Week 3:
Monopoly and Duopoly

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Reading: 1. Osborne Sections 3.1 and 3.2;
2. Snyder & Nicholson, chapters 14 and 15;
3. Sydsæter & Hammond,
Essential Mathematics for Economics Analysis, Section 4.6.

With thanks to Peter J. Hammond.

Outline

1. Monopoly
2. Cournot's model of quantity competition
3. Bertrand's model of price competition

Cost, Demand and Revenue

Consider a firm that is the only seller of what it produces.
Example: a patented medicine, whose supplier enjoys a monopoly.

Assume the monopolist's total costs are given by the quadratic function $C = \alpha Q + \beta Q^2$ of its output level $Q \geq 0$, where α and β are positive constants.

For each Q, its selling price P is assumed to be determined by the linear "inverse" demand function $P = a - bQ$ for $Q \geq 0$, where a and b are constants with $a > 0$ and $b \geq 0$.

So for any nonnegative Q, total revenue R is given by the quadratic function $R = PQ = (a - bQ)Q$.
As a function of output Q, profit is given by

$$\pi(Q) = R - C = (a - bQ)Q - \alpha Q - \beta Q^2 = (a - \alpha)Q - (b + \beta)Q^2$$

Profit Maximizing Quantity

Completing the square (no need for calculus!),
we have $\pi(Q) = \pi^M - (b+\beta)(Q - Q^M)^2$, where

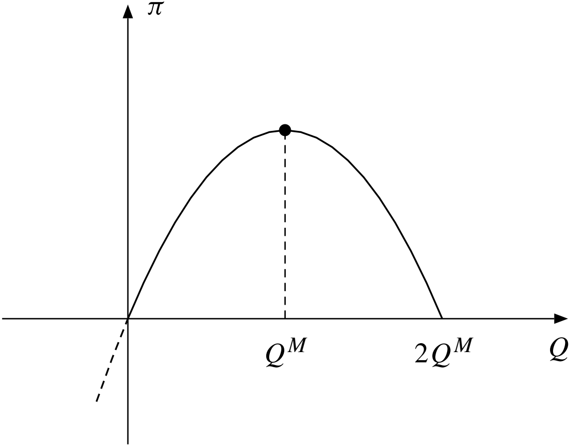
$$Q^M = \frac{a-\alpha}{2(b+\beta)} \quad \text{and} \quad \pi^M = \frac{(a-\alpha)^2}{4(b+\beta)}$$

So the monopolist has a profit maximum at $Q = Q^M$,
with maximized profit equal to π^M.
The monopoly price is

$$P^M = a - bQ^M = a - b\frac{a-\alpha}{2(b+\beta)} = \frac{b(a+\alpha)+2a\beta}{2(b+\beta)}$$

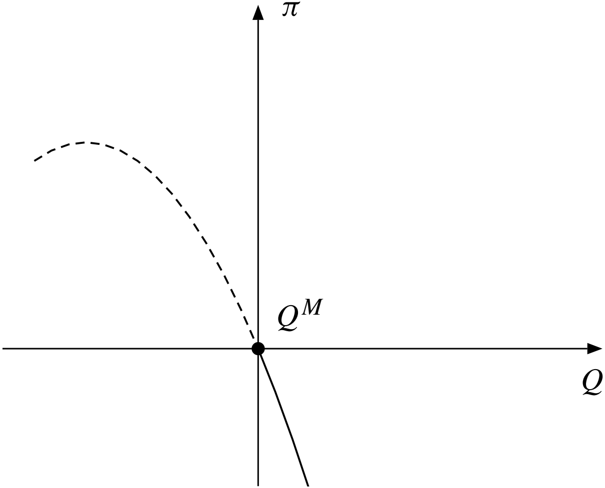
This is valid if $a > \alpha$;
if $a \leq \alpha$, the firm will not produce,
but will have $Q^M = 0$ and $\pi^M = 0$.
The two cases are illustrated in the next two slides.

Positive Output



Optimal output is positive if $a > \alpha$.

Zero Output



Optimal output is zero if $a \leq \alpha$.

Profit Maximizing Price

The inverse demand function is $P = a - bQ$ for $Q \geq 0$.
As a function of price P, the output quantity is given by the demand function is $Q = \frac{1}{b}(a - P)$ for $0 \leq P \leq a$.
So revenue is $R = PQ = \frac{1}{b}P(a - P)$ for $0 \leq P \leq a$.
And cost is given by

$$C = \alpha Q + \beta Q^2 = \frac{\alpha}{b}(a - P) + \frac{\beta}{b^2}(a - P)^2$$

This implies that, as a function of the price P, profit is

$$\pi = \frac{1}{b}(P - \alpha)(a - P) - \frac{\beta}{b^2}(a - P)^2$$

This time we find the maximum from the first-order condition

$$0 = \frac{d\pi}{dP} = \frac{1}{b}(a + \alpha - 2P) + \frac{\beta}{b^2}2(a - P)$$

Again, the monopoly price is $P^M = \frac{b(a+\alpha)+2a\beta}{2(b+\beta)}$ if $a > \alpha$;
if $a \leq \alpha$, the firm will choose $P^M \geq a$ to stifle all demand.

Surplus Maximization

Recall the monopolist's
inverse demand function $P = a - bQ$ for $Q \geq 0$,
and demand function $Q = \frac{1}{b}(a - P)$ for $0 \leq P \leq a$.
When price is P, consumer surplus CS is measured by the integral

$$\text{CS} = \int_0^Q (a - bq - P)dq = \Big|_0^Q \left[(a - P)q - \tfrac{1}{2}bq^2\right]$$

above the price line and below the inverse demand curve. So

$$\text{CS} = (a - P)Q - \tfrac{1}{2}bQ^2 = \tfrac{1}{2}bQ^2 = \tfrac{1}{2b}(a - P)^2$$

Total surplus TS is the sum of this and profit, which is

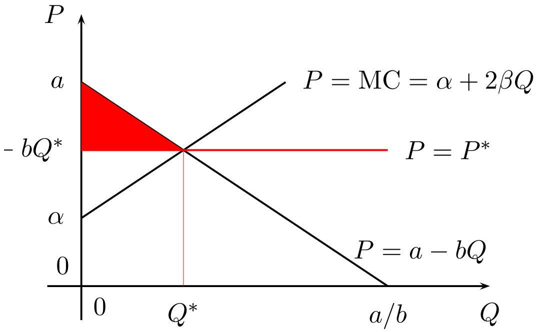
$$\text{TS} = \text{CS} + \pi = \tfrac{1}{2}bQ^2 + Q(a - bQ) - \alpha Q - \beta Q^2.$$

This reduces to $\text{TS} = (a - \alpha)Q - (\frac{1}{2}b + \beta)Q^2$.
It is maximized when the output quantity and price are

$$Q^* = \frac{a - \alpha}{b + 2\beta} \quad \text{and} \quad P^* = a - \frac{(a - \alpha)b}{b + 2\beta} = \frac{2\beta a + \alpha b}{b + 2\beta}.$$

Surplus Diagram

Surplus is maximized when $Q = Q^* = \dfrac{a - \alpha}{b + 2\beta}$.



Price Taking

Suppose the firm could be induced to act like a perfect competitor, by choosing an output along its marginal cost curve

$$\mathrm{MC} = \frac{d}{dQ}\alpha Q + \beta Q^2 = \alpha + 2\beta Q.$$

This intersects the demand curve $P = a - bQ$ where

$$a - bQ = \alpha + 2\beta Q \quad \text{and so} \quad Q = Q^* = \frac{a - \alpha}{b + 2\beta}$$

provided that $a > \alpha$.

The last slide showed that this competitive output level Q^* is the one that maximizes total surplus.
In fact, moving from Q^M to Q^* would pass a compensation test.
Is there a tax or subsidy that can persuade the firm to move to Q^*?

Taxing Monopoly Output

Suppose the monopolist is charged
a specific tax of t per unit of output.
The tax payment tQ is extra cost, so the new total cost function is

$$C = \alpha Q + \beta Q^2 + tQ = (\alpha + t)Q + \beta Q^2$$

To find the new profit maximizing quantity choice Q_t^M,
carry out the same calculations as before,
but with α replaced by $\alpha + t$; this gives

$$Q_t^M = \frac{a - \alpha - t}{2(b + \beta)} \quad \text{if } a > \alpha + t$$

but $Q_t^M = 0$ if $a \leq \alpha + t$.

Subsidizing Monopoly Output

The objective of remedial policy should be to maximize surplus by having the firm choose $Q_t^M = Q^*$. This requires

$$Q_t^M = \frac{a - \alpha - t}{2(b + \beta)} = Q^* = \frac{a - \alpha}{b + 2\beta}$$

provided that $a > \alpha$, with $Q_t^M = 0$ if $a \leq \alpha$. Clearing fractions, we obtain $(b + 2\beta)(a - \alpha - t) = 2(b + \beta)(a - \alpha)$ whose solution is

$$t^* = -\frac{(a - \alpha)b}{b + 2\beta} < 0 \quad \text{if } a > \alpha$$

So it is desirable to subsidize the monopolist's output in order to encourage additional production.

Taxing Monopoly Profits

Subsidizing monopolists is usually felt to be unjust,
and many additional complications need to be considered carefully
before formulating a desirable policy for dealing with monopolists.

Still the previous analysis suggests
that if justice requires lowering a monopolist's price or profit,
this is much better done directly than by taxing output.

Cournot Duopoly Example: Costs

Suppose two identical firms, labelled 1 and 2,
sell bottled mineral water.
Assume that there are no fixed costs,
but each firm i's variable costs of producing quantity q_i
are given by the quadratic cost function

$$c_i(q_i) = q_i^2 \text{ for } i \in \{1, 2\}.$$

Competitive Price-Taking

The assumption in a competitive environment
is that each firm will take a market price p as given,
believing that its behaviour cannot influence the market price.
COMMENT: Or will act as if it so believes.

Each firm i chooses a quantity q_i to maximize its profits.
So the firms face the identical profit maximization problem

$$\max_{q_i} \pi_i(q_i) := p \cdot q_i - c_i(q_i) = p \cdot q_i - q_i^2.$$

Note that $\pi_i'(q_i) = \frac{d}{dq_i}(p \cdot q_i - q_i^2) = p - 2q_i$
so $\pi_i'(q_i) > 0 \iff q_i < \frac{1}{2}p$ and $\pi_i'(q_i) < 0 \iff q_i > \frac{1}{2}p$.
Therefore, each firm will choose its supply
as a function $q_i(p) = \frac{1}{2}p$ of the price p.

Elementary Economics Interpretation

Economics students should know that
in a perfectly competitive market,
a firm's marginal cost as a function of its quantity
will represent its supply curve.

Here each firm i's marginal cost
is the derivative $C_i'(q_i) = 2q_i$ of its cost function.
Equating marginal cost to price gives $p = 2q_i$ or $q_i(p) = \frac{1}{2}p$,
exactly the supply function derived above.

Market Demand

Suppose the total consumer demand is given by

$$p(q) = 100 - q,$$

where q represents the total quantity demanded,
and $p(q)$ is the resulting price consumers are willing to pay.
Alternatively, we can write market demand
as a function $q_D = 100 - p$ of the price.

Note: Recall that, following Cournot himself (1838),
economists often "invert" the demand function
and write price as a function of quantity,
rather than quantity as a function of price.
This allows for some nice graphical analyses that we will also use.

Price-Taking Equilibrium I

The price-taking equilibrium occurs at a price p
where the total output $q_1(p) + q_2(p)$ of both firms' outputs equals the demand $q_D(p)$ at the same price.
Algebraically this is trivial:
we need to find a price that solves

$$q_1(p) + q_2(p) = 2 \cdot \frac{1}{2} p = q_D(p) = 100 - p.$$

This yields $p = 50$, and each firm produces $q_i = \frac{1}{2} p = 25$.

Price-Taking Equilibrium II

For those used to derive the competitive equilibrium graphically from supply and demand curves,
we add up the two firms' supply curves "horizontally",
and the equilibrium
is where total supply intersects the demand curve.

In this "competitive" price-taking equilibrium
each firm maximizes profits
when it takes the equilibrium price $p = 50$ as given
and chooses $q_i = \frac{1}{2}p = 25$.

Then each firm's profits
are $\pi_i = 50 \cdot 25 - 25^2 = 1250 - 625 = 625$.

Price Manipulation

Suppose firm 1 becomes more sophisticated
and realizes that reducing its output will raise the market price.
Specifically, suppose it produces 24 units instead of 25.
For demand to equal the new restricted supply,
the price must change to $p(q) = 100 - q = 51$,
where $q = 24 + 25 = 49$ instead of 50.
Firm 1's profits will then be
$\pi_i = 51 \cdot 24 - 24^2 = 1224 - 576 = 648 > 625$.
Of course, once firm 1 realizes its power over price,
it should not just set $q_1 = 25$ but look for its best choice.
However, that choice depends on firm 2's output quantity
— what will that be?
Clearly, firm 2 should be just as sophisticated.
Thus, we have to look for a solution
that considers both actions and counter-actions
of these rational and sophisticated (or well advised) firms.

Cournot Best Responses

Consider the Cournot duopoly game
with inverse demand $p = 100 - q$
and cost functions $c_i(q_i) = c_i \cdot q_i$ for firms $i \in \{1, 2\}$.

Firm i's profit is $u_i(q_i, q_j) = (100 - q_i - q_j) \cdot q_i - c_i \cdot q_i$.
Given its belief that its opponent's quantity is q_j,
its best response is $q_i^*(q_j) = \max\{0, 50 - \frac{1}{2}(q_j + c_i)\}$.

A (Cournot) Nash equilibrium occurs
at a pair of quantities (q_1, q_2)
that are mutual best responses.

So equilibrium requires $q_1 = q_1^*(q_2)$ and $q_2 = q_2^*(q_1)$.
We must solve both these best response equations simultaneously.

Zero Cost Case

In the easy case with each $c_i = 0$
the two best response functions are $q_1^*(q_2) = \max\{0, 50 - \frac{1}{2}q_2\}$
and $q_2^*(q_1) = \max\{0, 50 - \frac{1}{2}q_1\}$.

These two equations imply that $q_i^* \leq 50$ for $i = 1, 2$, hence
their solution satisfies both $q_1^* = 50 - \frac{1}{2}q_2^*$ and $q_2^* = 50 - \frac{1}{2}q_1^*$.

Subtracting one equation from the other
gives $q_1^* - q_2^* = -\frac{1}{2}(q_2^* - q_1^*)$, so $q_1^* = q_2^*$.
The unique Cournot equilibrium is $q_1^* = q_2^* = 33\frac{1}{3}$.

Intersecting Best Responses

In the general case when each $c_i < 50$
you should draw a diagram in the $q_1 - q_2$ plane
of the two best response
functions $q_1^*(q_2) = \max\{0, 50 - \frac{1}{2}(q_2 + c_1)\}$
and $q_2^*(q_1) = \max\{0, 50 - \frac{1}{2}(q_1 + c_2)\}$.

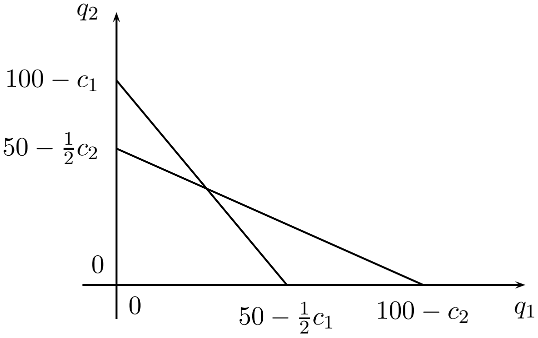
Part of each graph lies along the line segment $q_i^* = 0$.
The other parts are line segments joining:

1. $(0, 100 - c_1)$ to $(50 - \frac{1}{2}c_1, 0)$;
2. $(0, 50 - \frac{1}{2}c_2)$ to $(100 - c_2, 0)$.

The only equilibrium occurs at the intersection
of these two line segments, where $q_i > 0$
while $q_1 = 50 - \frac{1}{2}(q_2 + c_1)$ and $q_2 = 50 - \frac{1}{2}(q_1 + c_2)$.

Graph of Best Responses

Cournot equilibrium occurs at the intersection of the two firms' best response functions.



General Cournot Equilibrium

Finding the unique equilibrium requires solving the equations $q_1 = 50 - \frac{1}{2}(q_2 + c_1)$ and $q_2 = 50 - \frac{1}{2}(q_1 + c_2)$.

Adding the two equations implies that $q_1 + q_2 = 100 - \frac{1}{2}(q_2 + q_1) - \frac{1}{2}(c_1 + c_2)$ and so $q_1 + q_2 = 66\frac{2}{3} - \frac{1}{3}(c_1 + c_2)$.

Subtracting the second equation from the first implies that $q_1 - q_2 = -\frac{1}{2}(q_2 - q_1) - \frac{1}{2}(c_1 - c_2)$ and so $q_1 - q_2 = c_2 - c_1$.

These equations imply that $q_i = 33\frac{1}{3} + \frac{1}{3}c_j - \frac{2}{3}c_i$ for $i = 1, 2, j = 1, 2, i \neq j$.

For $c_2 = c_1$ we have $q_i = 33\frac{1}{3} - \frac{1}{3}c_i$ for $i = 1, 2$.

Bertrand Duopoly

In the Cournot model firms choose quantities,
and the price adjusts to clear market demand.

Joseph Bertrand in 1883 modelled duopoly
when firms set prices and consumers choose where to purchase.

Consider the alternative game where the two firms
each post their own price for their identical goods.
Assume that demand is given by $p = 100 - q$
and cost functions are $c_i(q_i) = c_i \, q_i$ for firms $i \in \{1, 2\}$.
Assume too that $0 \leq c_1 \leq c_2 < 100$.

Bertrand Competition

The firm that posts the lower price
obviously attracts all the consumers.
If both prices are the same, we assume
that the market is split equally between the two firms.

First, consider what happens if firm i is a monopolist.
It will choose quantity q
to maximize $(100 - q)q - c_i q = (100 - c_i)q - q^2$.
The monopoly quantity is $q_i^M = 50 - \frac{1}{2}c_i$.
The corresponding price is $p_i^M = 100 - q_i^M = 50 + \frac{1}{2}c_i$.

Note how $c_i < 100$ implies that $c_i < 50 + \frac{1}{2}c_i = p_i^M$ for $i \in \{1, 2\}$.

The Bertrand Normal Form

Here is the normal form:

Players: $N = \{1, 2\}$;

Strategy sets: The firms $i \in N$ choose prices $p_i \in S_i = [0, \infty)$;

Payoffs: The quantities are given by

$$q_i(p_i, p_j) = \begin{cases} 100 - p_i & \text{if } p_i < p_j \\ 0 & \text{if } p_i > p_j \\ \frac{1}{2}(100 - p_i) & \text{if } p_i = p_j \end{cases}$$

and the payoffs by $u_i(p_i, p_j) = (p_i - c_i)\, q_i(p_i, p_j)$.

Bertrand Best Responses

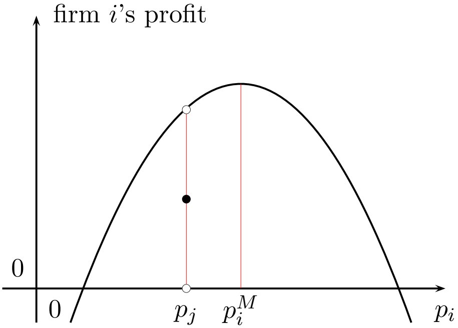
Case 1: If $p_j < c_i$ then firm i does best
to stay out of the market by charging $p_i \in BR_i(p_j) = (p_j, \infty)$.

Case 2: If $p_j = c_i$ then firm i does not want
to capture the whole market with a lower price,
but it earns zero profit for all $p_i \in BR_i(p_j) = [p_j, \infty)$.

Case 3: If $p_j > c_i$ and in fact $c_i < p_j \leq p_i^M = 50 + \frac{1}{2} c_i$,
then as long as firm i charges a price p_i satisfying $c_i < p_i < p_j$,
it captures the whole market
and earns positive profit $\hat{u}_i(p_i) = (p_i - c_i)(100 - p_i)$.
Also, $\hat{u}_i'(p_i) = 100 - 2p_i + c_i = 2(p_i^M - p_i) > 0$ in this interval.
But by increasing the price to $p_i = p_j$, firm i
gives up half the market, so its profit drops discontinuously.
There is no best response, so $BR_i(p_j) = \emptyset$.

No Profit Maximum

Firm i's profit is not maximized when $p_i = p_j$.



More Bertrand Best Responses

Case 4: Suppose $p_j > p_i^M = 50 + \frac{1}{2}c_i > c_i$ because $c_i < 100$.
As long as $p_i < p_j$, profits are $\hat{u}_i(p_i) = (p_i - c_i)(100 - p_i)$.
Therefore $\hat{u}_i'(p_i) = 100 - 2p_i + c_i \gtrless 0$ as $p_i \lesseqgtr 50 + \frac{1}{2}c_i = p_i^M$.
So $BR_i(p_j) = \{p_i^M\} = \{50 + \frac{1}{2}c_i\}$.

Bertrand Best Responses: Summary

To summarize these four cases:

$$BR_i(p_j) = \begin{cases} (p_j, \infty) & \text{if } p_j < c_i \\ [p_j, \infty) & \text{if } p_j = c_i \\ \emptyset & \text{if } c_i < p_j \leq p_i^M \\ \{p_i^M\} & \text{if } p_j > p_i^M \end{cases}$$

Recall that $c_1 < 100$, so $c_i < 50 + \frac{1}{2}c_i = p_i^M$ for $i \in \{1, 2\}$.
The Bertrand equilibrium set is as follows:

Case A: If $p_1^M = 50 + \frac{1}{2}c_1 \leq c_2$,
then there is a set of monopoly equilibria
with $p_1 = p_1^M = 50 + \frac{1}{2}c_1$ and $p_2 \in (p_1^M, \infty)$.

Case B: If $c_1 < c_2 < p_1^M = 50 + \frac{1}{2}c_1$,
then Bertrand equilibrium does not exist.

Case C: In the symmetric case with $c_1 = c_2 = c$,
there is a unique Bertrand equilibrium
with $p_1 = p_2 = c$, as under perfect competition.

Analysing Non-Existence

Why is there no equilibrium in Case B,
when $c_1 < c_2 < p_1^M = 50 + \frac{1}{2}c_1$?

Suppose the lower cost firm 1 enters the market first,
and charges the monopoly price $p_1^M = 50 + \frac{1}{2}c_1$.

In Case A this would lead to equilibrium with $p_2 > p_1^M$.

But in Case B firm 2 can profitably undercut and charge,
for instance, $p_2 = \frac{1}{2}(c_2 + p_1^M)$.
So there is no monopoly equilibrium.

In case B the only possible equilibrium price pairs (p_1, p_2)
would satisfy $p_2 \in [c_2, 50 + \frac{1}{2}c_1) \subset (c_1, 50 + \frac{1}{2}c_1)$.
But then $BR_1(p_2) = \emptyset$, so there can be no equilibrium.

Summary

1. A monopoly firm typically has too small an output relative to the perfectly competitive ideal.
   Correcting this distortion would require a subsidy, not a tax.
   This subsidy can be paid out of the firm's monopoly profits.
2. When there is imperfect competition, one case occurs when firms each choose their output as best responses to the other firms' output choices.
   This is the Cournot model of quantity competition.
3. A second case of imperfect competition occurs when firms each choose their price as best responses to the other firms' price choices.
   This is the Bertrand model of price competition.
   Sometimes it leads to a monopoly outcome; sometimes to a perfectly competitive outcome; and sometimes there is no equilibrium at all.