probability $\overline{p}$ 2 will participate and thus make the decision by voting according to his signal. In this case, with probability η, $\sigma^1 = \sigma^2$, in which case 1's payoff will be $(1-\lambda)\chi+\lambda$, as the better alternative is chosen according to u with probability 1 (as $\sigma^2 = \sigma^1 = a^1$), and the better alternative is chosen according to v with probability χ as $(\Pr(s = K\,|\sigma^2 = \sigma^1 = K|) = \chi$. Alternatively, with probability $1-\eta$, $\sigma^1 \neq \sigma^2$, in which case 1's payoff will be $(1-\lambda)0.5 + \lambda.0 = (1-\lambda)0.5$, as the better alternative is chosen according to u with probability 0 (as $\sigma^2 \neq \sigma^1 = a^1$), and the better alternative is chosen according to v with probability 0.5 (as $(\Pr(s = K\,|\sigma^2 = K,\ \sigma^1 = L|) = 0.5$. Alternatively, with probability $1-\overline{p}$, 2 will not participate, and thus the decision is made by random choice of A or B, giving 1 an overall expected payoff of 0.5. The formula for u_v is derived in a similar way, noting in particular (i) that if both vote, with probability $1-\eta$ the signals of both voters disagree, in which case there is a tie and the decision is by random selection, giving rise to the term $\overline{p}(1-\eta)0.5$; (ii) if only 1 votes, she will clearly vote with both her signal and PV, as they agree. Thus, the benefit of voting when $a^1 = \sigma^1$ is

$$b_A = u_v - u_n = \overline{p}(1-\eta)\lambda 0.5 + (1-\overline{p})((1-\lambda)\,(q-0.5)+\lambda-0.5)$$

Case 2: $a^1 \neq \sigma^1$.

Here, the payoffs to not voting and voting are

$$\begin{aligned} u_n &= \overline{p}(\eta(1-\lambda)\chi+(1-\eta)((1-\lambda)0.5+\lambda))+(1-\overline{p})0.5 \\ u_v &= \overline{p}(\eta(1-\lambda)\chi+(1-\eta)0.5)+(1-\overline{p})(1-\lambda)q \end{aligned}$$

Here we note in particular, that if 2 does not participate and 1 does, 1 will vote with his signal, rather[15] than his PV. Thus, the benefit of voting when $a^1 \neq \sigma^1$ is

$$b_D = u_v - u_n = -\overline{p}(1-\eta)\lambda 0.5 + (1-\overline{p})((1-\lambda)q - 0.5)$$

Finally, c_A, c_D are determined by the equations $b_A = c_A,\ b_D = c_D$.

[15] This is because 1 is pivotal, so the the gain to voting with his signal over his PV is $(1-\lambda)q-\lambda$. This is positive if $\lambda < q/(1+q)$. But by assumption, $\lambda < \hat{\lambda} = (q-0.5)/q$, and by simple calculation, $(q-0.5)/q < q/(1+q)$ for $q \in (0.5, 1)$.

(b) $\lambda > \hat{\lambda}$. In this case, the unique voting equilibrium in the subgame is where voting is according to PVs. W.l.o.g., we calculate the expected benefit to participation for agent 1.

Case 1: $a^1 = \sigma^1$.

Here, the payoffs to not voting and voting are

$$\begin{aligned} u_n &= p_D\left(\eta(1-\lambda)(1-\chi) + (1-\eta)((1-\lambda)0.5+1)\right) \\ &\quad + p_A(\eta((1-\lambda)\chi+\lambda) + (1-\eta)0.5(1-\lambda)) + (1-p_D-p_A)0.5 \\ u_v &= p_D(\eta 0.5 + (1-\eta)(0.5(1-\lambda)+\lambda)) \\ &\quad + p_A(\eta((1-\lambda)\chi+\lambda) + (1-\eta)0.5) + (1-p_D-p_A)((1-\lambda)q+\lambda) \end{aligned}$$

So, the gain to participation is

$$\begin{aligned} b_A &= p_D\eta((1-\lambda)(\chi-0.5)+\lambda 0.5) - p_A(1-\eta)\lambda 0.5 \\ &\quad + (1-p_D-p_A)((1-\lambda)q+\lambda-0.5) \end{aligned}$$

Case 2: $a^1 \neq \sigma^1$.

Here, the payoffs to not voting and voting are

$$\begin{aligned} u_n &= p_D(\eta((1-\lambda)(1-\chi)+\lambda) + (1-\eta)(1-\lambda)0.5) \\ &\quad + p_A(\eta((1-\lambda)\chi + (1-\eta)(0.5(1-\lambda)+\lambda)) + (1-p_D-p_A)0.5 \\ u_v &= p_D(\eta((1-\lambda)(1-\chi)+\lambda) + (1-\eta)0.5) \\ &\quad + p_A(\eta 0.5 + (1-\eta)(0.5(1-\lambda)+\lambda)) + (1-p_D-p_A)((1-\lambda)(1-q)+\lambda) \end{aligned}$$

Thus, the benefit of voting when $a^1 \neq \sigma^1$ is

$$\begin{aligned} b_D &= u_v - u_n = p_D(1-\eta)\lambda 0.5 + p_A\eta((1-\lambda)(0.5-\chi)+\lambda 0.5) \\ &\quad + (1-p_D-p_A)((1-\lambda)(1-q)+\lambda-0.5) \end{aligned}$$

Finally, c_A, c_D are determined by the equations $b_A = c_A,\ b_D = c_D$. □

Proof of Proposition 7. (a) $\lambda > \hat{\lambda}$. As before, we begin by defining the ex ante payoff to any citizen (i.e. prior to observing a^i, σ^i, c^i), conditional on a given cutoff strategy $(\hat{c}_D, \hat{c}_A)$. First, note that as $\lambda > \hat{\lambda}$, PV voting occurs. In this case, his ex-ante payoff is

$$\begin{aligned} w(\hat{c}_D, \hat{c}_A) &= 0.5 + \int_0^{\hat{c}_D} \left[\frac{b_D(\hat{c}_D, \hat{c}_A) + b_A(\hat{c}_D, \hat{c}_A)}{2} - c\right] f(c)dc \\ &\quad + \int_{\hat{c}_D}^{\hat{c}_A} \left[b_A(\hat{c}_D, \hat{c}_A) - c\right] f(c)dc \end{aligned}$$

So,

$$\frac{\partial w}{\partial \hat{c}_A} = \int_0^{\hat{c}_D} \frac{\frac{\partial b_D(\hat{c}_D,\hat{c}_A)}{\partial c_A} + \frac{\partial b_A(\hat{c}_D,\hat{c}_A)}{\partial c_A}}{2} f(c)dc$$
$$+ \int_{\hat{c}_D}^{\hat{c}_A} \frac{\partial b_A(\hat{c}_D,\hat{c}_A)}{\partial c_A} f(c)dc) + (b_A(\hat{c}_D,\hat{c}_A) - \hat{c}_A)$$

Now, $b_A(c_D, c_A) = c_A$ and therefore,

$$\frac{\partial w}{\partial \hat{c}_A}\Big|_{(\hat{c}_D = c_D, \hat{c}_A = c_A)} = \int_0^{c_D} \frac{\frac{\partial b_D(c_D,c_A)}{\partial c_A} + \frac{\partial b_A(c_D,c_A)}{\partial c_A}}{2} f(c)dc + \int_{c_D}^{c_A} \frac{\partial b_A(c_D,c_A)}{\partial c_A} f(c)dc)$$

Using the expression $b_D(c_D, c_A)$, derived in the proof for proposition 6, by computation, it follows that

$$\frac{\partial b_D(c_D,c_A)}{\partial c_A} = \left[\eta(1-\lambda)(0.5-\chi) + \frac{\eta\lambda}{2} - (1-\lambda)(1-q) - \lambda + \frac{1}{2}\right] F'(c_A)$$

which after simplification, yields

$$\frac{\partial b_D(c_D,c_A)}{\partial c_A} = [-\lambda(1-q)]\, F'(c_A) < 0$$

Using the expression $b_A(c_D, c_A)$, derived in the proof for proposition 6, by computation, it follows that

$$\frac{\partial b_A(c_D,c_A)}{\partial c_A} = \left[-\frac{(1-\eta)\lambda}{2} - ((1-\lambda)q + \lambda - 0.5)\right] F'(c_A)$$

Now, as $q > 0.5$, $(1-\lambda)q+\lambda > 0.5$ and therefore, $\frac{\partial b_A(c_D,c_A)}{\partial c_A} < 0$. But, then, it immediately follows that $\frac{\partial w}{\partial \hat{c}_A}\big|_{(\hat{c}_D=c_D,\hat{c}_A=c_A)} < 0$.

Next, note that

$$\frac{\partial w}{\partial \hat{c}_D} = \int_0^{\hat{c}_D} \frac{\frac{\partial b_D(\hat{c}_D,\hat{c}_A)}{\partial \hat{c}_D} + \frac{\partial b_A(\hat{c}_D,\hat{c}_A)}{\partial \hat{c}_D}}{2} f(c)dc + (\frac{b_D(\hat{c}_D,\hat{c}_A) + b_A(\hat{c}_D,\hat{c}_A)}{2} - \hat{c}_D)$$
$$+ \int_{\hat{c}_D}^{\hat{c}_A} \frac{\partial b_A(\hat{c}_D,\hat{c}_A)}{\partial \hat{c}_D} f(c)dc - (b_A(\hat{c}_D,\hat{c}_A) - \hat{c}_D)$$

Now, $b_D(c_D, c_A) = c_D$ and as $c_A > c_D$, $b_A(c_D, c_A) > c_D$ and as $c_A > c_D$, $b_A(c_D, c_A) > c_D$ and it follows that

$$(\frac{b_D(c_D,c_A) + b_A(c_D,c_A)}{2} - c_D) - (b_A(c_D,c_A) - c_D)$$
$$= -\frac{b_A(c_D,c_A) - c_D}{2} < 0.$$

Therefore,

$$\frac{\partial w}{\partial \hat{c}_D}\Big|_{(\hat{c}_D=c_D,\hat{c}_A=c_A)} = \int_0^{c_D} \frac{\frac{\partial b_D(c_D,c_A)}{\partial c_D}+\frac{\partial b_A(c_D,c_A)}{\partial c_D}}{2} f(c)dc + \int_{c_D}^{c_A} \frac{\partial b_A(c_D,c_A)}{\partial c_D} f(c)dc - \frac{b_A(c_D,c_A)-c_D}{2}$$

By computation, $\frac{\partial b_A(c_D,c_A)}{\partial c_D} = \left[-(1-\lambda)(q-\eta(\chi-0.5)) - \lambda\left(1-\frac{\eta}{2}\right)+\frac{1}{2}\right]F'(c_D)$. Also note that $q-\eta(\chi-0.5)=0.5$ while $1-\frac{\eta}{2}>0.5$ and therefore, $(1-\lambda)(q-\eta(\chi-0.5))+\lambda\left(1-\frac{\eta}{2}\right) > \frac{1}{2}$ which implies that $\frac{\partial b_A(c_D,c_A)}{\partial c_D} < 0$. However, $\frac{\partial b_D(c_D,c_A)}{\partial c_D}+\frac{\partial b_A(c_D,c_A)}{\partial c_D} = \left(\frac{2q-1}{2}+\lambda(1-q)\right)F'(c_D) > 0$. It follows that sign of $\frac{\partial w}{\partial \hat{c}_D}\big|_{(\hat{c}_D=c_D,\hat{c}_A=c_A)}$ is ambiguous although, in general; it is different from zero.

(b) $\lambda < \hat{\lambda}$. We begin by computing the payoff to any agent i if he does not participate given that the other agent participates with participation probabilities $(\hat{p}_D,\hat{p}_A)$. Let $v(m)$ be defined as in section 4 and let $v_0(\hat{p}_D,\hat{p}_A)$ be the expected values of $v(m)$ given non-participation for a given citizen i, given also that all $j\neq i$ participate with probabilities $(\hat{p}_D,\hat{p}_A)$. This is

$$v_0(\hat{p}_D,\hat{p}_A) = \frac{1}{2}v(0)\left[(1-\hat{p}_D)+(1-\hat{p}_A)\right]+\frac{1}{2}v(1)\left[\hat{p}_D+\hat{p}_A\right]$$

Let $[(1-\hat{p}_D)(1-\hat{p}_A)v(0)+\hat{p}_A(1-\hat{p}_D)v(1)+\hat{p}_D(1-\hat{p}_A)v(1)+\hat{p}_A\hat{p}_Dv(2)] = r(\hat{p}_D,\hat{p}_A)$. Let $v_{1,D}(\hat{p}_D,\hat{p}_A)$ be the expected values of $v(m)$ given participation, with probabilities $\hat{p}_D$, for a given citizen i, with $a^i\neq\sigma^i$, given also that all $j\neq i$ participate with probabilities $(\hat{p}_D,\hat{p}_A)$. This is

$$v_{1,D}(\hat{p}_D,\hat{p}_A) = \frac{1}{2}\left[(1-\hat{p}_D)^2 v(0)+2\hat{p}_D(1-\hat{p}_D)v(1)+\hat{p}_D^2 v(2)\right]+\frac{1}{2}r(\hat{p}_D,\hat{p}_A)$$

Let $v_{1,A}(\hat{p}_D,\hat{p}_A)$ be the expected values of $v(m)$ given participation, with probabilities $\hat{p}_A$, for a given citizen i, with $a^i=\sigma^i$, given also that all $j\neq i$ participate with probabilities $(\hat{p}_D,\hat{p}_A)$. This is

$$v_{1,A}(\hat{p}_D,\hat{p}_A) = \frac{1}{2}\left[(1-\hat{p}_A)^2 v(0)+2\hat{p}_A(1-\hat{p}_A)v(1)+\hat{p}_A^2 v(2)\right]+\frac{1}{2}r(\hat{p}_D,\hat{p}_A)$$

Let $v_1(\hat{p}_D,\hat{p}_A)$ be the expected values of $v(m)$ given participation, with probabilities $(\hat{p}_D,\hat{p}_A)$, for a given citizen i, given also that all $j\neq i$ participate with probabilities $(\hat{p}_D,\hat{p}_A)$. This is

$$v_1(\hat{p}_D,\hat{p}_A) = \frac{1}{2}v_{1,D}(\hat{p}_D,\hat{p}_A)+\frac{1}{2}v_{1,A}(\hat{p}_D,\hat{p}_A)$$

By computation, it follows that $v_0(\hat{p}_D,\hat{p}_A)$, $v_{1,D}(\hat{p}_D,\hat{p}_A)$, $v_{1,A}(\hat{p}_D,\hat{p}_A)$ (and therefore, $v_1(\hat{p}_D,\hat{p}_A)$) are all increasing in both $(\hat{p}_D,\hat{p}_A)$.

Now define the ex ante payoff to any citizen (i.e. prior to observing a^i, σ^i, c^i), conditional on a given cutoff strategy $(\hat{c}_D, \hat{c}_A)$ when $\lambda \leq \underline{\lambda}$. Given a cutoff strategy $(\hat{c}_D, \hat{c}_A)$, let $\hat{p}_D = 0.5F(\hat{c}_D)$ and $\hat{p}_A = F(\hat{c}_A) - 0.5F(\hat{c}_D)$. In this case, his ex ante payoff is

$$\begin{aligned} w(\hat{c}_D, \hat{c}_A) &= \lambda 0.5 + (1-\lambda) v_0(\hat{p}_D, \hat{p}_A) \\ &\quad + \int_0^{\hat{c}_D} \left[\frac{b_D(\hat{c}_D, \hat{c}_A) + b_A(\hat{c}_D, \hat{c}_A)}{2} - c \right] f(c) dc \\ &\quad + \int_{\hat{c}_D}^{\hat{c}_A} \left[b_A(\hat{c}_D, \hat{c}_A) - c \right] f(c) dc \end{aligned}$$

Now

$$\begin{aligned} \frac{\partial w}{\partial \hat{c}_A} &= (1-\lambda) \frac{\partial v_0(\hat{p}_D, \hat{p}_A)}{\partial \hat{p}_A} F'(c_A) + \int_0^{\hat{c}_D} \frac{\frac{\partial b_D(\hat{c}_D, \hat{c}_A)}{\partial c_A} + \frac{\partial b_A(\hat{c}_D, \hat{c}_A)}{\partial c_A}}{2} f(c) dc \\ &\quad + \int_{\hat{c}_D}^{\hat{c}_A} \frac{\partial b_A(\hat{c}_D, \hat{c}_A)}{\partial c_A} f(c) dc) + (b_A(\hat{c}_D, \hat{c}_A) - \hat{c}_A) \end{aligned}$$

Now, $b_A(c_D, c_A) = c_A$ and therefore,

$$\begin{aligned} \frac{\partial w}{\partial \hat{c}_A} \Big|_{(\hat{c}_D = c_D, \hat{c}_A = c_A)} &= (1-\lambda) \frac{\partial v_0(p_D, p_A)}{\partial p_A} F'(c_A) + \int_0^{c_D} \frac{\frac{\partial b_D(c_D, c_A)}{\partial c_A} + \frac{\partial b_A(c_D, c_A)}{\partial c_A}}{2} f(c) dc \\ &\quad + \int_{c_D}^{c_A} \frac{\partial b_A(c_D, c_A)}{\partial c_A} f(c) dc) \end{aligned}$$

$B_D(\hat{p}_D, \hat{p}_A) = b_D(F^{-1}(2\hat{p}_D), F^{-1}(\hat{p}_D + \hat{p}_A))$Let $B_A(\hat{p}_D, \hat{p}_A) = b_A(F^{-1}(2\hat{p}_D), F^{-1}(\hat{p}_D + \hat{p}_A))$. By definition, note that $B_D(\hat{p}_D, \hat{p}_A) + B_A(\hat{p}_D, \hat{p}_A) = (1-\lambda)(v_1(\hat{p}_D, \hat{p}_A) - v_0(\hat{p}_D, \hat{p}_A))$. It follows that

$$\frac{\partial b_D(c_D, c_A)}{\partial c_A} + \frac{\partial b_A(c_D, c_A)}{\partial c_A} = (1-\lambda)\left(\frac{\partial v_1(p_D, p_A)}{\partial p_A} - \frac{\partial v_0(p_D, p_A)}{\partial p_A}\right) F'(c_A)$$

Moreover, note that $B_A(\hat{p}_D, \hat{p}_A) = (1-\lambda)(v_{1,A}(\hat{p}_D, \hat{p}_A) - v_0(\hat{p}_D, \hat{p}_A))$. It follows that

$$\frac{\partial b_A(c_D, c_A)}{\partial c_A} = (1-\lambda)\left(\frac{\partial v_{1,A}(p_D, p_A)}{\partial p_A} - \frac{\partial v_0(p_D, p_A)}{\partial p_A}\right) F'(c_A)$$

and therefore,

$$\frac{\partial w}{\partial \hat{c}_A} \Big|_{(\hat{c}_D = c_D, \hat{c}_A = c_A)} = (1-\lambda) \left[\begin{array}{c} (1-p_A) \frac{\partial v_0(p_D, p_A)}{\partial p_A} + p_D \frac{\partial v_1(p_D, p_A)}{\partial p_A} \\ + (p_A - p_D) \frac{\partial v_{1,A}(p_D, p_A)}{\partial p_A} \end{array} \right] f(c_A) > 0.$$

Next, note that

$$\begin{aligned}\frac{\partial w}{\partial \hat{c}_D} &= (1-\lambda)\left[\frac{\partial v_0(\hat{p}_D,\hat{p}_A)}{\partial \hat{p}_D} - \frac{\partial v_0(\hat{p}_D,\hat{p}_A)}{\partial \hat{p}_A}\right]\frac{F'(\hat{c}_D)}{2} \\ &+ \int_0^{\hat{c}_D} \frac{\frac{\partial b_D(\hat{c}_D,\hat{c}_A)}{\partial \hat{c}_D} + \frac{\partial b_A(\hat{c}_D,\hat{c}_A)}{\partial \hat{c}_D}}{2} f(c)dc + (\frac{b_D(\hat{c}_D,\hat{c}_A)+b_A(\hat{c}_D,\hat{c}_A)}{2} - \hat{c}_D) \\ &+ \int_{\hat{c}_D}^{\hat{c}_A} \frac{\partial b_A(\hat{c}_D,\hat{c}_A)}{\partial \hat{c}_D} f(c)dc - (b_A(\hat{c}_D,\hat{c}_A) - \hat{c}_D)\end{aligned}$$

Notice that $\left[\frac{\partial v_0(\hat{p}_D,\hat{p}_A)}{\partial \hat{p}_D} - \frac{\partial v_0(\hat{p}_D,\hat{p}_A)}{\partial \hat{p}_A}\right] = 0$ and $b_D(c_D,c_A) = c_D$ and as $c_A > c_D$, $b_A(c_D,c_A) > c_D$ and therefore,

$$\begin{aligned}\frac{\partial w}{\partial \hat{c}_D}\Big|_{(\hat{c}_D=c_D,\hat{c}_A=c_A)} &= \int_0^{c_D} \frac{\frac{\partial b_D(c_D,c_A)}{\partial c_D} + \frac{\partial b_A(c_D,c_A)}{\partial c_D}}{2} f(c)dc + \int_{c_D}^{c_A} \frac{\partial b_A(c_D,c_A)}{\partial c_D} f(c)dc \\ &- \frac{b_A(c_D,c_A) - c_D}{2}\end{aligned}$$

By computation, it is also checked that $\frac{\partial b_D(c_D,c_A)}{\partial c_D} + \frac{\partial b_A(c_D,c_A)}{\partial c_D} = 0$ while

$$\frac{\partial b_A(c_D,c_A)}{\partial c_D} = -\frac{f(c_D)}{2}(1-\lambda)\left[\begin{array}{c} 2\left(v(1) - (1-p_A)\,v(0)\right) \\ +2p_A\left(v(2) - v(1)\right)\end{array}\right]$$

It follows that $\frac{\partial w}{\partial \hat{c}_D}\Big|_{(\hat{c}_D=c_D,\hat{c}_A=c_A)} < 0$. $\square$

Figure 1 : Multiple Symmetric Bayesian Equilibria

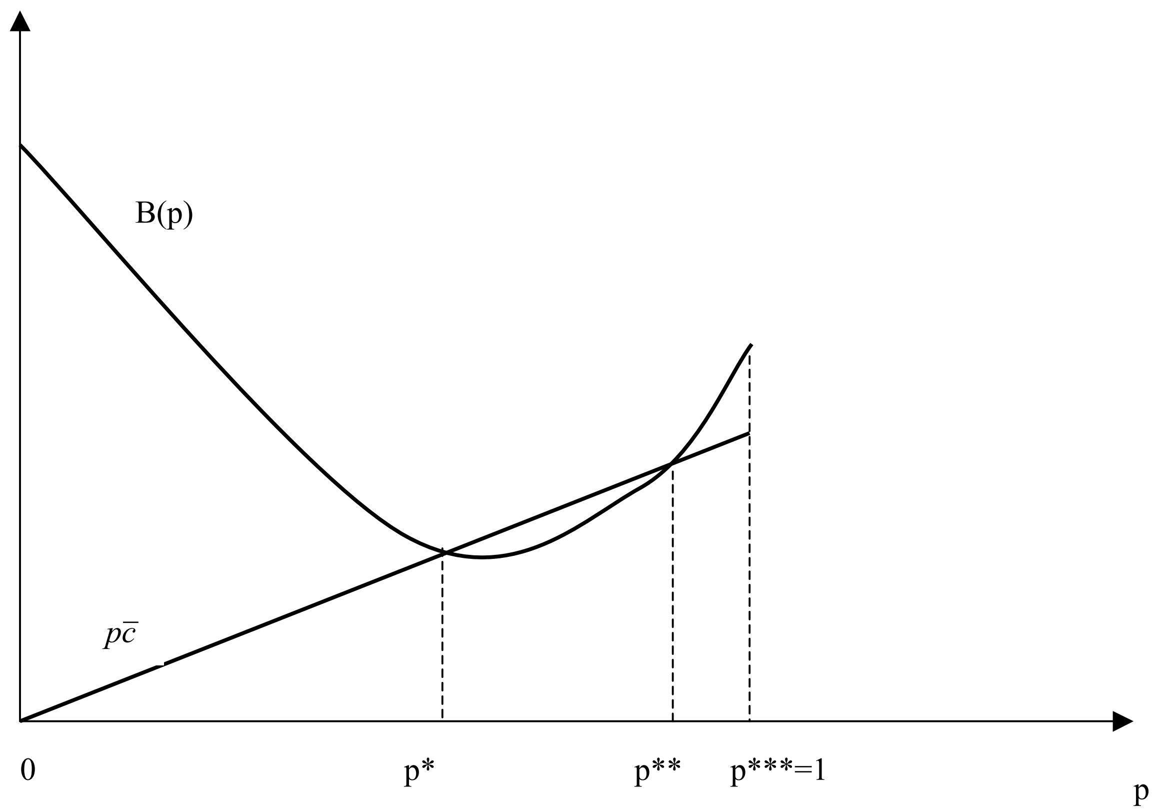

Figure 2 : Equilibrium Particpation Probabilities as λ Varies