Two-Dimensional Information Design *

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Abstract

I study information design in a two-dimensional setting, in which the ex-ante information correlation constrains the sender's ability to design information for each dimension. The receiver has to make two decisions. The sender can influence each of the decisions by providing information. I study two game forms - in the simultaneous game the receiver makes two decisions simultaneously; in the sequential game the receiver makes decisions sequentially. I show that if two dimensions are not negatively correlated, then in the sequential game the sender is able to overcome the correlation constraint, even if the sender is restricted to providing information only about one dimension at a time. I compare equilibrium payoffs in the simultaneous game and the sequential game with a constrained strategy set. I also completely solve the sequential game with an unconstrained strategy set for the sender and make relevant comparisons of equilibrium payoffs.

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1 Introduction

A defendant is accused of two crimes. The prosecutor can organize the investigation process to collect information about the crimes and by this persuade the judge that the defendant is guilty of the crimes. If the information about one crime also informs the judge about the second crime — for example if the defendant could have committed only one of the crimes — how should the prosecutor structure the investigation process? In particular, to maximize the probability of conviction for each of the crimes, when would the prosecutor prefer a simultaneous trial to sequential trials, or vice versa?

Now, consider a pharmaceutical company that seeks the approval of two drugs from a state agency. How should the company structure drug trials in order to maximize the probability of approval for each drug? It could test the effectiveness of both drugs simultaneously or sequentially. Which procedure would it prefer?

The current paper explores these questions. I consider a sender who wants to influence two decisions of the receiver by providing information. The sender provides information by conducting statistical tests. The receiver wants to match its decision with the payoff-relevant state of the world. For example, the judge would like to declare the defendant guilty if there is sufficient evidence that he or she is indeed guilty of the crime. The payoff-relevant states can be correlated. For example, if the state agency learns that one drug is of good quality, then it might be more likely that another drug is also of good quality. If a judge becomes convinced that a defendant committed one crime, then it might be regarded as less likely that the same defendant committed another crime. The sender may also care differently about the two decisions and the borders of proof to convince a decision-maker may vary across decisions.

I ask the following question: if the information provided about one payoff-relevant state also contains information about another payoff-relevant state, when should the sender allow this? Should the prosecutor allow the judge to learn about crime 2 from the evidence about crime 1? Or should the prosecutor investigate both crimes simultaneously? Separate (sequential) trials allow for distinct informational experiments, where information provided for the first dimension can influence decision-making about the second dimension, but not conversely. In contrast, with simultaneous trials, a single experiment provides information for both dimensions. Thus, sequential trials allow for indirect learning — the judge learns about crime 2 from the evidence about crime 1, while simultaneous trials do not.

I analyze this question by considering (1) a simultaneous game — consisting of a single stage and a single decision about both decision problems — and (2) a sequential game — a two-stage game, where the receiver makes decisions sequentially.

Implicit in my analysis is the assumption that the receiver's decision is not reversible. A real world analogue to this assumption would be double jeopardy in a criminal trial. Wikipedia says the following about this principle: "No one shall be liable to be tried or punished again for an offence for which he has already been finally convicted or acquitted in accordance with the law and penal procedure of each country."

For a simultaneous game, we first completely characterize the sender's equilibrium payoff when the receiver's threshold belief — the minimum belief in guilt at which the judge would declare the defendant guilty — for each of the decision problems is the same. I show that if two payoff-relevant states are sufficiently negatively correlated then the sender's equilibrium payoff decreases vis-à-vis the benchmark payoff — the payoff the sender would get if payoff-relevant states were independent.

Then we consider the sequential game — where the receiver makes decisions sequentially and the sender can provide information sequentially. First we conduct the analysis of sequential information provision by considering a constrained set of strategies for the sender. We assume that in the sequential game the sender directly provides information only about a single dimension of the payoff-relevant state. This assumption seems natural. For example, in separate trials, a prosecutor can only provide evidence about one crime at a time. In the sequential procedure, a drug company can run a trial only for a single drug. I characterize a necessary and sufficient condition for the sender to achieve the upper bound of payoff in the sequential game with this constrained set of strategies. I show, somewhat surprisingly, that even with the constrained set of strategies, if two payoff-relevant states are not negatively correlated, then the sender can always achieve the benchmark payoff in the sequential game, even if the two threshold beliefs differ from each other. When there is negative correlation, then given the same threshold beliefs, we show that if the sender cannot achieve the upper bound with the constrained set of strategies, then the sender prefers to provide first the information about the payoff-relevant state that is closer to the threshold belief.

Analyzing the sender's equilibrium payoff in the simultaneous and sequential games lets us compare these two games. I show that if payoff-relevant states are not negatively correlated then the sender prefers the sequential game, even with the constrained strategy set, to the simultaneous game. For negative correlation, I show that there are cases for which in the simultaneous game the sender's equilibrium payoff is less than its payoff from truthful revelation and in this case the sender's equilibrium payoff is strictly higher in the sequential game with the constrained strategy set. I also compare the sender's equilibrium payoffs in two games, when the upper bound is infeasible and identify conditions under which the sender is strictly better off in the sequential game with a constrained strategy set relative to the simultaneous game.

I completely solve the sequential game with an unconstrained strategy set. I show that if the sender is not able to achieve the upper bound payoff in the simultaneous game, then the sender is strictly better off in the sequential game with an unconstrained strategy set. I also compare the sender's equilibrium payoffs in the sequential game with and without constrained strategy sets. I identify a condition, in terms of the prior joint distribution, for which the sender's equilibrium payoff is the same in both types of the game, when the upper bound payoff remains out of reach.

2 Related Literature

The current work contributes to the literature about information design; see Bergemann and Morris [2016], Kamenica and Gentzkow [2011] and Aumann, Maschler, and Sterns [1995]. Ely and Szydlowski [2017] is most closely related. They analyze how a principal should design information to motivate an agent to exert effort on a project as long as possible, when the project's difficulty is uncertain. They show that the principal does better with dynamic information provision. Ely and Szydlowski [2017] focus on a single project, so that the correlation is not an issue; instead, dynamic information provision enables the principal to promise information as a reward for exerted effort.

The current work also contributes to the literature on communication in multistage games. This literature was initiated by Myerson [1986]. Recent contributions to information design in multistage games include Doval and Ely [2016] and Makris and Renou [2018]. These contributions consider how information provision influences strategic interaction among players. The current work concentrates solely on information design: there is no strategic interaction once information provision occurs.

3 Basic Setting

Let $u(a, \omega)$ denote the receiver's utility function, where $a \in A$ is the receiver's action and $\omega \in \Omega$ is the payoff-relevant state of the world. Both A and Ω are sets in R^2. Let $a = (a^1, a^2)$ denote an element of A, where $a^i \in \{0, 1\}$, for $i \in \{1, 2\}$. Similarly, let $\omega = (\omega^1, \omega^2)$ be an element of Ω, where $\omega^i \in \{0, 1\}$ for $i \in \{1, 2\}$. I assume that the receiver's utility is additively separable in actions; I also assume that for each dimension $i \in \{1, 2\}$ utility is only a function of action and dimension i of the state of the world, i.e., the utility function takes the form:

$$u(a, \omega) = u^1(a^1, \omega^1) + u^2(a^2, \omega^2) \tag{1}$$

Now we describe u^i - the receiver's utility for dimension i. The receiver's payoff from correctly matching the state of the world is 1 and it incurs a non-negative cost from mismatch that can vary with action. The following matrix summarizes the receiver's utility function for dimension $i \in \{1, 2\}$.

$a^i \backslash \omega^i$	1	0
1	1	$-\gamma^i$
0	$-\theta^i$	1

Figure 1: Receiver's utility function for dimension i

The interpretation is that the receiver has to make two decisions — 1 and 2 — and it would like to match its decision with the state of the world. For example, the receiver could be a judge who must decide if a defendant is guilty of two crimes. The judge wants to make a just decision for each crime separately. In the case of an unjust decision, the judge incurs a cost; this cost may differ if an innocent person is declared guilty than if a guilty person is declared innocent.

The sender's utility function is $v(a) = v^1(a^1) + v^2(a^2)$, i.e., it is only a function of the receiver's action and it is additively separable. The sender strictly prefers that for each dimension the receiver chooses 1. For the main part of the analysis I assume that the sender gives equal weights to both dimensions. Therefore, given the assumption that both decisions get equal weights, it is without loss of generality, to define the sender's utility function for dimension $i \in \{1, 2\}$ as:

$a^i \backslash \omega^i$	1	0
1	1	1
0	0	0

Figure 2: The sender's utility function for dimension i

Players share a common prior belief. Let p_{ij} denote the probability that the state is i for dimension 1 and j for dimension 2. Thus, p_{10} denotes the probability that the state is 1 for dimension 1 and 0 for dimension 2. Figure 3 describes players common prior distribution.

$\omega^1 \backslash \omega^2$	1	0
1	p_{11}	p_{10}
0	p_{01}	p_{00}

Figure 3: Common prior joint distribution

Let p^i denote the marginal probability that the state is 1 for dimension i. Thus, the probability of state 1 for dimension 1 is:

$$p^1 = p_{11} + p_{10}. \tag{2}$$

The sender can run a statistical test to influence the receiver's beliefs. I refer to the statistical test as a recommendation rule. The recommendation rule consists of a finite recommendation space ρ and a family of distributions $\{\pi(\cdot|\omega)\}_{\omega\in\Omega}$ over ρ. The sender chooses the recommendation rule and we seek to identify recommendation rules that maximize the sender's expected payoff. The solution concept is subgame perfect Nash equilibrium.

I analyze two game forms — simultaneous and sequential.

The simultaneous game. In the simultaneous game, the timing is as follows:

1: The sender chooses a recommendation rule that is observed by the receiver.

2: All players observe the recommendation and update their beliefs about the state of the world.

3: The receiver takes action $a = (a^1, a^2)$.

Thus, in the simultaneous game the receiver makes two decisions simultaneously. In the sequential game, as the name suggests, the receiver makes decisions sequentially.

The sequential game. The sequential game consists of two stages.

Stage I

1: The sender chooses a recommendation rule that the receiver observes.

2: All players observe the recommendation and update their beliefs about the state of the world.

3: The receiver takes an action a^i, $i \in \{1, 2\}$

Stage II

1: The sender chooses a recommendation rule that the receiver observes.

2: All players observe the recommendation and update their beliefs about the state of the world.

3: The receiver takes an action a^j, $j \in \{1, 2\}$, $j \neq i$.

The simultaneous game can be interpreted as a joint trial for two crimes and the sequential game as two separate trials.

I use $p^i(|\cdot)$ to denote the receiver's belief that the state of the world is 1 for dimension i by. Therefore, the receiver's expected utilities from choosing 1 and 0 respectively are:

$$p^i(|\cdot) - (1 - p^i(|\cdot))\gamma^i \tag{3}$$

$$-p^i(|\cdot)\theta^i + (1 - p^i(|\cdot)) \tag{4}$$

Observe that the receiver does best to choose 1 for dimension i if

$$p^i(|\cdot)(1 + \theta^i) \geq (1 - p^i(|\cdot))(1 + \gamma^i) \tag{5}$$

or equivalently if

$$\frac{p^i(|\cdot)}{1 - p^i(|\cdot)} \geq \frac{1 + \gamma^i}{1 + \theta^i}. \tag{6}$$

From expression (6), it follows that for any u^i there exists a threshold belief — α^i — such that if the receiver's belief exceeds this threshold, then she prefers to choose 1. For example if $\gamma^i = \theta^i$, then the receiver prefers to choose 1 if $p^i(|\cdot) \geq \frac{1}{2}$. We sometimes refer to the threshold belief as a decision threshold. I summarize this notation in the following definition.

Definition 1. *Decision threshold: the threshold belief α^i for dimension i.*

Assumption 1. *The prior joint distribution satisfies: $p^i < \alpha^i$.*

Assumption 1 implies that the sender needs to provide evidence to convince receiver to take action 1. The sole purpose of assumption 1 is to simplify exposition of arguments.

Before analysing simultaneous and sequential games, I show that the correlation represents an additional constraint for the sender, as compared to the benchmark case when the two dimensions are independent[1].

[1] I thank Phil Reny for pointing out my mistake in this regard.

4 Correlation as an additional constraint - describing the upper bound on sender payoffs

In our setting the receiver's optimal action for dimension i only depends on the marginal distribution of the payoff-relevant state - $p^i(|\cdot)$; and the sender's payoff is only a function of the receiver's action. These assumptions imply that any correlation between the two dimensions can only decrease the sender's optimal payoff. I next formalize this statement and then give a brief discussion of it. Before this, I give some definitions.

Definition 2. *Decision problem* (DP) *describes a setting in which the receiver faces two decision problems, with given decision thresholds* α^1 *and* α^2 *and the given prior joint distribution.*

I want to consider a transformed decision problem, where the two dimensions of the payoff-relevant states are independent.

Definition 3. *For a given* DP*, its independent transformation* (IDP) *is given by the transformed problem in which the prior distribution of* IDP *has the same marginal distributions as* DP*, but the two dimensions are now independent.*

Theorem 1 reveals that ex-ante information correlation can only hurt the sender.

Theorem 1. *For any* DP*, the upper bound on the sender's expected payoff is given by the equilibrium payoff of the sequential game of* IDP*.*

Theorem 1 formalizes the following observation: to influence receiver's decisions, in the current setting, the best the sender can do is to influence receiver's beliefs separately.

I defined a recommendation rule as a family of conditional distributions, where conditioning happens on the payoff-relevant state of the world. An equivalent definition of the recommendation rule is as a joint distribution of ρ and the payoff-relevant state, where the marginal distribution of the payoff-relevant state equals to prior distribution.

To prove Theorem 1, we observe that making a recommendation is equivalent to choosing a posterior belief. Therefore, a recommendation rule can be described as a distribution over posterior beliefs. Therefore the question: "what is the set of feasible recommendation rules"? is equivalent to the question: "what is the set of feasible distributions of posteriors"?

As a first step to the proof we provide a lemma, that describes the set of feasible recommendation rules. This logic builds on the argument of Kamenica and Gentzkow [2011].

Lemma 1. *A recommendation rule is feasible if and only if:*

It induces a distribution of posterior beliefs such that the expected posterior equals prior. This is equivalent to the following: there exists a joint distribution of ρ *and* Ω *such that the marginal distribution of the payoff-relevant state equals prior.*

The proof is an application of the law of total probability.

Proof. For a conditional distribution to be a valid probability distribution, it must satisfy two conditions. $0 \leq \pi(\cdot|\omega) \leq 1$ and $\sum_i \pi(\rho_i|\omega) = 1$. The second condition is equivalent to $\sum_i p(\rho_i \cap \omega) = p(\omega)$, because $\pi(\rho_i|\omega) = \frac{p(\rho_i \cap \omega)}{p(\omega)}$.

If: When the condition of the lemma is satisfied, then:

$$p(\omega) = p(\omega|\rho_1)p(\rho_1) + ... + p(\omega|\rho_n)p(\rho_n) = p(\omega \cap \rho_1) + ... + p(\omega \cap \rho_n)\ \forall \omega \in \Omega, \quad (7)$$

where the second equality follows from the Bayes' rule.

Thus, the recommendation rule is feasible.

Only if: Suppose the suggested distribution of posteriors does not satisfy the condition of the lemma. Then: $\exists\ \omega \in \Omega$ such that

$$p(\omega) \neq p(\omega|\rho_1)p(\rho_1) + ... + p(\omega|\rho_n)p(\rho_n) = p(\omega \cap \rho_1) + ... + p(\omega \cap \rho_n).$$

Therefore, there $\exists\ \omega$ for which $\sum_i \pi(\rho_i|\omega) \neq 1$, i.e., the suggested recommendation rule is not feasible.

□

If one thinks about the recommendation rule as a distribution over posteriors, then Lemma 1 states that a recommendation rule is feasible if and only if the expected posterior equals the prior. If one thinks about the recommendation rule as a joint distribution over recommendations and payoff-relevant states, then the recommendation rule is feasible if and only if the marginal distribution of the payoff-relevant states of the world, for the suggested joint distribution, equals to the prior. In our analysis we will use both arguments. That is, sometimes we will think about the recommendation rule as choosing a distribution of posteriors, and sometimes as a joint distribution of recommendation rules (ρ) and payoff-relevant states (Ω).

We will also require that recommendations satisfy an obedience constraint, i.e., given the recommendation, it is optimal for the receiver to follow it. Formally, the obedience constraint means that:

$$E_{\rho_{\bar{\alpha}}} u(\bar{\alpha}, \omega) \geq E_{\rho_{\bar{\alpha}}} u(\alpha, \omega) \; \forall \; a, \; \forall \; \rho_{\bar{\alpha}} \tag{8}$$

With Lemma 1 in hand, we now prove Theorem 1.

Proof. The sender's optimization problem reduces to choosing a distribution of marginal posteriors for decision problems 1 and 2. From lemma 1 we know that when choosing a distribution of posteriors, the single constraint that this distribution should satisfy is that the expected posterior must equal the prior.

Because of lemma 1, one can regard prior distribution as a constraint parameter of the optimization problem. Therefore, optimisation in stage I implies that the sender chooses a recommendation rule that makes recommendations conditional only on dimension i of the payoff-relevant state of the world. This is so, because by this step the sender can to reduce the number of parameters describing the prior distribution from 4 to 2.

Therefore, let the sender choose $\rho^i \in \left\{\pi(\cdot|\omega^i)\right\}_{\omega^i \in \Omega}$ in stage I and say $\hat{\rho^i}$ solves the sender's optimization problem for dimension i in stage I. Because dimensions i and j are independent by construction, from this follows that $\hat{\rho^i}$ does not affect the marginal distribution of dimension j in stage II. But because the feasible set of the posterior distributions is only a function of the prior distribution, this implies that $\hat{\rho}^i$ does not affect the set of feasible recommendation rules for dimension j.

□

Theorem 1 says that if the two dimensions are independent than the sender is able to separate the optimization problems for the two dimensions. In particular, when it solves the unconstrained optimization problem for dimension i in stage I, this does not affect the optimization problem for dimension j in stage II.

I now derive the upper bound explicitly to describe the optimal distribution of posteriors that the sender will try to induce. Theorem 1 says that to do this it suffices to consider IDP and let the sender choose recommendation rules sequentially.

First consider a DP:

$\omega^1 \backslash \omega^2$	1	0
1	p_{11}	p_{10}
0	p_{01}	p_{00}

with decision thresholds α^1 and α^2.

Now consider the associated IDP, as given in Figure 4.

$\omega^1 \backslash \omega^2$	1	0
1	$p^1 p^2$	$p^1(1-p^2)$
0	$(1-p^1)p^2$	$(1-p^1)(1-p^2)$

Figure 4: Independent Transformation

The sender chooses recommendation rules for dimensions 1 and 2 separately. We describe now the optimal recommendation rule for dimension i, when the two dimensions are independent. The sender chooses among the following set of feasible joint distributions:

$\rho^i \backslash \omega^i$	1	0
ρ^i_1	a	b
ρ^i_0	c	d

where $a + c = p^i$, $b + d = 1 - p^i$ and $\frac{a}{a+b} \geq \alpha^i$. The first two equalities express the condition that in each state of the world the sender will make one of the two recommendations. The inequality is the obedience constraint: when the sender recommends the action 1, then it must be optimal for the receiver to follow this recommendation. The sender's optimal recommendation rule maximizes the probability ρ_1^i among the set of feasible joint distributions. Remember that we assume $p^i < \alpha^i$, so the receiver chooses 0 if the sender does not provide some evidence that the state of the world is 1. The optimal recommendation rule is given below.

$\rho^i \backslash \omega^i$	1	0
ρ_1^i	p^i	$p^i \frac{1-\alpha^i}{\alpha^i}$
ρ_0^i	0	$(1 - p^i) - p^i \frac{1-\alpha^i}{\alpha^i}$

Figure 5: Optimal Recommendation Rule for dimension i

When the recommendation ρ_1^i is made, then the receiver's posterior belief is $\frac{p^i}{p^i + p^i \frac{1-\alpha^i}{\alpha^i}} = \frac{\alpha^i}{\alpha^i + 1 - \alpha^i} = \alpha^i$. Let's interpret what the optimal recommendation rule accomplishes.The sender wants to maximize $p(\rho_1^i)$. Therefore, it recommends action 1 with probability 1 if $\omega^i = 1$ and it recommends action 1 if $\omega = 0$ such that when 1 is recommended, the receiver is indifferent between choosing 1 and 0. Given the recommendation ρ_1^i, the receiver follows the recommendation if and only if the posterior is at least α^i. So the posterior cannot be less than α^i. Posteriors that exceed α^i cannot be optimal for the sender because this implies that the sender can increase the probability of recommending action 1 and the receiver would still follow the recommendation.

Now we want to describe the recommendation rule as a distribution of posteriors. This distribution is: the posterior is α^i with probability $\frac{p_1^i}{\alpha^i}$ and the posterior is 0 with complementary probability. The following Remark summarizes this discussion and provides an upper bound on the sender's payoff.

Remark 1. *For arbitrary DP, an upper bound on the sender's payoff is* $\frac{p_1^1}{\alpha^1} + \frac{p_1^2}{\alpha^2}$.

Theorem 1 states the problem we want to analyse. Our goal is to understand the implications of ex-ante information correlation for information design. The roadmap for the analysis is as follows. First we analyse the simultaneous recommendation rule and characterize a necessary and sufficient condition for the simultaneous recommendation rule to achieve the upper bound. Then we analyse sequential recommendation rules, by considering constrained strategy set of the sender and characterize a necessary and sufficient condition for the sequential recommendation procedure to achieve the upper bound. These characterizations help us understand the different trade-offs related to different approaches to the information design — static and dynamic.

5 Optimal Simultaneous Recommendation Rules

The simultaneous game is a standard Bayesian persuasion problem. Therefore, we can directly use insights from the literature about Bayesian persuasion. For example, following Kamenica and Gentzkow [2011] and Bergemann and Morris [2016] we can restrict our analysis to direct recommendation rules.

Finding optimal simultaneous recommendation rules requires solving a linear programming problem. Were the dimension of state-space less than 4, it would be straightforward to use the concavification approach, implemented by Kamenica and Gentzkow [2011], to characterize the sender's equilibrium payoff in the simultaneous game. The concavification approach simplifies the search for optimal recommendation rules by drawing sender's value function. The geometric picture makes it straightforward to find the optimal distribution of posteriors, which is equivalent to finding the optimal recommendation rule. When the dimension is exceeds 3, then the task of drawing a geometric picture to visualise the sender's value function ceases to be straightforward.[2] Therefore we will approach the problem indirectly, by referring to Theorem 1. In this section we consider the special case where the decision thresholds are equal.

Assumption 2. $\alpha^1 = \alpha^2 = \alpha$.

To see why correlation can decrease the sender's payoff, consider the following class of DPs with perfect negative correlation:

$\omega^1 \backslash \omega^2$	1	0
1	0	x
0	$1-x$	0

[2] Gentzkow and Kamenica [2016] discuss this issue.

In essence if the defendant committed one crime, she could not have committed the other crime. Given $\alpha^1 = \alpha^2 = \alpha$, it follows from Remark 1 that the upper bound on the sender's payoff for this class of problems is $\frac{x}{\alpha} + \frac{1-x}{\alpha} = \frac{1}{\alpha}$. Note that this payoff strictly exceeds 1 as long as $\alpha < 1$.

What is the maximum that a sender can achieve in the simultaneous game? Can the sender achieve the upper bound? It turns out, that for this class of problems, the simultaneous recommendation not only fails to achieve the upper bound, but the sender is unable to garble, i.e. it is forced to reveal states truthfully. Proposition 1 summarizes this argument.

Proposition 1. *Consider the class of DPs with* $p_{11} = p_{00} = 0$. *Then, the sender's expected payoff in the simultaneous game is* 1, *which is the sender's payoff from revealing information truthfully.*

Proof. It follows from lemma 1 that a feasible recommendation rule that maximizes the sender's expected payoff should have the following form:

$\rho \backslash \Omega$	11	10	01	00
ρ_{11}	0	a	b	0
ρ_{10}	0	c	d	0
ρ_{01}	0	e	f	0
ρ_{00}	0	g	h	0

Optimality implies that $g = h = 0$ — sender never recommends (00). If not, then the sender could decrease $p(\rho_{00})$ and increase the probability of at least one of the other recommendations without violating the obedience constraints.

Now we show that the obedience constraint for ρ_{11} implies that $a = b = 0$. Say $p(\rho_{11}) > 0$. Then the obedience constraint means: $\frac{a}{a+b} \geq \alpha$ and $\frac{b}{a+b} \geq \alpha$. These two inequalities can be rewritten respectively as $a\frac{1-\alpha}{\alpha} \geq b$ and $b\frac{1-\alpha}{\alpha} \geq a$. Because we assume that $p^i < \alpha$, for $i \in \{1, 2\}$, this means that for the considered problem $\alpha > \frac{1}{2}$, which implies $\frac{1-\alpha}{\alpha} < 1$. This shows that the obedience constraint for ρ_{11} cannot be satisfied. Therefore, for any optimal recommendation rule, $p(\rho_{10}) + p(\rho_{01}) = 1$.

□

From Proposition 1, it follows that in the simultaneous game a necessary condition for the sender to achieve the higher payoff than truthfully revealing the states is: Either $p_{11} > 0$, or $p_{00} > 0$.

Now consider the class of decision problems with perfect positive correlation:

$\omega^1 \backslash \omega^2$	1	0
1	x	0
0	0	$1-x$

In essence, the defendant has either committed both crimes or neither of them, where the probability of committing both crimes is x. The upper bound of sender's payoff for this class of problems is $\frac{2x}{\alpha}$. For this class of problems there always exists a simultaneous recommendation rule that achieves the upper bound: If $p_{10} = p_{01} = 0$ and the decision thresholds are the same, then the persuasion problem effectively collapses to a single persuasion problem. The optimal simultaneous recommendation rule consists of recommendations ρ_{11} and ρ_{00}, that we now derive. First we note that the feasibility constraint implies that the recommendation rule should have the following form:

$\rho \backslash \Omega$	11	10	01	00
ρ_{11}	a	0	0	b
ρ_{10}	c	0	0	d
ρ_{01}	e	0	0	f
ρ_{00}	g	0	0	h

The obedience constraint implies that the sender cannot recommend (10) or (01), so $c = d = e = f = 0$; because whatever the recommendation, the receiver will always take the same action for both dimensions. Therefore, the optimal recommendation rule is: $a = x$, $g = 0$, $b = x\frac{1-\alpha}{\alpha}$ and $h = 1 - x - x\frac{1-\alpha}{\alpha}$. We note that for perfect positive correlation, with equal decision thresholds, the optimal recommendation rule replicates the logic of the optimal recommendation rule for a single dimension. When the recommendation ρ_{11} is made, then the receiver's posterior belief for dimension i is $\frac{x}{x+x\frac{1-\alpha}{\alpha}} = \frac{\alpha}{\alpha+1-\alpha} = \alpha$.

The examples above reveal that in the simultaneous game with equal decision thresholds the sender's equilibrium payoff is given by the truthful-revelation payoff if the two dimensions are perfectly negatively correlated and it equals the upper-bound payoff if two dimensions are perfectly positively correlated.

Because of this, one might think that there exists a threshold in terms of covariance/correlation, such that for given decision thresholds, if covariance/correlation is (below) above this threshold the upper bound is (not) achievable. It turns out that this reasoning is not correct. Before establishing this, we characterize a necessary and sufficient condition for a recommendation rule to achieve the upper bound.

For the class of problems that we considered, the feasibility of the upper seems to depend on the following: can the sender pool states (10) and (01) effectively with states (11) and (00)? With perfectly negative correlation there was no way to achieve this pooling, whereas with perfectly positive correlation there was no requirement for it. As it turns out, this observation is correct when the decision thresholds are the same.

Before we make this observation precise, we make following normalization assumption:

Assumption 3. $p_{01} \leq p_{10}$.

The only purpose of Assumption 3 is to simplify notation in the following proposition. It helps us to specify the value of $\min\{p_{10}, p_{01}\}$.

Proposition 2. *In the simultaneous game, the sender's equilibrium payoff attains the upper bound payoff if and only if:*

(i) $p_{11} \geq p_{01}\frac{2\alpha-1}{1-\alpha}$, *or*

(ii) $p_{10} + p_{01} \leq p_{00}\frac{\alpha}{1-\alpha} + 2p_{11}\frac{1-\alpha}{2\alpha-1}$.

We first discuss conditions (i) and (ii) of Proposition 2. Condition (i) says, that the sender can achieve the upper bound if α and p_{01} are small relative to p_{11}. The right hand side of condition (i) is increasing in α, which means that the sender has to reveal information more truthfully, i.e. it has to say less often that the state is 1 for dimension i, when the state is 0. Condition (i) is always satisfied for $\alpha \leq \frac{1}{2}$. For $\alpha > \frac{1}{2}$, if α increases, then p_{01} should be even less relative to p_{11} for condition (i) to hold. If condition (i) is violated, then condition (ii) says that the sum of p_{10} and p_{01} should be less than the weighted sum of p_{00} and p_{11}, where the weights respectively are: $\frac{\alpha}{1-\alpha}$ and $2\frac{1-\alpha}{2\alpha-1}$. Because one can express the covariance of two Bernoulli random variables as $p_{11}p_{00} - p_{10}p_{01}$, one can see that the feasibility of the upper bound of the sender's payoff is related to the covariance. In particular, one sees that if the two dimensions are perfectly negatively correlated, $p_{11} = p_{00} = 0$, then both conditions are violated, whereas if the two dimensions are perfectly positively correlated, $p_{10} = p_{01} = 0$, then both conditions are satisfied.

Before proving proposition 2, we characterize the joint distribution that achieves the upper bound. The proof of Proposition 2 consists of showing that the suggested recommendation rule is feasible if and only if either condition (i) or (ii) is satisfied.

Remember that we seek to construct a recommendation rule that induces a distribution of joint posteriors that leads to the desired distribution of marginals. Therefore, the support of marginal posteriors should be $\{0, \alpha\}$. That is, any given marginal posterior is equivalent to the recommendation; i.e., if recommendation is ρ_{11}, then the obedience constraints imply that the marginal posterior beliefs are at least α for each of the decision problems, and optimization by the sender implies that when recommendation ρ_{11} is made, then posterior marginal beliefs should be exactly α. We summarize this observation in the following lemma.

Lemma 2. *A simultaneous recommendation rule achieves the upper bound payoff, if and only if the following joint distribution is feasible:*

$\rho\backslash\Omega$	11	10	01	00
ρ_{11}	p_{11}	b	b	c
ρ_{10}	0	$p_{10}-b$	0	d
ρ_{01}	0	0	$p_{01}-b$	e
ρ_{00}	0	0	0	f

If 1 *is recommended for a dimension i, then the marginal posterior for dimension i, given the recommendation, is α, if* 0 *is recommended, then given the recommendation the marginal posterior is* 0*.*

Proof. $p(\rho_{11}\cap 11)$ must equal p_{11}. Were it less than p_{11}, then feasibility of a joint distribution implies that there is a recommendation, where 0 is recommended for at least one decision problem and the marginal posterior for this decision problem is not 0. Similarly, $p(\rho_{11}\cap 10) = p(\rho_{11}\cap 01)$. Were $p(\rho_{11}\cap 10) \neq p(\rho_{11}\cap 01)$, then there exists a recommendation, ρ_{11}, such that 1 is recommended for a dimension and the marginal posterior for this dimension is not α. □

We now prove Proposition 2.

Proof.

From Lemma 2, it follows that we want to find conditions on the prior joint distribution such that we can fill in the values for the suggested recommendation rule.

If:

Say condition (i) $p_{11} \geq p_{01}\frac{2\alpha-1}{1-\alpha}$ holds. Then following recommendation rule is feasible: $b = p_{01}$, $c = p_{11}\frac{1-\alpha}{\alpha} + p_{01}\frac{1-2\alpha}{\alpha}$, $d = [p_{10} - p_{01}]\frac{1-\alpha}{\alpha}$, $e = 0$ and $f = p_{00} - c - d$.

First we show that the suggested recommendation rule satisfies the obedience constraints.

$$p^i(|\rho_{11}) = \frac{p_{11}+p_{01}}{p_{11}+p_{01}+p_{01}+p_{11}\frac{1-\alpha}{\alpha}+p_{01}\frac{1-2\alpha}{\alpha}} = \frac{p_{11}+p_{01}}{(p_{11}+p_{01})\frac{1}{\alpha}} = \alpha \tag{9}$$

$$p^1(|\rho_{10}) = \frac{p_{10} - p_{01}}{(p_{10} - p_{01}) + (p_{10} - p_{01})\frac{1-\alpha}{\alpha}} = \frac{p_{10} - p_{01}}{(p_{10} - p_{01})\frac{1}{\alpha}} = \alpha \tag{10}$$

Equation (9), for example, shows that given the recommendation ρ_{11} the sender's marginal posterior belief for dimension i is α.

Now we show that the suggested recommendation rule is also a feasible joint distribution. To do this we must show that $c + d + e \leq p_{00}$. Expressing p_{00} as a complementary probability, yields $p_{00} = 1 - p_{11} - p_{10} - p_{01}$.

Therefore, $c + d + e \leq p_{00} \iff$

$$p_{11}\frac{1-\alpha}{\alpha} + p_{01}\frac{1-2\alpha}{\alpha} + [p_{10} - p_{01}]\frac{1-\alpha}{\alpha} \leq 1 - p_{11} - p_{10} - p_{01}. \tag{11}$$

This expression simplifies to:

$$p_{11}\frac{1}{\alpha} + p_{10}\frac{1}{\alpha} \leq 1, \tag{12}$$

which holds, since we assume that the prior marginal p^i is less than the decision threshold, α.

Now suppose that (i) is violated, but condition $(ii) : p_{10} + p_{01} \leq p_{00}\frac{\alpha}{1-\alpha} + 2p_{11}\frac{1-\alpha}{2\alpha-1}$ is satisfied. Then following recommendation rule is feasible: $b = p_{11}\frac{1-\alpha}{2\alpha-1}$, $c = 0$, $d = (p_{10} - p_{11}\frac{1-\alpha}{2\alpha-1})(\frac{1-\alpha}{\alpha})$ and $e = (p_{01} - p_{11}\frac{1-\alpha}{2\alpha-1})(\frac{1-\alpha}{\alpha})$.

First, we show that obedience constraints are satisfied.

$$p^i(|\rho_{11}) = \frac{p_{11} + p_{11}\frac{1-\alpha}{2\alpha-1}}{p_{11} + p_{11}\frac{1-\alpha}{2\alpha-1} + p_{11}\frac{1-\alpha}{2\alpha-1}} = \alpha \tag{13}$$

$$p^1(|\rho_{10}) = \frac{p_{10} - p_{11}\frac{1-\alpha}{2\alpha-1}}{p_{10} - p_{11}\frac{1-\alpha}{2\alpha-1} + (p_{10} - p_{11}\frac{1-\alpha}{2\alpha-1})\frac{1-\alpha}{\alpha}} = \frac{1}{1 + \frac{1-\alpha}{\alpha}} = \alpha \tag{14}$$

$$p^2(|\rho_{01}) = \frac{p_{01} - p_{11}\frac{1-\alpha}{2\alpha-1}}{p_{01} - p_{11}\frac{1-\alpha}{2\alpha-1} + (p_{01} - p_{11}\frac{1-\alpha}{2\alpha-1})\frac{1-\alpha}{\alpha}} = \frac{1}{1 + \frac{1-\alpha}{\alpha}} = \alpha \tag{15}$$

Feasibility means that $d + e \leq p_{00}$, i.e.:

$$(p_{10} - p_{11}\frac{1-\alpha}{2\alpha-1})(\frac{1-\alpha}{\alpha}) + (p_{01} - p_{11}\frac{1-\alpha}{2\alpha-1})(\frac{1-\alpha}{\alpha}) \leq p_{00} \tag{16}$$

which is equivalent to condition (ii):

$$p_{10} + p_{01} \leq p_{00}\frac{\alpha}{1-\alpha} + 2p_{11}\frac{1-\alpha}{2\alpha-1}. \tag{17}$$

Only if:

If both conditions (i) and (ii) do not hold, then, for any feasible recommendation rule, at least one of the obedience constraints does not bind. This is so, because if all constraints bind, then $c + d + e > p_{00}$, i.e., the recommendation rule is not a feasible joint distribution. We now prove this claim.

Consider the following recommendation rule:

$b = p_{11}\frac{1-\alpha}{2\alpha-1}$, $c = 0$, $d = (p_{10} - p_{11}\frac{1-\alpha}{2\alpha-1})(\frac{1-\alpha}{\alpha})$ and $e = (p_{01} - p_{11}\frac{1-\alpha}{2\alpha-1})(\frac{1-\alpha}{\alpha})$.

We have shown in equations $13 - 15$, that the suggested recommendation rule satisfies the obedience constraints. The feasibility constraint becomes: $c + d + e \leq p_{00}$, which upon substituting for $c + d + e$ yields

$$(p_{10} - p_{11}\frac{1-\alpha}{2\alpha-1})(\frac{1-\alpha}{\alpha}) + (p_{01} - p_{11}\frac{1-\alpha}{2\alpha-1})(\frac{1-\alpha}{\alpha}) \leq p_{00}. \tag{18}$$

(18) is equivalent to condition (ii), which is a contradiction.

Now we show that there does not exist any other feasible recommendation rule that achieves the upper bound. Thus, we consider a recommendation rule with $c = c' > 0$. Then following is true:

$$c' + d' + e' > c + d + e > p_{00}, \tag{19}$$

where $c' > 0$ and d' and e' are the values of the suggested new recommendation rule and $c = 0$, $d = (p_{10} - p_{11}\frac{1-\alpha}{2\alpha-1})(\frac{1-\alpha}{\alpha})$ and $e = (p_{01} - p_{11}\frac{1-\alpha}{2\alpha-1})(\frac{1-\alpha}{\alpha})$. Condition (19) holds, because for any recommendation rule that achieves the upper bound, b is decreasing in c, whereas d and e are decreasing in b. Say $c = c' > 0$, then $b' < b = p_{11}\frac{1-\alpha}{2\alpha-1}$. If not, then

$$p^i(|\rho_{11}) = \frac{p_{11} + p_{11}\frac{1-\alpha}{2\alpha-1}}{p_{11} + p_{11}\frac{1-\alpha}{2\alpha-1} + p_{11}\frac{1-\alpha}{2\alpha-1} + c'} < \alpha \tag{20}$$

and the obedience constraint for ρ_{11} is violated. If a recommendation rule achieves the upper bound, then $d' = (p_{10} - b')(\frac{1-\alpha}{\alpha})$ and $e' = (p_{01} - b')(\frac{1-\alpha}{\alpha})$, i.e., the obedience constraints for ρ_{10} and ρ_{01} bind. This completes the argument.

This means that if both conditions are violated, then there does not exist a feasible recommendation rule that induces a distribution of marginal posteriors with support $\{0, \alpha\}$.

□

Now that we described a necessary and sufficient condition for the simultaneous recommendation rule to achieve the upper bound, we can show that for a given decision thresholds, covariance/correlation alone is not sufficient to determine whether the upper bound is feasible.

Consider example A:

$\omega^1 \backslash \omega^2$	1	0
1	0.15	0.40
0	0.40	0.05

Figure 6: Example A: $\alpha = 0.9$

Then the left-hand side of condition (i) is $p_{11} = 0.15$, which is less than the right-hand side of condition (i), $p_{01}\frac{2\alpha-1}{1-\alpha} = 3.2$; while the left-hand-side of condition (ii) is $p_{10} + p_{01} = 0.8$, which exceeds the right-hand side of condition (ii) $p_{00}\frac{\alpha}{1-\alpha} + 2p_{11}\frac{1-\alpha}{2\alpha-1} = 0.4875$. This shows that for example A neither of the conditions of Proposition 2 is satisfied. Therefore, the upper bound is not feasible.

Now consider example B. For example B the left-hand side of condition (ii) of Proposition 2 is 0.8 and the right-hand side exceeds 1. Therefore, there exists a simultaneous recommendation rule that achieves the upper bound.

$\omega^1 \backslash \omega^2$	1	0
1	0.05	0.40
0	0.40	0.15

Figure 7: Example B: $\alpha = 0.9$

If one writes the covariance of two Bernoulli random variables as:

$$p_{11}p_{00} - p_{10}p_{01}, \tag{21}$$

one sees that covariance/correlation is the same for both examples A and B. In example A, for each dimension, state 1 is more likely than in example B — $0.55 > 0.45$. When the two dimensions are independent, then the sender recommends 1 conditional on the state being 0 more often if the state being 1 is more likely. This is so, because the optimal recommendation rule that achieves the upper bound for dimension i implies

$$\pi(\rho_1^i|\omega^i) = \frac{p^i \frac{1-\alpha}{\alpha}}{1-p^i}. \tag{22}$$

So, as p^i increases, the sender would prefer, conditional on the state being 0, to reveal the state of the world less truthfully. But the conflict between the two dimensions, as expressed in the covariance/correlation, constrains the sender's ability to manipulate the information.

Now we want to describe optimal simultaneous recommendation rules and the sender's maximum payoff, when a simultaneous recommendation rule that achieves the upper bound does not exist.

When the simultaneous recommendation rule fails to achieve the upper bound, the problem is that p_{00} is not big enough to pool states (10) and (01) efficiently. It follows that when the upper bound is out of reach, then recommendation ρ_{00} is never made, i.e. $p(\rho_{00}) = 0$. The next lemma formalizes this argument.

Lemma 3. *Consider DP for which there does not exist a simultaneous recommendation rule that achieves the upper bound. Then for any optimal simultaneous recommendation rule* $p(\rho_{00}) = 0$.

Proof. Say $p(\rho_{00}) > 0$. Because the upper bound is not feasible, there exists a recommendation other than ρ_{00}, for which the obedience constraint does not bind. Therefore, the sender can decrease $p(\rho_{00})$ and increase the probability of the recommendation, for which the obedience constraint does not bind. This would increase the sender's payoff, a contradiction.

□

The sender's expected utility in the simultaneous game is $2p(\rho_{11}) + p(\rho_{10}) + p(\rho_{01})$. When the upper bound is not feasible, Lemma 3 implies that $p(\rho_{11}) + p(\rho_{10}) + p(\rho_{01}) = 1$. From this, it follows that when the upper bound is not feasible, the sender's expected payoff in the simultaneous game can be written as $1 + p(\rho_{11})$. We can now describe the optimal simultaneous recommendation rule and the sender's equilibrium payoff when the the upper bound is not feasible. Using the notation introduced in Lemma 2, we have:

Proposition 3. *Suppose there does not exist a simultaneous recommendation rule that achieves the upper bound. Then in the equilibrium of the simultaneous game the following are true:*

a) $p(\rho_{11} \cap (11)) = p_{11}$, $b = p_{11}\frac{1-\alpha}{2\alpha-1}$, $c = 0$.

b) The sender's payoff is $1 + p_{11}\frac{1}{2\alpha-1}$.

Proof. It follows from Lemma 3 that the sender's equilibrium payoff in the simultaneous game, when the upper bound is not feasible, is $1 + p(\rho_{11})$. Part a) follows from maximizing $p(\rho_{11})$ and part b) follows from direct calculation.

□

Discussion:

Propostions 2 and 3 completely characterize optimal recommendation rules and hence a sender's optimal expected payoff in a simultaneous game. The ex-ante information correlation introduces an additional constraint for the sender. With a single decision problem, the sender pools states (1) and (0) optimally together. With two decision problems, the question is how to pool states (10) and (01) optimally. The optimal simultaneous recommendation rule, for the same decision thresholds, is as follows: First, ρ_{11} is recommended, which means recommending (11) with probability 1 if the state is (11) and then maximizing probability of this recommendation for states (10) and (01). If condition (i) of Proposition 2 is satisfied, then (11) is also recommended when the state is (00). The remaining task is then straightforward and it has the same logic as providing the information for a single decision problem. If condition (i) of Proposition 2 does not hold, then the optimal recommendation rule pools states (10) and (01) with the state (11) as much as possible. For the remaining of times states (10) and (01) should be pooled with the state (00). Now, if the condition (ii) of Proposition 2 is satisfied, then the optimal pooling achieves the first best, whereas if this condition is violated then the sender has to reveal more information than it would do were the two dimensions independent. Thus, the correlation imposes a constraint that the sender cannot overcome when it provides information in a static way. In particular, for the sender to achieve a payoff that is higher than simply truthfully revealing the state of the world, it must be the case that either p_{10} or p_{01} (or both) is positive. If not, the sender is unable to pool states (10) and (01).

6 The Sequential Game — Dynamic Information Provision

In the simultaneous game, the sender induces a distribution of marginal posteriors by choosing the distribution of joint posteriors. The correlation constrains the sender's ability to design information in this setting. We now consider the sequential game, in which the receiver makes one decision at a time and the sender can provide information sequentially.

We first consider the following constrained set for the sender's strategies: in each stage of the game, the sender directly provides information only about one dimension of the state:

Assumption 4. *In each stage of the sequential game the sender chooses among the following recommendation rules:* $\left\{\pi(\cdot|\omega^i)\right\}_{\omega^i\in\Omega}$.

We refer to recommendation rules of both stages of the sequential game as a sequential recommendation rule. Assumption 4 says that in the sequential game, when the sender can provide information dynamically, the sender can directly inform only about one dimension of the payoff-relevant state. For example, with separate trials the prosecutor investigates one crime at a time. Or if the state agency decides about each drug separately, then the trial directly informs only about the drug that has to be decided. I first consider this constrained set of the sender's strategies. Section 8 then relaxes this restriction.

Thus, in a sequential game the sender directly chooses the distribution of marginals one at a time.

Before analysing the sequential game, I introduce some definitions.

Definition 4. $p^i(|\rho^i_-)$: *posterior for dimension* i*, when the recommendation for* i *is* ρ^i_-.

Definition 5. $p^j(|\rho^i_-)$: *posterior for dimension* j*, following a recommendation for dimension* i *of* ρ^i_-.

From the law of total probability and the fact that the recommendation for i directly informs only about dimension i, the receiver updates its belief about dimension j as follows:

$$p^j(|\rho^i_-) = p(i=1|j=1)p^i(|\rho^i_-) + p(i=1|j=0)(1-p^i(|\rho^i_-)) \tag{23}$$

Given the recommendation about dimension i, the new posterior of dimension j is the expectation of conditional probabilities, where the weights are given by the new posterior distribution of dimension i.

The law of total probability also implies that the expected posterior equals the prior. So, for example, when the first recommendation is made for dimension i, the expected posterior of j is the prior of j, i.e.,

$$E_{\rho^i} p^j(|\rho^i) = p^j. \tag{24}$$

We say that a recommendation rule for dimension i is optimal if it achieves the upper bound for the single dimension i.

I now ask the following question: Can the sender gain from dynamic information provision, vis-à-vis the static information provision? The following example answers this question:

Example C

$\omega^1 \backslash \omega^2$	1	0
1	0.0	0.5
0	0.5	0.0

Figure 8: Example C

For example C, Proposition 1 yields that in the simultaneous game the sender cannot achieve a higher payoff than truthfully revealing the payoff-relevant states — which gives the payoff 1. But in the sequential game the sender's equilibrium payoff is $1 + 0.5\frac{1-\alpha}{(\alpha)^2}$. This payoff is achieved by the following sequential recommendation rule. In stage I, the sender provides information about dimension 1. The optimal recommendation for dimension 1 induces following distribution of posteriors for dimension 1: α with probability $\frac{0.5}{\alpha}$ and 0 with probability $1 - \frac{0.5}{\alpha}$. We calculate $p^2(|\rho^1_-)$ by using equation (23): $p^2(|\rho^1_1) = 0 \cdot \alpha + 1 \cdot (1-\alpha) = 1 - \alpha$ and $p^2(|\rho^1_0) = p(\omega^2 = 1|\omega^1 = 0) = 1$. Note that when the receiver's belief is greater than α, then the sender does not gain from revealing any information. Therefore, the sender's expected payoff in the sequential game is: $(\frac{0.5}{\alpha})[1 + \frac{1-\alpha}{\alpha}] + (1 - \frac{0.5}{\alpha})[1] = 1 + 0.5\frac{1-\alpha}{(\alpha)^2}$. On the left-hand side, the first part is the sender's expected payoff from recommending action 1 for dimension 1 in stage I of the sequential game. So, for $\alpha < 1$, the sender can gain from providing the information sequentially.

I next characterize necessary and sufficient conditions for the feasibility of the upper bound on the sender's payoff in the sequential game. Recall that the upper bound on the sender's payoff is given by the independent case. What is special about the independent case is that the recommendation about dimension 1 does not contain any information about dimension 2. As noted in Theorem 1, the distribution of posteriors is only a function of a prior. When the two dimensions are correlated, then after a recommendation for dimension 1, the posterior of dimension 2 might change, which could affect the set of feasible distributions of posteriors. Thus, the question is how small must the change in posterior for dimension 2 be to achieve the upper bound.

I only consider direct recommendations. One cannot directly extend Kamenica and Gentzkow [2011] and Bergemann and Morris [2016] to a dynamic setting. But, in the spirit of Bergemann and Morris [2016], one can think about the sender as a mediator who makes recommendations to the receiver. Sugaya and Wolitzky [2017] show that it is without loss of generality to consider direct recommendations in multi-stage games if one uses perfect Bayesian equilibrium as a solution concept.

I now characterize a necessary and sufficient condition for a sequential recommendation to achieve the upper bound. In essence, the proposition says that following the first optimal recommendation, the posterior belief for the other problem cannot be too high.

Proposition 4. *Consider arbitrary DP:*

(a) *If the two dimensions are positively correlated, then there exists a sequential recommendation rule that achieves the upper bound if and only if there exists a recommendation order, such that when the first optimal recommendation is made for i,* $p^j(|\rho^i_1) \leq \alpha^j$.

(b) *If the two dimensions are negatively correlated, then there exists a sequential recommendation rule that achieves the upper bound if and only if there exists a recommendation order such that when the first optimal recommendation is made for i,* $p^j(|\rho^i_0) \leq \alpha^j$.

Before proving Proposition 4, we briefly explain its content. It says that posterior of j should not exceed its decision threshold. With positive correlation, this can only happen for the recommendation ρ^i_1. In contrast, with negative correlation posterior can exceed the decision threshold only when the recommendation is ρ^i_0, given that recommendation ρ^i is optimal.

We prove Proposition 4 in two steps. First we describe the constraints on the distribution of posteriors of j that must be satisfied to achieve the upper bound for dimension j, when the first recommendation is made for i.

Lemma 4. *Consider an arbitrary DP. Say the first recommendation is produced for dimension i. There exists a sequential recommendation rule that induces the optimal distribution of posteriors of j, when the first recommendation is made for i, if and only if:*

$p^j(|\rho^i_-) \leq \alpha^j$ *for any* ρ^i_-*. That is, the support of the distribution of* j*'s posteriors induced by the recommendation rule for* i *belongs to the interval* $[0, \alpha^j]$*.*

Proof. When choosing the distribution of posteriors, the sole constraint is that the expected posterior equals the prior (Lemma 1). From this it follows that for any posterior of j below the decision threshold — i.e., for any $p^j(|\rho^i_-) \leq \alpha^j$, the sender can choose the distribution of posteriors with support α^j and 0. Then it follows from the law of total probability that, in expectation, the sender is choosing the desired distribution of posteriors.

Say now $\exists\ \rho^i_-$, such that $p(\rho^i_-) > 0$ and $p^j(|\rho^i_-) > \alpha^j$. Then for this posterior of j the sender will not be able to choose a distribution of posteriors with support α^j and 0 - because again from the law of total probability the distribution of posteriors has to contain an element that exceeds α^j.

□

From the law of total probability it also follows, given Assumption 1, that $p^j(|\rho^i_-)$ can exceed α^j only for a single recommendation, since otherwise the expected posterior would exceed the prior.

We want to look at the support of the distribution of j's posteriors, when the recommendation rule for i is optimal, i.e., when the sender induces the distribution of posteriors for i with support $\{0, \alpha^i\}$.

The next Lemma shows that if the two dimensions are positively correlated then $p^j(|\rho^i_-) > \alpha^j$ only if $\rho^i_- = \rho^i_1$. If the two dimensions are negatively correlated then $p^j(|\rho^i_-) > \alpha^j$ only if $\rho^i_- = \rho^i_0$.

Lemma 5. (a) *Suppose that the two dimensions are positively correlated and the sender chooses the optimal recommendation rule for dimension* i*. Then the posterior of* j *can exceed the decision threshold only if* 1 *is recommended for dimension* i*.*

(b) *Suppose that the two dimensions are negatively correlated and the sender chooses the optimal recommendation rule for dimension* i*. Then the posterior of* j *can exceed the decision threshold only if* 0 *is recommended for dimension* i*.*

Proof. $p(j = 1|i = 1) > p^j$ if and only if the covariance is positive and $p(j = 1|i = 0) > p^j$ if and only if the covariance is negative. Recall that

$$p^j(|\rho_1^i) = p(i = 1|i = 1)\alpha^i + p(j = 1|i = 0)(1 - \alpha^i) \tag{25}$$

and

$$p^j(|\rho_0^i) = p(i = 1|i = 0). \tag{26}$$

Let's write the prior of dimension j as

$$p^j = p(i = 1|i = 1)p^i + p(j = 1|i = 0)(1 - p^i). \tag{27}$$

The equality follows from the law of total probability. Remember that by Assumption 1, $p^i < \alpha^i$ and $p^j < \alpha^j$. Therefore, (25) > (27) only if i and j are positively correlated and (26) > (27) only if i and j are negatively correlated.

□

Now we can prove the Proposition 4.

Proof. The proof follows from combining arguments of Lemmas 4 and 5 and noting that the sequential recommendation rule can achieve the upper bound only if the recommendation for dimension i is optimal.

□

After describing a necessary and sufficient condition for a sequential recommendation rule to achieve the upper bound, we want to say something about the existence of recommendation rules that achieve the upper bound. Note that if $p_{11} = p_{00} = 0$, then there does not exist a sequential recommendation rule that achieves the upper bound.[3] This is because, regardless of the order of recommendations, $p^j(|\rho_0^i) > \alpha^j$. The claim then follows from Proposition 4.

[3] I thank Sergiu Hart for providing an example where the sequential procedure fails to achieve the upper bound.

Thus, one sees that if the two dimensions are negatively correlated then there are DPs for which the sequential procedure fails to achieve the upper bound. Now suppose the two dimensions are positively correlated.

$\omega^1 \backslash \omega^2$	1	0
1	0.40	0.25
0	0.10	0.25

Figure 9: Example D: positive correlation, $\alpha^i = \alpha^j = 0.7$

From Proposition 4 if a sequential recommendation achieves the upper bound, then one can find an order, such that $p^j(|\rho_1^i) < \alpha^j$.

If the first recommendation is made for dimension 2, then applying (23) one gets:

$$p^1(|\rho_1^2) = \frac{0.40}{0.40 + 0.10}0.7 + \frac{0.25}{0.25 + 0.25}0.3 = 0.71. \tag{28}$$

Similarly, if the first recommendation is made for dimension 1:

$$p^1(|\rho_1^1) = \frac{0.40}{0.40 + 0.25}0.7 + \frac{0.10}{0.10 + 0.25}0.3 = 0.516. \tag{29}$$

Therefore, a sequential recommendation rule can achieve the upper bound if the first recommendation is made for dimension 1, but not if the first recommendation is made for dimension 2. This example shows that order can matter and by choosing the order of recommendations properly the sender can achieve the upper bound.

Starting from this example one could ask: fixing the decision threshold, does there exist a common prior joint distribution, where the two dimensions are not negatively correlated and yet the sequential procedure fails to achieve the upper bound? If not, maybe there is something special about the given threshold and therefore one could consider arbitrary thresholds and ask the question: does there exist DP, where dimensions are not negatively correlated and the sequential procedure fails to achieve the upper bound? Somewhat surprisingly, the answers to all of these questions are negative. That is, as long as two dimensions are not negatively correlated, then there always exists a sequential procedure that achieves the upper bound.

Proposition 5. *If two dimensions are positively correlated, then there exists a sequential recommendation rule that achieves the upper bound.*

First I show that if two dimensions are positively correlated, then there always exists i such that when the recommendation rule for i is optimal, $p^j(|\rho_1^i) < \alpha^j$.

Lemma 6. *Suppose that the two dimensions are positively correlated and the recommendation rule in stage I is optimal. Then there exist i and $j \in \{1, 2\}$, $i \neq j$ such that $p^j(|\rho_1^i) < \alpha^j$.*

Proof. We split the proof in two parts. First we consider the case when decision thresholds are the same: $\alpha^1 = \alpha^2 = \alpha$.

If the first recommendation is made for dimension 1, then if posterior of dimension 2 exceeds the decision threshold:

$$p^2(|\rho_1^1) = \frac{p_{11}}{p_{11} + p_{10}}\alpha + \frac{p_{01}}{p_{01} + p_{00}}(1 - \alpha) > \alpha \tag{30}$$

One can rearrange this expression as:

$$\frac{p_{11} + p_{10}}{\alpha} \frac{1 - \alpha}{p_{01} + p_{00}} > \frac{p_{10}}{p_{01}} \tag{31}$$

By symmetry, one gets a similar expression for the case when the first recommendation is made for dimension 2:

$$\frac{p_{11} + p_{01}}{\alpha} \frac{1 - \alpha}{p_{10} + p_{00}} > \frac{p_{01}}{p_{10}} \tag{32}$$

To see that both inequalities (31) and (32) cannot be satisfied simultaneously, note that by Assumption 1, the left hand side of both inequalities is strictly smaller than 1. However at least one of the ratios on the right hand sides of (31) and (32) is at least one.

This proves the claim when decision thresholds are the same. Now consider different decision thresholds: $\alpha^1 \neq \alpha^2$. To simplify presentation suppose that $\alpha^1 < \alpha^2$.

Note that from inequalities (31) and (32) the posterior can exceed the decision threshold only if the first recommendation is made for a dimension with the smaller prior. If $p^1 \geq p^2$, then from the previous argument it follows that the claim is true. This is so, because from the argument above $p^2(|\rho_1^1) \leq \alpha^1$. Now we consider the more interesting case, where $p^1 < p^2$.

For the sequential recommendation rule to fail the upper bound, the following two inequalities must hold:

$$\frac{p_{11}}{p_{11}+p_{10}}\alpha^1 + \frac{p_{01}}{p_{01}+p_{00}}(1-\alpha^1) > \alpha^2 \tag{33}$$

$$\frac{p_{11}}{p_{11}+p_{01}}\alpha^2 + \frac{p_{10}}{p_{10}+p_{00}}(1-\alpha^2) > \alpha^1 \tag{34}$$

It is impossible for inequalities (33) and (34) to hold simultaneously. To show this I use Mathematica. The command Reduce gives False as an outcome.

□

Now we prove Proposition 5.

Proof. The proof follows from Proposition 4 and Lemma 6. □

I now want to characterize optimal sequential recommendation rules when the upper bound is infeasible. It follows from Propositions 4 and 5 that the two dimensions must be negatively correlated and that $p^j(|\rho_0^i) > \alpha^j$, for any pair i, j. I describe optimal sequential recommendation rules for the case when decision thresholds are equal: $\alpha^1 = \alpha^2 = \alpha$.

Proposition 6. *Consider a class of DPs, for which the sequential procedure fails to achieve the upper bound and $\alpha^1 = \alpha^2$. Suppose the first recommendation achieves maximum payoff for the single dimension and $p^i \geq p^j$. Then the sender's payoff is maximized by first providing information about dimension i.*

To prove Proposition 6 I solve for the loss from the sequential procedure, defined as the deviation from the upper bound payoff.

Proof. Say the first recommendation is for dimension 1. Then the sender's expected loss is:

$$[1 - \frac{p_{11} + p_{10}}{\alpha}][\frac{p_{01}}{p_{01} + p_{00}}\frac{1}{\alpha} - 1], \tag{35}$$

where the expression in the first bracket is the probability of ρ_0^1 and the expression in the second bracket is the loss, which is positive by assumption. It is a loss, because $p^2(|\rho_0^1) = \frac{p_{01}}{p_{01}+p_{00}}$ and the upper bound would have been achieved if sender's payoff were $\frac{p_{01}}{p_{01}+p_{00}}\frac{1}{\alpha}$, when it is only 1.

Now, if the first recommendation is for dimension 2, then the sender's expected loss, by symmetry, is:

$$[1 - \frac{p_{11} + p_{01}}{\alpha}][\frac{p_{10}}{p_{10} + p_{00}}\frac{1}{\alpha} - 1]. \tag{36}$$

Now $p^1 \gtreqless p^2 \iff p_{10} \gtreqless p_{01}$. So, if $p_{10} > p_{01}$, then (36) $>$ (35), i.e., the sender's loss is greater from first providing the information about the dimension 2. This completes the proof.

□

Proposition 6 shows the sender's maximum payoff in the sequential game with the constrained strategy set, when the upper bound is not feasible and the sender chooses the optimal recommendation rule for dimension i in stage I. The sender could choose an alternative recommendation rule at stage I such that $p^j(|\rho_0^i) = \alpha$, i.e., it is not an optimal recommendation rule for dimension i. By this procedure the sender achieves the upper bound for dimension j but fails to achieve the upper bound for dimension i. For the examples I checked, the payoff from this procedure is always less than the payoff the sender gets from choosing the optimal recommendation rule for dimension i. I conjecture that Proposition 6 describes the sender's optimal payoff in the sequential game with constrained strategy set, when the upper bound payoff is not feasible.

7 Comparing Simultaneous and Sequential Games

Having described the sender's equilibrium payoffs in the simultaneous and sequential games, we want to compare these two game forms. We are interested when the equilibrium payoff in the sequential game strictly exceeds the equilibrium payoff in the simultaneous game.

I first define a preference relation between simultaneous and sequential games:

Definition 6. *The sequential game is preferred to the simultaneous one for a class of DPs if for this class of DPs, the sender's equilibrium payoff in the sequential game is always at least as high as the equilibrium payoff in the simultaneous game.*

We first consider settings where the two dimensions are positively correlated. It follows immediately, for this class of decision problems, the sequential game is preferred to the simultaneous one.

Theorem 2. *If the two dimensions are positively correlated, then the sequential game is preferred to the simultaneous one. In particular:*

(i) *If decision thresholds are equal – $\alpha^1 = \alpha^2$ – then the simultaneous and sequential games are equivalent, i.e., the equilibrium payoffs are the same and achieve the upper bound payoff.*

(ii) *If the decision thresholds differ – $\alpha^1 \neq \alpha^2$ – then the sender's equilibrium payoff in the sequential game is the upper bound payoff and the sequential game is preferred to the simultaneous one.*

Proof. Proposition 5 says that there always exists a sequential recommendation rule that achieves the upper bound. To prove the theorem it remains to show that for the same decision thresholds the equilibrium payoff in the simultaneous game always equals the upper bound payoff, whereas for different decision thresholds it sometimes falls short.

(i) Say $\alpha^1 = \alpha^2 = \alpha$. Then for a simultaneous recommendation rule to achieve the upper bound, one of the conditions of proposition 2 has to be satisfied. Condition (i) is always satisfied for $\alpha \leq 0.5$. If $\alpha > 0.5$, then condition (ii)

cannot be violated if the two dimensions are positively correlated. To establish this I use Mathematica, using the Reduce command.

(ii) Say $\alpha^1 < \alpha^2$ and consider the following class of decision problems:

$\omega^1 \backslash \omega^2$	1	0
1	x	0
0	0	$1-x$

Then the sender can make one of the two recommendations: it can recommend ρ_{11} and get payoff $\frac{2x}{\alpha^2}$ or recommend ρ_{10} and get payoff $\frac{x}{\alpha^1}$. But note that in both of these cases the sender's payoff is less than the upper bound of $\frac{x}{\alpha^1} + \frac{x}{\alpha^2}$.

□

Theorem 2 shows that even when two states are perfectly aligned, the simultaneous recommendation rule of the simultaneous game lacks the flexibility of the sequential game. Different decision thresholds introduce additional tension between decision problems, that static information provision cannot overcome. Now we will analyze the case when decision problems are negatively correlated.

We want to compare static and dynamic information provision when the upper bound is not feasible. But first we note that there are cases, when one of the procedures achieves the upper bound, but the other fails. For example, if decision thresholds are equal and do not exceed 0.5, then condition *(i)* of proposition 6 is always satisfied and therefore a simultaneous recommendation rule always achieves the upper bound. I conjecture that this result also holds for different decision thresholds, as long as the highest threshold does not exceed 0.5. For negative correlation, one can also construct examples when the equilibrium payoff in the sequential game achieves the upper bound and the simultaneous recommendation rule fails. The following analysis assumes the same decision thresholds for both decision problems.

Theorem 3. *Consider the class of decision problems for which the equilibrium payoffs both in the simultaneous or sequential games fall are less than the upper bound payoff. Then for an arbitrary decision threshold α there exist joint prior* *distributions with p_{11} small enough such that the sender prefers the sequential game to the simultaneous one.*

Proof. If the upper bound is not feasible, then it follows from Proposition 3 that the equilibrium payoff in the simultaneous game is:

$$1 + p_{11}\frac{1}{2\alpha - 1} \tag{37}$$

Say $p_{10} \geq p_{01}$. Then Proposition 6 implies that the equilibrium payoff in the sequential game is:

$$\frac{p_{11}+p_{10}}{\alpha}[1 + (\frac{p_{11}}{p_{11}+p_{10}}\alpha + \frac{p_{01}}{p_{01}+p_{00}}(1-\alpha))\frac{1}{\alpha}] + 1 - \frac{p_{11}+p_{10}}{\alpha} \tag{38}$$

Expression (38) simplifies to:

$$1 + p_{11}\frac{1}{\alpha} + \frac{p_{11}+p_{10}}{p_{01}+p_{00}}p_{01}\frac{1-\alpha}{(\alpha)^2} \tag{39}$$

(39) $\geq$ (37) $\iff$

$$(p_{11}+p_{10})\frac{p_{01}}{p_{01}+p_{00}} \geq p_{11}\frac{\alpha}{2\alpha-1} \tag{40}$$

Thus, as $p_{11} \to 0$, the payoff in the simultaneous game approaches 1, whereas the payoff in the sequential game is strictly bounded away from 1 for $\alpha \neq 1$. □

8 The Sequential Game with Unrestricted Strategy Set

Until now, the analysis of the sequential game considered constrained set of strategies for the sender. While Assumption 4 is plausible, I now drop this assumption and analyze the sender's payoff in the sequential game for an unconstrained strategy set.

First note that it is without loss of generality to consider the constrained set of signals, as defined by the Assumption 4, in the stage 2 of the sequential game. This is because stage 2 is the last stage of the game and the receiver has only one decision to make. We seek DPs, for which the sender fails to achieve the upper bound in the simultaneous game or the sequential game with the constrained strategy set.

I first describe the class of DPs, for which the sender's payoff in the sequential game does not decrease from considering the constrained set of strategies.

Proposition 7. *Consider the class of DPs for which $p_{11} = p_{00} = 0$ and $\alpha^1 = \alpha^2 = \alpha$. Then the sender's equilibrium payoff in the sequential game are the same when the sender can directly only inform about a single dimension at a time, as when the sender has an unconstrained set of strategies.*

Proof. Say without loss of generality that $p_{10} \geq p_{01}$. It follows from expression (39) that the sender's equilibrium payoff in the sequential game with the constrained strategy set is:

$$1 + p_{10}\frac{1-\alpha}{(\alpha)^2} \tag{41}$$

Say now the sender chooses the following sequential game: in stage I the receiver decides decision problem 1. Thus, in stage I the sender chooses a recommendation rule that recommends 1 for decision problem 1. If not, then the sender's maximum payoff is $p_{01}\frac{1}{\alpha} < 1$, which is less than the payoff from (41). Note that in stage I the sender chooses among the following recommendation rules:

$\rho\backslash\Omega$	(11)	(10)	(01)	(00)
ρ_{11}	0	0	0	0
ρ_{10}	0	$p_{10} - y$	x	0
ρ_{01}	0	y	$p_{01} - x$	0
ρ_{00}	0	0	0	0

Therefore, the sender's expected payoff is either:

$$p(\rho_{10})(1 + \frac{1}{\alpha}\frac{x}{p_{10} - y + x}) + (1 - p(\rho_{10})) = 1 + \frac{x}{\alpha} \tag{42}$$

or

$$p(\rho_{10})(1+\frac{1}{\alpha}\frac{x}{p_{10} - y + x})+(1-p(\rho_{10}))(\frac{1}{\alpha}\frac{p_{01} - x}{p_{01} - x + y}) = p(\rho_{10}) + \frac{x}{\alpha} + \frac{p_{01} - x}{\alpha} \tag{43}$$

Equation (42) denotes the sender's expected payoff in the sequential game if the recommendation rule in stage I is such that $p^2(|\rho_{01}) > \alpha$. Equation (43) denotes the sender's expected payoff if $p^2(|\rho_{01}) < \alpha$. Rewrite equations (42) and (43) as:

$$p_{10} + p_{01} + \frac{x}{\alpha} \tag{44}$$

$$p_{10} - y + x + \frac{p_{01}}{\alpha} \tag{45}$$

(45) > (44) $\iff$

$$(p_{01} - x)\frac{1 - \alpha}{\alpha} > y \tag{46}$$

But (46) implies $p^2(|\rho_{01}) > \alpha$, a contradiction. Therefore, the sender's payoff is maximized by the following sequential recommendation rule: $p^1(|\rho_{10}) \geq \alpha$ and $p^2(|\rho_{01}) > \alpha$, which means maximizing $1 + \frac{x}{\alpha}$. The sender's optimal payoff for this optimization problem is given exactly by equation 41. Now we developed this argument for the game when the receiver makes the first decision about dimension 1. It follows from the symmetry that if the receiver first decided about dimension 2, then the sender's optimal payoff would be $1+p_{01}\frac{1-\alpha}{(\alpha)^2}$. This completes the proof.

□

Proposition 7 described a class of DPs for which the constrained strategy set does not decrease the sender's payoff in the sequential game. Example B' shows that this is not always the case.

$\omega^1 \backslash \omega^2$	1	0
1	0.05	0.40
0	0.40	0.15

Figure 10: Example B': $\alpha = 0.7$

For this example, it follows from Propositions 2 and 4 that the sender is not able to achieve the upper bound if it chooses the simultaneous game or the sequential game with restricted strategy set. Now we construct the sequential recommendation rule that achieves the upper bound.

In stage I the receiver makes a decision about dimension 1 and the sender chooses the following recommendation rule:

$\rho \backslash \Omega$	(11)	(10)	(01)	(00)
ρ_{11}	0.0500	0.0375	0.0375	0
ρ_{10}	0	0.3625	0.1553	0
ρ_{01}	0	0	0.2071	0.0887
ρ_{00}	0	0	0	0.0612

In stage II the sender will produce informative recommendation rule only if in stage I the recommendation was ρ_{10}, because $p^2(|\rho_{10}) = 0.3$. Otherwise $p^2(|\rho_{-})$ is either 0.7 or 0. This sequential recommendation rule gives sender the following payoff:

$$2p(\rho_{11}) + p(\rho_{10})(1 + \frac{0.3}{0.7}) + p(\rho_{01}) = 1.2857 = 2\frac{0.45}{0.7}. \tag{47}$$

After having considered cases when the sender's equilibrium payoff in the sequential game with unconstrained strategy set is (not) the upper bound payoff, to complete the analysis, we want to answer the following questions: what are the conditions on the recommendation rules to achieve the upper bound? When are these conditions satisfied and what is the sender's equilibrium payoff when the upper bound is unattainable.

Before answering these questions, I clarify the use of notation I am adopting in this section. In stage I of the game the sender makes the recommendation only for dimension i. But, since we are dropping Assumption 4, I use the same notation for the recommendation rule in stage I, as I used when discussing simultaneous recommendation rules. For an illustration, consider the following example: in stage I the receiver decides about dimension 1 and the sender makes the recommendation ρ_{11}. The interpretation is that given recommendation ρ_{11} the receiver chooses 1 for dimension 1 in stage I; in stage II the sender does not provide any information, or always recommends action 1 for dimension 2, so that the receiver chooses 1 for dimension 2 in stage II.

Now I describe a necessary and sufficient condition to achieve the upper bound payoff, in terms of the recommendation rule in stage I.

Proposition 8. *The sender achieves the upper bound payoff if and only if there exists a recommendation order, s.t. when in stage I the recommendation is made for dimension i, the following is true:*

(*i*) *support of distribution of marginal posteriors for dimension i is* 0 *and* α*; and*

(*ii*) *support of distribution of marginal posteriors for dimension j is weakly smaller than* α*.*

Before proving Proposition 8, I compare it with Proposition 2 to point out the fact that in the sequential game with unconstrained strategy set the sender faces a relaxed problem of the simultaneous game. When proving Proposition 2, I showed in Lemma 2 that the feasibility of the upper bound required support of distribution of marginal posteriors to be 0 and α for both dimensions. In contrast, in the sequential game with unconstrained strategy set one relaxes this condition for dimension j, by requiring the support to belong to the interval $[0, \alpha]$.

The proof of Proposition 8 is an application of the law of total probability.

Proof. To simplify the exposition of the proof, let's consider the case when in stage I the receiver decides about dimension 1.

if: If condition (i) is satisfied, then the sender's expected payoff for dimension 1 in stage I is:

$$p(\rho_{11}) + p(\rho_{10}), \tag{48}$$

probability of recommending action 1 for dimension 1. Following Remark 1, we want to show that

$$p(\rho_{11}) + p(\rho_{10}) = \frac{p_{11} + p_{10}}{\alpha} \tag{49}$$

From Lemma 1, following is true:

$$p_{11} = p(11|\rho_{11})p(\rho_{11}) + p(11|\rho_{10})p(\rho_{10}) + p(11|\rho_{01})p(\rho_{01}) + p(11|\rho_{00})p(\rho_{00}) \tag{50}$$

$$p_{10} = p(10|\rho_{11})p(\rho_{11}) + p(10|\rho_{10})p(\rho_{10}) + p(10|\rho_{01})p(\rho_{01}) + p(10|\rho_{00})p(\rho_{00}) \tag{51}$$

Adding (50) to (51) and remembering that condition (i) holds, one gets:

$$\begin{aligned} p_{11} + p_{10} &= [p(11|\rho_{11}) + p(10|\rho_{11})]p(\rho_{11}) + [p(11|\rho_{10}) + p(10|\rho_{10})]p(\rho_{10}) \\ &= \alpha p(\rho_{11}) + \alpha p(\rho_{10}) = \alpha(p(\rho_{11}) + p(\rho_{10})) \end{aligned} \tag{52}$$

The second equality follows from condition (i): when 1 is recommended for dimension 1 marginal posterior is α and when 0 is recommended then marginal posterior is 0.

We want to show that if condition (ii) is satisfied, then the sender's expected payoff in two stages of the game for dimension 2 is:

$$\frac{p_{11} + p_{01}}{\alpha}, \tag{53}$$

the upper bound payoff, as defined in Remark 1.

First, let's write the sender's expected payoff in the sequential game for dimension 2:

$$p(\rho_{11})min\{1, \Big(\frac{p(\rho_{11}\cap 11)}{p(\rho_{11})}+\frac{p(\rho_{11}\cap 01)}{p(\rho_{11})}\Big)\frac{1}{\alpha}\}+p(\rho_{10})min\{1, \Big(\frac{p(\rho_{10}\cap 11)}{p(\rho_{10})}+\frac{p(\rho_{10}\cap 01)}{p(\rho_{10})}\Big)\frac{1}{\alpha}\}+ \\ p(\rho_{01})min\{1, \Big(\frac{p(\rho_{01}\cap 11)}{p(\rho_{01})}+\frac{p(\rho_{01}\cap 01)}{p(\rho_{01})}\Big)\frac{1}{\alpha}\}+p(\rho_{00})min\{1, \Big(\frac{p(\rho_{00}\cap 11)}{p(\rho_{00})}+\frac{p(\rho_{00}\cap 01)}{p(\rho_{00})}\Big)\frac{1}{\alpha}\} \tag{54}$$

Let's briefly explain expression (54): if recommendation is ρ_{11} in stage I, then the sender's expected payoff is either marginal posterior of dimension 2 times $\frac{1}{\alpha}$ or 1; in particular if the marginal posterior is bigger than α, then the expected payoff is 1.

Now, if condition (ii) holds, then marginal posterior is never bigger than α, therefore expression (54) becomes:

$$p(\rho_{11})\Big(\frac{p(\rho_{11}\cap 11)}{p(\rho_{11})}+\frac{p(\rho_{11}\cap 01)}{p(\rho_{11})}\Big)\frac{1}{\alpha}+p(\rho_{10})\Big(\frac{p(\rho_{10}\cap 11)}{p(\rho_{10})}+\frac{p(\rho_{10}\cap 01)}{p(\rho_{10})}\Big)\frac{1}{\alpha}+ \\ p(\rho_{01})\Big(\frac{p(\rho_{01}\cap 11)}{p(\rho_{01})}+\frac{p(\rho_{01}\cap 01)}{p(\rho_{01})}\Big)\frac{1}{\alpha}+p(\rho_{00})\Big(\frac{p(\rho_{00}\cap 11)}{p(\rho_{00})}+\frac{p(\rho_{00}\cap 01)}{p(\rho_{00})}\Big)\frac{1}{\alpha}= \\ =[p(\rho_{11}\cap 11)...+...p(\rho_{00}\cap 01)]\frac{1}{\alpha}=\frac{p_{11}+p_{01}}{\alpha}, \tag{55}$$

where the last equality follows from Lemma 1.

only if: if condition (i) is violated, then adding equation (50) to (51) gives:

$$p_{11}+p_{10}=[p(11|\rho_{11})+p(10|\rho_{11})]p(\rho_{11})+[p(11|\rho_{10})+p(10|\rho_{10})]p(\rho_{10})+ \\ [p(11|\rho_{01})+p(10|\rho_{01})]p(\rho_{01})+[p(11|\rho_{00})+p(10|\rho_{00})]p(\rho_{00}) \tag{56}$$

Since condition (i) is violated, from this follows that marginal posterior for dimension 1 is either bigger than α when 1 is recommended for dimension 1 or it is bigger than 0, when 0 is recommended for dimension 1. But then note, that equation (56) implies:

$$p(\rho_{11}) + p(\rho_{10}) < \frac{p_{11} + p_{10}}{\alpha}. \tag{57}$$

If condition (ii) is violated for dimension 2, then from this follows that there exists a recommendation in stage 1, s.t. function $min\{1, \cdot\}$ uniquely evaluates to 1. It follows from expression (55) that the sender's expected payoff for dimension 2 is smaller than the upper bound payoff.

□

Proposition 8 makes the search for optimal sequential recommendation rules tractable. For ease of exposition I maintain Assumption 3, that says that $p_{01} \leq p_{10}$.

Proposition 9. *Sender's equilibrium payoff in the sequential game with unconstrained strategy set achieves the upper bound if and only if in stage I the receiver makes decision about dimension* 1 *and the following recommendation rule is feasible in stage I:*

$\rho\backslash\Omega$	(11)	(10)	(01)	(00)
ρ_{11}	0	0	0	0
ρ_{10}	p_{11}	p_{10}	a	b
ρ_{01}	0	0	$p_{01} - a$	c
ρ_{00}	0	0	0	d

and a, b *and* c *are such that* $p^1(|\rho_{10}) = p^2(|\rho_{01}) = \alpha$.

Proof.

if: it is straightforward that conditions of Proposition 8 are satisfied.

Only if: Note that the only reason why the suggested recommendation rule can not be feasible is the following: there does not exist a, b and c for which

$p^2(|\rho_{01}) = \alpha$. This is so because by Assumption 1, $p_{11}+p_{10} < \alpha$, therefore there always exist a and b, s.t. $p^1(|\rho_{10}) = \alpha$. $p^2(|\rho_{01}) = \alpha$ implies $c = (p_{01} - a)\frac{1-\alpha}{\alpha}$, because $p^2(|\rho_{01}) = \frac{p_{01}-a}{p_{01}-a+c}$. From this follows that c is decreasing in a. By the similar logic, one sees that a is increasing in p_{10}. From this follows that if the suggested recommendation rule is not feasible, then similar recommendation rule would not be feasible if the first recommendation were made for dimension 2, because then the following condition has to be satisfied: $p^1(|\rho_{10}) = \frac{p_{10}-a}{p_{10}-a+c}$ and $p_{10} - a \geq p_{01} - a$.

□

After having described the recommendation rule that achieves the upper bound, now I will describe the optimal recommendation rule and the sender's equilibrium payoff when the upper bound is not feasible.

Proposition 10. *When the upper bound is not feasible, then in stage I the receiver decides about dimension* 1 *and the sender's equilibrium payoff is:* $1 + \frac{p_{11}}{\alpha} + (p_{11} + p_{10})\frac{1-\alpha}{(\alpha)^2}$.

Proof. When the upper bound is not feasible, then following the argument of Lemma 3, $p(\rho_{00}) = 0$. Therefore, the sender's expected payoff is:

$$1 + \frac{p_{11}}{\alpha} + (p_{11} + p_{10})\frac{1-\alpha}{(\alpha)^2} \tag{58}$$

if the receiver decides about dimension 1 in stage I and

$$1 + \frac{p_{11}}{\alpha} + (p_{11} + p_{01})\frac{1-\alpha}{(\alpha)^2} \tag{59}$$

if the receiver decides about dimension 2 in stage I.

Expression 58, for example, follows from the following: The sender's expected payoff is: $p(\rho_{10})[1 + \frac{p_{11}+a}{p(\rho_{10})}\frac{1}{\alpha}] + p(\rho_{01})[1] = 1 + \frac{p_{11}+a}{\alpha}$ and $a = (p_{11} + p_{10})\frac{1-\alpha}{\alpha}$. The first equality follows from the fact that $p(\rho_{00}) = 0$ and the second equality follows from the fact that $(p_{11} + p_{10})\frac{1-\alpha}{\alpha} < p_{01}$, if the upper bound is not feasible.

□

9 Comparing the Equilibrium Payoff in the Sequential Game with Unconstrained Strategies with Equilibrium Payoffs in other Games

After having solved the sequential game with unconstrained strategies, now I want to compare the payoffs in this game with other equilibrium payoffs. First I show that if the sender does not achieve the equilibrium payoff in the simultaneous game, then the sender's equilibrium in the sequential game is strictly higher.

Proposition 11. *If the sender's equilibrium payoff in the simultaneous game is smaller than the upper bound payoff, then the sender's equilibrium payoff in the sequential game with unconstrained strategy set is strictly bigger than the sender's equilibrium payoff in the simultaneous game.*

Proof. If the sender can not achieve the upper bound payoff in the simultaneous game, then it follows from Proposition 3 that for the optimal simultaneous recommendation rule obedience constraint does not bind either for ρ_{10} or ρ_{01}. It also follows from Lemma 3 that the optimal simultaneous recommendation rule has the following form:

$\rho\backslash\Omega$	(11)	(10)	(01)	(00)
ρ_{11}	a	b	c	d
ρ_{10}	e	f	g	h
ρ_{01}	i	j	k	l
ρ_{00}	0	0	0	0

First, consider the case when neither of the obedience constraints bind, i.e., $p^1(|\rho_{10}) > \alpha$ and $p^2(|\rho_{01}) > \alpha$ and consider the following sequential recommendation rule:

$\rho\backslash\Omega$	(11)	(10)	(01)	(00)
ρ_{11}	a	b	c	d
ρ_{10}	e	f	$g+\epsilon$	h
ρ_{01}	i	j	$k-\epsilon$	l
ρ_{00}	0	0	0	0

Note that $\exists\ \epsilon > 0$ small enough such that the suggested recommendation rule satisfies the obedience constraints; therefore, the sender's equilibrium payoff in the sequential game with unconstrained strategies is at least:

$$\begin{aligned} 2p(\rho_{11}) + (p(\rho_{10}) + \epsilon)[1 + \frac{\epsilon}{p(\rho_{10}) + \epsilon}\frac{1}{\alpha}] + (p(\rho_{01}) - \epsilon) = \\ 2p(\rho_{11}) + p(\rho_{10}) + p(\rho_{01}) + (p(\rho_{10}) + \epsilon)\frac{\epsilon}{p(\rho_{10}) + \epsilon}\frac{1}{\alpha} \end{aligned} \tag{60}$$

where $p(\rho_{\_})$ is the probability of recommendation $\rho_{\_}$ of the simultaneous game and $\frac{\epsilon}{\alpha}$ is the gain from recommending 1 for dimension 2 in stage II, when 1 was recommended for dimension 1 in stage I.

Now consider the case when only one of the obedience constraints binds. Say, for the sake of argument, that $p^2(|\rho_{01}) > \alpha$ and $p^1(|\rho_{10}) = \alpha$. Then $\exists\ \epsilon > 0$, s.t. the following recommendation rule is feasible and satisfies obedience constraints in stage I.

$\rho \backslash \Omega$	(11)	(10)	(01)	(00)
ρ_{11}	a	b	c	d
ρ_{10}	e	f	$g + \epsilon$	$h - \epsilon$
ρ_{01}	i	j	$k - \epsilon$	$l + \epsilon$
ρ_{00}	0	0	0	0

In particular, $p^1(|\rho_{10}) = \alpha$ implies that there exists a recommendation rule for which $h > 0$. But then for the suggested recommendation rule the sender's equilibrium payoff in the sequential game with unconstrained strategies is at least:

$$\begin{aligned} 2p(\rho_{11}) + p(\rho_{10})[1 + \frac{\epsilon}{p(\rho_{10})}\frac{1}{\alpha}] + p(\rho_{01}) = \\ 2p(\rho_{11}) + p(\rho_{10}) + p(\rho_{01}) + p(\rho_{10})\frac{\epsilon}{p(\rho_{10})}\frac{1}{\alpha} \end{aligned} \tag{61}$$

This completes the argument.

□

Proposition 11 shows that if the sender is not able to achieve the upper bound payoff in the simultaneous game, then it is strictly better off in the sequential game with unconstrained strategies. Now I compare the sender's equilibrium payoffs in sequential games with and without constrained strategy sets, when the upper bound is out of reach.

Proposition 12. *Say the equilibrium payoff in any game is smaller than the upper bound payoff. If* $p_{00} = 0$*, then the sender's equilibrium payoff is the same in the sequential games with and without constrained strategy sets.*

Proof. Expression 39 describes the sender's equilibrium payoff in the sequential game with constrained strategies, when the upper bound is not feasible. It can be written in the following way:

$$1 + p_{11}\frac{1}{\alpha} + (p_{11} + p_{10})\frac{p_{01}}{p_{01} + p_{00}}\frac{1-\alpha}{(\alpha)^2} \tag{62}$$

Note that expression 62 equals the sender's equilibrium payoff in the sequential game with unconstrained strategy set if $p_{00} = 0$ and the upper bound can not be reached.

□

10 Conclusion

This paper investigated the role of ex-ante information correlation in information design. We analyzed a setting where the ex-ante information correlation can constrain a sender's ability to design information. For the same decision thresholds, we completely characterized the sender's equilibrium payoff in the simultaneous game. We provide a necessary and sufficient condition to achieve the upper bound. We showed that in the simultaneous game ex-ante information correlation can not only decrease the sender's equilbrium payoff, but the sender might not be able to achieve higher payoff than revealing the payoff-relevant states truthfully.

Then we considered the sequential game with a constrained strategy set. In which we showed that if decision problems are positively correlated, then the sender's equilibrium payoff is always the upper bound payoff. Moreover with negative correlation the sender can always get higher payoff than what it would get from truthfully revealing the states of the world.

Then we compared simultaneous and sequential games. If decision problems are positively correlated, then it matters whether the receiver sets the same decision thresholds in both problems or not. If the decision thresholds are the same, the sender achieves the upper bound in both games. If decision thresholds are not the same, then the equilibrium payoff in the simultaneous game can fall short of the upper bound payoff, whereas the sender always achieves the upper bound in the sequential game even with constrained strategy set. With negative correlation and the same decision thresholds, we compared sender's payoffs when the upper bound is out of reach. Here we showed that if the tension between decision problems is strong enough, then the sender strictly prefers the sequential game.

We also analyzed the sender's equilibrium payoff in the sequential game with unconstrained strategy set. Here we described the class of DPs, for which the sender's equilibrium payoff is not affected by considering the constrained strategy set. We also showed, by constructing an example, that this is not always the case

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