Self-fulfilling Beliefs, Terms-of-Trade Dynamics, and Economic Welfare *

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Abstract

I study an open economy overlapping generations model in which large transitory shocks can generate permanent changes in economic welfare. When traded goods are poor substitutes, the model displays multiple equilibria: this creates the possibility of self-fulfilling fluctuations. This is important for researchers interested in international business cycles because the same transitory shock can have either transitory or permanent effects, depending on agents' beliefs. I solve this model using a non-trivial application of Negishi's method. This method allows me to describe equilibria using a low-order dynamical system that depends on the number of countries rather than the number of goods. In numerical simulations, I show that large and transitory endowment shocks can cause a shift from one equilibrium to another. Shifts of this kind are associated with large and persistent fluctuations in the terms of trade and the real exchange rate, as well as substantial and long-lasting welfare effects.

Keywords: International business cycle; Multiple equilibria; Overlapping generations; Real exchange rate; Social planning problem; Terms of trade; Welfare.

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1. Introduction

An important question in economics is whether large transitory shocks can generate permanent changes in output and economic welfare (Morris and Yildiz, 2019; Cerra et al., Forthcoming). I study this question in a theoretical intertemporal general equilibrium model that displays multiple equilibria, with an application to international economics. By introducing multiple equilibria, I can investigate equilibrium shifts as the product of transitory shocks and self-fulfilling beliefs.[1] My theoretical model is motivated by the observation that occasionally large drops in output are associated with persistent fluctuations in the terms of trade and real exchange rate, and long-lasting effects on economic welfare.[2] Terms of trade fluctuations can account for around half of the observed variability in output and the real exchange rate (Mendoza, 1995).[3]

Much of the literature on international business cycles uses a representative agent infinite-horizon Dynamic Stochastic General Equilibrium (DSGE) model and implicitly assumes the existence of a unique steady state.[4] Models in this class describe economic fluctuations as the combination of a stochastic long-term trend and a business cycle that deviates from this trend (Aguiar and Gopinath, 2007; Garcia-Cicco et al., 2010). Consequently, their scope is focused to the study of the effects of small fundamental shocks in the neighborhood of a steady state.[5] In this theoretical framework, transitory shocks have no role in explaining long-lasting effects on macroeconomic variables. As a result, the high persistence observed in data might be wrongly attributed to permanent shocks or shocks to the trend. Identifying whether observed macroeconomic fluctuations are driven by permanent or transitory shocks has proven challenging under standard theoretical frameworks.

I develop a two-country multi-commodity general equilibrium exchange economy

[1] The idea of self-fulfilling beliefs originally appears in Azariadis (1981) and Cass and Shell (1983) to describe fluctuations that do not stem from changes in preferences, technology or endowments.

[2] Cerra and Saxena (2008) calculate that, on average, balance of payments crises and banking crises are associated with a persistent loss of about 5 and 10 percent in output, respectively. More recently, Aikman et al. (2022) estimate that large recessions are associated with a 4 percent drop in the level of GDP for an average advanced economy.

[3] DiPace et al. (2021) find similar magnitudes for an extended sample of emerging and developing economies and Ayres et al. (2020) document a substantial comovement between prices of primary commodities and real exchange rates for a sample of developed economies.

[4] This class of models derives from the seminal work of Backus et al. (1992), which extends the Real Business Cycle (RBC) model to international macroeconomics.

[5] Fundamental shocks are attributed to changes in preferences, endowments, and technology.

that contains multiple steady states. The multiplicity of steady states arises from assumptions on preferences and endowments rather than from financial frictions. When traded goods are poor substitutes and asymmetrically distributed across countries, and there is home bias in consumption, countries might be willing to trade at unfavorable prices, making multiple steady-state terms of trade possible (Chipman, 2010; Bodenstein, 2011).[6]

I show that, in this model, large transitory shocks can cause a shift from one equilibrium to another as the result of self-fulfilling beliefs about future terms of trade. Moreover, shifts of this kind have substantial and permanent effects on the real exchange rate and economic welfare.[7] This result is a consequence of globally indeterminate dynamics: agents in the model need to predict future terms of trade to make consumption decisions. When multiple steady states arise, there are multiple equilibrium paths for the terms of trade. As a consequence, the same transitory shock can have either transitory or permanent effects depending on the equilibrium path in which agents coordinate. In this sense, beliefs about future terms of trade are self-fulfilling.[8]

The first contribution of this paper is its characterization of the business cycle dynamics and equilibrium shifts that are generated by the interaction of large transitory shocks and self-fulfilling beliefs.

I depart from the assumption of infinitely-lived agents and I use the overlapping-generations model (OLG) as an expository device. This framework can capture differences in savings behavior between countries as a result of heterogeneity in time preferences.[9] In this paper, different time preferences between countries result in a steady-state current account surplus in the economy with the higher time preference.[10]

[6] In this model, multiplicity of steady states is not related to the existence of a rational bubble. In this sense, this economy is *classical*. See Geanakoplos and Polemarchakis (1991) for a classic survey in OLG models and Weil (2008) for a recent discussion about *classical* and *Samuelsonian* economies.

[7] In the data, real exchange rates are an order of magnitude more volatile than real macroeconomic variables, such as output, employment, consumption or productivity. They are also weakly correlated with these variables (Obstfeld and Rogoff, 2000; Itskhoki and Mukhin, 2021). This suggests that self-fulfilling beliefs may have a role to play in understanding exchange rate dynamics.

[8] Self-fulfilling fluctuations in this paper do not stem from a locally indeterminate steady state as in Benhabib and Farmer (1999), but from the existence of multiple steady states as in Kaplan and Menzio (2016) and Benhabib et al. (2018).

[9] See Gârleanu and Panageas (2015) for an exploration of other dimensions of preference heterogeneity in OLG models.

[10] This contrasts with economies populated by infinitely-lived agents in which the asymptotic distribution of wealth is a corner solution and consumption in the long-run converges to zero for the agent with the higher rate of time preference (Becker, 1980).

In other words, it reflects a stylized form of global imbalances, which provides a better description of modern open economies.[11]

Tractability of economic models decreases when there are many more goods each period than agents, as is the case in the model of this paper and in other models. The second contribution of this paper is methodological. To solve my model, I develop a non-trivial application of Negishi's method to a dynamically efficient n-good OLG exchange economy with two countries.[12] Negishi's method allows me to describe equilibria using a low-order dynamical system that depends on the number of countries rather than the number of goods. I show that, in my model, this method involves important computational efficiency gains compared with the competitive equilibrium computation.

I then use this method to numerically solve the theoretical model. Specifically, I consider the case where goods are aggregated into baskets using a Cobb-Douglas function. I calibrate the model under plausible values for home bias in consumption, a low elasticity of substitution between traded goods, and an asymmetric distribution of endowments across countries. These conditions generate multiple steady states. In addition, I allow for heterogeneous time preferences across countries.

To illustrate the role of transitory shocks and self-fulfilling beliefs in generating permanent effects on economic welfare, I conduct two numerical experiments. The first experiment is a combination of an unanticipated large and transitory negative endowment shock and a belief shock that shifts the global economy from one equilibrium path to another. This kind of transition can account for a persistent deterioration in the terms of trade, a long-lasting depreciation of the real exchange rate, current account reversals, and a contraction in consumption. In contrast to much of the literature on open economies, the long-run welfare implications of a shift of this kind can be substantial.[13]

[11] Global imbalances refer to the substantial expansion of the current account deficit in the United States in recent decades, the rise in the current account surpluses of many emerging-market economies, and a worldwide decline in long-term real interest rates (see Bernanke et al., 2007 and Obstfeld and Rogoff, 2009).

[12] Negishi's method is an alternative way of computing Pareto-efficient equilibria using the solution of a social planning problem (Negishi, 1960). An equilibrium is dynamically efficient in exchange economies if aggregate endowments are finite at equilibrium prices (Geanakoplos and Polemarchakis, 1991). This need not to be the case in all OLG economies, as for instance, in *Samuelsonian* economies, where the competitive equilibrium is not Pareto-efficient.

[13] In RBC models with a unique steady state, the welfare cost of business cycles is typically small

Two historical cases where the theoretical model can fit the data are Japan during the 1990s and Argentina after the 1982 debt crisis. For example, the financial crisis in Japan in the early 1990s was initially associated with a substantial drop in output and, in the aftermath, with a persistent deterioration in the terms of trade, a long-lasting depreciation of the real exchange rate, and a reduction in economic growth (Obstfeld, 2010). Similarly, Argentina's debt default in 1982 is associated with a large drop in output followed by a long-lasting period of lower terms of trade and a depreciation of the real exchange rate (Dornbusch and De Pablo, 1989; Adler et al., 2018).[14]

In the second experiment, I ask under what conditions a transition of the form found in the first experiment can be interrupted. I simulate a second unanticipated self-fulfilling belief shock and I search for the last period where this second shock can shift the global economy back to the original steady state. I find that this equilibrium shift is possible only during periods immediately after the large shock hits, as only one equilibrium path is admissible thereafter. If belief shocks are correlated with public signals, such as speeches or announcements by governments or economic authorities, this experiment suggests that policy interventions should be enacted immediately when a large shock hits the economy, in order to prevent an equilibrium shift caused by self-fulfilling beliefs.[15] For instance, Fratzscher et al. (2019) find that interventions in foreign exchange markets are more effective when made public and supported via communication.

These exercises are a proof of concept. When my model admits multiple steady states, equilibrium paths are associated with different beliefs that the agents have about future terms of trade. As a consequence, in the theoretical model, beliefs can be so important that they affect long-run economic outcomes.

In Section 2, I show how my model differs from previous theoretical models in the open economy and in the OLG literature. In section 3, I describe the Negishi method in a static economy, and in Section 4 I show how to apply Negishi's method in an

(Lucas, 1987). That result also holds in an influential class of models that exploit financial frictions in small open economies such as Mendoza (2002) and Bianchi (2011).

[14] Persistent terms of trade fluctuations are frequent. Using a sample of 150 countries between 1960 and 2015, Adler et al. (2018) identify several episodes of large terms-of-trade level shifts. Specifically, they identify 59 episodes of terms of trade bust and 81 episodes of booms.

[15] In a stylized way, this interpretation resembles public interventions such as Mario Draghi's "whatever it takes ..." speech. Indeed, motivated by that episode, Morris and Yildiz (2019) study equilibrium shifts in coordination games under incomplete information.

OLG economy with multiple goods. In section 5, I offer a convenient redefinition of the equilibrium characterization and I describe the solution method of the model in this environment. Using the machinery of section 5, in section 6 I construct a two-country OLG model with multiple steady states. I also show that the model displays globally indeterminate dynamics. Section 7 provides an international business cycle interpretation of the model with multiple goods and heterogeneous time preferences between countries. I show the welfare properties of the steady states. I outline the numerical experiments in section 8 and I describe the results. Section 9 concludes.

2. Related Literature

This paper relates to the literature on OLG models and on multiple equilibria in international economics.[16] Computing equilibria in OLG models is often difficult (Kehoe, 1991). It is even more difficult when there are many goods per period, as there are in models of international trade. Models with multiple commodities can be simplified by a solving social planning problem when there is a finite number of agents (Negishi, 1960; Kehoe and Levine, 1985). In OLG models, however, there is an infinite number of agents. Kehoe et al. (1992) point out the possibility of characterizing OLG economies as the solution to a social planning problem when the competitive equilibrium is Pareto efficient. Building on this idea, Brumm and Kubler (2013) show how to formulate the equilibrium of an OLG economy recursively, given the sequence of welfare weights associated with newborn agents.

My application differs from these papers as follows. Rather than solving for the law of motion of the consumption shares as in Brumm and Kubler (2013), I compute the complete sequence of welfare weights that solve the individual transfer functions of all agents. This is important because a recursive equilibrium might not exist in OLG models with multiple equilibria (Kubler and Polemarchakis, 2004). Instead, my method to compute an equilibrium in OLG models can be applied to environments characterized by multiple equilibria.

Multiplicity of steady states and self-fulfilling beliefs are useful to explain interna-

[16] Surveys on the former are Geanakoplos and Polemarchakis (1991) and Weil (2008), and on the later, Masson (1999).

tional crises in stylized finite-horizon models with financial frictions (Krugman 1999; Chang and Velasco 2001). In infinite-horizon models, Bodenstein (2011) finds that in the class of international business cycles models nested in the framework of Corsetti et al. (2008) multiple steady states exist under home bias in consumption, a low trade elasticity, and zero net foreign asset positions.[17] Similarly, Schmitt-Grohé and Uribe (2021) show the possibility of multiple equilibria and self-fulfilling financial crises in a small open economy model with flow collateral constraints.[18]

This paper differs from previous open economy models of multiple equilibria in three ways. First, I use a two-period lived OLG model.[19] This allows me to model global imbalances as the result of heterogeneity in time preferences.[20] In this regard, this paper deviates from the literature that uses financial frictions to model global imbalances.[21] My model, however, is distinct from most applications that use the OLG framework in two aspects: i) I explicitly allow for multiple steady states and self-fulfilling beliefs to generate permanent welfare effects, and ii) in this setting the steady states are real (non-monetary) and saddle-point stable (locally determinate).[22]

Second, as a result of steady-state multiplicity, my model can display substantial welfare effects without appealing to frictions in goods or asset markets. Multiplicity of steady states in this paper is derived from assumptions on preferences and endow-

[17] Bodenstein (2010) explores these conditions in the model of Backus et al. (1992) and finds multiple isolated equilibria converging to a unique steady state.

[18] That paper finds that their results are also possible in the model of Bianchi (2011) under reasonable variations of the parameter configuration.

[19] Two-period lives entail no loss of generality as any finite lifetime can be redefined in terms of two-period lives with multiple goods and consumers (see Balasko et al. 1980).

[20] Gale (1974) is one of the first to show steady-state trade imbalances in OLG economies. Later, Buiter (1981) explored this idea in a Diamond-type production economy with time preference heterogeneity between countries. Ghironi et al. (2008) show in the perpetual youth model that differences in time preference across countries can have an important effect on consumption dynamics. Similarly, Eugeni (2015) offers an explanation for the trade imbalances between the U.S and China based on changes in savings behavior related to aging and the pension system. More recently, Auclert et al. (2021) quantify the general equilibrium effects of population aging on wealth accumulation, asset returns, and global imbalances.

[21] For instance, Mendoza et al. (2009) argue that large and persistent global imbalances could be the result of international financial integration among countries with different levels of domestic financial markets development. Caballero et al. (2008) argue that growth differentials and heterogeneity in the capacity to generate financial assets from real investments can explain the sustained rise in the US current account deficit, the decline in long-run real rates, and the rise in the share of US assets in the global portfolio.

[22] For instance, monetary OLG models such as Platonov (2019) and Bambi and Eugeni (2021) exploit locally indeterminate steady states and sunspots to study excess volatility in the exchange rate and in the risk premium.

ments. In this sense, my work deviates from the literature that displays a unique steady state and frictions that amplify the effects of fundamental shocks, such as the credit and financial friction literature that followed Mendoza (2002), and the literature that stresses the role that nominal rigidities and monetary shocks have in explaining international business cycle fluctuations, such as Gali and Monacelli (2005).[23]

My work also deviates from recent models that display equilibrium shifts in settings with incomplete information and endogenous growth with non-linearities as in Morris and Yildiz (2019) and Cerra et al. (Forthcoming), respectively. By focusing on preferences and endowments, I isolate the role of self-fulfilling beliefs in generating equilibrium shifts.

Third, my model exploits global indeterminacy. This means that, given some initial conditions, there exists a region in the state space of the model where isolated perfect foresight equilibria can converge to different steady states. In these situations, beliefs act as coordination devices and select the equilibrium path that materializes. Such dynamics have been useful in explaining unemployment fluctuations and credit cycles in the search theoretical models of Kaplan and Menzio (2016) and Branch and Silva (2021), and in the monetary model of Benhabib et al. (2018), respectively. I show that they can help to explain terms of trade and real exchange rate dynamics.

3. The Negishi Approach in a Static Economy

To gain intuition about the Negishi method, I present in this section the case of a static economy. I leave for next section the extension to the OLG framework.

Negishi's method consists in defining a set of individual transfer functions that indicate the extent to which the allocations derived from a social planning problem violate the budget constraints of the agents (Negishi, 1960).[24] When there is a finite number of infinitely-lived agents, the Negishi approach replaces an infinite set of market

[23] Itskhoki and Mukhin (2021) explore the role of fundamental shocks under different frictions in the goods and the asset market to study exchange rate dynamics. Their work suggests that monetary shocks under sticky prices are not sufficient to explain the behavior of exchange rates. Instead, they propose a model where shocks in the financial markets are the main driver to explain exchange rate puzzles. That work abstracts from non-fundamental shocks.

[24] The intuition follows from the observation that, when competitive equilibrium is Pareto-efficient, a social planning problem satisfies all the optimality conditions of the competitive equilibrium except the individual budget constraints.

clearing equations by a finite set of transfer functions (Kehoe and Levine, 1985). This approach is commonly considered useful only when the number of goods is larger than the number of agents. In the OLG framework, however, there is a double infinity of goods and agents. To deal with this complication, I exploit the structure of the transfer functions, which is recursive. In this way, equilibria are characterized by a system of difference equations in welfare weights.[25] Consequently, this approach replaces the problem of finding a sequence of prices that make the excess demand functions equal to zero to the problem of finding a sequence of welfare weights that make the transfers functions equal to zero. With n goods and two countries, the competitive equilibrium computation involves a system of n second-order difference equations in prices (Kehoe and Levine, 1985), whereas my application of Negishi's method involves a system of two first-order difference equations in welfare weights.

I start with the static case. Consider a two-person exchange economy with n goods. There is no uncertainty. I refer to the two agents as domestic and foreign. Variables associated with the foreign agent are denoted with a star superscript. Consider the following definition of a competitive equilibrium of this economy.

Problem 1 *Given endowments and prices, each agent maximizes utility subject to her budget constraint,*

$$\max_{x} \ \mathcal{L} = \mathcal{U}(x) + \mu[p^\top(x - e)]$$
$$\max_{x^*} \ \mathcal{L}^* = \mathcal{U}^*(x^*) + \mu^*[p^\top(x^* - e^*)]$$

where x, x^*, e and e^* are elements of $\mathbb{R}^n_+$, μ and μ^* are non-negative real numbers, p is an element of $S \in \{\mathbb{R}^n_+ \mid \sum_i p_i = 1\}$, and $\{\mathcal{U}, \mathcal{U}^*\} : \mathbb{R}^n_+ \to \mathbb{R}$ are increasing and quasi-concave functions.

In Problem 1, $\mathcal{U}$ and $\mathcal{U}^*$ are utility functions and the variables x, e and x^*, e^* denote the consumption and endowment vectors of the domestic and the foreign agent, respectively. The vector p denotes prices, and, μ and μ^* represent the Lagrange multipliers on the agents' budget constraints. The solution to problem 1 is a pair of excess demand functions $\{f(p, m), f^*(p, m^*)\} : S \times \mathbb{R}_+ \to \mathbb{R}$ and a pair of functions $\{\psi, \psi^*\} : S \times \mathbb{R}_+ \to \mathbb{R}$, where $m(p) \equiv p^\top e$ and $m^*(p) \equiv p^\top e^*$ are the incomes of

[25] In the context of infinitely-lived agents, recursive welfare weights as the endogenous state have shown to be useful when agents have heterogeneous beliefs (Beker and Espino, 2011).

the two people expressed as functions of p, and $\mu = \psi(p, m)$, $\mu^* = \psi^*(p, m^*)$ are the values of the Lagrange multipliers that solve this problem expressed as functions of prices and income.

Lastly, define the aggregate excess demand function, $F : S \to \mathbb{R} \equiv f(p, m(p)) + f^*(p, m^*(p))$. In equilibrium, prices are such that the goods market clears. Then, an equilibrium price vector of this economy is a vector $\hat{p}$ such that $F(\hat{p}) = 0$. Importantly, note that $F(\hat{p}) = 0$ corresponds to a set of n equations in n prices. Moreover, as a consequence of Walras's Law, $\hat{p}^\top F(\hat{p}) = 0$. Note that, as excess demand functions are homogeneous of degree zero in prices and Walras's law holds, then the equilibrium conditions of this economy can be viewed as a set of $n-1$ equations in $n-1$ prices.

Equipped with the definition of a competitive equilibrium, I now proceed to explain the Negishi method for the same static economy. Consider the following social planning problem.

Problem 2 *Given welfare weights and aggregate endowments, a social planner maximizes a social welfare function subject to the resource constraints of the economy,*

$$\max_{x,\, x^*} \mathcal{L}^s = \alpha\, \mathcal{U}(x) + \alpha^*\, \mathcal{U}^*(x^*) + \lambda^\top (x + x^* - e - e^*)$$

where α and α^* are non-negative real numbers such that $\alpha + \alpha^* = 1$, and λ is an n dimensional vector of non-negative real numbers.

In Problem 2, the social welfare function corresponds to a weighted sum of individual utilities. Let λ represent the vector of Lagrange multipliers on the resource constraints of this problem. Define the pair of functions $\{\tilde{x}(\lambda, \alpha), \tilde{x}^*(\lambda, \alpha^*)\} : \mathbb{R}^{n+1}_+ \to \mathbb{R}^n_+$ as implicit solutions to the first-order conditions,

$$\alpha\, \mathcal{U}_x(x) = \lambda \quad \text{and} \quad \alpha^*\, \mathcal{U}^*_{x^*}(x^*) = \lambda. \tag{1}$$

The solution to Problem 2 is characterized by the pair of functions $\{\hat{x}(\lambda, \alpha), \hat{x}^*(\lambda, \alpha^*)\} : \mathbb{R}^{n+1}_+ \to \mathbb{R}^n$ such that $\lambda \in \{\mathbb{R}^n_+ \mid \tilde{x}(\lambda, \alpha) + \tilde{x}^*(\lambda, \alpha^*) = e + e^*\}$. Note that $\{\hat{x}(\lambda, \alpha), \hat{x}^*(\lambda, \alpha^*)\}$ defines a continuum of Pareto-efficient allocations indexed by welfare weights, α^* and α^*, since λ depends on both welfare weights as well. The solution to Problem 2 satisfies all the optimality conditions of the competitive equilibrium except the individual budget constraints.[26] Negishi's method consists in defining a set of in-

[26] Consider the first-order conditions of the competitive equilibrium $\mathcal{U}_x(x) = \mu p$ and $\mathcal{U}^*_{x^*}(x^*) = \mu^* p$.

dividual transfer functions that indicate the extent to which the allocations indexed by the welfare weights, α^* and α^*, violate the individual budget constraints. Define the transfer functions $\{\mathcal{T}(\lambda, \alpha, \alpha^*, e), \mathcal{T}^*(\lambda, \alpha, \alpha^*, e^*)\} : \mathbb{R}_+^{2n+2} \to \mathbb{R}$ using the solution to Problem 2 as allocations and the Lagrange multiplier λ as the price vector. Then, $\mathcal{T}(\lambda, \alpha, \alpha^*, e) = \lambda^\top[\hat{x}(\lambda, \alpha, \alpha^*) - e]$ and $\mathcal{T}^*(\lambda, \alpha, \alpha^*, e^*) = \lambda^\top[\hat{x}^*(\lambda, \alpha, \alpha^*) - e^*]$. In other words, Negishi's method to compute the competitive equilibrium consists in solving for the welfare weights such that the the transfer functions are equal to zero. More formally,

Theorem 1 (Negishi) *For a given endowment pattern $\{e, e^*\}$, a solution to Problem 2 is a competitive equilibrium for values of α and α^* that solve the set of scalar equations,*

$$\mathcal{T}(\lambda(\alpha, \alpha^*), \alpha, \alpha^*, e) = 0$$
$$\mathcal{T}^*(\lambda(\alpha, \alpha^*), \alpha, \alpha^*, e) = 0.$$

Here, I make explicit that $\lambda(\alpha, \alpha^*)$ to emphasize the dimensionality reduction of this approach. While the competitive equilibrium involves solving for n prices in n excess demand functions, Negishi's method involves solving for two welfare weights in two transfer functions. Therefore, the dimensionality reduction goes from the number of goods to the number of agents. Note that this problem can be viewed as the solution of one welfare weight in one transfer function since $\mathcal{T}(\alpha, e) = 0$ implies $\mathcal{T}^*(\alpha^*, e^*) = 0$ as a consequence of Walras's law and $\alpha + \alpha^* = 1$.

4. The Negishi Approach in a Dynamic Economy

In this section, I extend the analysis of the previous section to an infinite-horizon economy populated by overlapping generations. I show that this method allows me to develop a low-order dynamical system to describe equilibria that depends on the number of countries rather than the number of goods.

Consider a pure exchange economy with two countries and n goods in each time period. There is no uncertainty. Time proceeds in a sequence of periods indexed by $t =$

Notice that if we set $\mu = 1/\alpha$ and $\mu^* = 1/\alpha^*$, then the competitive allocations solve Problem 2 (see equation (1)). This is the first welfare theorem: a competitive equilibrium is Pareto-efficient. Notice that the resource constraints faced by the social planner correspond to the goods market-clearing conditions.

$\{1, 2, ..., \infty\}$.[27] Each country is populated by a sequence of identical generations, each of which lives for two periods. In the first period, there is one agent in each country that only lives in that period. I refer to these agents as the initial old. Generations have time-separable preferences over goods and discount future consumption by a constant factor that may differ across countries but not across generations. Preferences and endowments are stationary. Each consumer faces a single budget constraint in the two periods of their life. There is neither storage nor fiat money. The economy has no trade frictions, and I assume that preferences and endowments are such that the equilibria of this economy are dynamically efficient.[28]

Consider the following definition of a competitive equilibrium in a two-country OLG exchange economy.

Problem 3 *Given endowments and prices, each agent maximizes utility subject to her budget constraint,*

$$\max_{x_t^t,\, x_{t+1}^t} \mathcal{L} = \mathcal{U}\left(x_t^t\right) + \beta\, \mathcal{U}\left(x_{t+1}^t\right) + \mu_t \left[p_t^\top \left(x_t^t - e_t^t\right) + p_{t+1}^\top \left(x_{t+1}^t - e_{t+1}^t\right)\right]$$

$$\max_{x_t^{*,t},\, x_{t+1}^{*,t}} \mathcal{L}^* = \mathcal{U}^*\left(x_t^{*,t}\right) + \beta^*\, \mathcal{U}^*\left(x_{t+1}^{*,t}\right) + \mu_t^* \left[p_t^\top \left(x_t^{*,t} - e_t^{*,t}\right) + p_{t+1}^\top \left(x_{t+1}^{*,t} - e_{t+1}^{*,t}\right)\right]$$

where $\{x_t^t, x_{t+1}^t, x_t^{*,t}, x_{t+1}^{*,t}\}$ and $\{e_t^t, e_{t+1}^t, e_t^{*,t}, e_{t+1}^{*,t}\}$ are elements of $\mathbb{R}_+^n$, μ_t and μ_t^* are non-negative real numbers, $p_t = (p_t^1, ..., p_t^n)$ denotes the prices prevailing in period t, and $\mathcal{U}$ and $\mathcal{U}^*$ are increasing and quasi-concave. Discount factors β and $\beta^* \in (0, 1)$.

In this setting, the time superscript denotes the period when a generation is born and the time subscript denotes the current period. Star superscripts denote the foreign country variables. For example, $x_t^{*,t-1}$ denotes the demand for good x in period t by the foreign consumer born in period $t-1$. The functions $\mathcal{U}$ and $\mathcal{U}^*$ represent utility, and μ_t and μ_t^* are the Lagrange multipliers of the budget constraints of the consumers born in period t in each country, respectively.

As in Kehoe and Levine (1985), consumption and savings decisions of consumers across countries from generation t are aggregated into excess demand functions $\tilde{y}(p_t, p_{t+1})$ in period t (the *young*) and $\tilde{z}(p_t, p_{t+1})$ in period $t+1$ (the *old*), where $\{\tilde{y}, \tilde{z}\} : \mathbb{R}_+^n \to \mathbb{R}_+^n$

[27] Alternatively, we can think of the first period summarizing all previous history from period $-\infty$.

[28] An equilibrium is dynamically efficient in an exchange economies if aggregate endowments are finite at equilibrium prices Geanakoplos and Polemarchakis (1991).

are smooth functions. The economy begins in period one. The aggregate excess demand of initial old people in period one is $\tilde{z}_0(p_1)$.

A *perfect foresight equilibrium* is a sequence of non-negative vectors $\{p_t\}_{t=1}^{\infty}$ such that,

$$\tilde{z}_0(p_1) + \tilde{y}(p_1, p_2) = 0 \tag{2}$$

and

$$\tilde{z}(p_{t-1}, p_t) + \tilde{y}(p_t, p_{t+1}) = 0, \qquad \forall t > 1. \tag{3}$$

Once p_1 and p_2 are determined, equation (3) acts as a nonlinear difference equation determining all future prices. Assumptions made at the beginning of the section imply that excess demand functions are continuously differentiable for all strictly positive prices (p_t, p_{t+1}), homogeneous of degree zero in (p_t, p_{t+1}) and they obey Walras's Law.

I now proceed to characterize the competitive equilibrium of this economy using the Negishi method. As in the previous section, the idea is to define a set of individual transfer functions that indicate the extent to which the allocations indexed by the welfare weights violate the individual budget constraints. The difference, however, is that in the OLG setting there is an infinite number of individual transfer functions and welfare weights. In what follows, I explain how to exploit the recursive structure of the transfer functions to deal with this issue.

Let $\mathcal{W}(\tilde{\alpha})$ be a utilitarian social welfare function in period one defined as a weighted sum of individual utility functions across countries, and let $\tilde{\alpha} = \{\alpha_t, \alpha_t^*\}_{t=0}^{\infty}$ be the vector of *all* welfare weights.

Problem 4 *The social planner chooses allocations* $\{x_1^0, x_1^{*,0}\}$ *for the initial old and the sequence* $\{x_t^t, x_t^{t-1}, x_t^{*,t}, x_t^{*,t-1}\}_{t=1}^{\infty}$ *to maximize,*

$$\begin{aligned} \mathcal{W}(\tilde{\alpha}) \; &\equiv \alpha_0 \, \mathcal{U}_0\left(x_1^0\right) \; + \; \alpha_0^* \, \mathcal{U}_0^*\left(x_1^{*,0}\right) \; + \; \sum_{t=1}^{\infty} \alpha_t \left[\sum_{s=0}^{1} \beta^s \, \mathcal{U}_{t+s}^t\left(x_{t+s}^t\right)\right] \\ &+ \; \sum_{t=1}^{\infty} \alpha_t^* \left[\sum_{s=0}^{1} \beta^{*s} \, \mathcal{U}_{t+s}^{*,t}\left(x_{t+s}^{*,t}\right)\right] \end{aligned}$$

subject to the resource constraints,

$$x_t^t + x_t^{t-1} + x_t^{*,t} + x_t^{*,t-1} = e_t^t + e_t^{t-1} + e_t^{*,t} + e_t^{*,t-1} \equiv E_t, \qquad \forall t$$

where $\tilde{\alpha}$ *are non-negative real numbers such that* $\sum_{t=0}^{\infty} \alpha_t < \infty$ *and* $\sum_{t=0}^{\infty} \alpha_t^* < \infty$, *and* E_t *are* *elements of* $\mathbb{R}^n_+$.

The social welfare function $\mathcal{W}(\tilde{\alpha})$ is the sum of four terms. The first two terms represent the weighted utilities of the initial old in each country and are weighted by $\{\alpha_0, \alpha_0^*\}$. The last two terms represent the weighted utility of all consumers born from period one onward who live for two periods. In other words, under $\mathcal{W}(\tilde{\alpha})$, the planner discounts individual utilities using coefficients indexed by the time period in which a consumer is born. As mentioned before, equilibria in this economy are assumed to be dynamically efficient. This means that in the competitive equilibrium prices do not explode ($\sum_{t=1}^{\infty} p_t < \infty$). As a consequence, welfare weights are summable and $\mathcal{W}(\tilde{\alpha})$ is well defined at an equilibrium sequence of welfare weights.[29]

Let λ_t represent the vector of Lagrange multipliers on the resource constraints in Problem 4. Define the functions $\{\tilde{x}_t^t(\lambda_t, \alpha_t), \tilde{x}_t^{t-1}(\lambda_t, \alpha_{t-1}), \tilde{x}_t^{*,t}(\lambda_t, \alpha_t^*), \tilde{x}_t^{*,t-1}(\lambda_t, \alpha_{t-1}^*)\}$: $\mathbb{R}^{n+1}_+ \to \mathbb{R}^n_+$ as implicit interior solutions to the first-order conditions

$$\begin{aligned}
\alpha_t\, \mathcal{U}_{x_t^t}(x_t^t) = \lambda_t \quad &\text{and} \quad \alpha_t^*\, \mathcal{U}^*_{x_t^{*,t}}(x_t^{*,t}) = \lambda_t\,, \quad \forall\, t \\
\alpha_{t-1}\, \beta\, \mathcal{U}_{x_t^{t-1}}(x_t^{t-1}) = \lambda_t \quad &\text{and} \quad \alpha_{t-1}^*\, \beta^*\, \mathcal{U}^*_{x_t^{*,t-1}}(x_t^{*,t-1}) = \lambda_t\,, \quad \forall\, t \geq 1 \\
\alpha_0\, \mathcal{U}_{x_1^0}(x_1^0) = \lambda_1 \quad &\text{and} \quad \alpha_0^*\, \mathcal{U}^*_{x_1^{*,0}}(x_1^{*,0}) = \lambda_1.
\end{aligned}$$

The solution to Problem 4 is characterized by functions

$$\{\hat{x}_t^t(\lambda_t, \alpha_t), \hat{x}_t^{t-1}(\lambda_t, \alpha_{t-1}), \hat{x}_t^{*,t}(\lambda_t, \alpha_t^*), \hat{x}_t^{*,t-1}(\lambda_t, \alpha_{t-1}^*)\} : \mathbb{R}^{n+1}_+ \to \mathbb{R}^n_+ \tag{4}$$

such that $\lambda_t \in \{\mathbb{R}^n_+ \mid \tilde{x}_t^t(\lambda_t, \alpha_t) + \tilde{x}_t^{t-1}(\lambda_t, \alpha_{t-1}) + \tilde{x}_t^{*,t}(\lambda_t, \alpha_t^*) + \tilde{x}_t^{*,t-1}(\lambda_t, \alpha_{t-1}^*) = E_t\}$, $\forall\, t$.

Define (in *bold*) the vector of consecutive welfare weights, $\boldsymbol{a}_t = (\alpha_{t-1}, \alpha_t, \alpha_{t-1}^*, \alpha_t^*)$. Notice that $\boldsymbol{a}_t \cap \boldsymbol{a}_{t+1} = \{\alpha_t, \alpha_t^*\}$. Recall that the Lagrange multiplier λ_t that solves Problem 4 depends on both $\boldsymbol{a}_t$ and the aggregate endowment, E_t, $\{\lambda_t\} : \mathbb{R}^{n+4}_+ \to \mathbb{R}^n_+$. To simplify notation, redefine the functions in (4) as $\{\hat{x}_t^t(\boldsymbol{a}_t), \hat{x}_t^{t-1}(\boldsymbol{a}_t), \hat{x}_t^{*,t}(\boldsymbol{a}_t), \hat{x}_t^{*,t-1}(\boldsymbol{a}_t)\}$: $\mathbb{R}^{n+4}_+ \to \mathbb{R}^n_+$. Notice that (4) defines allocations of consumers of different generations that are alive in the same period. Recall that the budget constraint considers the allocations of one consumer in her two periods of life. Define the transfer function of the

[29] OLG economies typically have many equilibria that are not dynamically efficient (Kehoe and Levine, 1985). In such cases, $\mathcal{W}(\tilde{\alpha})$ would not converge at an equilibrium (Kehoe et al., 1992). An overtaking criterion can be helpful to extend this analysis to those cases. However, that analysis is beyond the scope of this paper.

initial old as

$$\mathcal{T}_0(\lambda_1, \boldsymbol{a}_1, e_1^0) = \lambda_1^\top \left[\hat{x}_1^0(\boldsymbol{a}_1) - e_1^0\right]$$
$$\mathcal{T}_0^*(\lambda_1, \boldsymbol{a}_1, e_1^{*,0}) = \lambda_1^\top \left[\hat{x}_1^{*,0}(\boldsymbol{a}_1) - e_1^{*,0}\right]$$

for the initial young as

$$\mathcal{T}_1(\lambda_1, \lambda_2, \boldsymbol{a}_1, \boldsymbol{a}_2, e_1^1, e_2^1) = \lambda_1^\top \left[\hat{x}_1^1(\boldsymbol{a}_1) - e_1^1\right] + \lambda_2^\top \left[\hat{x}_2^1(\boldsymbol{a}_2) - e_2^1\right]$$
$$\mathcal{T}_1^*(\lambda_1, \lambda_2, \boldsymbol{a}_1, \boldsymbol{a}_2, e_1^{*,1}, e_2^{*,1}) = \lambda_1^\top \left[\hat{x}_1^{*,1}(\boldsymbol{a}_1) - e_1^{*,1}\right] + \lambda_2^\top \left[\hat{x}_2^{*,1}(\boldsymbol{a}_2) - e_2^{*,1}\right]$$

and for a consumer born in periods $t \geq 2$ as

$$\mathcal{T}_t(\lambda_t, \lambda_{t+1}, \boldsymbol{a}_t, \boldsymbol{a}_{t+1}, e_t^t, e_{t+1}^t) = \lambda_t^\top \left[\hat{x}_t^t(\boldsymbol{a}_t) - e_t^t\right] + \lambda_{t+1}^\top \left[\hat{x}_{t+1}^t(\boldsymbol{a}_{t+1}) - e_{t+1}^t\right]$$
$$\mathcal{T}_t^*(\lambda_t, \lambda_{t+1}, \boldsymbol{a}_t, \boldsymbol{a}_{t+1}, e_t^{*,t}, e_{t+1}^{*,t}) = \lambda_t^\top \left[\hat{x}_t^{*,t}(\boldsymbol{a}_t) - e_t^{*,t}\right] + \lambda_{t+1}^\top \left[\hat{x}_{t+1}^{*,t}(\boldsymbol{a}_{t+1}) - e_{t+1}^t\right].$$

Recall that this set contains an infinite number of individual transfer functions, one for each agent in an OLG economy. Notice that the individual transfer functions are indexed by *consecutive* welfare weights. As in the static economy, they indicate the extent to which the allocations of the social planner violate the individual budget constraints.

Notice that $\{\mathcal{T}_0(\cdot), \mathcal{T}_0^*(\cdot)\} : \mathbb{R}_+^{n+4} \to \mathbb{R}$ and $\{\mathcal{T}_1(\cdot), \mathcal{T}_1^*(\cdot), \mathcal{T}_t(\cdot), \mathcal{T}_t^*(\cdot)\} : \mathbb{R}_+^{2n+6} \to \mathbb{R}$, $\forall t$. In other words, the dimensionality of the problem is associated with the number of countries rather than with the number of goods. Two remarks are required. First, there are four consumers each period. Second, the first-order conditions of the initial old are different as they do not discount the future. Because a young consumer born in period one coincides with the initial old, and preferences and endowments are stationary, her allocations differ from those of the rest of the young consumers. For periods $t \geq 2$, the transfer functions are summarized by $\mathcal{T}_t(\cdot)$ and $\mathcal{T}_t^*(\cdot)$.

Theorem 2 (OLG-Negishi) *For a given distribution of endowments defined by* $\{e_1^0, e_1^{*,0}\}$ *and* $\{e_t^t, e_{t+1}^t, e_t^{*,t}, e_{t+1}^{*,t}\}_{t=1}^\infty$*, a solution to Problem 4 is a perfect foresight equilibrium for the summable sequence of* $\tilde{\alpha} = \{\alpha_t, \alpha_t^*\}_{t=0}^\infty$ $(\sum_{t=0}^\infty \alpha_t < \infty, \sum_{t=0}^\infty \alpha_t^* < \infty)$ *that solves the set of second-order difference equations,*

$$\mathcal{T}_0(\lambda_1, \boldsymbol{a}_1, e_1^0) = 0; \qquad \mathcal{T}_0^*(\lambda_1, \boldsymbol{a}_1, e_1^{*,0}) = 0 \tag{5}$$

$$\mathcal{T}_1(\lambda_1, \lambda_2, \boldsymbol{a}_1, \boldsymbol{a}_2, e_1^1, e_2^1) = 0; \qquad \mathcal{T}_1^*(\lambda_1, \lambda_2, \boldsymbol{a}_1, \boldsymbol{a}_2, e_1^{*,1}, e_2^{*,1}) = 0 \tag{6}$$

$$\mathcal{T}_t(\lambda_t, \lambda_{t+1}, \boldsymbol{a}_t, \boldsymbol{a}_{t+1}, e_t^t, e_{t+1}^t) = 0; \quad \mathcal{T}_t^*(\lambda_t, \lambda_{t+1}, \boldsymbol{a}_t, \boldsymbol{a}_{t+1}, e_t^{*,t}, e_{t+1}^{*,t}) = 0, \quad \forall\, t \geq 2 \tag{7}$$

Proof: In a competitive equilibrium, agents solve problem 3 and the goods market clears. By construction, the solution of problem 4 is feasible and Pareto-efficient. It remains to show that a solution of problem 4 satisfies the first-order conditions of the competitive equilibrium and the budget constraints. Recall that μ_t and μ_t^* are the Lagrange multipliers of the budget constraint of an agent born in period t in each country, respectively. Notice that if we set $\alpha_t = 1/\mu_t$ and $\alpha_t = 1/\mu_t^*$ the first-order conditions of problems 3 and 4 coincide for $p_t = \lambda_t$. Dynamic efficiency ensures that the aggregate endowment is finite, that prices do not explode ($\sum_{t=0}^{\infty} p_t < \infty$), and that therefore the social welfare function $\mathcal{W}(\tilde{\alpha})$ is well defined with summable welfare weights. Let $\lambda_t(\boldsymbol{a}_t)$ and $\{\hat{x}_t^t(\boldsymbol{a}_t), \hat{x}_t^{t-1}(\boldsymbol{a}_t), \hat{x}_t^{*,t}(\boldsymbol{a}_t), \hat{x}_t^{*,t-1}(\boldsymbol{a}_t)\}$ be the solutions of the first-order conditions and the resource constraints of the planner's problem in terms of welfare weights. Quasi-concavity of the utilities, $\mathcal{U}$ and $\mathcal{U}^*$, ensures that set of transfer functions $\{\mathcal{T}_0(\cdot), \mathcal{T}_0^*(\cdot), \mathcal{T}_1(\cdot), \mathcal{T}_1^*(\cdot), \mathcal{T}_t(\cdot), \mathcal{T}_t^*(\cdot)\}$ are continuous and differentiable in welfare weights. Then, there exists a sequence $\tilde{\alpha} = \{\alpha_t, \alpha_t^*\}_{t=0}^{\infty}$ such that the transfer functions are equal zero, and therefore, satisfies the individual budget constraints.$\square$

Note that conditions (6), (7), and $\mathcal{T}_0(\lambda_1, \boldsymbol{a}_1, e_1^0) = 0$, imply $\mathcal{T}_0^*(\lambda_1, \boldsymbol{a}_1, e_1^{*,0}) = 0$ as a consequence of Walras's law. Once $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are determined, difference equations in (7) determine all future welfare weights. Similarly to excess demand functions, the transfer functions are continuously differentiable for all strictly positive welfare weights $(\boldsymbol{a}_t, \boldsymbol{a}_{t+1})$, homogeneous of degree zero in $(\boldsymbol{a}_t, \boldsymbol{a}_{t+1})$, and they obey an analogue to Walras's Law.[30]

Theorem 2 shows that the computation of a competitive equilibrium can be rewritten as a simpler problem using the insights of Negishi (1960). Using this method, I no longer have to solve for an infinite sequence of n prices in n budget constraints ($\{p_t\}_{t=1}^{\infty}, p_t \in \mathbb{R}_+^n$), but rather I can solve for an infinite sequence of h welfare weights

[30] See Appendix A.1 for the proof of the analogue of Walras's Law.

in h transfer functions per period ($\{\alpha_t\}_{t=1}^{\infty}, \alpha_t \in \mathbb{R}_+^h$), where h is equal to the number of countries. Therefore, Negishi's method implies important computational efficiency gains when there are many more goods each period than agents, as there are for example, in models of international trade.

5. Making the System Stationary

In this section, I define a set of new variables that characterize a stationary solution to Problem 4 and I show that this stationary characterization provides further computational efficiency gains by expressing the set of transfer functions in terms of first-order non-linear difference equations. This characterization also simplifies the dynamic analysis of equilibria through a redefinition of the steady states. I then describe the solution method under this redefinition and I discuss its advantages.

Define variables $\kappa_t = \alpha_t/\alpha_{t-1}$, $\kappa_t^* = \alpha_t^*/\alpha_{t-1}^*$, and $q_t = \alpha_t^*/\alpha_t$. The first two represent rates of growth of country-specific welfare weights, while q_t corresponds to the ratio of contemporaneous welfare weights across countries, or alternatively, the relative welfare weight in period t.

Recall that the Lagrange multiplier λ_t that solves Problem 4 depends on $\boldsymbol{a}_t$ and the aggregate endowment, E_t, $\{\lambda_t\} : \mathbb{R}_+^{n+4} \to \mathbb{R}_+^n$. Allocations also depend on consecutive welfare weights, $\{\hat{x}_t^t(\boldsymbol{a}_t), \hat{x}_t^{t-1}(\boldsymbol{a}_t), \hat{x}_t^{*,t}(\boldsymbol{a}_t), \hat{x}_t^{*,t-1}(\boldsymbol{a}_t)\} : \mathbb{R}_+^{n+4} \to \mathbb{R}_+^n$. Redefine the vector $\boldsymbol{a}_t = (\alpha_{t-1}, \alpha_t, \alpha_{t-1}^*, \alpha_t^*)$ as $\boldsymbol{a}_t = \left(1,\ \alpha_t/\alpha_{t-1},\ \alpha_{t-1}^*/\alpha_{t-1},\ \alpha_t^*/\alpha_{t-1}\right)$. Using the new variables κ_t^*, κ_t and q_t we can rewrite $\boldsymbol{a}_t$ as $\boldsymbol{a}_t = (1,\ \kappa_t,\ q_{t-1},\ \kappa_t^* \cdot q_{t-1})$.

Theorem 3 *For a given distribution of endowments defined by $\{e_1^0, e_1^{*,0}\}$ and $\{e_t^t, e_{t+1}^t, e_t^{*,t}, e_{t+1}^{*,t}\}_{t=1}^{\infty}$, a stationary solution to Problem 4 is a stationary perfect foresight equilibrium for the sequence of relative welfare weights $\{q_t\}_{t=0}^{\infty}$ and rates of growth $\{\kappa_t^*, \kappa_t\}_{t=1}^{\infty}$ that solve the set of first-order difference equations,*

$$\mathcal{T}_0(\kappa_1^*, \kappa_1, q_0, e_1^0) = 0; \qquad \mathcal{T}_0^*(\kappa_1^*, \kappa_1, q_0, e_1^{*,0}) = 0 \tag{8}$$

$$\mathcal{T}_1(\kappa_2^*, \kappa_2, q_1, \kappa_1^*, \kappa_1, q_0, e_1^1, e_2^1) = 0; \qquad \mathcal{T}_1^*(\kappa_2^*, \kappa_2, q_1, \kappa_1^*, \kappa_1, q_0, e_1^{*,1}, e_2^{*,1}) = 0 \tag{9}$$

$$\mathcal{T}_t\left(\kappa_{t+1}^*, \kappa_{t+1}, q_t, \kappa_t^*, \kappa_t, q_{t-1}, e_t^t, e_{t+1}^t\right) = 0; \qquad \mathcal{T}_t^*\left(\kappa_{t+1}^*, \kappa_{t+1}, q_1, \kappa_t^*, \kappa_t, q_{t-1}, e_t^{*,t}, e_{t+1}^{*,t}\right) = 0 \tag{10}$$

$$q_t = \frac{\kappa_t^*}{\kappa_t} q_{t-1}, \qquad \forall\, t \geq 1 \tag{11}$$

where q_0 is an initial condition and equations in (10) are defined $\forall\, t \geq 2$.

Proof: Given a sequence ($\sum_{t=0}^{\infty} p_t < \infty$) that solve the competitive equilibrium, there exists a sequence $\tilde{\alpha} = \{\alpha_t, \alpha_t^*\}_{t=0}^{\infty}$ ($\sum_{t=0}^{\infty} \alpha_t < \infty$, $\sum_{t=0}^{\infty} \alpha_t^* < \infty$) such that the solution to problem 4 is a competitive equilibrium (Theorem 2). Redefine the solution to problem 4 using $\kappa_t = \alpha_t/\alpha_{t-1}$, $\kappa_t^* = \alpha_t^*/\alpha_{t-1}^*$, and $q_t = \alpha_t^*/\alpha_t$. Because the equilibrium sequence $\tilde{\alpha} = \{\alpha_t, \alpha_t^*\}_{t=0}^{\infty}$ solves for the competitive equilibrium, then the sequence $\{\kappa_t^*, \kappa_t, q_t\}_{t=0}^{\infty}$ constructed from $\tilde{\alpha} = \{\alpha_t, \alpha_t^*\}_{t=0}^{\infty}$ solves for the competitive equilibrium in the redefined problem as well.$\square$

The transformation of the system has three advantages. First, it provides a simple intuition about the evolution of the welfare weights that characterize a competitive equilibrium. Equation (11) shows the relative welfare weight at period t as a function of its previous value and the ratio of the rates of growth of each welfare weight. Therefore, given an initial relative welfare weight, q_0, the dynamics of the system is characterized by the rates of growth κ_t^* and κ_t. Second, this transformation allows me to define the steady state of the dynamical system as a vector of scalars rather than as a path of welfare weights. This simplifies the local analysis of the dynamical system as I can use the steady state as a fixed point to approximate the non-linear system of difference equations.[31] Third, the equilibrium is characterized by a system of first-order non-linear difference equations in relative welfare weights and growth rates, implying computational efficiency gains on two fronts. First, dimensionality is reduced from a second-order to a first-order difference equation for the non-linear model, which is also an advantage over the competitive equilibrium computation in problem 3.[32] Second, under the redefinition, $\{\mathcal{T}_0(\cdot), \mathcal{T}_0^*(\cdot)\} : \mathbb{R}_+^{n+3} \to \mathbb{R}$ and $\{\mathcal{T}_1(\cdot), \mathcal{T}_1^*(\cdot), \mathcal{T}_t(\cdot), \mathcal{T}_t^*(\cdot)\} : \mathbb{R}_+^{2n+5} \to \mathbb{R}$, $\forall t$. This imply that, independently of the number of goods, the dynamical system can be defined as a first-order system of

[31] Notice that if $\tilde{\alpha} = \{\alpha_t, \alpha_t^*\}_{t=0}^{\infty}$ represent a vector summable sequences, then both α_t and α_t^* converge asymptotically to zero as time approaches infinity. While each of these two sequences approach zero, I show that the relative welfare weight q_t is well defined and converges to a scalar as time approaches infinity.

[32] When the non-linear system of difference equations in problem 3 is linearized around a point is simple to express it as first-order system of linear difference equations using the *companion* form. Using my method, the reduction of order takes place in the non-linear system. This is important when the analysis is performed outside the neighborhood of the steady state as local approximations might not be valid, as in models with multiple steady states.

non-linear equations in dimensions $(\kappa_t^*, \kappa_t, q_{t-1})$.

5.1 Solution Method

I now proceed to describe the solution method. To reduce notation, I define the vector $V_t = (\kappa_t^*, \kappa_t, q_{t-1})$ and I collapse equations (10) and (11) into the three dimensional system of first-order difference equations,

$$F(V_t, V_{t+1}) = \begin{bmatrix} \mathcal{T}_t\left(V_t, V_{t+1}\right) \\ \mathcal{T}_t^*\left(V_t, V_{t+1}\right) \\ q_t - \frac{\kappa_t^*}{\kappa_t} q_{t-1} \end{bmatrix} = 0, \qquad \forall\, t \geq 2. \tag{12}$$

Similarly, the equilibrium conditions in period one are characterized by

$$F_1(V_1, V_2) = \begin{bmatrix} \mathcal{T}_1\left(V_1, V_2\right) \\ \mathcal{T}_1^*\left(V_1, V_2\right) \\ q_1 - \frac{\kappa_1^*}{\kappa_1} q_0 \end{bmatrix} = 0 \quad \text{and} \quad F_0(V_1) = \begin{bmatrix} \mathcal{T}_0\left(V_1\right) \\ \mathcal{T}_0^*\left(V_1\right) \end{bmatrix} = 0. \tag{13}$$

The focus of this paper is whether large transitory shocks can generate permanent changes in economic welfare. For this reason, I constrain the analysis in what follows to equilibrium paths that converge to a steady state.[33]

Definition 1 *For a given constant endowment pattern, $\{e_y, e_o, e_y^*, e_o^*\}$, a steady state is a pair (κ, q) such that,*

$$\mathcal{T}\left(\kappa, \kappa, \kappa, \kappa, q, e_y, e_o\right) = 0; \quad \mathcal{T}^*\left(\kappa, \kappa, \kappa, \kappa, q, e_y^*, e_o^*\right) = 0. \tag{14}$$

A steady state in this model corresponds to a pair (κ, q) such that, when the rates of growth of the welfare weights are constant and equal to κ, the relative welfare weight is constant and equal to q and the transfers functions are equal to zero in every period.[34] In this case, the allocations of the planner are constant and coincide with the

[33] The solution of a dynamical systems might also converge to a limit cycle, where the economy fluctuates permanently in the absence of exogenous shocks. That type of solutions are beyond the scope of this paper. For a recent study on this topic in modern macroeconomic models see Beaudry et al. (2020).

[34] We can see in equation (11) that κ_t^* and κ_t must converge to κ. Otherwise, the relative welfare weight q_t converges to either zero or infinity.

stationary allocations of the competitive equilibrium.

The solution method consists in a two-step algorithm. The steady state is interpreted as a terminal condition of the system $F(V_t, V_{t+1})$ in 12. Given a steady state, the first is step is to find V_2 such that V_t converges to the steady state. Given V_2, the second step consist in finding V_1 such that the equilibrium conditions in period one are satisfied, i.e, to verify that equations in (13) hold. Using this algorithm, the computation of equilibrium corresponds to find the sequence $\{\kappa_t^*, \kappa_t, q_{t-1}\}_{t=1}^{\infty}$ such that all transfer functions are equal to zero.

5.1.1 A Remark on Steady State Multiplicity

In exchange economies with two agents and two goods multiple equilibria are possible.[35] In a dynamic economy, the conditions associated with multiplicity in static economies might imply multiplicity of steady states. In definition 1, for instance, if the steady state is unique, then the pair (κ, q) corresponds to a pair of scalars. When there are multiple steady states, as in the main example of this paper, (κ, q) corresponds to pair of vectors with multiple values for κ and q. An useful tool to diagnose multiplicity is the index theorem. Appendix A.2 offers a discussion of the sufficient conditions for multiplicity of steady states using transfer functions and describe the advantages of the Negishi's method for computing the index theorem.

6. An Example with Isoelastic Preferences and Multiple Steady States

In this section I use the Negishi approach developed in the previous section to compute equilibria in a two-country OLG economy with multiple steady states. In this economy, multiple steady states are associated with different levels of consumption and therefore of economic welfare. I study the global dynamics of this model using numerical simulations and I show numerically that this model displays global indeterminacy. This means that, given some initial conditions, there exists a region in the

[35] For a survey on the conditions for the existence of multiple equilibria in *finite* general equilibrium models see Kehoe (1985) and Mas-Colell et al. (1991).

state space of the model where isolated perfect foresight equilibria can converge to different steady states.

6.1 Static case

To gain intuition for the multiplicity of equilibria, consider a static two-country exchange economy with two traded goods, x and y. When preferences and endowments are symmetric, a relative price of traded goods equal to unity is an equilibrium price. If this equilibrium is unstable, by the index theorem there are at least two more stable equilibria. Intuitively, if the aggregate excess demand function crosses zero from below at an equilibrium price, then it must cross zero from above for at least two other prices.[36]

Suppose the domestic economy has more of good x than good y and, with symmetry, the foreign economy has more of good y than good x. When the relative price of y in terms of x is low, agents in the domestic economy are wealthier than foreign agents. If the elasticity of substitution between goods is low, then foreign consumers are willing to trade most of their abundant good (y) to import some of the scarce good (x), while the domestic country consumes most of both goods. Conversely, when the terms of trade favor the foreign economy, meaning that the price of good (x) is now low, domestic agents are willing to trade most of their abundant good x in exchange of some of good y. Both extreme cases, then, are equilibria.[37]

Consider the following specification for preferences:

$$\mathcal{U}(x,y) = \frac{(1-\gamma)^{\sigma} x^{1-\sigma} + \gamma^{\sigma} y^{1-\sigma}}{1-\sigma} \quad \text{and} \quad \mathcal{U}^*(x^*,y^*) = \frac{\gamma^{\sigma} x^{*1-\sigma} + (1-\gamma)^{\sigma} y^{*1-\sigma}}{1-\sigma},$$

where $0 < \gamma < 1$ determines the utility weights and $1/\sigma$ is the elasticity of substitution between good x and good y. Endowments of goods x and y of the domestic agent are $1-\varepsilon$ and ε, respectively, while those for the foreign agent are ε and $1-\varepsilon$, where $0 < \varepsilon < 1$.

Consider a social planning problem of this economy as in problem 2. Define the domestic transfer function as $\mathcal{T}(q) = \lambda(q)[x(q) - (1-\varepsilon)] + \phi(q)[y(q) - \varepsilon]$, where $x(q)$

[36] See Appendix A.2 for a formal treatment of the index theorem.

[37] Toda and Walsh (2017) and Chipman (2010) offer examples of static exchange economies where conditions for multiplicity are derived using the aggregate excess demand.

and $y(q)$ are the allocations of the planner for the domestic economy, q is the relative welfare weight between the two countries, and λ and ϕ represent the Lagrange multipliers of the resource constraints of good x and good y, respectively. Due to symmetry, a relative welfare weight equal to unity is an equilibrium welfare weight. Thus, for multiplicity to arise, is sufficient to show that $\partial\mathcal{T}(q)/\partial q\mid_{q=1} > 0$.[38] In Appendix B.1, I show that in this economy there are three equilibria when

$$\frac{1}{\sigma} < 1 - \frac{1}{2}\left(\frac{\varepsilon}{\gamma} + \frac{1-\varepsilon}{1-\gamma}\right). \tag{15}$$

This sufficient condition for multiplicity relates utility weights, the elasticity of substitution, and the distribution of endowments. Intuitively, the right hand side of the inequality represents the degree of heterogeneity between countries. If countries are homogeneous, no multiplicity is possible as the elasticity of substitution takes non-negative values. When countries are heterogeneous, then a low enough elasticity of substitution between traded goods is needed to support trade at extreme prices. As a consequence, multiplicity of equilibrium is associated with different levels of welfare in each country at different equilibrium prices.

In the rest of the section I focus on understanding the global dynamics of this model. In the next section I apply the dynamic analysis of this section to an open economy model with endowment shocks and self-fulfilling fluctuations.

6.2 Dynamic case

Now I extend this case with two goods to a dynamic economy with OLG as in Problem 4.[39] In the dynamic case, the savings behavior between countries is captured by the discount factors β and β^*. Agents in the country with the lower discount factor are relatively more impatient than agents in the other country, and, all else equal, value consumption when young relatively more that their foreign counterparts. One advantage of the OLG assumption is that it allows me to exploit heterogeneity in discount factors, which provides a better description of modern open economies, as countries can be

[38] Intuitively, if the transfer function of the domestic country crosses zero from below at an equilibrium welfare weight, then it must cross zero from above for at least two other welfare weights.

[39] Appendix B.2 shows the full derivation for the general formulation of this model in terms of Problem 4. Appendix C presents the model as a three-dimensional first-order dynamical system using the insights of section 5.

described as net savers or net borrowers.[40] In contrast, discount factor heterogeneity with infinitely-lived agents results in an asymptotic distribution of wealth where the consumption of the relatively more impatient agent approaches zero (Becker, 1980).

Consider a calibration of parameters for the dynamic economy such that condition (15) is satisfied. In the dynamic case, this condition generates multiple steady states when heterogeneity in the discount factor across countries, β and β^*, is not too high.[41] All else equal, in what follows I refer to the two-country dynamic economy with $\beta = \beta^*$ as the homogeneous case and with $\beta \neq \beta^*$ as the heterogeneous case. In the rest of the section I focus on the homogeneous case as its graphical representation is simpler. In the next section I study an example with heterogeneous discount factors.

Figure 1 plots the steady-state transfer functions for the homogeneous case ($\beta = \beta^*$). The figure shows the existence of three steady states, as the curves intersect at zero three times. In this example, the rates of growth of the welfare weights are equal to the homogeneous discount factor ($\kappa = \beta$) in each steady state, and the relative welfare weight, q, is equal to unity in the middle steady state, just as in the static case.[42]

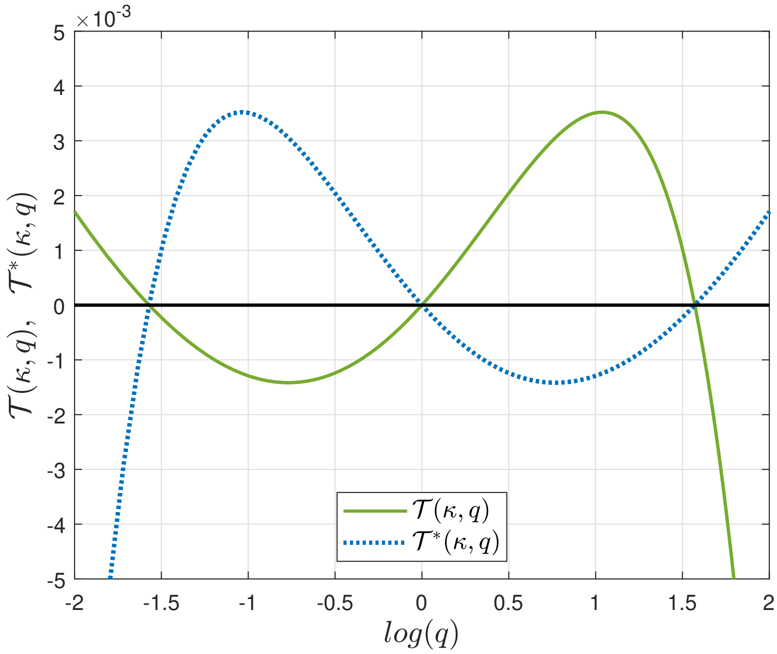
Recall that an equilibrium of the dynamical system corresponds to the sequence $\{\kappa_t^*, \kappa_t, q_{t-1}\}_{t=1}^{\infty}$ such that all transfer functions are equal to zero. To compute the equilibrium sequence I exploit the two-step algorithm developed in the previous section using the steady state as a terminal condition. When the dynamical system admits only one steady state (and no externalities) there is a unique perfect foresight equilibrium for any initial relative welfare weight, q_0, as the steady state is typically a saddle (see next subsection). In that case, we say that the global dynamics of the model are determinate. When the dynamical system admits multiple steady states, the characterization of the set of perfect foresight equilibria is more involved as not only the local dynamic properties of the steady states might change with respect to the case with an unique steady state but also the dynamics away from the steady state. The analysis

[40] For instance, Ghironi et al. (2008) show that discount factor heterogeneity is useful to model persistent consumption dynamics in the U.S. economy.

[41] When discount factors are heterogeneous, there is no analytical condition for multiplicity. In such cases, I use the index theorem to assess the existence of multiplicity. See Appendix B.4 for a discussion on the index theorem. Intuitively, if heterogeneity in β and β^* is high enough, then the patient agent is not willing to trade at the extreme unfavorable price when young.

[42] If $\beta \neq \beta^*$, then homogeneity is lost, q is no longer equal to unity in the middle steady state, and each value of q is associated with a different value of κ that satisfy the transfer functions in the steady state.

Figure 1: Transfer Functions in Steady State under $\beta = \beta^*$



Note: The figure shows the transfer functions, $\mathcal{T}(\kappa, \kappa, \kappa, \kappa, q)$ and $\mathcal{T}^*(\kappa, \kappa, \kappa, \kappa, q)$ against different values of the logarithm of the relative welfare weight q when $\kappa = \beta$ and condition (15) is satisfied. Given κ, the intersections of the curves at zero map into equilibrium values of q. In this example, there are three such steady-state values. Due to homogeneity in discount factors, the middle steady-state value for q is unity (and zero in logarithm). For notation simplicity, the legend in the figure is denoted as depending only on two arguments: (κ, q).

of the global dynamics of this economy consist of two parts. First, I need to characterize the solutions of the dynamical system $F(V_t, V_{t+1})$ around each steady state, i.e., their local determinacy properties. Second, I need to characterize the solutions of the dynamical system away from the steady states, i.e., the global determinacy properties. For the first part I use standard techniques and for the second part I rely on numerical solutions to show that this model display global indeterminacy.

6.2.1 Local Determinacy

Let J_1 and J_2 denote the Jacobians of $F(V_t, V_{t+1})$ with respect to V_t and V_{t+1}, respectively, where $V_t = (\kappa_t^*, \kappa_t, q_{t-1})$. In this system, q_t is a predetermined variable and κ_t, κ_t^* are non-predetermined. The behavior of this system in a neighborhood of the steady state is governed by the eigenvalues of the matrix $J = -J_2^{-1} J_1$ evaluated in the steady state (κ, κ, q).[43] If exactly two roots of the matrix $J = -J_2^{-1} J_1$ are outside the unit circle, then the steady state is a saddle and there exists one and only one trajectory —the stable manifold— that converges toward (κ, κ, q).[44] Although not reported in this paper, for most calibrations such that the dynamical system displays multiple steady states, I find that the extreme steady states are saddle-point stable (locally determinate), while the middle steady state is unstable. Moreover, in most calibrations, the middle steady state displays a pair of complex conjugates.[45] When $\beta = \beta^*$, one of the eigenvalues of the intermediate steady state is always $1/\beta > 1$ as discussed in Kehoe and Levine (1985).

6.2.2 Global Indeterminacy

When multiple steady states exist, the economy can display global indeterminacy despite saddle-point stability of the extreme steady states. The global behavior of $F(V_t, V_{t+1})$ depends on the shape of the stable manifolds associated with the extreme steady states and on whether the middle steady state is a source or a sink (recall footnote 44). Additionally, the values of $V_2 = (\kappa_2^*, \kappa_2, q_1)$ such that the sequence $V_t = \{\kappa_t^*, \kappa_t, q_{t-1}\}_{t=2}^{\infty}$ converges to the steady-state values of $F(V_t, V_{t+1})$ must be such that the initial conditions in period one are satisfied (see equation (13)).[46]

Because the model does not have an analytical solution, I explore its global dynamics numerically. Figure 2 illustrates the solutions of the dynamical system $F(V_t, V_{t+1})$ converging to a steady state for the homogeneous case ($\beta = \beta^*$) using the same calibration as in Figure 1. To find the trajectories I solve a two-boundary problem where the initial condition is given by the relative welfare weight q_1 and the terminal condi-

[43] Because J_2 might be singular, the generalized eigenvalues of the matrix J are computed using the generalized Schur decomposition. This routine produces two matrices, $J_1 = QTZ'$ and $J_2 = QSZ'$, where Q and Z are orthonormal matrices and S and T are upper triangular matrices. The generalized eigenvalues are defined as $\lambda_i^{eig} = T_{i,i}/S_{i,i}$.

[44] If more than two roots of J are outside the unit circle, the steady state is unstable (a *source*). If only one one root is outside, the steady state is locally indeterminate (a *sink*) and there is a continuum of equilibrium paths that all converge to the steady state (Blanchard and Kahn, 1980). OLG economies can display any type of steady states (Kehoe and Levine, 1985).

[45] Complex unstable eigenvalues might be related with a limit cycle around the unstable steady state as in the model of Kaplan and Menzio (2016). However, in this paper I do not pursuit that exploration.

[46] It is possible that some equilibrium paths that depart from the unstable steady state converge to either one of the extreme saddle points, or alternatively, to a limit cycle. Although interesting in itself, I leave that analysis for future research.

tion is the corresponding steady state.[47] The solid lines represent the stable manifolds converging to the lower steady state (blue circle) and the upper steady state (green diamond), respectively. Recall that, under discount factor homogeneity, the rates of growth of welfare weights, κ_t^* and κ_t converge to the discount factor, β, in each steady state. Thus, each steady state differs only in the coordinate q (as in Figure 1). The middle steady state is denoted by a (magenta) square and in this example its coordinates corresponds to $(\kappa^*, \kappa, q) = (\beta, \beta, 1)$. This is the symmetric steady state. Recall that the steady states are associated with different levels of consumption, thus convergence to one or another imply substantial changes in economic welfare.

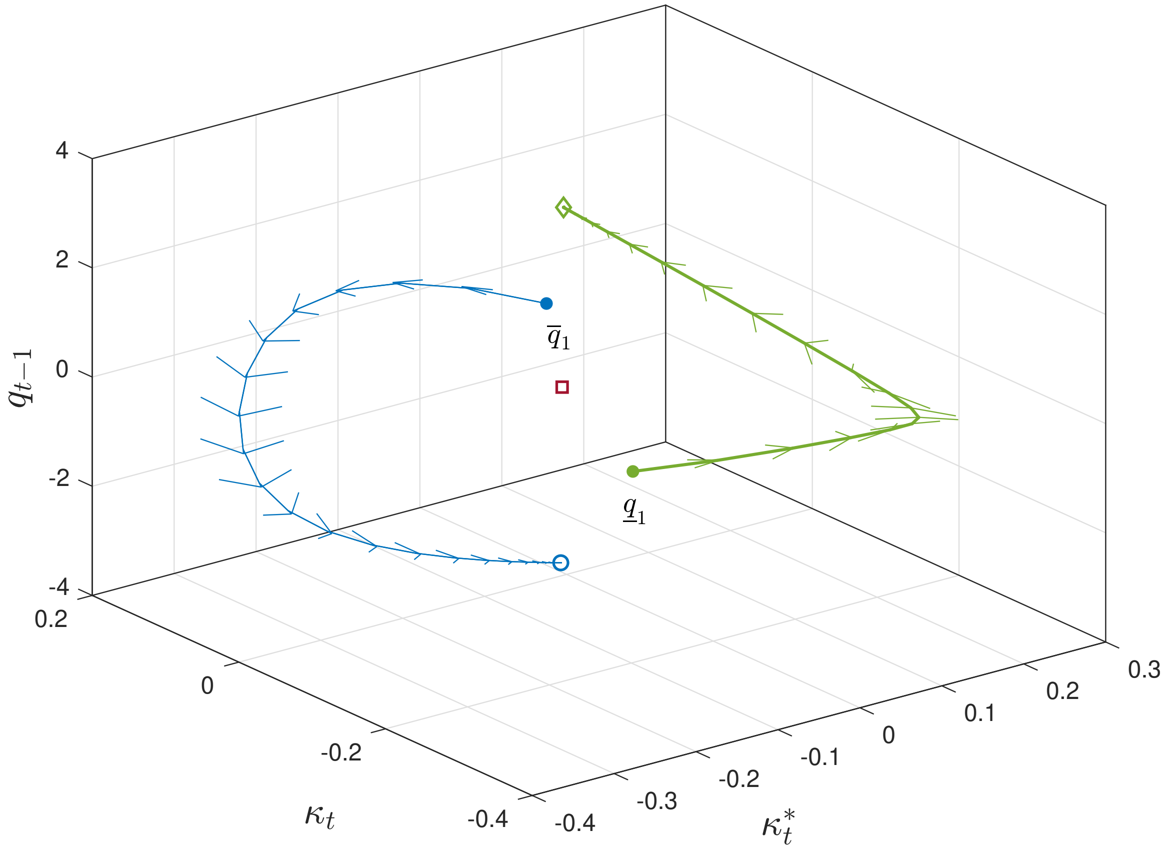
To characterize the equilibrium paths of this model, I solve numerically for the initial conditions in the relative welfare weight q_1 such that the sequence $V_t = \{\kappa_t^*, \kappa_t, q_{t-1}\}_{t=2}^{\infty}$ converges to a steady state. In Figure 2, we can identify three regions of initial conditions for values of q_1. The lower region corresponds to the values of q_1 where $q_1 < \underline{q}_1$. In this region, the solutions of the dynamical system with initial condition q_1 converge to the lower steady state (blue circle). In such cases, the domestic country enjoys more favorable terms of trade. The upper region corresponds to the values of q_1 where $q_1 > \overline{q}_1$ and the solutions of the dynamical system with initial condition q_1 converge to the upper steady state (green diamond). In such cases, the foreign country enjoys more favorable terms of trade. However, in the region around the unstable steady state (square), denoted by values of $q_1 \in \left[\underline{q}_1, \overline{q}_1\right]$, the system might be subject to global indeterminacy in the sense that there are trajectories converging to either the lower or upper steady state. To find the indeterminacy region $\left[\underline{q}_1, \overline{q}_1\right]$, I apply a shooting algorithm to search the highest feasible $\overline{q}_1$ as an initial condition for the transition to the lower steady state, and, conversely, the lowest feasible $\underline{q}_1$ as an initial condition for the transition to the upper steady state.[48]

The intuition for global indeterminacy is the following. Agents in the model need

[47] I compute the deterministic paths converging to a steady state by solving the simultaneous system of non-linear equations $F(V_t, V_{t+1})$ with a finite number of periods. Specifically, I use Newton-type methods available in DYNARE such as Juillard et al. (1996). See Adjemian et al. (2011) for a detailed description of the methods.

[48] Computing initial conditions is sensitive to the choice of algorithm. Moreover, even when multiple initial conditions converge to the same steady state, shooting algorithms can be numerically unstable. When searching for initial conditions I found this type of instability. Although I do not pursue further on this issue, a conjecture is that this instability might be related with to the existence of a limit cycle around the unstable steady state.

Figure 2: Solutions of the Dynamical System under $\beta = \beta^*$



Note: The figure shows the solutions of the dynamical system $F(V_t, V_{t+1})$ using the calibration of the static economy under discount factor homogeneity . The upper steady state is denoted with a diamond (green), the middle steady state with a square (magenta), and the lower steady state with a circle (blue). The solid lines are the stable manifolds converging to the lower and upper steady state, respectively. The arrows in each manifold represent the direction of the endogenous variables in three dimensions. The points, $\underline{q}_1$ and $\overline{q}_1$, represent the initial conditions. Values are in logarithms.

to predict future prices (or terms of trade) to make consumption decisions. When multiple steady states arise, there are multiple paths of such future prices. In that environment, which equilibrium path materializes depends on agents' beliefs about future outcomes. Since agents coordinate in one possible outcome, beliefs about future prices (or terms of trade) become self-fulfilling.

To complete the computation of the perfect foresight equilibrium, the transfer functions in period one must hold as well. Take any initial vector $V_2 = (\kappa_2^*, \kappa_2^*, q_1)$ that belongs to some stable manifold of $F(V_t, V_{t+1})$ and solve (numerically) for the vector $V_1 = (\kappa_1^*, \kappa_1^*, q_0)$ such that the transfer functions of agents alive in the first period are equal to zero.[49] The model displays global indeterminacy when, given the relative welfare weight between the initial old q_0, there is more than one value for the vector $V_2 = (\kappa_2^*, \kappa_2^*, q_1)$ with $q_1 \in \left[\underline{q}_1, \overline{q}_1\right]$ that solves the transfer functions in the first period. As a consequence, some values of $q_1 \in \left[\underline{q}_1, \overline{q}_1\right]$ might be ruled out if they do not satisfy the transfer functions in the first period. This means that the initial condition q_0 is associated with two equilibrium paths that converge to either the lower or the upper steady state, and therefore, self-fulfilling beliefs act as an independent driver of economic fluctuations.[50]

The existence of multiple steady states implies that the behavior of this economy is determined not solely by fundamentals (i.e., preferences and endowments) but also by the agents' beliefs about which equilibrium path materializes. The coexistence of equilibria converging to different steady states has the interpretation that agents' beliefs can be important enough to determine the long-run outcomes of the economy, and not simply the path that the economy follows in the short run.[51] As I show in the next section, in contrast to models with a unique steady state, a model with multiple steady states and self-fulfilling fluctuations provides a rationale to explain permanent welfare effects of transitory shocks.

7. International Business Cycles and Global Imbalances

In this section and the next, I study a version of the open economy model presented in section 6 in which countries have heterogeneous time preferences and where goods x and y represent each a basket of n composite goods. I calibrate the model to display multiple steady states and I discuss the empirical relevance of such calibrations. I focus in this section on their welfare properties and I leave for the next section the study of the effects of large transitory shocks and self-fulfilling beliefs.

There are two types of goods, e and w, which represent the overall differences in

[49] Recall from Appendix A.1 that one budget constraint is redundant because of Walras's Law.
[50] The middle steady state is unstable. The only way of reaching this steady state is by starting there.
[51] I only consider perfect foresight equilibria. It might be possible to construct more equilibria, such as rational expectations equilibria where fundamental shocks display a bimodal distribution where the economy oscillates between two steady states, or Markov-switching rational expectations equilibria as in Kaplan and Menzio (2016). I leave those extensions for future research.

endowments between countries, and there are n goods of each type.[52] Countries are endowed with both types of goods, but each good within a type represents a fixed proportion of the aggregate endowment of that good. For instance, the young agent in the domestic country is endowed with $(1-\varepsilon)$ units of each good of type e and with ε units of each good of type w.

Baskets of consumption goods are type-specific and defined as a Cobb-Douglas aggregator over n goods with coefficient ρ_i, where $i = 1, ..., n$. The basket x is composed by the n goods of type e and the basket y is composed by the n goods of type w. Agents have the same preferences as in the previous sections but they demand type-specific baskets rather than single goods. In Appendix D, I show the full derivation of the social planning problem of this model applying Negishi's method.

Using the competitive equilibrium allocations derived from the solution to the social planning problem in Appendix D, I compute the terms of trade, real exchange rate, and current account balance. Define p_t to be the terms of trade across countries, i.e., the relative price of basket y in terms of basket x,

$$p_t = \frac{\Phi_t}{\Lambda_t} = \left(\frac{\gamma}{(1-\gamma)}\frac{x_t^t}{y_t^t}\right)^{\sigma}, \tag{16}$$

where the Lagrange multipliers of the planner's resource constraint, Φ_t and Λ_t, represent the competitive prices of the baskets x and y, respectively. The second equality is derived from the equilibrium conditions.[53] Under this definition, an increase in p_t corresponds to a terms of trade deterioration for the domestic economy as the country needs to export more in order to afford the same quantity of imports. The consumer aggregate price levels in each country are,

$$\begin{aligned} P_t &= \left((1-\gamma) + \gamma(p_t)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \\ P_t^* &= \left((1-\gamma)(p_t)^{\frac{\sigma-1}{\sigma}} + \gamma\right)^{\frac{\sigma}{\sigma-1}}. \end{aligned} \tag{17}$$

Notice that the price levels depend only on preferences and the terms of trade. The real exchange rate measures the relative price levels across countries and therefore corresponds to,

$$\xi_t = \frac{P_t^*}{P_t}, \tag{18}$$

[52] This is more general than country-specific goods as one country might be endowed with both types of goods.
[53] See Appendix D.1 for the optimality conditions of the problem

which is also a function of the relative price.[54] Finally, the current account in each country corresponds to the savings levels of the country. Therefore, the current account of the domestic economy is equivalent to its trade balance,

$$CA_t = \left(e_t^t - x_t^t + e_t^{t-1} - x_t^{t-1}\right) + p_t\left(w_t^t - y_t^t + w_t^{t-1} - y_t^{t-1}\right). \tag{19}$$

where e_t^t and w_t^t correspond to aggregations of type e and type w endowments (see Appendix D.2).

Many modern economies are characterized by global imbalances. This is a popular term to refer to the substantial expansion of the current account deficit in the United States in recent decades and the rise in the current account surpluses of many emerging-market economies (Bernanke et al., 2007). In this model, a global imbalance corresponds to the case in which a country's steady state current account is not zero.[55] In contrast to models with infinitely-lived agents, in OLG models, a global imbalance is possible when countries have different time preferences (Buiter, 1981; Ghironi et al., 2008). In this regard, this paper deviates from the literature using financial frictions to model global imbalances, and exploits heterogeneous time preferences to capture differences in savings behavior between countries.

Let the domestic economy be the impatient country ($\beta < \beta^*$). In this model, there is no trade between generations in the same country, but there is trade between young consumers across countries due to heterogeneity in endowments and preferences.[56] In the competitive equilibrium, heterogeneous time preferences mean that the impatient young consumer in the domestic economy is willing to borrow from the foreign patient young saver and pay back with interest when old. Because the interest rate is positive in dynamically efficient economies and there is no growth in endowments, an impatient old agent reduces the value of her consumption by more than what the newborn young borrows, as she must pay her debt back. As a consequence, in the steady state, the domestic country, whose residents are more impatient to consume than their

[54] Notice that the consumer aggregate price levels are weighted averages of the terms of trade, thus, by construction, the terms of trade are more volatile than the real exchange rate, which is counterfactual. Itskhoki and Mukhin (2021) show that pricing complementarities in domestic production improves the fit of the data in this regard. However, Ayres et al. (2020) argue that this relation actually fit the data when terms of trade are considered over primary commodities.

[55] Because the model features only two countries, $CA + CA^* = 0$

[56] Recall that assumptions on endowments and preferences imply that this is a *classical* economy and equilibria are dynamically efficient. See footnote 12.

international trading partners, experiences a long-run current account surplus.[57] As in every period there is an old agent paying back the debt acquired when young, the domestic economy is indebted in perpetuity against the foreign economy. In the social planning problem, this behavior is expressed through the planner's allocations at the welfare weights that solve the transfer functions.

7.1 Calibration

For a class of international real business cycles the possibility of multiple equilibria collapses to the empirical relevance of low values of the elasticity of substitution (Bodenstein, 2011). Consider the condition for multiplicity, equation (15), rewritten here for simplicity,

$$\frac{1}{\sigma} < 1 - \frac{1}{2}\left(\frac{\varepsilon}{\gamma} + \frac{1-\varepsilon}{1-\gamma}\right).$$

Suppose that goods are produced either domestically or overseas and therefore the parameter ε is close to unity.[58] Suppose further that there is home bias in consumption that manifests itself as a low value of γ. Then, the above condition is satisfied if $1/\sigma$ is low enough. Intuitively, to sustain trade at extreme equilibrium prices, traded goods must be poor substitutes.

Table 1 presents a calibration of the model that exhibits both multiple steady states and global imbalances. Since $\beta < \beta^*$, the domestic economy runs a current account surplus in the steady state.

While there is relative consensus in the literature regarding home bias in consumption, there is considerable uncertainty regarding the estimates of the trade price elasticity at the aggregate level.[59] Bodenstein (2010, 2011) provides a review of the empirical literature on the trade elasticity. Three observations emerge. First, estimates from aggregate data range from 0 to 1.5 and estimates from lower levels of aggregation are above unity. Second, while most calibrations in applied macroeconomics are between

[57] In the the steady state, impatient young agents in the domestic economy run a trade balance deficit, $TB_y < 0$, as they consume more than the value of their endowments. When old, some of their resources are used to pay the debt back, therefore old agents run a trade balance surplus, $TB_o > 0$. Because the interest rate is positive, $TB_o = -(1+r)TB_y$. Then, $CA = TB_y + TB_o = -rTB_y$. Because $TB_y < 0$, the domestic country runs a current account surplus.

[58] I exclude unity to avoid boundary concerns in demand functions.

[59] See McCallum (1995) and Obstfeld and Rogoff (2000) for a discussion on the home-bias-in-trade puzzle.

Table 1: Calibration for Multiple Steady States and Global Imbalances

Parameter	Value
Home bias in consumption	$(1-\gamma) = 0.83$
Elasticity of substitution between traded goods	$1/\sigma = 0.37$
Intertemporal elasticity of substitution	$1/\sigma = 0.37$
Endowment distribution	$\varepsilon = 0.01$
Domestic discount factor	$\beta = 0.84$
Foreign discount factor	$\beta^* = 0.87$

Note: This calibration satisfies condition (15) and computes an $index(\kappa, q) = -1$ for the middle steady state (see Appendix A.2).

1 and 1.5, some studies suggest that values below 0.5 provide a better performance in matching key moments in data. Third, estimations of DSGE models using Bayesian techniques suggest that values below 0.5 can better fit the data.[60] Importantly, Corsetti et al. (2008) show that multiplicity of steady states might arise for values of the trade elasticity around unity when distribution costs in terms on non-traded goods are included in standard international business cycle models.

7.2 Steady States

Table 2 shows the values for the key variables in the three steady states of the model using the calibration in Table 1. Notice that, because of time preference heterogeneity, the relative welfare weight, q, is not unity anymore in the middle steady state, but is tilted towards the more impatient economy because the terms of trade are more favorable to the domestic economy ($q < 1$).[61] Unlike the homogeneous case ($\beta = \beta^*$), the rates of growth of welfare weights, κ and κ^*, are different in each steady state.

[60] The first group of papers includes Taylor (1993) and Hooper et al. (2000) for aggregate estimates and Broda and Weinstein (2006) for lower levels. The second group includes Heathcote and Perri (2002) and Benigno and Thoenissen (2008). The third group includes Lubik and Schorfheide (2005) and Rabanal and Tuesta (2010).

[61] Recall the relative welfare weight is $q = \alpha^*/\alpha$. Because under $\beta < \beta^*$ the foreign economy would enjoy a higher marginal utility at $q = 1$, the social planner adjusts the welfare weights towards the domestic economy to keep the marginal utility ratios equal across countries. See Appendix B.3 for the first-order conditions of the simpler planner's problem with two goods.

Recall, however, that κ and κ^* converge to the same scalar as otherwise they converge either to zero or infinity.

The difference in the steady state values reveals the welfare consequences of transitioning from one equilibrium to another. Because I am interested in the permanent effects of transitory shocks, I consider a welfare measure in the steady state rather than in the transition path. Define $\mathcal{U}^i_{\mathcal{W}}$ and $\mathcal{U}^{*,i}_{\mathcal{W}}$ as the indirect utility function in the steady state for the domestic and the foreign economies, respectively, as,

$$\begin{aligned} \mathcal{U}^i_{\mathcal{W}} &= \mathcal{U}(x^i_y, y^i_y) + \beta\, \mathcal{U}(x^i_o, y^i_o) \\ \mathcal{U}^{*,i}_{\mathcal{W}} &= \mathcal{U}^*(x^{*,i}_y, y^{*,i}_y) + \beta^*\, \mathcal{U}(x^{*,i}_o, y^{*,i}_o), \end{aligned} \tag{20}$$

where $\{x_y, y_y, x^*_y, y^*_y\}$ and $\{x_o, y_o, x^*_o, y^*_o\}$ are the steady state allocations when young and old in the $i = \{1, 2, 3\}$ steady state where one refers to the lower steady state and three to the upper one. Because the middle steady state is dynamically unstable, I focus on the lower and upper steady states. As in the static economy, in the lower steady state, agents in the domestic economy are wealthier than foreign agents due to beneficial terms of trade, even though they are more impatient. This translates to higher levels of consumption of both goods and thus to a higher level of welfare. A transition to the upper steady steady state implies lower levels of consumption for those agents and thus a reduction in their economic welfare. The reverse is true for the foreign economy.

While quantitatively small, time preference heterogeneity manifests itself in global imbalances as the domestic current account differs from zero in every steady state. In particular, the domestic economy displays a current account surplus in every steady state and therefore a perpetual transfer of resources to the foreign economy. This magnitude is amplified in the upper steady state since the increase in borrowing when young at the new terms of trade more than offset the reduction in the steady state rate of interest. How large these transitions are depends on how far apart the steady states are (given some measure of this distance), which, in turn, depends on the calibration of the model.

Table 2: Steady State Values under Multiplicity

Variable	Lower	Middle	Upper
Domestic Current Account (CA)	0.03	0.05	0.07
Real Exchange Rate (ξ)	0.60	0.90	1.84
Terms of Trade (p)	0.45	0.85	2.56
Domestic Welfare Weights growth (κ)	0.85	0.85	0.86
Foreign Welfare Weights growth (κ^*)	0.85	0.85	0.86
Relative Welfare Weight (q)	0.29	0.77	4.17
Domestic Welfare ($\mathcal{U}_{\mathcal{W}}$)	0.40	0.35	0.22
Foreign Welfare ($\mathcal{U}^*_{\mathcal{W}}$)	0.24	0.32	0.41

Note: The table shows the steady state values of each variable under multiplicity. Calibration for multiplicity is in Table 1. Steady states are defined as lower, middle, and upper depending on the value of the relative welfare weight (q). The current account is multiplied by 100 as it is quantitatively small, but different from zero. Welfare computations are transformed using an exponential function as the coefficient σ is greater than one.

8. Experiments

In this section I study perfect foresight equilibrium dynamics in the model presented in the previous section. I show that large and unanticipated transitory endowment shocks can cause a shift from one equilibrium to another depending on agents' beliefs. Shifts of this kind have substantial and permanent effects on the terms of trade, the real exchange rate, and on economic welfare.

These dynamics are explained by global indeterminacy. This means that there exists a region in the state space of the model where isolated perfect foresight equilibria can converge to different steady states. Agents in the model need to predict future prices (or terms of trade) to make consumption decisions. When multiple steady states arise, there are multiple possible paths of such future prices. In that environment, the equilibrium path that materializes depends on agents' beliefs about future outcomes. Since agents coordinate on one possible outcome, beliefs about future prices (or terms of trade) become self-fulfilling.

To study the dynamics of the terms of trade, the real exchange rate, consumption, and the current account I conduct two numerical experiments. The first experiment shows that a large and unanticipated transitory endowment shock can shift the global economy from the lower steady state to the upper steady state in the presence of self-fulfilling beliefs. Consider the case when the economy is in the the lower steady state and the domestic economy enjoys favorable terms of trade. In period one, the domestic economy is perturbed by a large negative endowment shock that puts it into the indeterminacy region of the model. In this region, two equilibrium paths are possible. At the same time, a belief shock changes agents' expectations such that they coordinate on the upper steady state and an equilibrium path converging to that steady state materializes. A shift of this kind is associated with a persistent deterioration in the terms of trade, a depreciation of the real exchange rate, a current account reversal, and a contraction in consumption. Dynamics of this type resemble the observed persistent deterioration in the terms of trade and the depreciation of the real exchange rate experienced by Japan since the 1990s, as documented by Obstfeld and Rogoff (2009).

In the second experiment, I ask under what conditions a transition of the form found in the first experiment can be interrupted. In the absence of additional fundamental shocks, the transition to the upper steady state can only be interrupted if a second belief shock materializes in the indeterminacy region of the model, and agents coordinate on the equilibrium path converging to the lower steady state. In numerical simulations, this situation is only possible in periods immediately after the negative endowment shock hits the domestic economy, as later the global economy leaves the indeterminacy region, and after that, only one equilibrium path is admissible. If belief shocks are correlated with public signals such as speeches or announcements by governments or economic authorities, my second experiment suggests that those policy interventions should take place as quickly as a large shock hits the economy, to prevent an equilibrium shift caused by self-fulfilling beliefs.

Recall that a belief shock can produce an equilibrium shift only in the indeterminacy region of the model. As Figure 2 illustrates, outside this region only one equilibrium path is possible. If the starting point of the dynamical system is a steady state, then a sufficiently large or persistent fundamental shock is necessary to move the economy into the indeterminacy region. If the fundamental shock is too small, then the economy fluctuates around the original steady state and eventually returns to it. If instead the fundamental shock perturbs the economy above and beyond the indeterminacy region, then the only possible equilibrium path converges to the other steady state.

These experiments also highlight the importance of using non-linear solution methods. Linearization or higher-order perturbation methods that approximate the dynamical system around a deterministic steady state are not well suited for the kind of shocks just discussed. In particular, in the case of large shocks local approximation techniques are inappropriate, as they search for an equilibrium outside the neighborhood of the steady state where an equilibrium of the nonlinear model might not exist.

8.1 Self-fulfilling Beliefs and Steady State Transitions

Figures 3-7 illustrate the first experiment. Suppose the global economy is in the lower steady state in period zero, provided this is an equilibrium that satisfies the initial conditions discussed in section 6.2.2. In period one, marked with the dashed vertical line in the figures, a large and unanticipated negative endowment shock hits the domestic economy. I assume that, as a result, the young's endowment of basket x shrinks.[62] As a consequence of the negative endowment shock, the terms of trade improve on impact and the real exchange rate appreciates (a reduction in p_t and xi_t as plotted in Figure 3). While the negative supply shock improves the terms of trade for the domestic economy, the reduction in the endowment of the young agent is not offset by the price effect, and so her consumption falls.

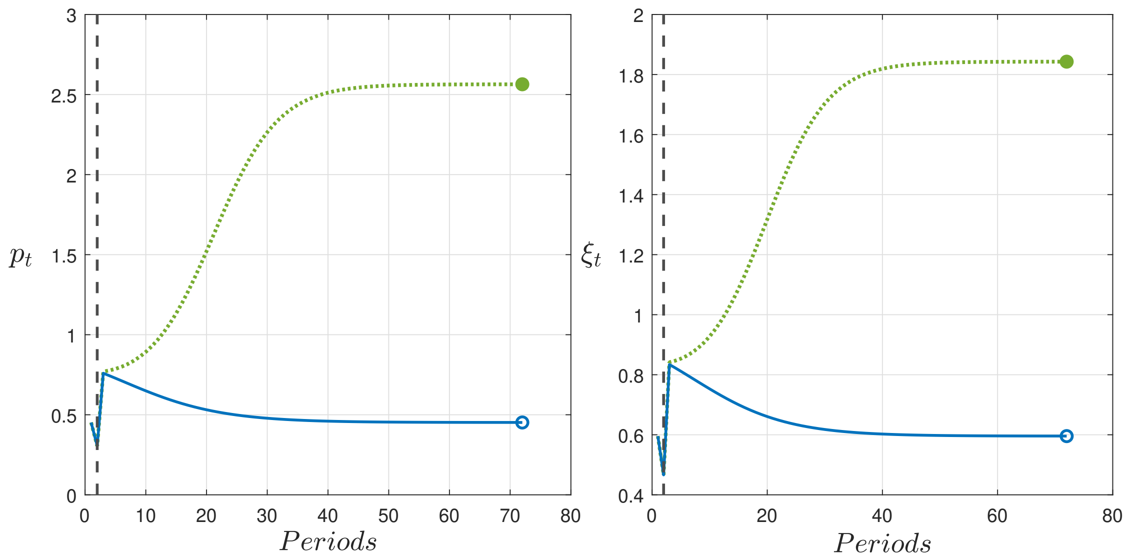
Figure 4 shows that consumption of both baskets by the domestic young decreases on impact.[63] However, consumption of basket x by the young in the domestic economy is more affected as there is no trade between generations. Notice that the consumption of both baskets by the old in period two is less affected by the shock. Although both the domestic young and old alive in period two enjoy the full endowment of basket x, they face worse terms of trade compared with the original steady state as prices adjust to the recovery in supply.[64] Due to the sharp contraction in consumption

[62] I assume that the endowments of all the goods that compose basket x shrink by the same proportion when the shock hits.

[63] Notice that the upper panel in Figure 4 illustrates the consumption sequence of young agents in each period. The lower panel depicts the sequence of consumption of the same consumer when old. Therefore each young in period t in the upper panel corresponds to the old in $t+1$ in the lower panel.

[64] For completeness, Appendix E.1 describes the evolution of consumption of the foreign economy.

Figure 3: Terms of Trade and Real Exchange Rate



Note: The left panel plots the terms of trade, p_t, and the right panel the real exchange rate, ξ_t. The vertical dashed line identifies the period when the fundamental shock materializes. The dotted (green) line shows the transition to the upper (green, solid circle) steady state. The solid (blue) line shows the transition to the lower (blue, circle) steady state.

of basket x by the domestic young, the domestic current account improves on impact (see Figure 5).[65]

Changes in endowments, and therefore in the terms of trade and consumption, map into the planner's solution as adjustment in the relative welfare weight, q_t, and in the rates of growth, κ_t^* and κ_t, such that the transfer functions are equal to zero in every period.[66] As a consequence of the negative endowment shock to the generation born in period one in the domestic economy, q increases on impact (see Figure 6). Similarly, Figure 7 shows the evolution of the rates of growth of welfare weights. While both rates of growth, κ_t^* and κ_t, decrease on impact, κ_t decreases by more as it reflects the shrink in endowments in the domestic economy (see the left panel).[67]

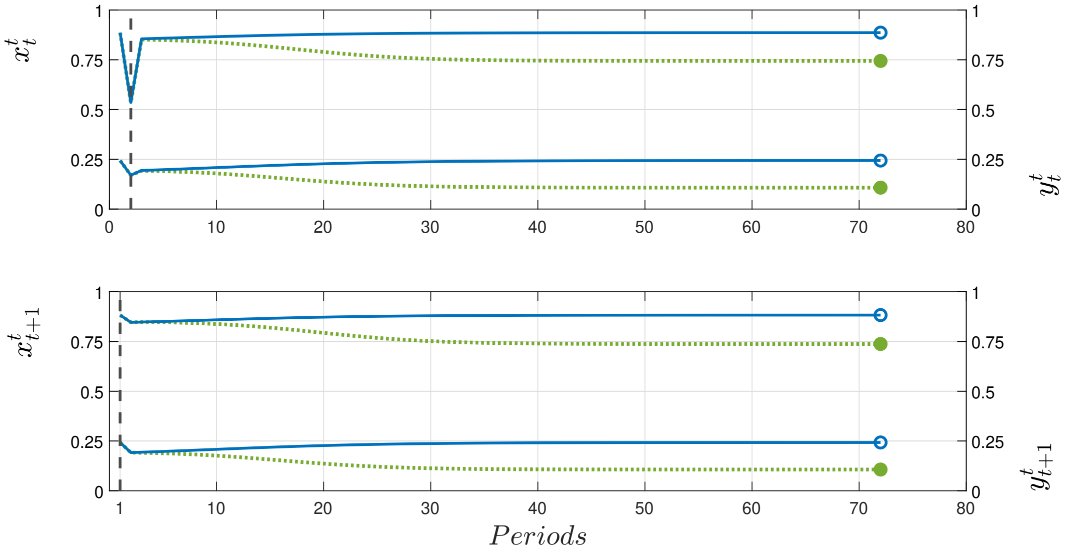
After the adjustments in terms of trade and consumption in periods two and three, the global economy enters the indeterminacy region. In this region two equilibrium

[65] The domestic current account computation considers consumption of the young in period t, x_t^t and y_t^t, and consumption of the old in period t, x_t^{t-1} and y_t^{t-1} (see equation (19)). For simplicity, I assume that the old alive in period one (born in period zero) is not affected by the shock in any form, and therefore, her allocations correspond to the steady state levels.

[66] Recall that the transfer functions indicate the extent to which the planner's allocations indexed by welfare weights violate the individual budget constraints.

[67] Recall that $q_t = \alpha_t^*/\alpha_t$, $\kappa_{t+1}^* = \alpha_{t+1}^*/\alpha_t^*$, and $\kappa_{t+1} = \alpha_{t+1}/\alpha_t$.

Figure 4: Domestic consumption



Note: The figure shows the domestic consumption allocations of the young and the old born in period t for goods x and y. The vertical dashed line identifies the period when the fundamental shock materializes. The upper panel plots the consumption allocations for the young, x_t^t and y_t^t. The two upper lines in that graph represent x_t^t. The dotted (green) line shows the transition to the upper (green, solid circle) steady state. The solid (blue) line shows the transition to the lower (blue, circle) steady state. The two lower lines in that graph represent y_t^t. The transition follows the same description. The lower panel plots the consumption allocations for the old in $t+1$, born in period t, i.e., x_{t+1}^t and y_{t+1}^t. The description of the two upper lines, the two lower lines, and the transition to the steady state follow the same description as before. Notice that the upper and the lower panel represent consumption by the same consumer but in different time periods. As a clarification, in a given period t, total domestic consumption of good x corresponds to $x_t^{t-1} + x_t^t$.

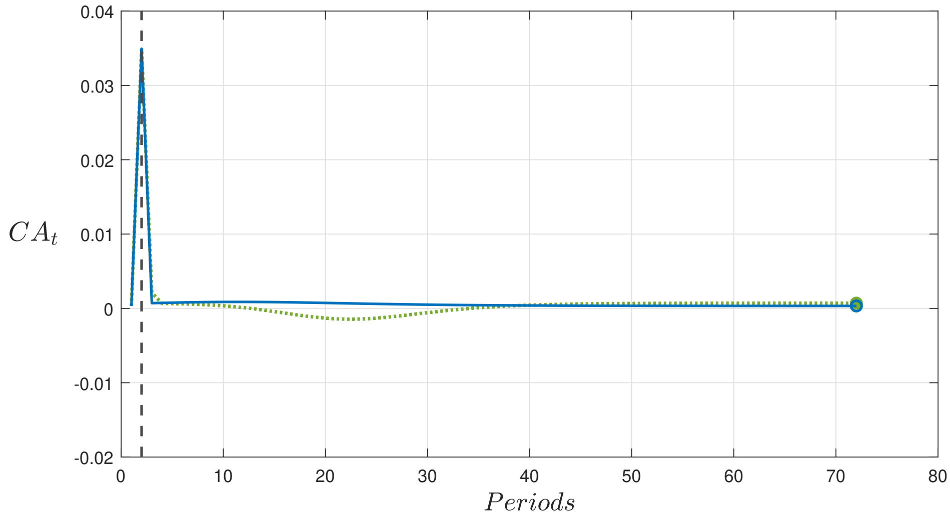
paths are possible, depending on agents' beliefs about future outcomes. In Figures 3-7, the solid (blue) line corresponds to the case when agents coordinate on the equilibrium path converging to the lower steady state, and the dotted (green) line corresponds to the case when agents coordinate on the equilibrium path converging to the upper steady state.[68]

The equilibrium path illustrated by the the solid (blue) line corresponds to the standard dynamics around the neighborhood of the lower steady state when a fundamental shock hits the economy. In this case, the economy behaves as if there is a unique steady state. Therefore, transitory shocks have only transitory effects.

In contrast, in the equilibrium path illustrated by the dotted (green) line in Figures

[68] Recall that the lower steady state is associated with more favorable terms of trade for the domestic economy. Conversely, terms of trade favor the foreign economy in the upper steady state.

Figure 5: Domestic Current Account



Note: The figure plots the domestic current account, CA_t. The vertical dashed line identifies the period when the fundamental shock materializes. The solid and dotted (green) lines shows the transition to the upper (green, solid circle) steady state. The solid and dot-dashed (blue) line shows the transition to the lower (blue, circle) steady state. $CA_t > 0$ means a surplus. Notice that the definition of the current account involves consumers born in different periods (see equation 19).

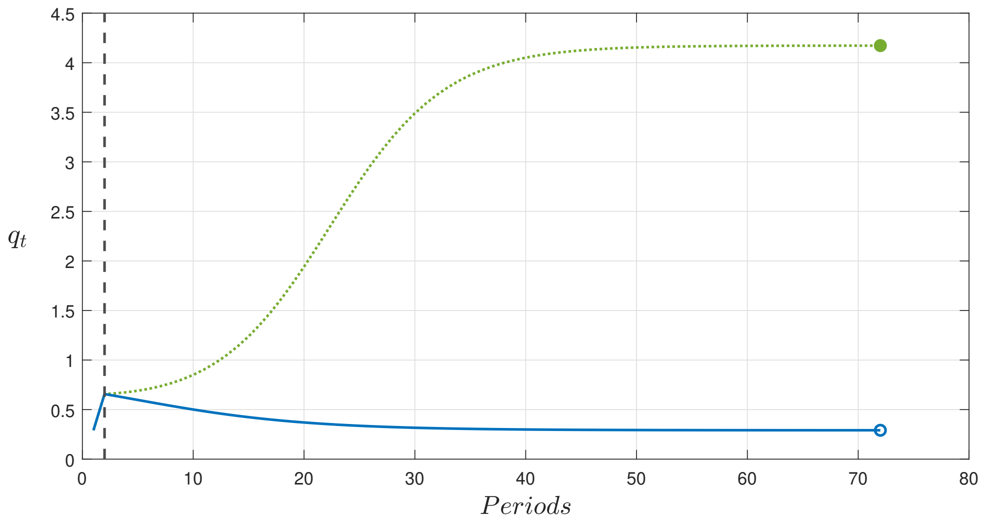
3-7, the transitory endowment shock has permanent effects as the global economy transitions to the upper steady state. Figure 6 shows how the equilibrium relative welfare weight, q, gradually adjusts from a value less than one to a value above one, as the foreign economy faces more favorable terms of trade. Instead, the rates of growth of the welfare weights, κ^* and κ, display a sharp and then non-monotonic adjustment to converge to a higher level in the upper steady state (see Figure 7).[69]

During the transition, the domestic economy becomes less wealthy and its consumption levels decrease smoothly from period three onward, until they converge to the lower steady state levels. Consequently, as Table 2 shows, the permanent reduction of the consumption levels in the domestic economy translates to lower economic welfare, while the foreign economy enjoys higher levels of consumption and welfare

[69] The steady-state rates, κ^* and κ, are associated with the discount factors, β and β^*. With homogeneous time preferences, $\kappa^* = \kappa = \beta$, and with heterogeneous time preferences, $\beta < \kappa^* = \kappa < \beta^*$. In the competitive equilibrium, this corresponds to the case where the world international interest rate is between the time preferences of each country. With multiple steady states, the rates of growth are tilted to the discount factor of the wealthier economy. Alternatively, in the competitive equilibrium, the interest rates are tilted to the time preference of the wealthier economy. Therefore, due to heterogeneous time preferences, there is borrowing and lending in each steady state.

as the terms of trade in the upper steady state are now more favorable.

Figure 6: Relative Welfare Weight



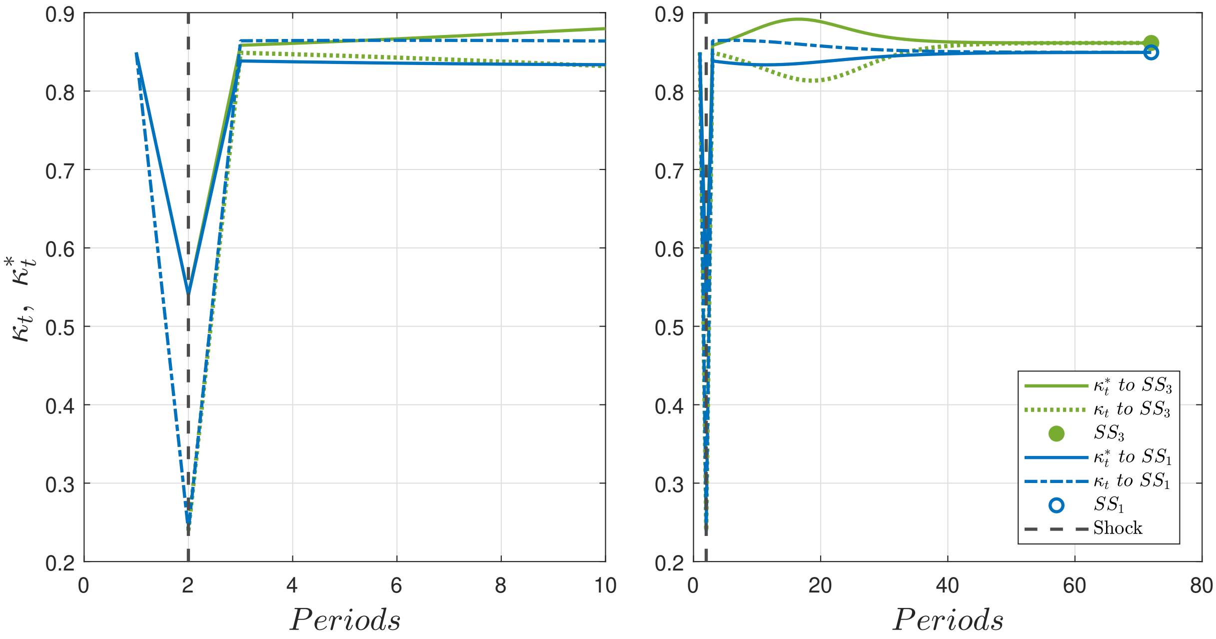
Note: The figure plots the the relative welfare weight, q_t. The vertical dashed line identifies the period when the fundamental shock materializes. The dotted (green) line shows the transition to the upper (green, solid circle) steady state. The solid (blue) line shows the transition to the lower (blue, circle) steady state.

After the endowment and the belief shock, the domestic economy displays a current account reversal, as seen in the equilibrium path converging to the upper steady state (see Figure 5). This pattern implies that the behavior of the old consumer dominates that of the young. Specifically, after the belief shock hits, the income of the old agent falls faster than the reduction in her consumption, as less favorable terms of trade materialize. This implies a deterioration in the current account. However, as the terms of trade converge to the upper steady state level, a reduction in consumption and spending by the old agent offsets the loss of income, and therefore the current account improves. Agents in the domestic economy are relatively more impatient than in the foreign economy, which translates to a steady state current account surplus, or global imbalance, as the old agent in the domestic economy transfers a fixed amount of resources that offset the amount of resources received by the new young generation entering the economy.[70] Notice that the current account in the upper steady state (green solid circle) converges to a slightly larger surplus than in the lower steady state (blue circle). Although quantitatively small, the current account transition also shows

[70] See footnote 57 for the intuition of a steady state current account surplus.

a permanent effect of a transitory shock.[71]

Figure 7: Rates of Growth of Welfare Weights



Note: The figure plots the rates of growth of welfare weights, κ_t^* and κ_t. The vertical dashed line identifies the period when the fundamental shock materializes. The solid and dotted (green) lines shows the transition to the upper (green, solid circle) steady state. The solid and dot-dashed (blue) line shows the transition to the lower (blue, circle) steady state. Notice κ^* and κ are equal in each steady state (see equation 11). The left panel shows the adjustments until period ten, while the right panel completes the transition.

8.2 An Interrupted Transition

In this subsection, I explore the conditions under which a transition of the form found in the first experiment can be interrupted. I address this question conducting a simulation, in which I search for the maximum distance between the vector $V_t = (\kappa_t^*, \kappa_t, q_{t-1})$ in the manifold converging to the upper steady state and its corresponding value in the lower steady state, such that a second belief shock can interrupt the transition to the upper steady state. Recall that a belief shock can produce an equilibrium shift only in the indeterminacy region of the model. In other words, I search for the last time period before the economy leaves the indeterminacy region. After that period, only the

[71] As explained in footnote 69, the upper steady state is related to a lower interest rate in the competitive equilibrium, and therefore with more borrowing and lending.

equilibrium path converging to the upper steady state is possible, and therefore, in the absence of additional fundamental shocks, a transition to the lower steady state is ruled out.[72]

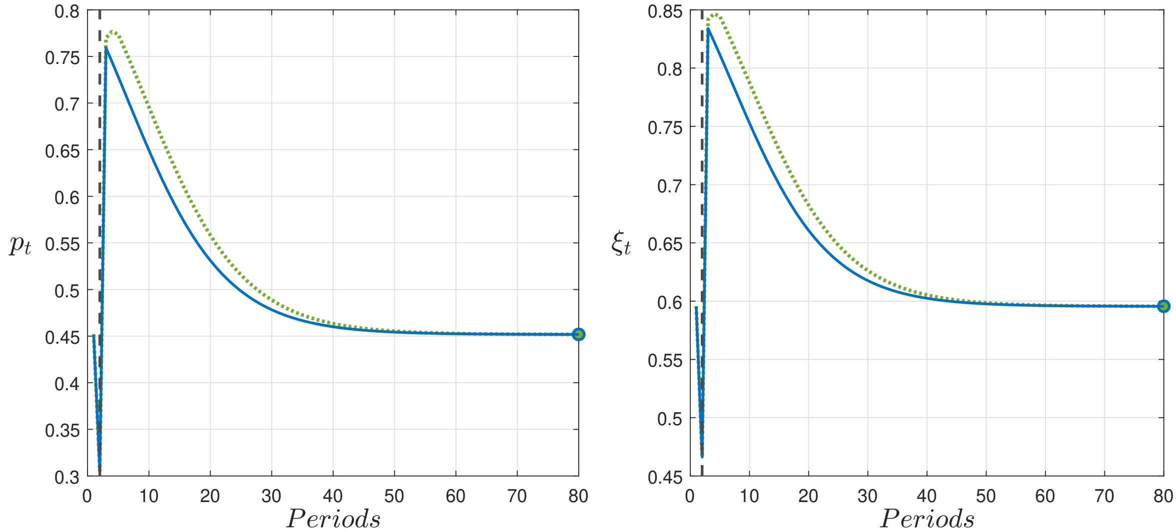
Consider a transition to the upper steady state of the form of experiment one. As in the first experiment, starting from period two, the domestic economy displays a deterioration of the terms of trade and a depreciation of the real exchange rate. Suppose that the domestic government worries about a further deterioration of the terms of trade and a depreciation of the real exchange rate, as the the result of a self-fulfilling belief. While in the indeterminacy region, in period four, the domestic government makes an unexpected public announcement, a "whatever it takes" type of speech, that manifests itself as a second shock to beliefs. If this shock results in agents coordinating on the lower steady state, then the transition to the upper steady state is interrupted (see Figure 8). Under this interpretation, the public signal prevents a self-fulfilling permanent deterioration in the terms of trade and the real exchange rate. Therefore, it avoids a reduction in welfare for the domestic economy. Notice that this is not possible in period five, as for this simulation, the economy is outside the indeterminacy region in that period and only one equilibrium path is possible. In this sense, the authority must react quickly enough to impede the transition to the upper steady state.

Figure 8 also shows that convergence in the terms of trade and the real exchange rate in the transition related with the second belief shock (dotted green line) is slower than the transition associated with the first belief shock (solid blue line). This means that, although there are no permanent changes as the economy converges to the original steady state, there is still a temporary loss of welfare in the domestic economy, as agents alive during the transition face less favorable terms of trade and therefore lower economic welfare.[73]

[72] For instance, in a perfect foresight exercise, De Ferra et al. (2020) model a sustained expansion in the current account as the result of an one-off unanticipated shock and then its reversal using a second one-off unanticipated shock.

[73] For completeness, Appendix E.2 illustrates the dynamics for the rest of the variables of the model, as in the first experiment.

Figure 8: Terms of Trade and Real Exchange Rate



Note: The left panel plots the terms of trade, p_t, and the right panel the real exchange rate, ξ_t. The vertical dashed line identifies the period when the fundamental shock materializes. The dotted (green) line shows the transition to the lower (green, solid) steady state after the second belief shock. The solid (blue) line shows the transition to the lower (circle, blue) steady state.

9. Conclusion

I provide an explanation for large and persistent fluctuations in the terms of trade and the real exchange rate, based on the idea of equilibrium shifts. I propose a theoretical model in which large and transitory endowment shocks can generate permanent effects on economic welfare as the result of self-fulfilling beliefs.

To solve this model, I develop an application of Negishi's method to an n-good OLG exchange economy with two countries. I show that, using this approach, the computation of a competitive equilibrium can be rewritten as a simpler problem. Specifically, the method allows me to describe equilibria using a low-order dynamical system that depends on the number of countries rather than the number of goods.

Using this method, I conduct two numerical experiments. The first experiment shows that transitory shocks can generate permanent effects on the terms of trade, the real exchange rate, and economic welfare. The second experiment shows that policy interventions in the form of public speeches might prevent self-fulfilling fluctuations if taken quickly.

These exercises are a proof of concept. They show that, in the theoretical model, beliefs can be sufficiently important as to affect long-run economic outcomes and therefore, large transitory shocks can generate permanent changes in economic welfare.

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Mathematical Appendix

A. Appendix A

A.1 Walras's Law using Negishi's method

This appendix shows an analogue of Walras's law in the OLG setting. When the welfare theorems hold, and the transfer functions of all but one agent are equal to zero, then the remaining one is redundant. In what follows, I modify the notation compared with the rest of the text to generalize this proposition to an environment with n goods and h countries.

Let $\lambda_t = \{\lambda_t^1, \lambda_t^2, \lambda_t^3, ..., \lambda_t^n\}$ be an n dimensional vector of Lagrange multipliers associated with the resource constraint of good $i = 1, 2, 3, ..., n$ (i.e., the competitive price of good i) in a social planning problem. Define $y_t^j = \{y_t^{j,1}, y_t^{j,2}, y_t^{j,3}, ..., y_t^{j,n}\}$ as the n dimensional vector of excess demands for good i of a young agent of country $j = 1, 2, 3, ..., h$ born in period t derived from the social planning problem. Variables $z_t^j = \{z_t^{j,1}, z_t^{j,2}, z_t^{j,3}, ..., z_t^{j,n}\}$ collects the excess demands of an old agent of country j at time t (i.e., born in $t-1$) derived from the social planning problem. Denote by $Y_t = \sum_{j=1}^{h} y_t^j$ and $Z_t = \sum_{j=1}^{h} z_t^j$ the n dimensional vector of *aggregate* excess demands for good i of the young and old alive in period t, respectively. This set *aggregate* excess demands depends on the social planner welfare weights. Let $p_t = \{p_t^1, p_t^2, p_t^3, ..., p_t^n\}$ be an n dimensional vector of competitive prices of the associated competitive equilibrium.

Proposition 1 *If the competitive equilibrium of an exchange economy with n goods and populated by h types of two-period lived OLG consumers satisfies the property* $\sum_{t=1}^{\infty} p_t < \infty$ *then the transfer functions derived from the social planning problem associated with this economy satisfy Walras's law.*

Proof: Let the transfer function of an agent from country j born in period $t \geq 1$ be

$$\mathcal{T}_t^j = \lambda_t^\top \cdot y_t^j + \lambda_{t+1}^\top \cdot z_{t+1}^j \tag{A.1}$$

Suppose that all the individual transfer functions of agents born from period one onward hold, i.e., $\mathcal{T}_t^j = 0, \forall t \geq 1$. Notice that all agents alive in the same period face the same prices. Therefore, we can aggregate them as,

$$\sum_{t=1}^{\infty} \mathcal{T}_t^j \equiv \mathcal{T}_t \equiv \lambda_t^\top \cdot Y_t + \lambda_{t+1}^\top \cdot Z_{t+1} = 0 \qquad \forall t \geq 1 \tag{A.2}$$

Then, we can express the aggregate transfer functions until period T as,

$$\begin{aligned}
\lambda_1^\top \cdot Y_1 + \lambda_2^\top \cdot Z_2 &= 0 \\
\lambda_2^\top \cdot Y_2 + \lambda_3^\top \cdot Z_3 &= 0 \\
\lambda_3^\top \cdot Y_3 + \lambda_4^\top \cdot Z_4 &= 0 \\
&\vdots \\
\lambda_{T-1}^\top \cdot Y_{T-1} + \lambda_T^\top \cdot Z_T &= 0
\end{aligned} \tag{A.3}$$

The solution of a social planning problem is always feasible, therefore,

$$\sum_{j=1}^{h} z_t^j + \sum_{j=1}^{h} y_t^j = 0 \tag{A.4}$$

or simply,

$$Y_t + Z_t = 0 \tag{A.5}$$

We can add up all the transfer functions in (A.3) and use the fact that $\lambda_t^\top (Y_t + Z_t) = 0$ $\forall\, t$ from (A.5). The resulting expression contains only the first and the last element of (A.3) as the rest offset each other. This is

$$\lambda_1^\top \cdot Y_1 + \lambda_T^\top \cdot Z_T = 0 \tag{A.6}$$

We assume that in the competitive equilibrium aggregate endowments are finite at equilibrium prices. Therefore, competitive prices do not explode. We can compute the limit of expression (A.6) as

$$\lambda_1^\top \cdot Y_1 + \lim_{t\to\infty} \lambda_T^\top \cdot Z_T = 0 \tag{A.7}$$

If the equilibrium of this economy satisfies the property $\sum_{t=1}^{\infty} p_t < \infty$, then from the first-order conditions of the planner, $\sum_{t=1}^{\infty} \lambda_t < \infty$. As a result, $\lim_{t\to\infty} \lambda_T^\top \cdot Z_T = 0$ and therefore $\lambda_1^\top \cdot Y_1 = 0$.

To complete the proof, we need to show that if $\lambda_1^\top \cdot Y_1 = 0$ and $h-1$ transfer functions in that period hold, then the remaining one holds as well.

Define $Z_1 = \sum_{j=1}^{h} z_1^j$. This expression represents the *aggregate* excess demands of all the initial old (born in period $t = 0$). If $h-1$ transfer functions of the initial old hold, then we can add them without loss of generality as $\lambda_1^\top \cdot Z_1 - \lambda_1^\top \cdot z_1^1 = 0$. Together with $\lambda_1^\top \cdot Y_1 = 0$ and the feasibility constraint in period $t = 1$, then $\lambda_1^\top \cdot z_1^1 = 0$. $\square$

It is important to note that this proof relies on the property $\sum_{t=1}^{\infty} p_t < \infty$. This condition implies that the value of the aggregate endowment is finite at the equilibrium prices, and therefore the competitive equilibrium is Pareto efficient. OLG economies with valued *fiat money* do not satisfy this property, although they might satisfy more general Pareto-efficiency criteria, such as, $\sum_{t=1}^{\infty} ||p_t||^{-1} = \infty$, where $||p_t|| = \left(p_t^\top p_t\right)^{1/2}$ (see Kehoe, 1989).

A.2 Multiplicity of Steady States

When there are many goods in each period and more than one consumer in each generation, multiple steady states might exist. Typically, sufficient conditions on the uniqueness of the steady state are derived from aggregate excess demand functions in exchange economies using the index theorem (see Kehoe and Levine, 1984). Similar sufficient conditions on uniqueness can be derived by applying the index theorem to the transfer functions to diagnose steady state multiplicity in (14). In this case, the Negishi method also reduce dimensionality of the computation on the index by defining steady state conditions and their respective Jacobians that depend on the number of agents rather than on the number of goods. As the index theorems involves computing the determinant of a matrix, the Negishi method also results in computational efficiency gains in computing the indexes in each steady state.

Definition: The system of equations (14) is regular if its Jacobian with respect to (κ, q), denoted by $\mathcal{J}_T(\kappa, q)$, has full rank whenever (κ, q) solve (14).

Following Kehoe and Levine (1984), if the system (14) is regular, we can define *index* (κ, q) to be $+1$ or -1 according to the *sign of the determinant of the negative of* $\mathcal{J}_T(\kappa, q)$. A standard argument in this literature posits that when summed over all equilibria, $\sum$ *index* $(\kappa, q) = +1$. In a non-monetary economy, this implies there is an odd number of steady states, and specifically a unique steady state when *index* $(\kappa, q) = +1$ at every possible steady state. In other words, to assert the existence of multiple steady states it suffices to show that *index* $(\kappa, q) = -1$ in some steady state. When this is the case, then at least another two steady states must exist, each of them with *index* $(\kappa, q) = +1$.

B. Appendix B

B.1 Static Case

Consider the following specification for preferences:

$$\mathcal{U}(x,y) = \frac{(1-\gamma)^{\sigma} x^{1-\sigma} + \gamma^{\sigma} y^{1-\sigma}}{1-\sigma} \quad \text{and} \quad \mathcal{U}^*(x^*,y^*) = \frac{\gamma^{\sigma} x^{*1-\sigma} + (1-\gamma)^{\sigma} y^{*1-\sigma}}{1-\sigma},$$

where $0 < \gamma < 1$ determines the utility weights and $1/\sigma$ is the elasticity of substitution between good x and good y. Endowments of goods x and y of the domestic agent are $1-\varepsilon$ and ε, respectively, while those for the foreign agent are ε and $1-\varepsilon$, where $0 < \varepsilon < 1$.

Given welfare weights and aggregate endowments, a social planner maximizes a social welfare function subject to the resource constraints of the economy,

$$\max_{x,y\ x^*,y^*} \ \mathcal{L}^s = \alpha\, \mathcal{U}(x,y) + \alpha^*\, \mathcal{U}^*(x^*,y^*) + \lambda(x + x^* - 1) + \phi(y + y^* - 1)$$

where α and α^* are non-negative real numbers such that $\alpha + \alpha^* = 1$, and λ and ϕ are the Lagrange multipliers of the resource constraints.

The allocations of the planner for the domestic economy correspond to

$$x = \frac{(1-\gamma)}{(1-\gamma) + \gamma q^{\frac{1}{\sigma}}} \quad \text{and} \quad y = \frac{\gamma}{\gamma + (1-\gamma) q^{\frac{1}{\sigma}}} \tag{B.1}$$

where $q = \alpha^*/\alpha$.

Using the Lagrange multipliers as functions of the welfare weights (see problem 2), I define the domestic transfer function as

$$\mathcal{T}(q) = \lambda(q)[x(q) - (1-\varepsilon)] + \phi(q)[y(q) - \varepsilon] \tag{B.2}$$

Due to symmetry, a relative welfare weight equal to unity is an equilibrium welfare weight. Thus, for multiplicity to arise, is sufficient to show that $\partial \mathcal{T}(q)/\partial q\,|_{q=1} > 0$. In this economy there are three equilibria when

$$\frac{1}{\sigma} < 1 - \frac{1}{2}\left(\frac{\varepsilon}{\gamma} + \frac{1-\varepsilon}{1-\gamma}\right). \tag{B.3}$$

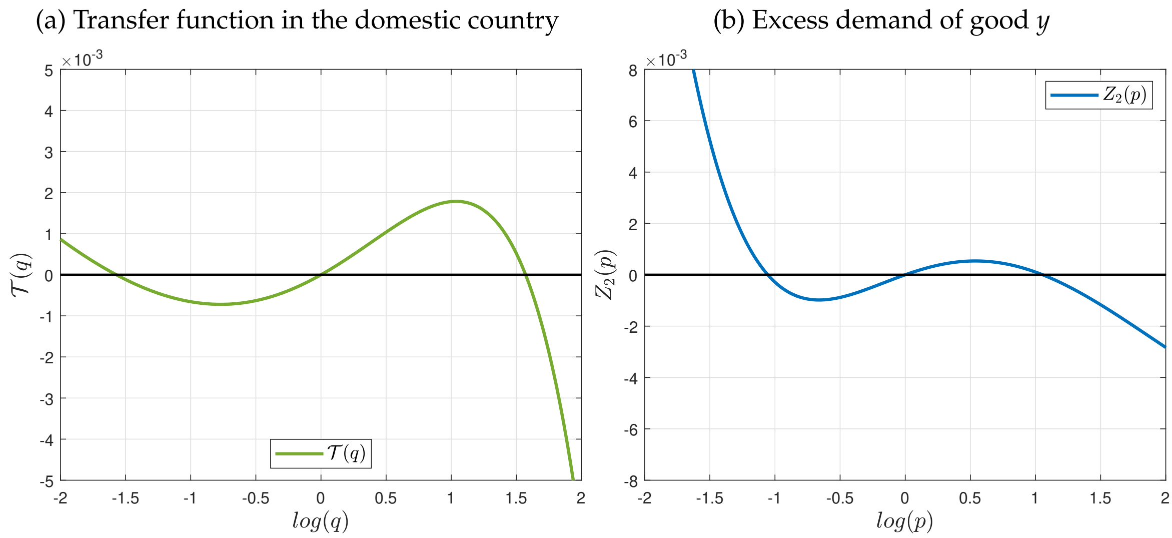
Toda and Walsh (2017) derive conditions for multiplicity in this economy using the aggregate excess demand approach.

Define the aggregate excess demand of good y as $z(p) = y(p) + y^*(p) - 1$, where $y(p)$ and $y^*(p)$ are the individual demands for good y and p is the relative price of good y in terms of good x. Thus, for multiplicity to arise, is sufficient to show that $\partial z(p)/\partial p \mid_{p=1} > 0$. In this economy there are three equilibria when

$$\frac{1}{\sigma} < 1 - \frac{1}{2}\left(\frac{\varepsilon}{\gamma} + \frac{1-\varepsilon}{1-\gamma}\right). \tag{B.4}$$

Figure 9 shows both approaches for calibrations that satisfy the previous inequality. Note that both q and p are equal to unity in the middle equilibrium.

Figure 9: Two approaches for multiplicity



Note: The Figure shows two approaches to derive sufficient conditions for multiplicity in static exchange economies. Note that the x-axis is expressed in logarithmic scale.

B.2 Dynamic Case

Let $\mathcal{W}(\tilde{\alpha})$ be a utilitarian social welfare function in period one defined as a weighted sum of individual utility functions across countries, and $\tilde{\alpha} = \{\alpha_t\}_{t=0}^{\infty}$ be the vector of *all* welfare weights. The social planner chooses allocations $\{x_1^0, y_1^0, x_1^{*,0}, y_1^{*,0}\}$ for the *initial old* and the sequence $\{x_t^t, y_t^t, x_t^{t-1}, y_t^{t-1}, x_t^{*,t}, y_t^{*,t}, x_t^{*,t-1}, y_t^{*,t-1}\}_{t=1}^{\infty}$ to maximize,

$$
\begin{aligned}
\mathcal{W}(\tilde{\alpha}) \; = & \; \alpha_0 \, \mathcal{U}_0 \left(x_1^0, y_1^0\right) \; + \; \alpha_0^* \, \mathcal{U}_0^* \left(x_1^{*,0}, y_1^{*,0}\right) \; + \; \sum_{t=1}^{\infty} \alpha_t \left[\sum_{s=0}^{1} \beta^s \, \mathcal{U}_{t+s}^t \left(x_{t+s}^t, y_{t+s}^t\right) \right] \\
& + \sum_{t=1}^{\infty} \alpha_t^* \left[\sum_{s=0}^{1} \beta^{*s} \, \mathcal{U}_{t+s}^{*,t} \left(x_{t+s}^{*,t}, y_{t+s}^{*,t}\right) \right]
\end{aligned}
$$

subject to the resource constraints,

$$
\begin{aligned}
x_t^t + x_t^{t-1} + x_t^{*,t} + x_t^{*,t-1} = e_t^t + e_t^{t-1} + e_t^{*,t} + e_t^{*,t-1} = E_t, \quad \forall t \\
y_t^t + y_t^{t-1} + y_t^{*,t} + y_t^{*,t-1} = w_t^t + w_t^{t-1} + w_t^{*,t} + w_t^{*,t-1} = W_t, \quad \forall t.
\end{aligned}
$$

Time superscript denotes the period when the generation is born, while the time subscript denotes the current period. Star superscript denotes the foreign country.[74] Coefficients $\{\alpha_t, \alpha_t^*\}_{t=0}^{\infty}$ are positive welfare weights for the domestic and foreign consumers, respectively, and they account for the period when a generation is born. Endowments are stationary with the same distribution of the static economy. Denoting by e the endowment of good x and by w the endowment of good y, the endowment distribution is parametrized by ε as follows: $e_t^t = e_t^{t-1} = w_t^{*,t} = w_t^{*,t-1} = 1 - \varepsilon$, and $e_t^{*,t} = e_t^{*,t-1} = w_t^t = w_t^{t-1} = \varepsilon$, $\forall \, t$. Thus, aggregate endowments for goods x and y, namely, E_t and W_t, respectively, are equal to two units.

Individual utilities in the intertemporal setting are,

$$
\begin{aligned}
\mathcal{U}_{t+s}^t \left(x_{t+s}^t, y_{t+s}^t\right) &= \left(\frac{(1-\gamma)^{\sigma} {x_{t+s}^t}^{1-\sigma} + \gamma^{\sigma} {y_{t+s}^t}^{1-\sigma}}{1-\sigma} \right) \\
\mathcal{U}_{t+s}^{*,t} \left(x_{t+s}^{*,t}, y_{t+s}^{*,t}\right) &= \left(\frac{\gamma^{\sigma} {x_{t+s}^{*,t}}^{1-\sigma} + (1-\gamma)^{\sigma} {y_{t+s}^{*,t}}^{1-\sigma}}{1-\sigma} \right).
\end{aligned}
$$

Under this specification both the intertemporal elasticity of substitution and the elasticity of substitution between good x and good y are equal to $1/\sigma$.[75] Parameters β and β^* are the subjective discount factors and denote the inverse of the time preference. $\beta < \beta^*$ means that domestic consumers are relatively more patient than foreign consumers.

[74] For example, $x_t^{*,t-1}$ denotes the demand for good x in period t of the foreign consumer born in period $t-1$.

[75] This simplification corresponds to a CES consumption aggregator with elasticity of substitution $1/\sigma$ nested into a isoelastic utility function with an intertemporal substitution equal to $1/\sigma$.

B.3 Optimality Conditions and Allocations

Let $\mathcal{L}$ be the Lagrangean of the planner's program in (5) and define λ_t and ϕ_t as Lagrange multipliers of the resource constraints of good x and good y, respectively. The optimality conditions of the planner are,

$$\frac{\partial \mathcal{L}}{\partial x_t^t} : \alpha_t (1-\gamma)^{\sigma} \frac{1}{(x_t^t)^{\sigma}} - \lambda_t = 0; \qquad \frac{\partial \mathcal{L}}{\partial x_t^{*,t}} : \alpha_t^* \gamma^{\sigma} \frac{1}{(x_t^{*,t})^{\sigma}} - \lambda_t = 0 \tag{B.5}$$

$$\frac{\partial \mathcal{L}}{\partial x_t^{t-1}} : \alpha_{t-1} (1-\gamma)^{\sigma} \beta \frac{1}{(x_t^{t-1})^{\sigma}} - \lambda_t = 0; \qquad \frac{\partial \mathcal{L}}{\partial x_t^{*,t-1}} : \alpha_{t-1}^* \gamma^{\sigma} \beta^* \frac{1}{(x_t^{*,t-1})^{\sigma}} - \lambda_t = 0 \tag{B.6}$$

$$\frac{\partial \mathcal{L}}{\partial y_t^t} : \alpha_t \gamma^{\sigma} \frac{1}{(y_t^t)^{\sigma}} - \phi_t = 0; \qquad \frac{\partial \mathcal{L}}{\partial y_t^{*,t}} : \alpha_t^* (1-\gamma)^{\sigma} \frac{1}{(y_t^{*,t})^{\sigma}} - \phi_t = 0 \tag{B.7}$$

$$\frac{\partial \mathcal{L}}{\partial y_t^{t-1}} : \alpha_{t-1} \gamma^{\sigma} \beta \frac{1}{(y_t^{t-1})^{\sigma}} - \phi_t = 0; \qquad \frac{\partial \mathcal{L}}{\partial y_t^{*,t-1}} : \alpha_{t-1}^* (1-\gamma)^{\sigma} \beta^* \frac{1}{(y_t^{*,t-1})^{\sigma}} - \phi_t = 0 \tag{B.8}$$

Then, we can write the following relationships. Combining (B.5) and (B.6),

$$x_t^{t-1} = \beta^{\frac{1}{\sigma}} \left(\frac{\alpha_{t-1}}{\alpha_t}\right)^{\frac{1}{\sigma}} x_t^t; \qquad x_t^{*,t-1} = \beta^{*\frac{1}{\sigma}} \left(\frac{\alpha_{t-1}^*}{\alpha_t^*}\right)^{\frac{1}{\sigma}} x_t^{*,t} \tag{B.9}$$

combining (B.7) and (B.8),

$$y_t^{t-1} = \beta^{\frac{1}{\sigma}} \left(\frac{\alpha_{t-1}}{\alpha_t}\right)^{\frac{1}{\sigma}} y_t^t; \qquad y_t^{*,t-1} = \beta^{*\frac{1}{\sigma}} \left(\frac{\alpha_{t-1}^*}{\alpha_t^*}\right)^{\frac{1}{\sigma}} y_t^{*,t} \tag{B.10}$$

combining (B.5) and (B.7),

$$x_t^{*,t} = \left(\frac{\gamma}{1-\gamma}\right) \left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} x_t^t; \qquad y_t^{*,t} = \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} y_t^t \tag{B.11}$$

combining (B.6) and (B.11), and (B.8) and (B.11),

$$x_t^{*,t-1} = \left(\frac{\gamma}{1-\gamma}\right) \beta^{*\frac{1}{\sigma}} \left(\frac{\alpha_{t-1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} x_t^t; \qquad y_t^{*,t-1} = \left(\frac{1-\gamma}{\gamma}\right) \beta^{*\frac{1}{\sigma}} \left(\frac{\alpha_{t-1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} y_t^t \tag{B.12}$$

Using conditions (B.9) to (B.11) into the resource constraints,

$$x_t^t \left(1 + \beta^{\frac{1}{\sigma}} \left(\frac{\alpha_{t-1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right) \left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right) \beta^{*\frac{1}{\sigma}} \left(\frac{\alpha_{t-1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}}\right) = e_t^t + e_t^{t-1} + e_t^{*,t} + e_t^{*,t-1} = E_t \tag{B.13}$$

$$y_t^t \left(1 + \beta^{\frac{1}{\sigma}} \left(\frac{\alpha_{t-1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right) \beta^{*\frac{1}{\sigma}} \left(\frac{\alpha_{t-1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}}\right) = w_t^t + w_t^{t-1} + w_t^{*,t} + w_t^{*,t-1} = W_t \tag{B.14}$$

Then, the allocations of the social planner as functions of welfare weights are,

$$x_t^t = \frac{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} E_t}{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right) \left(\frac{\alpha_t^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}} \left(\frac{\gamma}{1-\gamma}\right) \left(\frac{\alpha_{t-1}^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}}};\ y_t^t = \frac{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} W_t}{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{\alpha_t^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}} \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{\alpha_{t-1}^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}}}$$

$$x_{t+1}^t = \frac{\beta^{\frac{1}{\sigma}} E_{t+1}}{\left(\frac{\alpha_{t+1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right) \left(\frac{\alpha_{t+1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}} \left(\frac{\gamma}{1-\gamma}\right) \left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}};\ y_{t+1}^t = \frac{\beta^{\frac{1}{\sigma}} W_{t+1}}{\left(\frac{\alpha_{t+1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{\alpha_{t+1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}} \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}}$$

for the domestic country, while for the foreign country,

$$x_t^{*,t} = \frac{\left(\frac{\gamma}{1-\gamma}\right) \left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} \left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} E_t}{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right) \left(\frac{\alpha_t^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}} \left(\frac{\gamma}{1-\gamma}\right) \left(\frac{\alpha_{t-1}^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}}};\ y_t^{*,t} = \frac{\left(\frac{1-\gamma}{\gamma}\right) \left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} \left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} W_t}{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{\alpha_t^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}} \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{\alpha_{t-1}^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}}}$$

$$x_{t+1}^{*,t} = \frac{\left(\frac{\gamma}{1-\gamma}\right) \beta^{*\frac{1}{\sigma}} \left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} E_{t+1}}{\left(\frac{\alpha_{t+1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right) \left(\frac{\alpha_{t+1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}} \left(\frac{\gamma}{1-\gamma}\right) \left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}};\ y_{t+1}^{*,t} = \frac{\left(\frac{1-\gamma}{\gamma}\right) \beta^{*\frac{1}{\sigma}} \left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} W_{t+1}}{\left(\frac{\alpha_{t+1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{\alpha_{t+1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}} \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}}$$

B.4 Transfer Functions

For notation simplicity, define (in *bold*) the vector of one period consecutive welfare weights, $\boldsymbol{a}_t = (\alpha_{t-1}, \alpha_t, \alpha_{t-1}^*, \alpha_t^*)$. The set of transfer functions of all the consumers in the economy takes the form,

$$\mathcal{T}_0(\boldsymbol{a}_1) = \lambda_1(\boldsymbol{a}_1) \left[x_1^0(\boldsymbol{a}_1) - e_1^0\right] \ + \ \phi_1(\boldsymbol{a}_1) \left[y_1^0(\boldsymbol{a}_1) - w_1^0\right] \tag{B.15}$$

$$\mathcal{T}_0^*(\boldsymbol{a}_1) = \lambda_1(\boldsymbol{a}_1) \left[x_1^{*,0}(\boldsymbol{a}_1) - e_1^{*,0}\right] \ + \ \phi_1(\boldsymbol{a}_1) \left[y_1^{*,0}(\boldsymbol{a}_1) - w_1^{*,0}\right] \tag{B.16}$$

for the initial old in period $t = 1$,

$$\begin{aligned} \mathcal{T}_1(\boldsymbol{a}_2, \boldsymbol{a}_1) = &\ \lambda_1(\boldsymbol{a}_1) \left[x_1^1(\boldsymbol{a}_1) - e_1^1\right] \ + \ \lambda_2(\boldsymbol{a}_2) \left[x_2^1(\boldsymbol{a}_2) - e_2^1\right] \\ &+ \phi_1(\boldsymbol{a}_1) \left[y_1^1(\boldsymbol{a}_1) - w_1^1\right] \ + \ \phi_2(\boldsymbol{a}_2) \left[y_2^1(\boldsymbol{a}_2) - w_2^1\right] \end{aligned} \tag{B.17}$$

$$\mathcal{T}_1^*(\boldsymbol{a}_2, \boldsymbol{a}_1) = \lambda_1(\boldsymbol{a}_1)\left[x_1^{*,1}(\boldsymbol{a}_1) - e_1^{*,1}\right] + \lambda_2(\boldsymbol{a}_2)\left[x_2^{*,1}(\boldsymbol{a}_2) - e_2^{*,1}\right] + \phi_1(\boldsymbol{a}_1)\left[y_1^{*,1}(\boldsymbol{a}_1) - w_1^{*,1}\right] + \phi_2(\boldsymbol{a}_2)\left[y_2^{*,2}(\boldsymbol{a}_2) - w_2^{*,1}\right] \tag{B.18}$$

for the young consumer in period $t = 1$, and,

$$\mathcal{T}(\boldsymbol{a}_{t+1}, \boldsymbol{a}_t) = \lambda_t(\boldsymbol{a}_t)\left[x_t^t(\boldsymbol{a}_t) - e_t^t\right] + \lambda_{t+1}(\boldsymbol{a}_{t+1})\left[x_{t+1}^t(\boldsymbol{a}_{t+1}) - e_{t+1}^t\right] + \phi_t(\boldsymbol{a}_t)\left[y_t^t(\boldsymbol{a}_t) - w_t^t\right] + \phi_{t+1}(\boldsymbol{a}_{t+1})\left[y_{t+1}^t(\boldsymbol{a}_{t+1}) - w_{t+1}^t\right] \tag{B.19}$$

$$\mathcal{T}^*(\boldsymbol{a}_{t+1}, \boldsymbol{a}_t) = \lambda_t(\boldsymbol{a}_t)\left[x_t^{*,t}(\boldsymbol{a}_t) - e_t^{*,t}\right] + \lambda_{t+1}(\boldsymbol{a}_{t+1})\left[x_{t+1}^{*,t}(\boldsymbol{a}_{t+1}) - e_{t+1}^{*,t}\right] + \phi_t(\boldsymbol{a}_t)\left[y_t^{*,t}(\boldsymbol{a}_t) - w_t^{*,t}\right] + \phi_{t+1}(\boldsymbol{a}_{t+1})\left[y_{t+1}^{*,t}(\boldsymbol{a}_{t+1}) - w_{t+1}^{*,t}\right] \tag{B.20}$$

for a consumer born in period $t \geq 2$.

C. Appendix C

C.1 Planner's Allocations with Redefined Variables

Redefine the vector $\boldsymbol{a}_t = (\alpha_{t-1}, \alpha_t, \alpha_{t-1}^*, \alpha_t^*)$ as $\boldsymbol{a}_t = \left(1, \frac{\alpha_t}{\alpha_{t-1}}, \frac{\alpha_{t-1}^*}{\alpha_{t-1}}, \frac{\alpha_t^*}{\alpha_{t-1}}\right)$. In addition, we define the following new variables: $\kappa_t = \frac{\alpha_t}{\alpha_{t-1}}$, $\kappa_t^* = \frac{\alpha_t^*}{\alpha_{t-1}^*}$, and $q_t = \frac{\alpha_t^*}{\alpha_t}$. Then, the allocations are,

$$x_t^t = \frac{\kappa_t^{\frac{1}{\sigma}} E_t}{\kappa_t^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)(\kappa_t^* q_{t-1})^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{\gamma}{1-\gamma}\right) q_{t-1}^{\frac{1}{\sigma}}}; \; y_t^t = \frac{\kappa_t^{\frac{1}{\sigma}} W_t}{\kappa_t^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)(\kappa_t^* q_{t-1})^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{1-\gamma}{\gamma}\right) q_{t-1}^{\frac{1}{\sigma}}}$$

$$x_{t+1}^t = \frac{\beta^{\frac{1}{\sigma}} E_{t+1}}{\kappa_{t+1}^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)(\kappa_{t+1}^* q_t)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{\gamma}{1-\gamma}\right) q_t^{\frac{1}{\sigma}}}; \; y_{t+1}^t = \frac{\beta^{\frac{1}{\sigma}} W_{t+1}}{\kappa_{t+1}^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)(\kappa_{t+1}^* q_t)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{1-\gamma}{\gamma}\right) q_t^{\frac{1}{\sigma}}}$$

for the domestic country, and,

$$x_t^{*,t} = \frac{\left(\frac{\gamma}{1-\gamma}\right) q_t^{\frac{1}{\sigma}} \kappa_t^{\frac{1}{\sigma}} E_t}{\kappa_t^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)(\kappa_t^* q_{t-1})^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{\gamma}{1-\gamma}\right) q_{t-1}^{\frac{1}{\sigma}}}; \; y_t^{*,t} = \frac{\left(\frac{1-\gamma}{\gamma}\right) q_t^{\frac{1}{\sigma}} \kappa_t^{\frac{1}{\sigma}} W_t}{\kappa_t^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)(\kappa_t^* q_{t-1})^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{1-\gamma}{\gamma}\right) q_{t-1}^{\frac{1}{\sigma}}}$$

$$x_{t+1}^{*,t} = \frac{\left(\frac{\gamma}{1-\gamma}\right) \beta^{*\frac{1}{\sigma}} q_t^{\frac{1}{\sigma}} E_{t+1}}{\kappa_{t+1}^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)(\kappa_{t+1}^* q_t)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{\gamma}{1-\gamma}\right) q_t^{\frac{1}{\sigma}}}; \; y_{t+1}^{*,t} = \frac{\left(\frac{1-\gamma}{\gamma}\right) \beta^{*\frac{1}{\sigma}} q_t^{\frac{1}{\sigma}} W_{t+1}}{\kappa_{t+1}^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)(\kappa_{t+1}^* q_t)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{1-\gamma}{\gamma}\right) q_t^{\frac{1}{\sigma}}}$$

for the foreign country. Because preferences of the initial old are different from the rest, their allocations (x_1^0, $x_1^{*,0}$) and, as a consequence, the allocations of the young in the first period (x_1^1, $x_1^{*,1}$) do not involve the discount factor (β,β^*).

C.2 Dynamical System

Define $V_t = (\kappa_t^*, \kappa_t, q_{t-1})$. The transformed dynamical system defined in (10)-(11) is,

$$\begin{aligned} \mathcal{T}\left(V_t, V_{t+1}\right) &= 0 \\ = \frac{(1-\gamma)^\sigma}{x_t^t(V_t)^\sigma}\left[x_t^t(V_t) - (1-\varepsilon)\right] &+ \frac{\gamma^\sigma}{y_t^t(V_t)^\sigma}\left[y_t^t(V_t) - \varepsilon\right] + \\ \frac{(1-\gamma)^\sigma \beta}{x_{t+1}^t(V_{t+1})^\sigma}\left[x_{t+1}^t(V_{t+1}) - (1-\varepsilon)\right] &+ \frac{\gamma^\sigma \beta}{y_{t+1}^t(V_{t+1})^\sigma}\left[y_{t+1}^t(V_{t+1}) - \varepsilon\right] \end{aligned} \tag{C.1}$$

$$\begin{aligned} \mathcal{T}^*\left(V_t, V_{t+1}\right) &= 0 \\ = \frac{\gamma^\sigma}{x_t^{*,t}(V_t)^\sigma}\left[x_t^{*,t}(V_t) - \varepsilon\right] &+ \frac{(1-\gamma)^\sigma}{y_t^{*,t}(V_t)^\sigma}\left[y_t^{*,t}(V_t) - (1-\varepsilon)\right] + \\ \frac{\gamma^\sigma \beta^*}{x_{t+1}^{*,t}(V_{t+1})^\sigma}\left[x_{t+1}^{*,t}(V_{t+1}) - \varepsilon\right] &+ \frac{(1-\gamma)^\sigma \beta^*}{y_{t+1}^{*,t}(V_{t+1})^\sigma}\left[y_{t+1}^{*,t}(V_{t+1}) - (1-\varepsilon)\right] \end{aligned} \tag{C.2}$$

and,

$$q_t = \frac{\kappa_t^*}{\kappa_t} q_{t-1} \tag{C.3}$$

In addition, we can define the system of (C.1), (C.2), and (C.3),

$$F(V_t, V_{t+1}) = \begin{bmatrix} \mathcal{T}\left(V_t, V_{t+1}\right) \\ \mathcal{T}^*\left(V_t, V_{t+1}\right) \\ q_t - \frac{\kappa_t^*}{\kappa_t} q_{t-1} \end{bmatrix} = 0, \qquad \forall\, t \geq 2 \tag{C.4}$$

This is (12) in the main text. Lastly, the initial conditions of this economy are,

$$\begin{aligned} \mathcal{T}_0\left(V_1\right) &= 0 \\ = \frac{(1-\gamma)^\sigma}{x_1^0(V_1)^\sigma}\left[x_1^0(V_1) - (1-\varepsilon)\right] &+ \frac{\gamma^\sigma}{y_1^0(V_1)^\sigma}\left[y_1^0(V_1) - \varepsilon\right] \end{aligned} \tag{C.5}$$

$$\begin{aligned} \mathcal{T}_0^*\left(V_1\right) &= 0 \\ = \frac{\gamma^\sigma}{x_1^{*,0}(V_1)^\sigma}\left[x_1^{*,0}(V_1) - \varepsilon\right] &+ \frac{(1-\gamma)^\sigma}{y_1^{*,0}(V_1)^\sigma}\left[y_1^{*,0}(V_1) - (1-\varepsilon)\right] \end{aligned} \tag{C.6}$$

$$
\begin{aligned}
&\mathcal{T}_1\left(V_1, V_2\right) = 0 \\
= \ &\frac{(1-\gamma)^\sigma}{x_1^1(V_1)^\sigma}\left[x_1^1(V_1) - (1-\varepsilon)\right] + \frac{\gamma^\sigma}{y_1^1(V_1)^\sigma}\left[y_1^1(V_1) - \varepsilon\right] + \\
&\frac{(1-\gamma)^\sigma \beta}{x_2^1(V_2)^\sigma}\left[x_2^1(V_2) - (1-\varepsilon)\right] + \frac{\gamma^\sigma \beta}{y_2^1(V_2)^\sigma}\left[y_2^1(V_2) - \varepsilon\right]
\end{aligned} \tag{C.7}
$$

$$
\begin{aligned}
&\mathcal{T}_1^*\left(V_1, V_2\right) = 0 \\
= \ &\frac{\gamma^\sigma}{x_1^{*,1}(V_1)^\sigma}\left[x_1^{*,1}(V_1) - \varepsilon\right] + \frac{(1-\gamma)^\sigma}{y_1^{*,1}(V_1)^\sigma}\left[y_1^{*,1}(V_1) - (1-\varepsilon)\right] + \\
&\frac{\gamma^\sigma \beta^*}{x_2^{*,1}(V_2)^\sigma}\left[x_2^{*,1}(V_2) - \varepsilon\right] + \frac{(1-\gamma)^\sigma \beta^*}{y_2^{*,1}(V_2)^\sigma}\left[y_2^{*,1}(V_2) - (1-\varepsilon)\right]
\end{aligned} \tag{C.8}
$$

$$
q_1 - \frac{\kappa_1^*}{\kappa_1} q_0 = 0 \tag{C.9}
$$

Note that the allocations in (C.7) and (C.8) are different from the ones in equations (C.1) and (C.2). This is because the planner's allocation to the young in period one depends on the preferences of the initial old, which are different from the rest.

D. Appendix D

D.1 Optimality Conditions and Allocations

Let $\mathcal{W}(\tilde{\alpha})$ be a utilitarian social welfare function in period one defined as a weighted sum of individual utility functions across countries, and $\tilde{\alpha} = \{\alpha_t\}_{t=0}^{\infty}$ be the vector of *all* welfare weights. The social planner chooses allocations $\{x_1^0, y_1^0, x_1^{*,0}, y_1^{*,0}\}$ for the *initial old* and the sequence $\{x_t^t, y_t^t, x_t^{t-1}, y_t^{t-1}, x_t^{*,t}, y_t^{*,t}, x_t^{*,t-1}, y_t^{*,t-1}\}_{t=1}^{\infty}$ to maximize,

$$\begin{aligned}\mathcal{W}(\tilde{\alpha}) &= \alpha_0\, \mathcal{U}_0\left(x_1^0, y_1^0\right) + \alpha_0^*\, \mathcal{U}_0^*\left(x_1^{*,0}, y_1^{*,0}\right) + \sum_{t=1}^{\infty} \alpha_t \left[\sum_{s=0}^{1} \beta^s\, \mathcal{U}_{t+s}^t\left(x_{t+s}^t, y_{t+s}^t\right)\right] \\ &\quad + \sum_{t=1}^{\infty} \alpha_t^* \left[\sum_{s=0}^{1} \beta^{*s}\, \mathcal{U}_{t+s}^{*,t}\left(x_{t+s}^{*,t}, y_{t+s}^{*,t}\right)\right]\end{aligned}$$

where,

$$x_t^t = \prod_{i=1}^{n} x_{i,t}^{t\ \rho_i}; \quad x_t^{t-1} = \prod_{i=1}^{n} x_{i,t}^{t-1\rho_i}; \quad y_t^t = \prod_{i=1}^{n} y_{i,t}^{t\ \rho_i}; \quad y_t^{t-1} = \prod_{i=1}^{n} y_{i,t}^{t-1\rho_i}$$

$$x_t^{*,t} = \prod_{i=1}^{n} x_{i,t}^{*,t\rho_i}; \quad x_t^{*,t-1} = \prod_{i=1}^{n} x_{i,t}^{*,t-1\rho_i} \quad y_t^{*,t} = \prod_{i=1}^{n} y_{i,t}^{*,t\rho_i}; \quad y_t^{*,t-1} = \prod_{i=1}^{n} y_{i,t}^{*,t-1\rho_i}$$

subject to the resource constraints,

$$\begin{aligned} x_t^t + x_t^{t-1} + x_t^{*,t} + x_t^{*,t-1} &= \prod_{i=1}^{n} E_{i,t}{}^{\rho_i} = E_t, \quad \forall t \\ y_t^t + y_t^{t-1} + y_t^{*,t} + y_t^{*,t-1} &= \prod_{i=1}^{n} W_{i,t}{}^{\rho_i} = W_t, \quad \forall t \end{aligned}$$

and,

$$\begin{aligned} x_{i,t}^t + x_{i,t}^{t-1} + x_{i,t}^{*,t} + x_{i,t}^{*,t-1} &= e_{i,t}^t + e_{i,t}^{t-1} + e_{i,t}^{*,t} + e_{i,t}^{*,t-1} = E_{i,t}, \quad \forall\, i,t \\ y_{i,t}^t + y_{i,t}^{t-1} + y_{i,t}^{*,t} + y_{i,t}^{*,t-1} &= w_{i,t}^t + w_{i,t}^{t-1} + w_{i,t}^{*,t} + w_{i,t}^{*,t-1} = W_{i,t}, \quad \forall\, i,t \end{aligned}$$

Let $\mathcal{L}$ be the Lagrangean of the planner's program and define Λ_t and Φ_t as Lagrange multipliers of the resource constraints of good x and good y, respectively. The optimality conditions of the planner are,

$$\frac{\partial \mathcal{L}}{\partial x_t^t} : \ \alpha_t (1-\gamma)^{\sigma} \frac{1}{(x_t^t)^{\sigma}} - \Lambda_t = 0; \qquad \frac{\partial \mathcal{L}}{\partial x_t^{*,t}} : \ \alpha_t^* \gamma^{\sigma} \frac{1}{(x_t^{*,t})^{\sigma}} - \Lambda_t = 0 \tag{D.1}$$

$$\frac{\partial \mathcal{L}}{\partial x_t^{t-1}} : \alpha_{t-1}(1-\gamma)^{\sigma}\beta\frac{1}{(x_t^{t-1})^{\sigma}} - \Lambda_t = 0; \qquad \frac{\partial \mathcal{L}}{\partial x_t^{*,t-1}} : \alpha_{t-1}^{*}\gamma^{\sigma}\beta^{*}\frac{1}{(x_t^{*,t-1})^{\sigma}} - \Lambda_t = 0 \tag{D.2}$$

$$\frac{\partial \mathcal{L}}{\partial y_t^{t}} : \alpha_{t}\gamma^{\sigma}\frac{1}{(y_t^{t})^{\sigma}} - \Phi_t = 0; \qquad \frac{\partial \mathcal{L}}{\partial y_t^{*,t}} : \alpha_{t}^{*}(1-\gamma)^{\sigma}\frac{1}{(y_t^{*,t})^{\sigma}} - \Phi_t = 0 \tag{D.3}$$

$$\frac{\partial \mathcal{L}}{\partial y_t^{t-1}} : \alpha_{t-1}\gamma^{\sigma}\beta\frac{1}{(y_t^{t-1})^{\sigma}} - \Phi_t = 0; \qquad \frac{\partial \mathcal{L}}{\partial y_t^{*,t-1}} : \alpha_{t-1}^{*}(1-\gamma)^{\sigma}\beta^{*}\frac{1}{(y_t^{*,t-1})^{\sigma}} - \Phi_t = 0 \tag{D.4}$$

Then, we can write the following relationships. Combining (D.1) and (D.2),

$$x_t^{t-1} = \beta^{\frac{1}{\sigma}}\left(\frac{\alpha_{t-1}}{\alpha_t}\right)^{\frac{1}{\sigma}} x_t^t; \qquad x_t^{*,t-1} = \beta^{*\frac{1}{\sigma}}\left(\frac{\alpha_{t-1}^*}{\alpha_t^*}\right)^{\frac{1}{\sigma}} x_t^{*,t} \tag{D.5}$$

combining (D.3) and (D.4),

$$y_t^{t-1} = \beta^{\frac{1}{\sigma}}\left(\frac{\alpha_{t-1}}{\alpha_t}\right)^{\frac{1}{\sigma}} y_t^t; \qquad y_t^{*,t-1} = \beta^{*\frac{1}{\sigma}}\left(\frac{\alpha_{t-1}^*}{\alpha_t^*}\right)^{\frac{1}{\sigma}} y_t^{*,t} \tag{D.6}$$

combining (D.1) and (D.3),

$$x_t^{*,t} = \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} x_t^t; \qquad y_t^{*,t} = \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} y_t^t \tag{D.7}$$

combining (D.2) and (D.7), and (D.4) and (D.7),

$$x_t^{*,t-1} = \left(\frac{\gamma}{1-\gamma}\right)\beta^{*\frac{1}{\sigma}}\left(\frac{\alpha_{t-1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} x_t^t; \qquad y_t^{*,t-1} = \left(\frac{1-\gamma}{\gamma}\right)\beta^{*\frac{1}{\sigma}}\left(\frac{\alpha_{t-1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} y_t^t \tag{D.8}$$

Using conditions (D.5) to (D.7) into the resource constraints,

$$x_t^t\left(1+\beta^{\frac{1}{\sigma}}\left(\frac{\alpha_{t-1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)\beta^{*\frac{1}{\sigma}}\left(\frac{\alpha_{t-1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}}\right) = E_t \tag{D.9}$$

$$y_t^t\left(1+\beta^{\frac{1}{\sigma}}\left(\frac{\alpha_{t-1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)\beta^{*\frac{1}{\sigma}}\left(\frac{\alpha_{t-1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}}\right) = W_t \tag{D.10}$$

where $E_t = \prod_{i=1}^{n} {E_{i,t}}^{\rho_i}$ and $W_t = \prod_{i=1}^{n} {W_{i,t}}^{\rho_i}$. Then, the allocations of the social planner as functions of welfare weights are,

$$x_t^t = \frac{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} E_t}{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_{t-1}^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}}}; \quad y_t^t = \frac{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} W_t}{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_{t-1}^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}}}$$

$$x_{t+1}^{t} = \frac{\beta^{\frac{1}{\sigma}} E_{t+1}}{\left(\frac{\alpha_{t+1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_{t+1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}}; \; y_{t+1}^{t} = \frac{\beta^{\frac{1}{\sigma}} W_{t+1}}{\left(\frac{\alpha_{t+1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_{t+1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}}$$

for the domestic country, while for the foreign country,

$$x_{t}^{*,t} = \frac{\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} E_t}{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_{t-1}^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}}}; \; y_{t}^{*,t} = \frac{\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} W_t}{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_{t-1}^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}}}$$

$$x_{t+1}^{*,t} = \frac{\left(\frac{\gamma}{1-\gamma}\right)\beta^{*\frac{1}{\sigma}}\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} E_{t+1}}{\left(\frac{\alpha_{t+1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_{t+1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}}; \; y_{t+1}^{*,t} = \frac{\left(\frac{1-\gamma}{\gamma}\right)\beta^{*\frac{1}{\sigma}}\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} W_{t+1}}{\left(\frac{\alpha_{t+1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_{t+1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}}$$

Recall that the previous allocations are in terms of aggregate bundles. To obtain the demand for each good, we compute the following first-order conditions with $\lambda_{i,t}$ and $\phi_{i,t}$ representing the resource constraint of the $i = \{1, ..., n\}$ good.

$$\frac{\partial \mathcal{L}}{\partial x_{i,t}^{t}} : \alpha_t (1-\gamma)^{\sigma} \frac{1}{(x_t^t)^{\sigma}} \rho_i \frac{x_t^t}{x_{i,t}^t} = \lambda_{i,t}; \qquad \frac{\partial \mathcal{L}}{\partial x_{i,t}^{*,t}} : \alpha_t^* \gamma^{\sigma} \frac{1}{(x_t^{*,t})^{\sigma}} \rho_i \frac{x_t^{*,t}}{x_{i,t}^{*,t}} = \lambda_{i,t} \tag{D.11}$$

$$\frac{\partial \mathcal{L}}{\partial x_{i,t}^{t-1}} : \alpha_{t-1} (1-\gamma)^{\sigma} \beta \frac{1}{(x_t^{t-1})^{\sigma}} \rho_i \frac{x_t^{t-1}}{x_{i,t}^{t-1}} = \lambda_{i,t}; \qquad \frac{\partial \mathcal{L}}{\partial x_{i,t}^{*,t-1}} : \alpha_{t-1}^* \gamma^{\sigma} \beta^* \frac{1}{(x_t^{*,t-1})^{\sigma}} \rho_i \frac{x_t^{*,t-1}}{x_{i,t}^{*,t-1}} = \lambda_{i,t} \tag{D.12}$$

$$\frac{\partial \mathcal{L}}{\partial y_{i,t}^{t}} : \alpha_t \gamma^{\sigma} \frac{1}{(y_t^t)^{\sigma}} \rho_i \frac{y_t^t}{y_{i,t}^t} = \phi_{i,t}; \qquad \frac{\partial \mathcal{L}}{\partial y_{i,t}^{*,t}} : \alpha_t^* (1-\gamma)^{\sigma} \frac{1}{(y_t^{*,t})^{\sigma}} \rho_i \frac{y_t^{*,t}}{y_{i,t}^{*,t}} = \phi_{i,t} \tag{D.13}$$

$$\frac{\partial \mathcal{L}}{\partial y_{i,t}^{t-1}} : \alpha_{t-1} \gamma^{\sigma} \beta \frac{1}{(y_t^{t-1})^{\sigma}} \rho_i \frac{y_t^{t-1}}{y_{i,t}^{t-1}} = \phi_{i,t}; \qquad \frac{\partial \mathcal{L}}{\partial y_{i,t}^{*,t-1}} : \alpha_{t-1}^* (1-\gamma)^{\sigma} \beta^* \frac{1}{(y_t^{*,t-1})^{\sigma}} \rho_i \frac{y_t^{*,t-1}}{y_{i,t}^{*,t-1}} = \phi_{i,t} \tag{D.14}$$

To find the allocations for each good we follow the same procedure as before, but exploiting the first-order conditions in terms of the aggregators (D.5-D.8). Then, using conditions (D.11-D.14), (D.5-D.8), into the resource constraints,

$$x_{i,t}^{t}\left(1 + \beta^{\frac{1}{\sigma}}\left(\frac{\alpha_{t-1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)\beta^{*\frac{1}{\sigma}}\left(\frac{\alpha_{t-1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}}\right) = e_{i,t}^{t} + e_{i,t}^{t-1} + e_{i,t}^{*,t} + e_{i,t}^{*,t-1} = E_{i,t} \tag{D.15}$$

$$y_{i,t}^{t}\left(1 + \beta^{\frac{1}{\sigma}}\left(\frac{\alpha_{t-1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)\beta^{*\frac{1}{\sigma}}\left(\frac{\alpha_{t-1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}}\right) = w_{i,t}^{t} + w_{i,t}^{t-1} + w_{i,t}^{*,t} + w_{i,t}^{*,t-1} = W_{i,t} \tag{D.16}$$

Then, the allocations of the social planner as functions of welfare weights are,

$$x_{i,t}^t = \frac{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} E_{i,t}}{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_{t-1}^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}}};\ y_{i,t}^t = \frac{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} W_{i,t}}{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_{t-1}^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}}}$$

$$x_{i,t+1}^t = \frac{\beta^{\frac{1}{\sigma}} E_{i,t+1}}{\left(\frac{\alpha_{t+1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_{t+1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}};\ y_{i,t+1}^t = \frac{\beta^{\frac{1}{\sigma}} W_{i,t+1}}{\left(\frac{\alpha_{t+1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_{t+1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}}$$

for the domestic country, while for the foreign country,

$$x_{i,t}^{*,t} = \frac{\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} E_{i,t}}{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_{t-1}^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}}};\ y_{i,t}^{*,t} = \frac{\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} W_{i,t}}{\left(\frac{\alpha_t}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_{t-1}^*}{\alpha_{t-1}}\right)^{\frac{1}{\sigma}}}$$

$$x_{i,t+1}^{*,t} = \frac{\left(\frac{\gamma}{1-\gamma}\right)\beta^{*\frac{1}{\sigma}}\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} E_{i,t+1}}{\left(\frac{\alpha_{t+1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_{t+1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}};\ y_{i,t+1}^{*,t} = \frac{\left(\frac{1-\gamma}{\gamma}\right)\beta^{*\frac{1}{\sigma}}\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}} W_{i,t+1}}{\left(\frac{\alpha_{t+1}}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}} + \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_{t+1}^*}{\alpha_t}\right)^{\frac{1}{\sigma}} + \beta^{*\frac{1}{\sigma}}\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha_t^*}{\alpha_t}\right)^{\frac{1}{\sigma}}}$$

From the point of view of the planner, the allocations of goods are divided in the same way as the allocations of aggregated bundles. This is because the aggregator function is the same in every country for both x and y.

D.2 Transfer Functions with n-goods

For simplicity I express first the transfer functions of the domestic agent born in period $t \geq 2$ with n goods, then I define the rest of the transfer functions of the economy. For notation simplicity, define (in *bold*) the vector of one period consecutive welfare weights, $\boldsymbol{a}_t = (\alpha_{t-1}, \alpha_t, \alpha_{t-1}^*, \alpha_t^*)$

$$\begin{aligned}\mathcal{T}(\boldsymbol{a}_{t+1}, \boldsymbol{a}_t) = \sum_{i=1}^{n} \lambda_{i,t}(\boldsymbol{a}_t)\left[x_{i,t}^t(\boldsymbol{a}_t) - e_{i,t}^t\right] \;+\; \lambda_{i,t+1}(\boldsymbol{a}_{t+1})\left[x_{i,t+1}^t(\boldsymbol{a}_{t+1}) - e_{i,t+1}^t\right] \\ +\; \phi_{i,t}(\boldsymbol{a}_t)\left[y_{i,t}^t(\boldsymbol{a}_t) - w_{i,t}^t\right] \;+\; \phi_{i,t+1}(\boldsymbol{a}_{t+1})\left[y_{i,t+1}^t(\boldsymbol{a}_{t+1}) - w_{i,t+1}^t\right]\end{aligned} \tag{D.17}$$

From the first-order conditions, we can define:

$$\lambda_{i,t} e_{i,t}^t = \frac{\rho_i \Lambda_t x_t^t}{x_{i,t}^t} e_{i,t}^t \tag{D.18}$$

Then,

$$\begin{aligned}\sum_{i=1}^{n} \lambda_{i,t} e_{i,t}^{t} &= \sum_{i=1}^{n} \frac{\rho_i \Lambda_t x_t^t}{x_{i,t}^t} e_{i,t}^t \\ &= \Lambda_t x_t^t \sum_{i=1}^{n} \frac{\rho_i}{x_{i,t}^t} e_{i,t}^t \\ &= \Lambda_t \, E_t \underbrace{\sum_{i=1}^{n} \rho_i \frac{e_{i,t}^t}{E_{i,t}^t}}_{e_t^t}\end{aligned} \tag{D.19}$$

Additionally, $\sum_{i=1}^{n} \lambda_{i,t} x_{i,t}^t = \Lambda_t x_t^t$. Then, the transfer function of the domestic agent can be expressed as,

$$\begin{aligned}\mathcal{T}(\boldsymbol{a}_{t+1}, \boldsymbol{a}_t) = \Lambda_t(\boldsymbol{a}_t) \left[x_t^t(\boldsymbol{a}_t) - e_t^t\right] \, + \, \Lambda_{t+1}(\boldsymbol{a}_{t+1}) \left[x_{t+1}^t(\boldsymbol{a}_{t+1}) - e_{t+1}^t\right] \\ + \, \Phi_{i,t}(\boldsymbol{a}_t) \left[y_t^t(\boldsymbol{a}_t) - w_t^t\right] \, + \, \Phi_{t+1}(\boldsymbol{a}_{t+1}) \left[y_{t+1}^t(\boldsymbol{a}_{t+1}) - w_{t+1}^t\right]\end{aligned} \tag{D.20}$$

where w_t^t and w_{t+1}^t are defined similarly to e_t^t, and Λ_t and Φ_t represents the price of the aggregate bundles, E_t and W_t, respectively. Notice how e_t^t is defined from expression (D.20). Wealth of the domestic agent, $\sum_{i=1}^{n} \lambda_{i,t} x_{i,t}^t$, is defined as a weighted average of the value of the aggregate bundle, $\Lambda_t E_t$, where the weight of each individual endowment depends on i) the share of the individual endowment of good i on the aggregate endowment, $e_{i,t}^t / E_{i,t}^t$, and the share of good i, ρ_i, on the aggregate bundle, $E_t = \prod_{i=1}^{n} E_{i,t}{}^{\rho_i}$.

Under these definitions, the set of transfer functions of all the consumers in the economy takes the form,

$$\mathcal{T}_0(\boldsymbol{a}_1) = \Lambda_1(\boldsymbol{a}_1) \left[x_1^0(\boldsymbol{a}_1) - e_1^0\right] \, + \, \Phi_1(\boldsymbol{a}_1) \left[y_1^0(\boldsymbol{a}_1) - w_1^0\right] \tag{D.21}$$

$$\mathcal{T}_0^*(\boldsymbol{a}_1) = \Lambda_1(\boldsymbol{a}_1) \left[x_1^{*,0}(\boldsymbol{a}_1) - e_1^{*,0}\right] \, + \, \Phi_1(\boldsymbol{a}_1) \left[y_1^{*,0}(\boldsymbol{a}_1) - w_1^{*,0}\right] \tag{D.22}$$

for the initial old in period $t = 1$,

$$\begin{aligned}\mathcal{T}_1(\boldsymbol{a}_2, \boldsymbol{a}_1) = \Lambda_1(\boldsymbol{a}_1) \left[x_1^1(\boldsymbol{a}_1) - e_1^1\right] \, + \, \Lambda_2(\boldsymbol{a}_2) \left[x_2^1(\boldsymbol{a}_2) - e_2^1\right] \\ + \, \Phi_1(\boldsymbol{a}_1) \left[y_1^1(\boldsymbol{a}_1) - w_1^1\right] \, + \, \Phi_2(\boldsymbol{a}_2) \left[y_2^1(\boldsymbol{a}_2) - w_2^1\right]\end{aligned} \tag{D.23}$$

$$\mathcal{T}_1^*(\boldsymbol{a}_2, \boldsymbol{a}_1) = \Lambda(\boldsymbol{a}_1)\left[x_1^{*,1}(\boldsymbol{a}_1) - e_1^{*,1}\right] + \Lambda_2(\boldsymbol{a}_2)\left[x_2^{*,1}(\boldsymbol{a}_2) - e_2^{*,1}\right] + \Phi_1(\boldsymbol{a}_1)\left[y_1^{*,1}(\boldsymbol{a}_1) - w_1^{*,1}\right] + \Phi_2(\boldsymbol{a}_2)\left[y_2^{*,2}(\boldsymbol{a}_2) - w_2^{*,1}\right] \tag{D.24}$$

for the young consumer in period $t = 1$, and,

$$\mathcal{T}(\boldsymbol{a}_{t+1}, \boldsymbol{a}_t) = \Lambda_t(\boldsymbol{a}_t)\left[x_t^t(\boldsymbol{a}_t) - e_t^t\right] + \Lambda_{t+1}(\boldsymbol{a}_{t+1})\left[x_{t+1}^t(\boldsymbol{a}_{t+1}) - e_{t+1}^t\right] + \Phi_t(\boldsymbol{a}_t)\left[y_t^t(\boldsymbol{a}_t) - w_t^t\right] + \Phi_{t+1}(\boldsymbol{a}_{t+1})\left[y_{t+1}^t(\boldsymbol{a}_{t+1}) - w_{t+1}^t\right] \tag{D.25}$$

$$\mathcal{T}^*(\boldsymbol{a}_{t+1}, \boldsymbol{a}_t) = \Lambda_t(\boldsymbol{a}_t)\left[x_t^{*,t}(\boldsymbol{a}_t) - e_t^{*,t}\right] + \Lambda_{t+1}(\boldsymbol{a}_{t+1})\left[x_{t+1}^{*,t}(\boldsymbol{a}_{t+1}) - e_{t+1}^{*,t}\right] + \Phi_t(\boldsymbol{a}_t)\left[y_t^{*,t}(\boldsymbol{a}_t) - w_t^{*,t}\right] + \Phi_{t+1}(\boldsymbol{a}_{t+1})\left[y_{t+1}^{*,t}(\boldsymbol{a}_{t+1}) - w_{t+1}^{*,t}\right] \tag{D.26}$$

for a consumer born in period $t \geq 2$.

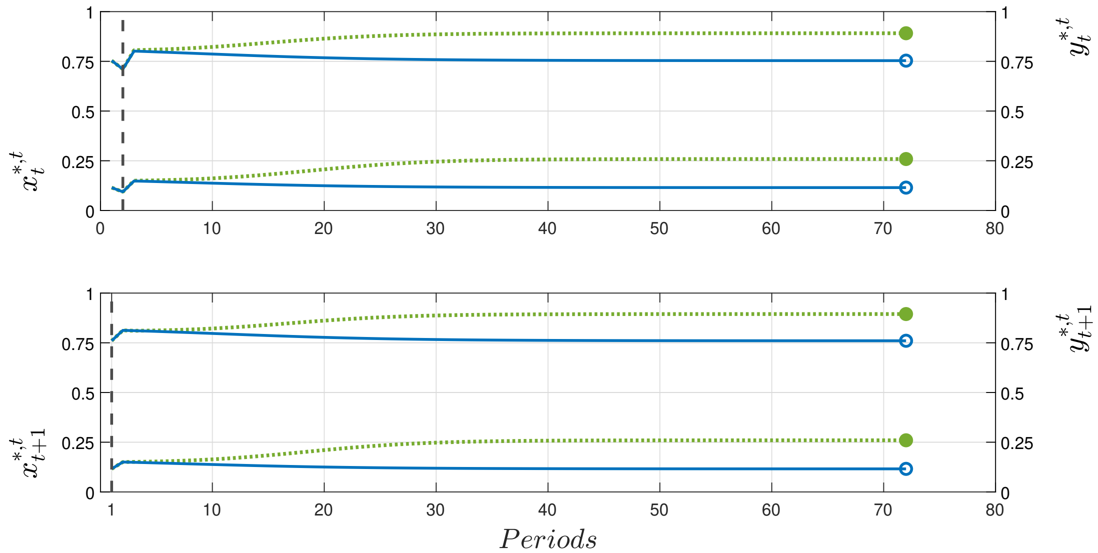
Equipped with the transfer functions, the dynamical system follows the same procedure as in C.2.

E. Appendix E

E.1 Additional Figures: Experiment One

Figure 10 shows that consumption of both goods when young decrease on impact in the foreign economy. The negative supply improves the terms of trade for the domestic economy, which makes the foreign consumers less wealthy. In addition, because of the reduction in the aggregate endowment, consumption is affected in both countries. Notice that the upper and the lower panel represent consumption by the same consumer but in different time periods. Because old agents on the foreign economy face more favorable terms of trade in period two, their consumption goes up in period three (marked with the dotted line in Figure 10).

Figure 10: Foreign consumption

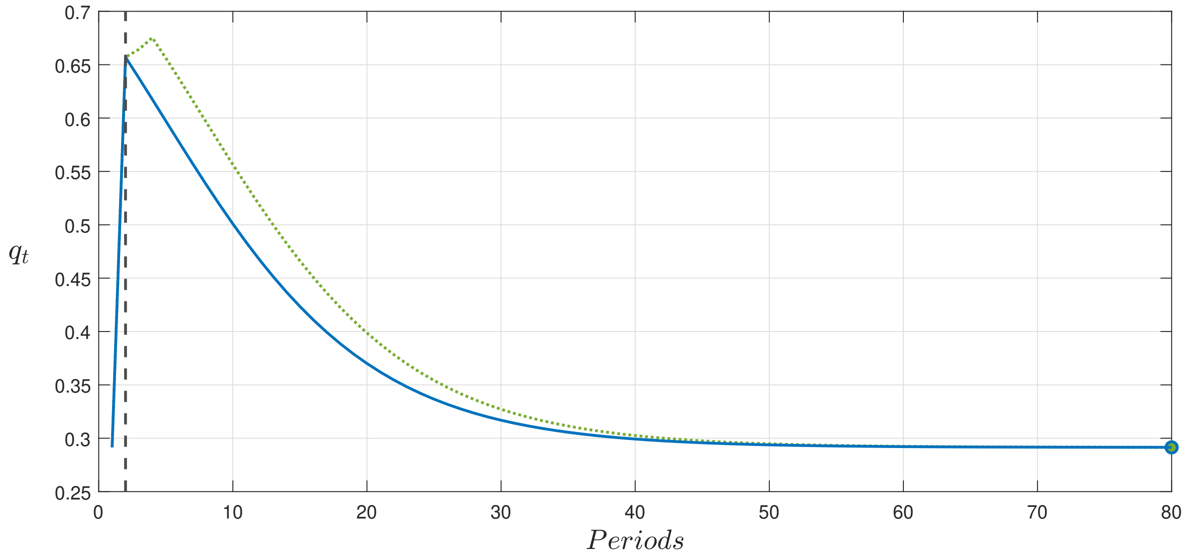


Note: The figure shows the foreign consumption allocations of the young and the old born in period t for goods x and y. The vertical dashed line identifies the period when the fundamental shock materializes. The upper panel plots the consumption allocations for the young, $x_t^{*,t}$ and $y_t^{*,t}$. The two upper lines in that graph represent $y_t^{*,t}$. The dotted (green) line shows the transition to the upper (green, solid) steady state. The solid (blue) line shows the transition to the lower (circle, blue) steady state. The two lower lines in that graph represent $x_t^{*,t}$. The transition follows the same description. The lower panel plots the consumption allocations for the old in $t+1$, born in period t, i.e., $x_{t+1}^{*,t}$ and $y_{t+1}^{*,t}$. The description of the two upper lines, the two lower lines, and the transition to the steady state follow the same description as before. Notice that the upper and the lower panel represent consumption by the same consumer but in different time periods. As a clarification, in a given period t, total foreign consumption of good x corresponds to $x_t^{*,t-1} + x_t^{*,t}$.

E.2 Additional Figures: Experiment Two

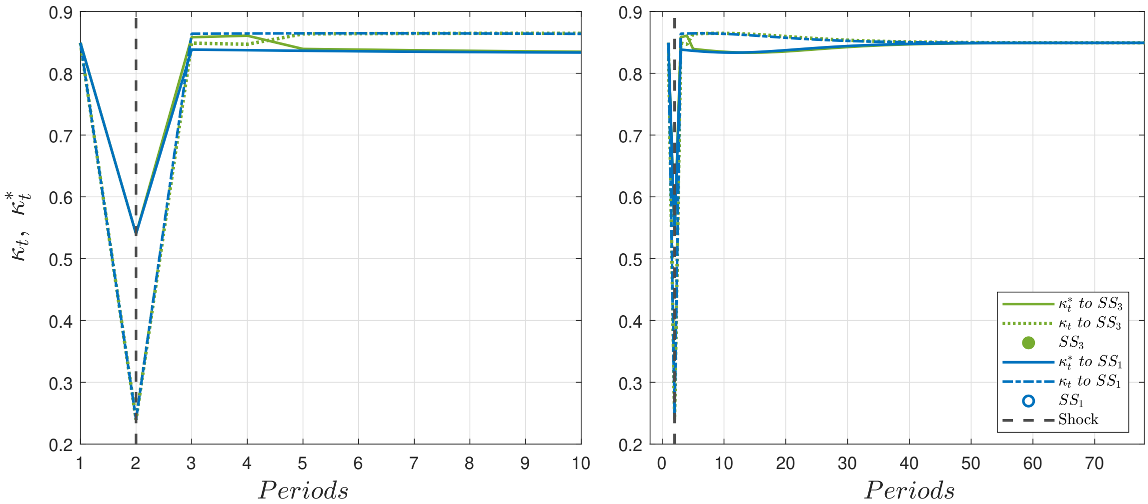
Figure 11 and 12 show the dynamics of the relative welfare weight, q_t, and the rates of growth of welfare weights, κ_t^* and κ_t in the second experiment.

Figure 11: Relative Welfare Weights



Note: The figure plots the the relative welfare weight, q_t. The vertical dashed line identifies the period when the fundamental shock materializes. The dotted (green) line shows the transition to the lower (green, solid circle) steady state after the second belief shock. The solid (blue) line shows the transition to the lower (blue, circle) steady state.

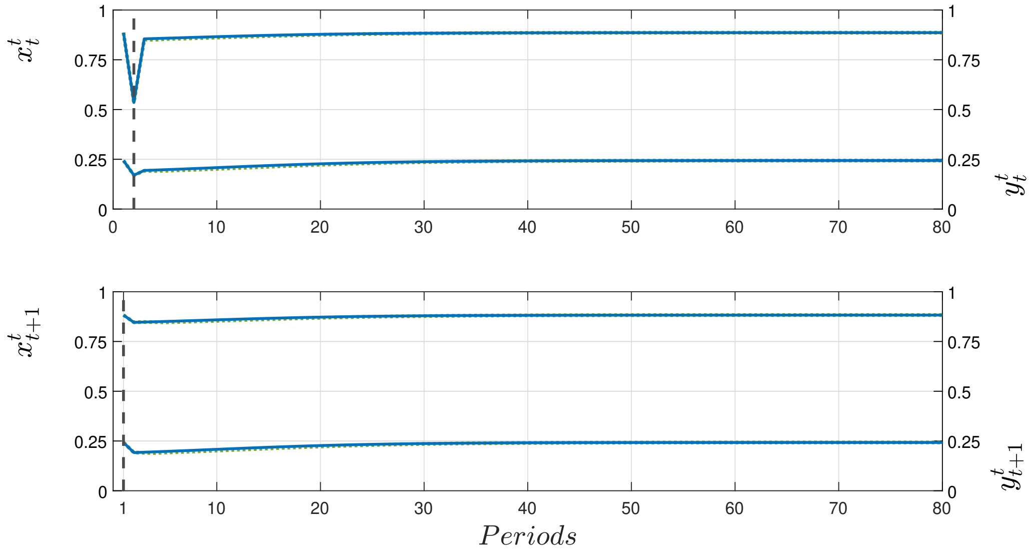
Figure 12: Rates of Growth of Welfare Weights



Note: The figure plots the rates of growth of welfare weights, κ_t^* and κ_t. The vertical dashed line identifies the period when the fundamental shock materializes. The solid and dotted (green) lines shows the transition to the lower (green, solid circle) steady state after the second belief shock. The solid and dot-dashed (blue) line shows the transition to the lower (blue, circle) steady state. Notice κ^* and κ are equal in each steady state (see equation (11)).

Figure 13 shows the consumption levels of both goods when young and old for the domestic economy. After a reduction and a recovery in consumption due to the fundamental shock, the economy slowly converges back to the original steady state. As the intervention is quickly, consumption dynamics after the announcement are almost identical to ones that would have materialized in the absence of the first belief shock (i.e., only in the presence of the endowment shock). Figure 14 show the consumption dynamics in experiment two for the foreign economy.

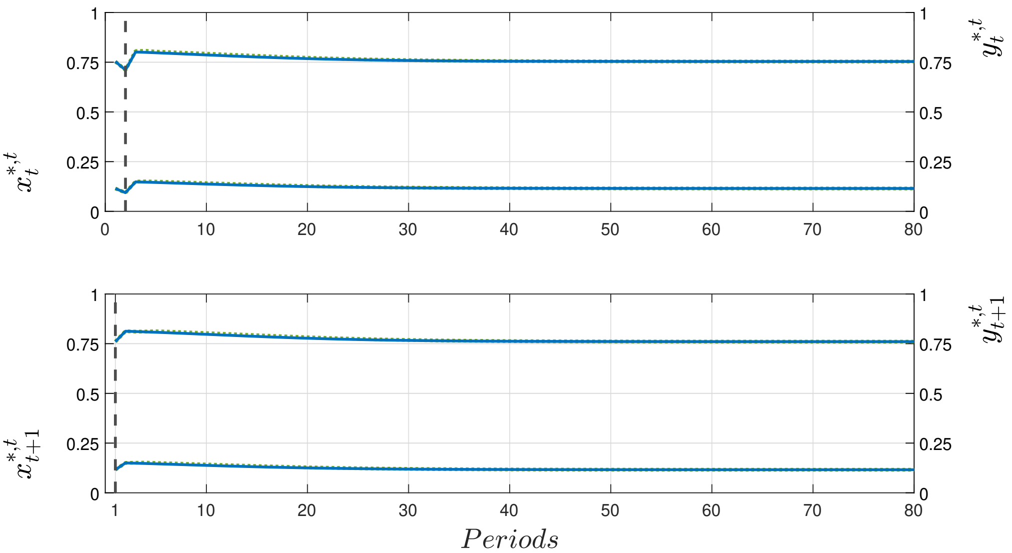
Figure 13: Domestic consumption



Note: The figure shows the domestic consumption allocations of the young and the old born in period t for goods x and y. The vertical dashed line identifies the period when the fundamental shock materializes. The upper panel plots the consumption allocations for the young, x_t^t and y_t^t. The two upper lines in that graph represent x_t^t. The dotted (green) line shows the transition to the lower (green, solid circle) steady state after the second belief shock. The solid (blue) line shows the transition to the lower (blue, circle) steady state. The two lower lines in that graph represent y_t^t. The transition follows the same description. The lower panel plots the consumption allocations for the old in $t+1$, born in period t, i.e., x_{t+1}^t and y_{t+1}^t. The description of the two upper lines, the two lower lines, and the transition to the steady state follow the same description as before. Notice that the upper and the lower panel represent consumption by the same consumer but in different time periods. As a clarification, in a given period t, total domestic consumption of good x corresponds to $x_t^{t-1} + x_t^t$.

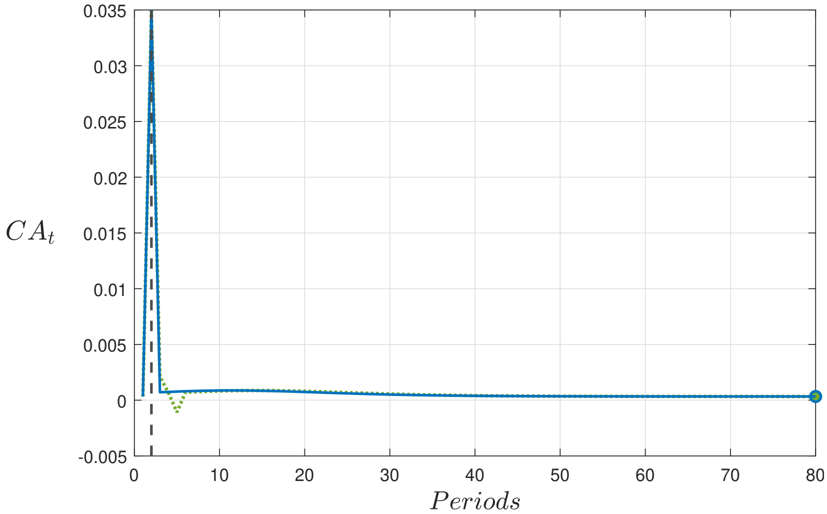
Figure 15 show a similar transition after the announcement takes place, with a small deterioration of the current account followed by a reversal in the equilibrium path where the public announcement materialized.

Figure 14: Foreign consumption



Note: The figure shows the foreign consumption allocations of the young and the old born in period t for goods x and y. The vertical dashed line identifies the period when the fundamental shock materializes. The upper panel plots the consumption allocations for the young, $x_t^{*,t}$ and $y_t^{*,t}$. The two upper lines in that graph represent $y_t^{*,t}$. The dotted (green) line shows the transition to the lower (green, solid circle) steady state after the second belief shock. The solid (blue) line shows the transition to the lower (blue, circle) steady state. The two lower lines in that graph represent $x_t^{*,t}$. The transition follows the same description. The lower panel plots the consumption allocations for the old in $t+1$, born in period t, i.e., $x_{t+1}^{*,t}$ and $y_{t+1}^{*,t}$. The description of the two upper lines, the two lower lines, and the transition to the steady state follow the same description as before. Notice that the upper and the lower panel represent consumption by the same consumer but in different time periods. As a clarification, in a given period t, total foreign consumption of good x corresponds to $x_t^{*,t-1} + x_t^{*,t}$.

Figure 15: Domestic Current Account



Note: The figure plots the domestic current account, CA_t. The vertical dashed line identifies the period when the fundamental shock materializes. The solid and dotted (green) lines shows the transition to the lower (green, solid circle) steady state after the second belief shock. The solid and dot-dashed (blue) line shows the transition to the lower (blue, circle) steady state. $CA_t > 0$ means a surplus. Notice that the definition of the current account involves consumers born in different periods (see equation 19).