Available online at www.sciencedirect.com
ScienceDirect
Journal of Economic Theory 145 (2010) 639–660
JOURNAL OF Economic Theory
www.elsevier.com/locate/jet

Competitive experimentation with private information: The survivor's curse ☆

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Received 22 July 2008; final version received 11 November 2009; accepted 13 November 2009

Available online 29 December 2009

Abstract

We study a winner-take-all R&D race between two firms that are privately informed about the arrival rate of an invention. Over time, each firm only observes whether the opponent left the race or not. The equilibrium displays a strong herding effect, that we call a 'survivor's curse.' Unlike in the case of symmetric information, the two firms may quit the race (nearly) simultaneously even when their costs and benefits for research differ significantly.

JEL classification: C73; D82

Keywords: Learning; Experimentation; Private information; Survivor's curse

☆ We thank Michele Boldrin, Chris Harris, Heidrun Hoppe, Paul Klemperer, David Levine, Tracy Lewis, Glenn Mac Donald, Meg Meyer, Mike Peters, Jennifer Reinganum, Jeroen Swinkels, Juuso Valimaki, and seminar audiences at Boston University, UC San Diego, Duke Fuqua Business School, Nuffield College University of Oxford, University of Bonn, University of Munich, Econometric Society Summer Meetings (2003), Society for Economic Dynamics (2003), ESSET at Gerzensee (2004), for valuable comments. The usual disclaimer applies.

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0022-0531/$ – see front matter
doi:10.1016/j.jet.2009.12.001

1. Introduction

Research and Development activities produce information — about the promise, feasibility and interim experimental results of a project — that is private to the researcher. There are no incentives to disclose such information, but rather to carefully protect it from industrial espionage.[2] Motivated by this observation, we theoretically investigate the effects of private information on R&D activities. We find a strong herding effect, that we dub a 'survivor's curse': firms may delay exit until observing that the opponent quit the race, and then exit in regret of not having left earlier. Unlike in the case of symmetric information, competing firms may abandon simultaneously a line of research even if their R&D costs and benefits differ significantly. As we later explain in details, our 'survivor's curse' is more severe than the winner's curse in standard auctions, all-pay interdependent-value auctions and than in wars of attrition models, see [14]. This fact underlines a key predictive distinction between our R&D game and wars of attrition.

Our analysis is staged in a simple analytical framework. We introduce private signals into the framework first introduced by Reinganum [21,22], and then studied by Choi [7] in the case of imperfect but symmetric information. Two firms challenge each other in a research race with fixed experimentation intensity and winner-take-all termination. At each point in time, each firm decides whether to quit the race or to keep paying a flow cost to stay in the race. Once a firm quits, prohibitive sunk costs make re-entry infeasible.[3] The prize can arrive to any player who is still in the race. Research costs and value of the prize may differ across firms. The prize arrival rate may change over time and comprises an idiosyncratic component (the firm's specific R&D efficiency) and a common unknown component (the 'promise' or feasibility of the research project). At the beginning of the race, a private signal partially informs each player about the project's common component of the arrival rate, larger signals correspond to more optimistic beliefs. After that, each firm only observes whether a prize has arrived and the opponent is still in the race. As time goes by and no prize arrives, beliefs about the common component of the project grow increasingly pessimistic.[4]

We analyze monotonic equilibria of this game: each firm optimally selects a quitting time that is increasing in her own private signal realization and depends also on whether the opponent is still in the race or not. Due to the interdependent-value nature of the game, the equilibrium displays a rather extreme 'winner's curse' property, or more precisely a 'survivor's curse.' Specifically, the firms herd on each other's participation in the race, rationally presuming that the opponent's signal realization is larger than the signal that would make the opponent quit in the

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[2] When private corporations sponsor university research, as a norm they require the faculty and graduate students involved to sign non-disclosure and exclusive-licensing agreements. Cohen et al. [8] conduct a survey questionnaire administered to 1478 R&D labs in the U.S. manufacturing sector in 1994. They find that firms typically protect the profits due to invention with secrecy and lead time advantages.

[3] We interpret quitting the R&D race as publicly abandoning a line of research, hence dismissing project-specific facilities and research teams. A temporary suspension is unlikely to be observed by competitors. Although irreversible exit is a strong simplifying assumption, in this context it appears more natural than the opposite assumption of costless re-entry. Indeed, R&D firms are seldom observed re-entering a line of research after dismissing it, possibly because of the large costs involved.

[4] A technically related model is the multi-armed bandit model studied by Keller et al. [12], who do not consider private information. In this strategic multi-armed bandit literature (see also [4]), equilibrium experimentation is sub-optimal because players cannot conceal their findings from each other. Essentially, under-experimentation consists of the under-provision of a public good. In contrast, we study R&D races where private information is carefully protected. Among other continuous-time models of multi-agent Bayesian learning, see also [2,3,13].

immediate future. Because of monotonicity of equilibrium stopping times in signals, when a firm quits, the 'survivor' discovers the quitter's actual signal realization and discontinuously revises her beliefs downward. If the opponent stops shortly before the player had planned to quit in case the opponent was still in the race, then this negative surprise makes the survivor immediately quit and regret not having quit the race earlier.[5]

To appreciate the effects of private information in R&D races, compare our results with the case of public information, which extends the analysis in [7]. If the firms' signals are public, then each firm quits the game when growing sufficiently pessimistic about the project's promise, at a time that depends on own cost and benefit from R&D. When costs and benefits of research differ across firms, they quit at different times. In our model, firms may quit simultaneously, rationally herding on each other's belief that the opponent's private assessment of the project is more favorable than it truly is. This result provides a novel implication of private information in R&D race models: When firms abandon simultaneously the same line of research, despite differences in costs, benefits and efficiency of research, an outside observer may detect our survivor's curse, and conclude that there is evidence of private information about interdependent values.[6] Private information is also detected when firms that appear very similar in R&D efficiency and potential returns are observed to quit at very different times.

Our work is related to several strands of literature. The benchmark are continuous-time R&D races, modeled as either differential or stopping games. The differential game approach is put forth in [21,22]. At each moment in time, each player selects an experimentation intensity that affects linearly the arrival rate of the invention, at a quadratic cost. The simpler approach where each player experiments with fixed intensity until quitting the race can be understood as a stopping game. Choi [7] adopts this approach to study the case of uncertain arrival rate of innovation with commonly known prior. This work is further extended by Malueg and Tsutsui [17] in a full-fledged differential game. We adopt the stopping game approach to address the effects of private information.[7]

Our study of private information in R&D races adapts and extends solution techniques from auction theory. But, while our game shares many elements of an all-pay ascending auction, or equivalently of a war of attrition with interdependent values, the models are significantly different. Consider, for the purpose of comparison, the symmetric version of our model, where firms have the same costs and benefits for research, and the same research efficiency. In this model, each player selects her stopping time acting as if the opponent's signal were *larger* than her own, whereas in a symmetric war of attrition she selects her stopping time acting as if the opponent's

[5] As a mirror image of this survivors's curse, a 'quitter's curse' arises if a firm quits too soon and the opponent does not follow suit. The quitter realizes having left the race too soon because, after it quits, the opponent holds perfect information. In a previous working paper version [20], we perform welfare analysis for the parametric example illustrated in Section 5. The two curses work in opposite directions towards making equilibrium R&D excessive or insufficient.

[6] In fact, the history of R&D displays several lines of research that attracted the effort of several research teams to then be abandoned as the promise of the projects did not materialize. As an example, consider the so-called fifth generation computers initiative, simultaneously undertaken in the mid 80s by Japan's Ministry of International Trade and Industry (MITI), by the Microelectronics and Computer Technology Corporation (MCTC) in the US, by the Alvey team in the UK, and by the European Strategic Program of Research in Information Technology (ESPRIT). This line of research, which intended to develop a computing hardware that used massive parallelism, was abandoned by the competing teams in the beginning of the early '90s, as the few machines produced did not meet commercial success.

[7] An alternative approach to modeling R&D competition is the "tug-of-war": firms take turns in making costly steps towards a "finish line" [10,11]. We are not aware of any analysis of private information in a tug-of-war.

signal were *smaller* than her own [14].[8] In the symmetric equilibrium of the war of attrition, in fact, observing that the opponent has not left the game, conveys bad news about the chance of winning, as the player with the higher signal will eventually win the contest. In our R&D race, instead, the random prize arrivals are conditionally independent across players, and observing that the opponent is still in the race only conveys good news on the player's prize arrival rate. As a result, players are willing to postpone exit much later in our R&D race than in wars of attrition.[9]

The paper is organized as follows. Section 2 lays out the model, Section 3 characterizes the monotonic equilibrium, Section 4 compares it with the equilibrium when signals are public information, Section 5 illustrates and extends the results of the general model in a canonical parametric example, Section 6 concludes. Appendix A contains the proofs.

2. The model

Two players, A and B, play the following stopping game. A prize $b_i > 0$ arrives to player $i = A, B$ at a random time $t_i \geqslant 0$, according to a cumulative distribution function $F_i(t_i|\lambda)$ that admits a full support continuous density $f_i(t_i|\lambda)$. We let the hazard rate of the prize be denoted by $\rho_i(t_i|\lambda) = f_i(t_i|\lambda)/[1 - F_i(t_i|\lambda)]$. The term λ describes the common component of the arrival process of the innovation (the promise of the project), but we allow for the arrival processes to differ in order to cover the possibility that firms have different efficiency. The common promise of either prize, $\lambda \in \Lambda$, is drawn by Nature, unobserved by the players, according to a distribution with full support continuous density $\pi(\lambda)$. Player i pays a flow cost $c_i > 0$ to stay in the race. We make a *winner-take-all* assumption: when a player receives his prize, the game ends. At each point in time, each player may irreversibly quit the race. Costs and prizes are discounted at rate r. Before starting to pay costs, each player i observes a private signal z_i distributed according to a cumulative distribution function $H_i(z_i|\lambda)$ that admits a continuous density $h_i(z_i|\lambda)$ on the support $Z_i = (\underline{z}_i, \bar{z}_i]$, where we assume that $\bar{z}_i$ is finite, but we make no assumptions on $\underline{z}_i$. The two private signals z_A, z_B are independent, conditionally on λ.

We let $\mathbb{E}^i_{t,t'}[\rho_i(\tau_i|\lambda)|x, y]$, $\mathbb{E}^i_{t,t'}[\rho_i(\tau_i|\lambda)|x, y-]$ and $\mathbb{E}^i_{t,t'}[\rho_i(\tau_i|\lambda)|x, y+]$ be the expected hazard rate at time τ_i, conditional on player i holding signals $z_i = x$, on knowing respectively that $z_j = y$, $z_j \leqslant y$, and $z_j \geqslant y$, and on projects i and j not having delivered a prize by times t and t', respectively. For future reference, we denote by $\pi^i_{t,t'}(\lambda|x, y)$, $\pi^i_{t,t'}(\lambda|x, y-)$, and $\pi^i_{t,t'}(\lambda|x, y+)$ the density functions of the posterior beliefs on λ, conditional on this information.

We now introduce and then maintain two regularity assumptions on the expected hazard rates. Under the first assumption, larger realizations x and y of the private signals make players more optimistic about the arrival of the prizes.

[8] Coincidentally, however, Bulow et al. [5] show that each bidder acts as if the opponent's signal were larger than her own, also when they are endowed with a share of the good auctioned off. In standard symmetric two-player common-value ascending second-price auctions [19], instead, each player acts as if the opponent's signal were equal to her own.

[9] Our work is also related to the literature on information aggregation in timing games, e.g., [6,9]. But these papers study coordination problems, where if players could share their private information, they would do so. In our R&D race, the winner-take-all assumption induces strong incentives to conceal private information (see, e.g., [15]). Lambrecht and Perraudin [16], Mariotti and Décamps [18] study continuous-time stopping games where private information is only of private value; whereas in our game it has interdependent value. More distantly related, Aoki and Reitman [1] study a two-stage model where firms first may invest to reduce their private-information production cost, and then compete à la Cournot.

Assumption 1 *(Private good news).* For any $i = A, B$, any $t, t' \geqslant 0$, $x, y \in Z_i$ the expected hazard rates $\mathbb{E}^i_{t,t'}[\rho_i(t|\lambda)|x, y-]$, $\mathbb{E}^i_{t,t'}[\rho_i(t|\lambda)|x, y]$ and $\mathbb{E}^i_{t,t}[\rho_i(t|\lambda)|x, y+]$ are continuously differentiable and strictly increasing in x and y.

As the expected hazard rate $\mathbb{E}^i_{t,t'}[\rho_i(t|\lambda)|x, y]$ increases in y, we now show that knowing that the opponent's signal realization exceeds y is better news than knowing it is exactly y, which in turn is better news than knowing it is less than y.

Lemma 1 *(Hazard rates order). For any $i = A, B$, any $t, t' \geqslant 0$, $x, y \in Z_i$*

$$\mathbb{E}^i_{t,t'}\big[\rho_i(t|\lambda)|x, y-\big] < \mathbb{E}^i_{t,t'}\big[\rho_i(t|\lambda)|x, y\big] < \mathbb{E}^i_{t,t'}\big[\rho_i(t|\lambda)|x, y+\big].$$

Under our second assumption, as time goes by and no prize materializes, the players become more and more pessimistic about the arrival rates of the prizes.

Assumption 2 *(No news is bad news).* For any $i = A, B$, any $t, t' \geqslant 0$, $x, y \in Z_i$, the expected hazard rates $\mathbb{E}^i_{t,t'}[\rho_i(t|\lambda)|x, y-]$, $\mathbb{E}^i_{t,t'}[\rho_i(t|\lambda)|x, y]$ and $\mathbb{E}^i_{t,t}[\rho_i(t|\lambda)|x, y+]$ are continuously differentiable and strictly decreasing in t and t'.

We now introduce our equilibrium notion. Each player's strategy depends only on her own private signal and on whether her opponent is still in the race. For each player $i = A, B$, a pure strategy in this game is a pair of functions $(\sigma_{1,i}, \sigma_{2,i})$, describing stopping behavior. The stopping time function given one's own private signal *and* given that the opponent is still in the race, is denoted by $\sigma_{1,i} : Z_i \to \overline{\mathbb{R}}_+$. For any signal $x \in Z_i$, the strategy $\sigma_{1,i}$ prescribes that player i stays in the race until time $\sigma_{1,i}(x)$ unless observing that the opponent has left the race at any time before $\sigma_{1,i}(x)$.[10] Note that the stopping time $\sigma_{1,i}(x) = 0$ prescribes that player i should not enter the race at all. The stopping strategy given one's own private signal and given that the opponent has already left the game, is denoted by $\sigma_{2,i} : Z_i \times \mathbb{R}_+ \to \overline{\mathbb{R}}_+$. Here, $\sigma_{2,i}(x, \tau) \geqslant \tau$ describes player i's stopping time when holding signal x and after the opponent has quit at time τ.[11]

As it is standard in Bayesian games with a continuum of types (e.g. auctions and timing games), we focus on *monotonic differentiable* equilibria. We denote equilibrium strategies as $(\sigma^*_{1,i}, \sigma^*_{2,i})$. Our monotonicity requirement is that there exist $\underline{x}^*_i$, $\bar{x}^*_i$ such that $\sigma^*_{1,i}(x) = 0$ if $x \leqslant \underline{x}^*_i$, $\sigma^*_{1,i}(x)$ is positive, differentiable and strictly increasing if $\underline{x}^*_i < x < \bar{x}^*_i$, and $\sigma^*_{1,i}(x) = \infty$ if $x \geqslant \bar{x}^*_i$. We denote by g^*_i the inverse function of $\sigma^*_{1,i}$ on the domain $(\underline{x}^*_i, \bar{x}^*_i)$.

In order to guarantee existence, the following additional conditions on the primitives of the model are sufficient (albeit not necessary):

[10] The choice of the extended positive real numbers $\overline{\mathbb{R}}_+$ as the range of $\sigma_{1,i}$ is made to allow for the possibility that a player may decide to stay in the game and wait for the prize forever, given some signal realization x, and given that the opponent is not leaving the game either.

[11] When $\sigma_{2,i}(x, \tau) = \tau$, the player leaves the race immediately after seeing that the opponent left at τ. Formally, this continuous stopping time is derived from a finite-time approximation where each period lasts Δ, and $\lim_{\Delta\to 0^+} \sigma_{2,i}(x, \tau) = \tau^+$. Metaphorically, in the moment that a player leaves the race, the clock is 'stopped for an instant' and the remaining player is left to choose whether to continue or follow suit. For a general treatment on how to construct stopping time strategies in continuous time games and on their interpretation, see [23].

Assumption 3 *(Hazard rate derivatives).* There exists $\overline{G}' < 0$ such that for any $i = A, B$, any $t, t' \geqslant 0$, $x, y \in Z_i$

$$\frac{d}{dy}E^i_{t,t}\big[\rho_i(t|\lambda)|x, y+\big] < \overline{G}'\frac{d}{dt}E^i_{t,t}\big[\rho_i(t|\lambda)|x, y+\big] < \frac{d}{dx}E^i_{t,t}\big[\rho_i(t|\lambda)|x, y+\big].$$

Finally, the following assumptions relate the expected hazard rates at the "boundaries" of the signal sets Z_i and of the time interval $[0, +\infty)$ to the cost benefit ratios c_i/b_i.

Assumption 4 *(Boundary conditions).* For each player $i = A, B$,

$$\lim_{t\to+\infty} E_{t,0}\big[\rho_i(t|\lambda)|\bar{z}_i, \bar{z}_i\big] < c_i/b_i,$$
$$\lim_{x\to\underline{z}_i} E_{0,0}\big[\rho_i(0|\lambda)|x, \bar{z}_i\big] < c_i/b_i < E_{0,0}\big[\rho_i(0|\lambda)|\bar{z}_i, \bar{z}_i\big].$$

The first inequality imposes that the expected hazard rate eventually becomes smaller than the cost benefit ratio, as a player waits longer and longer for the arrival of the prize, even in the case that private signals are as favorable as possible. This inequality will imply that for any signal x, the stopping times $\sigma^*_{1,i}(x)$ and $\sigma^*_{2,i}(x, \tau)$ are finite, so that $\bar{x}^*_i = \bar{z}_i$. The second inequality requires that the expected hazard rate is smaller than the cost benefit ratio when a firm's signal is sufficiently unfavorable. It will imply that $\underline{x}^*_i > \underline{z}_i$: for sufficiently low signals, firms do not enter the race. Conversely, the third inequality requires that the expected hazard rate exceeds the cost benefit ratio, in the case that signals are as favorable as possible, and that firms have not entered the race yet. It will imply that there are sufficiently favorable signals to convince the firms to enter the race, so that $\underline{x}^*_i < \bar{z}_i$.

3. Equilibrium analysis

3.1. Equilibrium play after the opponent quits

We calculate equilibrium strategies by backward induction, starting from the equilibrium strategy $\sigma^*_{2,i}$ of a player after the opponent left the race. Suppose that player j observes signal y, enters the game at time 0, and then quits first according to the equilibrium strategy $\sigma^*_{1,j}$ at time $\tau = \sigma^*_{1,j}(y) > 0$. In a monotonic equilibrium, the 'survivor,' player i, perfectly infers from τ the signal $y = g^*_j(\tau)$ privately held by her opponent j, and from that moment on she acts fully informed. At any time $t \geqslant \tau$, conditional on a true value of the prize hazard rate λ, unknown to the player, and on the fact that no prize has arrived to date, the expected value of planning to stop at some future date $\tau_2 \geqslant t$ equals

$$U^i_{2,t}(\tau_2|\lambda) = \int_t^{\tau_2} \frac{f_i(s|\lambda)}{1 - F_i(t|\lambda)}\left[\int_t^s (-c_i)e^{-r(v-t)}\,dv + e^{-r(s-t)}b_i\right]ds + \frac{1 - F_i(\tau_2|\lambda)}{1 - F_i(t|\lambda)}\int_t^{\tau_2}(-c_i)e^{-r(v-t)}\,dv.$$

The first term in the expression corresponds to the expected value conditional on the prize arriving before τ_2, the second term on the expected value conditional on the prize not arriving by τ_2.

At any time $t \geqslant \tau$, player i plans to quit at the time $\tau_2 \geqslant t$ that maximizes the expectation of this value $U_{2,t}$, conditional on all available information:

$$\tau^i_{2,t}\big(x, g^*_j(\tau)\big) = \arg\max_{\tau_2 \geqslant t}\left\{ V^i_{2,t}\big(\tau_2|x, g^*_j(\tau), \tau\big) = \int\limits_{\Lambda} U^i_{2,t}(\tau_2|\lambda)\pi_{t,\tau}\big(\lambda|x, g^*_j(\tau)\big)\, d\lambda \right\}.$$

By inspection, we see that the expected value $V^i_{2,t}(\tau_2|x, g^*_j(\tau), \tau)$ is continuously differentiable in τ_2 for every $\tau_2 \geqslant t$ and every x, τ, t. Hence, we can derive the optimal stopping time $\tau^i_{2,t}$ by differentiating $V^i_{2,t}(\tau_2|x, g^*_j(\tau), \tau)$ with respect to τ_2.[12] After differentiating $V^i_{2,t}(\tau_2|x, g^*_j(\tau), \tau)$, we substitute t with τ_2, and delete the subscript t in $\tau^i_{2,t}$. This is the requirement of time consistency: regardless of what she had planned earlier, player i must find it optimal to exit the race at time τ^i_2 when making her decision at time τ^i_2. As we show in Appendix A, the expected marginal value of waiting an extra instant before quitting takes the following simple form:

$$\frac{d}{d\tau_2} V^i_{2,t}\big(\tau_2|x, g^*_j(\tau), \tau\big)\Big|_{t=\tau_2} = b_i \mathbb{E}_{\tau_2,\tau}\big[\rho_i(\tau_2|\lambda)|x, g^*_j(\tau)\big] - c_i. \tag{1}$$

Intuitively, player i trades off the marginal cost of waiting c_i with the marginal benefit, which consists of the prize b_i multiplied by its expected hazard rate conditional on all information available at time τ_2.

The second derivative of $V^i_{2,t}(\tau_2|x, g^*_j(\tau), \tau)$ is negative because the expected hazard rate is decreasing in τ_2 by Assumption 2, so the right-hand side of Eq. (1) is decreasing in τ_2. Thus, player i quits at the earliest time τ_2 after τ when the right-hand side of Eq. (1) becomes negative. We thus obtain the following result.[13]

Proposition 2 *(Second quitter's stopping time). In any monotonic equilibrium, after the opponent j quits at any time $\tau > 0$, a player i with private signal x quits at time*

$$\sigma^*_{2,i}(x, \tau) = \min\left\{\tau_2 \geqslant \tau\colon\ \frac{c_i}{b_i} \geqslant \mathbb{E}_{\tau_2,\tau}\big[\rho_i(\tau_2|\lambda)|x, g^*_j(\tau)\big]\right\}.$$

If player j with signal y fails to join the game, then the remaining player i cannot perfectly infer the opponent's signal realization y, because the equilibrium strategy $\sigma^*_{1,j}$ is not invertible for $y \leqslant \underline{x}^*_j \equiv g^*_j(0)$, but only learns that $y \leqslant \underline{x}^*_j$. Calculations analogous to the ones leading to Proposition 2 yield the following result.

Proposition 3 *(Stopping time when racing alone). In any monotonic equilibrium, if the opponent j fails to join the game, player i with signal x stops at the time*

$$\sigma^*_{2,i}(x, 0) = \min\left\{\tau_2 \geqslant 0\colon\ \frac{c_i}{b_i} \geqslant \mathbb{E}_{\tau_2,0}\big[\rho_i(\tau_2|\lambda)|x, \underline{x}^*_j-\big]\right\}.$$

[12] Alternatively, we could differentiate the value $V^i_{2,t}(\tau_2|x, g^*(\tau), \tau)$ with respect to current time t and obtain a differential equation for the value, which is the continuous-time Hamilton–Jacobi–Bellman equation for this problem. We choose to proceed through the sequential formulation of the problem because technically simpler and more instructive.

[13] Note that we write min rather than inf because by Assumption 4 the expected hazard rate becomes smaller than c_i/b_i as $t \to \infty$, hence this stopping time for player i always exists.

Since quitting the game at any time $\tau > 0$ perfectly reveals player j's own private information y, while not joining the game at all only reveals an upper bound $\underline{x}_j^*$ to y, there is a natural discontinuity in the equilibrium strategy $\sigma_{2,i}^*(x,\tau)$ at $\tau = 0$. In fact, $\lim_{\tau\downarrow 0}\sigma_{2,i}^*(x,\tau) > \sigma_{2,i}^*(x,0)$, because $\mathbb{E}_{\tau_2,0}[\rho_i(\tau_2|\lambda)|x, g_j^*(0)-] < \lim_{\tau\to 0}\mathbb{E}_{\tau_2,\tau}[\rho_i(\tau_2|\lambda)|x, g_j^*(\tau)]$ by Lemma 1.

For future reference, let $\underline{x}_i^{**} \equiv \sup\{x\colon\ \sigma_{2,i}^*(x,0) = 0\}$, the lowest signal for which player i is willing to stay in the race, upon seeing that the opponent j did not enter the game. To summarize: if $x > \underline{x}_i^{**}$ then player i enters and stays in for some time regardless of what the opponent does; if $\underline{x}_i^* \leqslant x \leqslant \underline{x}_i^{**}$ then the player enters and quits immediately if and only if the opponent failed to join; if $x < \underline{x}_i^*$ then the player does not enter at all.[14]

The comparative statics of strategy $\sigma_{2,i}^*(x,\tau)$ follow directly from Assumptions 1 and 2. The quitting time $\sigma_{2,i}^*(x,\tau)$ increases in the signals, x, as conjectured, and $g_j^*(\tau)$; it decreases in the cost/benefit ratio c_i/b_i. The direct effect of the time τ the opponent stayed in the race without receiving the prize, ignoring the indirect effect of τ via $g_j^*(\tau)$, is to shorten the maximum time $\sigma_{2,i}^*(x,\tau)$ that player i is willing to spend alone in the race.

3.2. *Equilibrium play before the opponent quits*

The most complex part of the equilibrium characterization concerns the instance where both players are still in the game. Each player must plan an optimal stopping time based on the hypothesis that the opponent will quit later, and on the resulting information about the opponent's signal.

Proceeding as in the previous part of the section, we denote by $V_{1,t}^i(\tau_1|x)$ the value function of a player $i = A, B$ at any time $t > 0$ for quitting at time $\tau_1 \geqslant t$, conditional on the facts that opponent j has not quit yet at time τ_1 and is adopting a monotonic strategy $\sigma_{1,j}^*$, with associated inverse g_j^*. We report the expression of $V_{1,t}^i(\tau_1|x)$ in Appendix A. We then differentiate the expected value $V_{1,t}^i(\tau_1|x)$ with respect to the stopping time τ_1, and apply time consistency to substitute t with τ_1. As shown in Appendix A, we obtain:

$$\left.\frac{dV_{1,t}^i(\tau_1|x)}{d\tau_1}\right|_{t=\tau_1} = -c_i + b_i\mathbb{E}_{\tau_1,\tau_1}\big[\rho_i(\tau_1|\lambda)|x, g_j^*(\tau_1)+\big] + \mathbb{E}_{\tau_1,\tau_1}\left[\frac{h_j(g_j^*(\tau_1)|\lambda)}{1-H_j(g_j^*(\tau_1)|\lambda)}\frac{dg_j^*(\tau_1)}{d\tau_1}\middle| x, g_j^*(\tau_1)+\right] \times V_{2,\tau_1}^i\big(\sigma_{2,i}^*(x,\tau_1)|x, g_j^*(\tau_1), \tau_1\big). \tag{2}$$

The marginal value of waiting before quitting equals two flow expected benefit terms minus the flow cost c_i. The first benefits term is the prize b_i times the expected hazard rate of prize arrival. The second one is the expected hazard rate of the time when the opponent j quits, times the continuation value of remaining alone.

One key result of this section is a "survivor's curse." Suppose that in the event that player j remains in the race, player i plans to quit first at a time τ_1 which satisfies the first-order condition

[14] An alternative interpretation of results is that for $\underline{x}_i^* \leqslant x \leqslant \underline{x}_i^{**}$, player i enters the race if and only if observing player j entering the race. This form of imitation is very common in real world R&D races. Effectively, player i is uncertain of whether to enter the race or not, and is willing to invest in the project if and only if observing that player j invests in the project.

$dV^i_{1,t}(\tau_1|x)/d\tau_1|_{t=\tau_1}=0$. If j quits first at any time τ earlier than but close enough to τ_1, then i must also immediately leave the race, *regretting not having left earlier.*

The intuition behind this result is simple. When player i plans to leave the race at τ_1, she conditions on player j still being in the race and hence on the expectation $\mathbb{E}[y|y \geqslant g^*_j(\tau_1)]$ with respect to j's signal y. If it happens that j quits first at a time τ close but smaller than τ_1, then i suddenly realizes that j had observed signal $y = g^*_j(\tau)$, which is much smaller than $\mathbb{E}[y|y \geqslant g^*_j(\tau)]$. This induces a sudden pessimistic revision of i's beliefs with respect to the promise of the project. Accordingly, i quits immediately after j, regretting her previous over-optimistic expectation of the (rival's assessment of the) project's feasibility.[15]

Proposition 4 *(Survivor's curse). In any monotonic equilibrium, suppose that a player i with signal x plans to quit first at time $\tau_1 > 0$ solving $dV^i_{1,t}(\tau_1|x)/d\tau_1|_{t=\tau_1}=0$. If the opponent quits first at time $\tau < \tau_1$, and τ is close enough to τ_1, then player i's best response is to immediately follow suit*: $\sigma^*_{2,i}(x,\tau)=\tau$.

Proposition 4 immediately implies that $V_{2,\tau_1}(\sigma^*_{2,i}(x,\tau_1)|x, g^*_j(\tau_1), \tau_1) = 0$, so that we can delete the second line of Eq. (2), and derive the following equilibrium characterization. Recall that $g^*_j = \sigma^{*-1}_{1,j}$ is the inverse of player j's stopping strategy as the first quitter.

Proposition 5 *(First quitter's stopping time). In any monotonic equilibrium, the stopping time of a player i with signal x, conditional on the opponent j still being in the game, is*

$$\sigma^*_{1,i}(x) = \min\left\{\tau_1 \geqslant 0\colon \frac{c_i}{b_i} \geqslant E_{\tau_1,\tau_1}\big[\rho_i(\tau_1|\lambda)|x, g^*_j(\tau_1)+\big]\right\}. \tag{3}$$

This equilibrium stopping time $\sigma^*_{1,i}(x)$ is independent of r, decreases in the cost–benefit ratio c_i/b_i and increases in the private signal x.

Having characterized all possible equilibria, we prove in Appendix A:

Proposition 6 *(Existence of equilibrium). There exists a monotonic differentiable equilibrium $(\sigma^*_{1,i}, \sigma^*_{2,i})$, for $i = A, B$.*

The assumptions guarantee that, given any well-behaved strategy g_j by the opponent, as defined in Eq. (3), the value $V_{1,t}$ of continuing has a unique local and global maximum, a best response that is a continuously differentiable function of own signal x. The bounds on the slope of the expected hazard rate in Assumption 3 are sufficient to avoid multiple local peaks in V_1 and resulting discontinuities in the best response. For example, if $\rho_i(t|\lambda)$ was highly non-monotonic, a small change in x may cause a jump in the best response, as it may become profitable waiting much longer for a later increase in the hazard rate. In addition, the best response operator defined in Eq. (3) is continuous in g_j with respect to the sup norm, and preserves some regularity properties of a candidate strategy. This allows us to apply Schauder's fixed point theorem. Notice that the best response operator is not a contraction, so we cannot apply Banach fixed point theorem to also obtain uniqueness. The reason is the underlying strategic complementarity of the game:

[15] As a mirror image of this survivors's curse, a 'quitter's curse' arises if player j leaves the race at a time τ much earlier than τ_1, the planned stopping time of player i. In this case, player i remains in the race after τ_1, and player j regrets having left.

a very aggressive strategy by the opponent, waiting for the opponent to give up first, may be met by an even more aggressive best response, and so on.

4. Comparison with the symmetric information case

We now study the case where the signals x, y are public information. In equilibrium, the firms run the projects independently and exit when the updated beliefs on λ make remaining in the race unprofitable in expectation. In the following result, we denote as 'firm 1' the first firm that quits the race, and as 'firm 2' the second firm that quits the race.

Proposition 7 *(Public information equilibrium). In any equilibrium of the public information game, for every pair of signals x, y, the first firm quits the race at time*

$$T_1(x, y) = \min\left\{t \geqslant 0\colon\ \frac{c_1}{b_1} \geqslant E_{t,t}\left[\rho_1(t|\lambda)|x, y\right]\right\}$$

and the second firm at the time

$$T_2(x, y) = \min\left\{t \geqslant T_1(x, y)\colon\ \frac{c_2}{b_2} \geqslant E_{t,T_1(x,y)}\left[\rho_2(t|\lambda)|x, y\right]\right\}.$$

The two firms exit sequentially, $T_2(x, y) > T_1(x, y)$, for almost all parameter configurations such that $T_1(x, y) > 0$.

Whether firm A or firm B will be the first firm to quit the race depends on the efficiency of the two firms, expressed by the cost/benefit ratios c_i/b_i and on the shape of the hazard rate functions $\rho_i(\cdot|\lambda)$. The smaller is the cost ratio c_i/b_i and the larger is the hazard rate $\rho_i(\tau|\lambda)$, the more efficient is firm i. In the equilibrium of the public information game, the more efficient firm stays longer in the race.

Proposition 8 *(Exit sequence with public information). If $c_A/b_A > c_B/b_B$ and $\rho_A(\tau|\lambda) < \rho_B(\tau|\lambda)$ for all τ, λ, then firm A quits before firm B in any equilibrium of the public information game, when signals x, y are such that at least one firm enters the game.*

We now compare the monotonic equilibrium in the private information game with the equilibrium of the public information game. In the following discussion we assign signal x to player A and signal y to player B, and restrict attention to the case where $c_A/b_A > c_B/b_B$ and $\rho_A(\tau|\lambda) < \rho_B(\tau|\lambda)$ for all τ, λ, i.e. firm A is less efficient than firm B. For brevity, we only describe results for signals x and y such that both players enter the race, whether the information is private or public. The complete characterization is illustrated in Fig. 1, which is drawn for the parametric example of Section 5, but holds more generally in a qualitative sense.

There are three cases. First, when y is sufficiently larger than x, in the private information game player A exits before player B and regrets having left too early: $\sigma^*_{1,A}(x) < \sigma^*_{1,B}(y)$ and $\sigma^*_{1,A}(x) < \sigma^*_{2,B}(y, \sigma^*_{1,A}(x))$. When quitting at time $\sigma^*_{1,A}(x)$, player A underestimates the opponent's signal y and quits too soon with respect to the public information equilibrium. Second, when x is sufficiently larger than y, player B exits first, unlike in the public information case: $\sigma^*_{1,B}(y) < \sigma^*_{1,A}(x)$ and $\sigma^*_{1,B}(y) < \sigma^*_{2,A}(x, \sigma^*_{1,B}(y))$. Third, and most important, there is a set of signal realizations of positive (unconditional) probability where x, y are such that either $\sigma^*_{1,A}(x) > \sigma^*_{1,B}(y) = \sigma^*_{2,A}(x, \sigma^*_{1,B}(y))$, or $\sigma^*_{1,B}(y) > \sigma^*_{1,A}(x) = \sigma^*_{2,B}(y, \sigma^*_{1,A}(x))$. In this case,

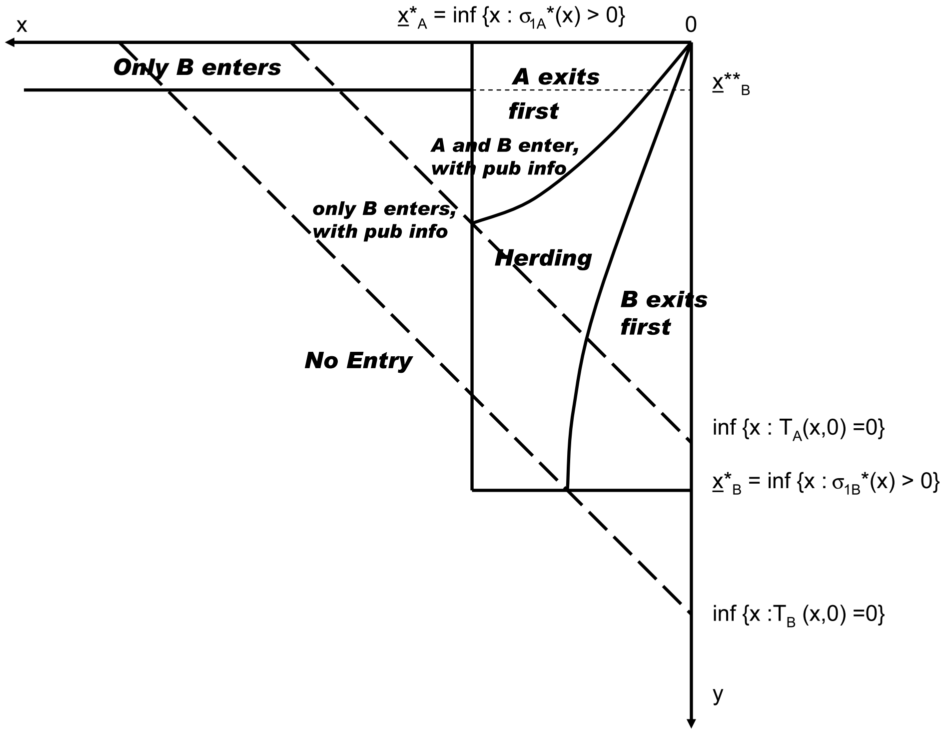

Fig. 1. Equilibrium outcomes in private signal space.

the survivor's curse occurs. The two players exit simultaneously despite the fact that their efficiency in conducting R&D is different. This is a non-generic outcome when information is public, as the least efficient firm will always leave the race earlier. Further, because of the survivor's curse, it must be the case that $\sigma^*_{1,A}(x) + \sigma^*_{2,A}(x, \sigma^*_{1,B}(y)) > T_A(x, y) + T_B(x, y)$, when $\sigma^*_{1,B}(y) > \sigma^*_{1,A}(x)$, and that $\sigma^*_{1,B}(y) + \sigma^*_{2,A}(x, \sigma^*_{1,B}(y)) > T_A(x, y) + T_B(x, y)$ when $\sigma^*_{1,A}(x) > \sigma^*_{1,B}(y)$. The aggregate durations conditional on no prize arrival are longer in the private information game than in the public information game.

The above results highlight the effects of private information on the equilibrium of the R&D race. Most importantly, for some signals (x, y), private information induces simultaneous exit. This possibility is generically ruled out when information is public. This result provides a simple test to identify private information of common value in R&D races. Further, when player A's signal x is sufficiently more optimistic than player B's signal y, the latter exits the race before player A. Again, this possibility is ruled out when information is public, in the case that player A is less efficient than player B.

5. The Gamma-exponential model

To conclude, we present a canonical parametric example to sharpen our characterization of equilibrium.

The arrival time is exponentially distributed, with $F_i(t_i|\lambda) = 1 - e^{-(\zeta_i+\lambda)t_i}$ and $f_i(t_i|\lambda) = (\zeta_i + \lambda)e^{-(\zeta_i+\lambda)t_i}$ for all $t_i \geqslant 0$, and hence the hazard rate of prize arrival is constant: $\rho_i(t_i|\lambda) =$

$\zeta_i + \lambda \geqslant 0$, with $\zeta_i \geqslant 0$ and $\lambda \geqslant 0$. The term ζ_i represents the firm specific component of the arrival rate, i.e. firm i's research efficiency, and is public information. For future reference, we let the adjusted cost/benefit ratio be $q_i = c_i/b_i - \zeta_i$; and without loss of generality assume that $q_A \geqslant q_B$. The $\lambda \geqslant 0$ component is uncertain and distributed according to a gamma distribution $\pi(\lambda) = e^{-\alpha\lambda}\lambda^{\beta-1}\alpha^\beta/\Gamma(\beta)$, for $\alpha > 0, \beta > 0$. Finally, the two private signals z_A and z_B are exponentially distributed: for every $z_i \leqslant 0$, $H(z_i|\lambda) = e^{\lambda z_i}$ with density $h(z_i|\lambda) = \lambda e^{\lambda z_i}$. Derivations available upon request show that this model satisfies Assumptions 1–3, and that Assumption 4 is satisfied whenever $\zeta_i < c_i/b_i < [\beta + 2]/\alpha + \zeta_i$.

The optimality conditions in Propositions 2 and 3 can be solved explicitly. The equilibrium strategy of a player i with signal x after the opponent quits at time $\tau > 0$ is:

$$\sigma^*_{2,i}(x, \tau) = \max\left\{\tau, \frac{1}{q_i}(\beta + 2) + x + g^*_j(\tau) - \alpha - \tau\right\},$$

whereas when the opponent fails to join the game:

$$\sigma^*_{2,i}(x, 0) = \max\left\{0, \frac{1}{q_i}(\beta + 1) + x + \underline{x}^*_j - \alpha\right\}.$$

By specializing Proposition 5 to this example, we show that the strategy of a player i with signal x, conditional on the opponent j being still in the game, is $\sigma^*_{1,i}(x) = \max\{0, \tau_1\}$ where τ_1 is a solution of

$$q_i = (\beta + 1)\frac{[(\alpha + 2\tau_1 - x)^{-\beta-2} - (\alpha + 2\tau_1 - x - g^*_j(\tau_1))^{-\beta-2}]}{[(\alpha + 2\tau_1 - x)^{-\beta-1} - (\alpha + 2\tau_1 - x - g^*_j(\tau_1))^{-\beta-1}]}$$

and, whenever positive, it is the inverse of g^*_i. For $\tau_1 = 0$ this implicit equation pins down the signal realization that makes player i indifferent between entering the race or not, $\underline{x}^*_i = g^*_i(0)$.

It can be shown that this equilibrium is unique (proof available upon request). Hence, in the symmetric Gamma-exponential model where $q_A = q_B$, there do not exist any monotonic asymmetric equilibria.

In the public information case, assume that firm A has a higher adjusted cost ratio $q_A > q_B$. Propositions 7 and 8 imply that firm A quits the race first, at time $T_A(x, y)$, and then firm B quits the race at time $T_B(x, y)$, where

$$T_A(x, y) = \max\left\{0, \frac{1}{2}\left[\frac{1}{q_A}(\beta + 2) + x + y - \alpha\right]\right\},$$
$$T_B(x, y) = \max\left\{0, \frac{1}{q_B}(\beta + 2) + x + y - \alpha - T_A(x, y)\right\}.$$

Inspection of these expressions immediately shows that the public information equilibrium is unique.

To compare the private and public information equilibrium, we focus on the case where both players receive sufficiently favorable private information to join the R&D race, whether the information is private or public, $x + y \geqslant \alpha - \frac{1}{q_A}(\beta + 2)$, $x > \underline{x}^*_A$ and $y > \underline{x}^*_B$. We have three subcases.

First, when y is sufficiently larger than x, player A quits first at time $\sigma^*_{1,A}(x)$, followed by player B at time $\sigma^*_{2,B}(y, \sigma^*_{1,A}(x)) > \sigma^*_{1,A}(x)$, and the aggregate durations conditional on no prize arrival are the same in the private and public information games:

$$\sigma^*_{2,B}\big(y, \sigma^*_1(x)\big) + \sigma^*_{1,A}(x) = \frac{1}{q_B}(\beta + 2) + x + y - \alpha = T_A(x, y) + T_B(x, y).$$

Second, when x is sufficiently larger than y, player B exits first, unlike in the public information case, followed by player A at time $\sigma^*_{2,A}(x, \sigma^*_{1,B}(y)) > \sigma^*_{1,B}(y)$, and the aggregate durations conditional on no prize arrival are smaller with private information than with public information:

$$\sigma^*_{2,B}\big(y, \sigma^*_1(x)\big) + \sigma^*_{1,A}(x) = \frac{1}{q_A}(\beta+2) + x + y - \alpha < \frac{1}{q_B}(\beta+2) + x + y - \alpha$$
$$= T_A(x,y) + T_B(x,y).$$

Finally, for signal realizations (x, y) such that the survivor suffers the curse and follows the opponent suit, aggregate durations conditional on no prize arrival are longer in the private information game than in the public information game.

6. Conclusions

This paper studies a winner-take-all R&D race where firms are privately informed about the arrival rate of the invention. Due to the interdependent-value nature of the problem, the equilibrium displays a strong herding effect. When the opponent is still in the race, each player presumes that her opponent's signal is larger than the signal that would induce exit in the immediate future. Hence, upon seeing that the opponent leaves the race, a player discontinuously revises her belief on the opponent's signal downwards. If the opponent exits at a time close to the time at which the player planned to stop conditional on the opponent being in the race, then the player will exit immediately after the opponent, in regret of not having left before. As a consequence of this survivor's curse, unlike models of symmetric information, firms may exit nearly simultaneously even when their cost and benefits for research differ significantly. This provides a simple test for private information in R&D races.

Appendix A

Proof of Lemma 1. Let p denote the density of the opponent j's signal z conditional on $z_i = x$, on $t_i \leqslant t$ and on $t_j \leqslant t'$. Note that:

$$\mathbb{E}_{t,t'}\big[\rho_i(t|\lambda)|x, y-\big] = \int_{\underline{z}_j}^{y} \frac{\mathbb{E}_{t,t'}[\rho_i(t|\lambda)|x,z]p(z|x,t,t')\,dz}{\int_{\underline{z}_j}^{y} p(z|x,t,t')\,dz},$$

$$\mathbb{E}_{t,t'}\big[\rho_i(t|\lambda)|x, y+\big] = \int_{y}^{\bar{z}_j} \frac{\mathbb{E}_{t,t'}[\rho_i(t|\lambda)|x,z]p(z|x,t,t')\,dz}{\int_{y}^{\bar{z}_j} p(z|x,t,t')\,dz}.$$

The result that $\mathbb{E}^i_{t,t'}[\rho_i(t|\lambda)|x, y-] < \mathbb{E}^i_{t,t'}[\rho_i(t|\lambda)|x, y] < \mathbb{E}^i_{t,t'}[\rho_i(t|\lambda)|x, y+]$ follows immediately from the assumption that $\mathbb{E}_{t,t'}[\rho_i(t|\lambda)|x, z]$ increases in z. □

Proof of Proposition 2. Differentiating $V^i_{2,t}(\tau_2|x, g^*_j(\tau), \tau)$ at $t = \tau_2$ we find:

$$\left.\frac{dV^i_{2,t}(\tau_2|x, g^*_j(\tau), \tau)}{d\tau_2}\right|_{t=\tau_2} = \int_{\Lambda} \frac{d}{d\tau_2} U^i_{2,t}(\tau_2|\lambda)\pi_{t,\tau}\big(\lambda|x, g^*_j(\tau)\big)\,d\lambda|_{t=\tau_2}$$

$$= \int_{\Lambda} \left[\frac{f_i(\tau_2|\lambda)}{1 - F_i(\tau_2|\lambda)} \left(\int_{\tau_2}^{\tau_2} (-c_i) e^{-r(v-\tau_2)}\,dv + e^{-r(\tau_2-\tau_2)} b_i \right) \right.$$

$$- \frac{f_i(\tau_2|\lambda)}{1-F_i(\tau_2|\lambda)} \int_{\tau_2}^{\tau_2} (-c_i) e^{-r(v-\tau_2)}\, dv$$

$$\left. + \frac{1-F_i(\tau_2|\lambda)}{1-F_i(\tau_2|\lambda)} (-c_i) e^{-r(\tau_2-\tau_2)} \right] \pi_{\tau_2,\tau}\big(\lambda|x, g_j^*(\tau)\big)\, d\lambda\, d\lambda$$

$$= b_i \mathbb{E}_{\tau_2,\tau}\left[\frac{f_i(\tau_2|\lambda)}{1-F_i(\tau_2|\lambda)} \middle| x, g_j^*(\tau) \right] - c_i. \qquad \square \tag{4}$$

Proof of Proposition 3. At time t, player i plans to stop at time

$$\tau_{2,t}^i\big(x, g_j^*(0)-\big) = \arg\max_{\tau_2 \geqslant t} \left\{ V_{2,t}^i\big(\tau_2|x, g_j^*(0)-, 0\big) = \int_\Lambda U_{2,t}^i(\tau_2|\lambda) \pi_{t,0}\big(\lambda|z_i = x, z_j \leqslant g_j^*(0)\big)\, d\lambda \right\}.$$

Proceeding as in Eq. (4), we obtain:

$$\left. \frac{dV_{2,t}^i(\tau_2|x, g_j^*(0)-, 0)}{d\tau_2} \right|_{t=\tau_2} = b_i \mathbb{E}_{\tau_2,0}\left[\frac{f_i(\tau_2|\lambda)}{1-F_i(\tau_2|\lambda)} \middle| x, g_j^*(0)- \right] - c_i. \qquad \square \tag{5}$$

Calculations leading to expression (2). The value function of a player $i = A, B$ at any time $t > 0$ for quitting at time $\tau_1 \geqslant t$, conditional on the facts that opponent j has not quit yet at time τ_1 and is adopting a monotonic strategy $\sigma_{1,j}^*$, with associated inverse g_j^*, is as follows:

$$V_{1,t}^i(\tau_1|x) = \int_\Lambda U_{1,t}^i(\tau_1|\lambda) \pi_{t,t}\big(\lambda|x, g_j^*(t)+\big)\, d\lambda,$$

where $U_{1,t}^i(\tau_1|\lambda)$ denotes the expected value at time t for planning to stop at time $\tau_1 > t$, conditional on the opponent being still in the game at time τ_1, and conditional on λ. Specifically:

$$U_{1,t}^i(\tau_1|\lambda) = \int_t^{\tau_1} \frac{f_i(s|\lambda)}{1-F_i(t|\lambda)} \frac{1-F_j(s|\lambda)}{1-F_j(t|\lambda)} \frac{1-H_j(g_j^*(s)|\lambda)}{1-H_j(g_j^*(t)|\lambda)}$$

$$\times \left[\int_t^s -c_i e^{-r(v-t)}\, dv + e^{-r(s-t)} b_i \right] ds$$

$$+ \int_t^{\tau_1} \frac{f_j(s|\lambda)}{1-F_j(t|\lambda)} \frac{1-F_i(s|\lambda)}{1-F_i(t|\lambda)} \frac{1-H_j(g_j^*(s)|\lambda)}{1-H_j(g_j^*(t)|\lambda)} \left[\int_t^s -c_i e^{-r(v-t)}\, dv \right] ds$$

$$+ \int_t^{\tau_1} \left(\frac{1-F_i(s|\lambda)}{1-F_i(t|\lambda)} \right) \left(\frac{1-F_j(s|\lambda)}{1-F_j(t|\lambda)} \right) \frac{h_j(g_j^*(s)|\lambda)\, dg_j^*(s)/ds}{1-H_j(g_j^*(t)|\lambda)}$$

$$\times \left[\int_t^s -c_i e^{-r(v-t)}\, dv + e^{-r(s-t)} V_{2,s}^i\big(\sigma_{2,i}^*(x,s)|x, g_j^*(s), s\big) \right] ds$$

$$+\left(\frac{1-F_i(\tau_1|\lambda)}{1-F_i(t|\lambda)}\right)\left(\frac{1-F_j(\tau_1|\lambda)}{1-F_j(t|\lambda)}\right)\frac{1-H_j(g_j^*(\tau_1)|\lambda)}{1-H_j(g_j^*(t)|\lambda)}\int_t^{\tau_1}-c_i e^{-r(v-t)}\,dv.$$

The first line expresses the possibility that player i's prize arrives at $t_i \in [t,\tau_1)$, before the prize arrives to the rival and before the opponent quits. In this case, player i wins the race, pays costs up to that time t_i and collects the prize b. The second line illustrates the case when player j's prize arrives at $t_j \in [t,\tau_1)$, before i's prize arrives and before i quits. As a result, j wins the race at time t_j and player i just pays costs. Third, player j quits at $\tau \in [t,\tau_1)$ before either prize arrives. Then the signal y is revealed to i by inverting $y = g_j^*(s)$. Player i pays costs and collects $V_{2,s}(\sigma_2^*(x,s)|x, g_j^*(s), s)$, the continuation value of going on alone optimally. Fourth and last, nothing happens in the time interval $[t,\tau_1)$: no one quits and no prize arrives. In this case player i quits at τ_1 and just pays costs.

Differentiating the value $V_{1,t}^i(\tau_1|x)$ with respect to τ_1, we obtain:

$$\begin{aligned}
\frac{d}{d\tau_1}V_{1,t}^i(\tau_1|x) = \int_\Lambda \Bigg[& \frac{f_i(\tau_1|\lambda)}{1-F_i(t|\lambda)}\frac{1-F_j(\tau_1|\lambda)}{1-F_j(t|\lambda)}\frac{1-H_j(g_j^*(\tau_1)|\lambda)}{1-H_j(g_j^*(t)|\lambda)} \\
& \times\left[\int_t^{\tau_1}-c_i e^{-r(v-t)}\,dv + e^{-r(\tau_1-t)}b_i\right] \\
& + \frac{f_i(\tau_1|\lambda)}{1-F_i(t|\lambda)}\frac{1-F_j(\tau_1|\lambda)}{1-F_j(t|\lambda)}\frac{1-H_j(g_j^*(\tau_1)|\lambda)}{1-H_j(g_j^*(t)|\lambda)}\left[\int_t^{\tau_1}-c_i e^{-r(v-t)}\,dv\right] \\
& + \left(\frac{1-F_i(\tau_1|\lambda)}{1-F_i(t|\lambda)}\right)\left(\frac{1-F_j(\tau_1|\lambda)}{1-F_j(t|\lambda)}\right)\frac{h_j(g_j^*(\tau_1)|\lambda)\,dg_j^*(\tau_1)/d\tau_1}{1-H_j(g_j^*(t)|\lambda)} \\
& \times\left[\int_t^{\tau_1}-c_i e^{-r(v-t)}\,dv + e^{-r(\tau_1-t)}V_{2,\tau_1}^i\big(\sigma_{2,i}^*(x,\tau_1)|x, g_j^*(\tau_1),\tau_1\big)\right] \\
& - \left(\frac{f_i(\tau_1|\lambda)}{1-F_i(t|\lambda)}\right)\left(\frac{1-F_j(\tau_1|\lambda)}{1-F_j(t|\lambda)}\right)\frac{1-H_j(g_j^*(\tau_1)|\lambda)}{1-H_j(g_j^*(t)|\lambda)}\int_t^{\tau_1}-c_i e^{-r(v-t)}\,dv \\
& - \left(\frac{1-F_i(\tau_1|\lambda)}{1-F_i(t|\lambda)}\right)\left(\frac{f_j(\tau_1|\lambda)}{1-F_j(t|\lambda)}\right)\frac{1-H_j(g_j^*(\tau_1)|\lambda)}{1-H_j(g_j^*(t)|\lambda)}\int_t^{\tau_1}-c_i e^{-r(v-t)}\,dv \\
& - \left(\frac{1-F_i(\tau_1|\lambda)}{1-F_i(t|\lambda)}\right)\left(\frac{1-F_j(\tau_1|\lambda)}{1-F_j(t|\lambda)}\right) \\
& \times\frac{h_j(g_j^*(\tau_1)|\lambda)\,dg_j^*(\tau_1)/d\tau_1}{1-H_j(g_j^*(t)|\lambda)}\int_t^{\tau_1}-c_i e^{-r(v-t)}\,dv \\
& - \left(\frac{1-F_i(\tau_1|\lambda)}{1-F_i(t|\lambda)}\right)\left(\frac{1-F_j(\tau_1|\lambda)}{1-F_j(t|\lambda)}\right) \\
& \times\frac{1-H_j(g_j^*(\tau_1)|\lambda)}{1-H_j(g_j^*(t)|\lambda)}c_i e^{-r(\tau_1-t)}\Bigg]\pi_{t,t}\big(\lambda|x, g_j^*(t)+\big)\,d\lambda.
\end{aligned}$$

Simplifying and specializing the expression at $t = \tau_1$ by the requirement of time consistency gives:

$$\begin{aligned}\frac{d}{d\tau_1} V^i_{1,t}(\tau_1|x)|_{t=\tau_1} = \int_\Lambda \bigg[& \frac{f_i(\tau_1|\lambda)}{1-F_i(\tau_1|\lambda)} \frac{1-F_j(\tau_1|\lambda)}{1-F_j(\tau_1|\lambda)} \frac{1-H_j(g^*_j(\tau_1)|\lambda)}{1-H_j(g^*_j(\tau_1)|\lambda)} b_i \\ & + \left(\frac{1-F_i(\tau_1|\lambda)}{1-F_i(\tau_1|\lambda)}\right)\left(\frac{1-F_j(\tau_1|\lambda)}{1-F_j(\tau_1|\lambda)}\right) \frac{h_j(g^*_j(\tau_1)|\lambda)\, dg^*_j(\tau_1)/d\tau_1}{1-H_j(g^*_j(\tau_1)|\lambda)} \\ & \times V^i_{2,\tau_1}\big(\sigma^*_{2,i}(x,\tau_1)|x, g^*_j(\tau_1), \tau_1\big) \\ & - \left(\frac{1-F_i(\tau_1|\lambda)}{1-F_i(\tau_1|\lambda)}\right)\left(\frac{1-F_j(\tau_1|\lambda)}{1-F_j(\tau_1|\lambda)}\right) \frac{1-H_j(g^*_j(\tau_1)|\lambda)}{1-H_j(g^*_j(\tau_1)|\lambda)} c_i \bigg] \\ & \times \pi_{\tau_1,\tau_1}\big(\lambda|x, g^*_j(\tau_1)+\big)\, d\lambda.\end{aligned}$$

We further simplify to obtain Eq. (2). □

Proof of Proposition 4. By definition of $\sigma^*_{2,i}(x,\tau)$ and by continuity, we only need to show that $c_i > b_i \mathbb{E}_{\tau_1,\tau_1}[\rho_i(\tau_1|\lambda)|x, g^*_j(\tau_1)]$. Since $\tau_1 > 0$ solves the first-order condition $dV^i_{1,t}(\tau_1|x)/d\tau_1|_{t=\tau_1} = 0$, using Eq. (2), we obtain:

$$\begin{aligned} 0 &= b_i \mathbb{E}_{\tau_1,\tau_1}\big[\rho_i(\tau_1|\lambda)|x, g^*_j(\tau_1)+\big] - c_i \\ &\quad + \mathbb{E}_{\tau_1,\tau_1}\left[\frac{h_j(g^*_j(\tau_1)|\lambda)}{1-H_j(g^*_j(\tau_1)|\lambda)} \frac{dg^*_j(\tau_1)}{d\tau_1}\bigg| x, g^*_j(\tau_1)+\right] V^i_{2,\tau_1}\big(\sigma^*_2(x,\tau_1)|x, g^*_j(\tau_1), \tau_1\big) \\ &\geqslant b_i \mathbb{E}_{\tau_1,\tau_1}\big[\rho_i(\tau_1|\lambda)|x, g^*_j(\tau_1)+\big] - c_i > b_i \mathbb{E}_{\tau_1,\tau_1}\big[\rho_i(\tau_1|\lambda)|x, g^*_j(\tau_1)\big] - c_i, \end{aligned}$$

where the first inequality follows because $\frac{h_j(g^*_j(\tau_1)|\lambda)}{1-H_j(g^*_j(\tau_1)|\lambda)} \frac{dg^*_j(\tau_1)}{d\tau_1} > 0$ and $V^i_{2,\tau_1}(\sigma^*_2(x,\tau_1)|x, g^*_j(\tau_1), \tau_1) \geqslant 0$, whereas the second inequality follows from Lemma 1: knowing that the opponent j's signal is exactly $g^*_j(\tau_1)$ is bad news with respect to knowing that it is at least $g^*_j(\tau_1)$. □

Proof of Proposition 5. In light of Proposition 4, for any x, the first-order condition $dV^i_{1,t}(\tau_1|x)/d\tau_1|_{t=\tau_1} = 0$ can be rewritten simply as:

$$c_i = b_i \mathbb{E}_{\tau_1,\tau_1}\big[\rho_i(\tau_1|\lambda)|x, g^*_j(\tau_1)+\big]. \tag{6}$$

The following lemma establishes two "corner properties" of any equilibrium strategy $\sigma^*_{1,i}(x)$. As a result, the equilibrium strategy $\sigma^*_{1,i}(x)$ must satisfy the first-order condition, Eq. (6), whenever $x \geqslant \underline{x}^*_i$ and must equal zero when $x < \underline{x}^*_i$.

Lemma A.1. *Suppose that player j plays a monotonic strategy $\sigma^*_{1,j}$ with inverse g^*_j. For any signal realization x observed by player i, there exists a time $\bar{\tau} > 0$ large enough that player i's marginal value of waiting $dV^i_{1,t}(\tau_1|x)/d\tau_1|_{t=\tau_1}$ is negative for any $\tau_1 \geqslant \bar{\tau}$. For any signal realization x such that $b_i E_{0,0}[\rho_i(0|\lambda)|x, g^*_j(0)+] > c_i$, there exists $\underline{\tau} > 0$ small enough that $dV^i_{1,t}(\tau_1|x)/d\tau_1|_{t=\tau_1} > 0$ for any $\tau_1 \leqslant \underline{\tau}$.*

Proof. To prove the first result, note that for τ_1 sufficiently large,

$$b_i \mathbb{E}_{\tau_1,\tau_1}\big[\rho_i(\tau_1|\lambda)|x, g^*_j(\tau_1)\big] - c_i \leqslant b_i \mathbb{E}_{\tau_1,\tau_1}\big[\rho_i(\tau_1|\lambda)|x, 0\big] - c_i < 0$$

because $\mathbb{E}_{\tau_1,\tau_1}[\rho_i(\tau_1|\lambda)|x,y] < c_i/b_i$ as $\tau_1 \to \infty$, for all x, y, by Assumption 4. So, $\sigma^*_{2,i}(x,\tau_1) = \tau_1$ and hence $V^i_{2,\tau_1}(\sigma^*_{2,i}(x,\tau_1)|x, g^*_j(\tau_1), \tau_1) = 0$. It follows that

$$\frac{d}{d\tau_1}V^i_{1,t}(\tau_1|x)\bigg|_{t=\tau_1} = b_i\mathbb{E}_{\tau_1,\tau_1}\big[\rho_i(\tau_1|\lambda)|x, g^*_j(\tau_1)+\big] - c_i$$
$$\leqslant b_i\mathbb{E}_{\tau_1,\tau_1}\big[\rho_i(\tau_1|\lambda)|x, 0+\big] - c_i < 0$$

for τ_1 large enough by Assumptions 2 and 4 combined.

The second result follows from

$$\frac{d}{d\tau_1}V^i_{1,t}(\tau_1|x)\bigg|_{t=\tau_1} \geqslant b_i\mathbb{E}_{\tau_1,\tau_1}\big[\rho_i(\tau_1|\lambda)|x, g^*_j(\tau_1)+\big] - c_i$$

together with continuity of $\mathbb{E}_{\tau_1,\tau_1}[\rho_i(\tau_1|\lambda)|x, g^*_j(\tau_1)+]$. □

To conclude that this strategy $\sigma^*_{1,i}$ identified by the first-order condition, Eq. (6), induces a monotonic equilibrium, we are only left to determine the optimal decision at the very beginning of the game, i.e. at time $t = 0$. The next lemma verifies that if the opponent j enters the game whenever $x > \underline{x}^*_j$, then it is optimal to enter the race if and only if $x > \underline{x}^*_i$.

Lemma A.2. *Suppose that player j plays the first-quitter stopping strategy $\sigma^*_{1,j}$. If player i holds a signal $x \leqslant \underline{x}^*_i$, then at time 0 she optimally chooses not to enter the game. Whereas if $x > \underline{x}^*_i$, then player i enters the game at time 0, and optimally selects the stopping time $\sigma^*_{1,i}(x)$.*

Proof. Consider the choice at time $t = 0$ of player i with a signal x. If she chooses not to enter the game and set $\tau_1 = 0$, her payoff is $V^i_{1,0}(0|x) = 0$. If she chooses to enter the game, and sets $\tau_1 > 0$, then she will observe whether j enters the game or not. This allows us to write i's expected payoff for playing any $\tau_1 > 0$ as

$$V^i_{1,0}(\tau_1|x) = \Pr\big(z_j \leqslant \underline{x}^*_j|x\big)\lim_{t\downarrow 0} V^i_{2,t}\big(\sigma^*_{2,i}(x,0)|x, \underline{x}^*_j-, 0\big)$$
$$+ \big[1 - \Pr(z_j \leqslant \underline{x}^*_j|x)\big]\lim_{t\downarrow 0} V^i_{1,t}(\tau_1|x).$$

Suppose first that x is such that $b_i E_{0,0}[\rho_i(0|\lambda)|x, \underline{x}^*_j+] \leqslant c_i$; and hence that the equilibrium prescription is $\sigma^*_{1,i}(x) = 0$. This implies that for any $\tau_1 > 0$, $dV^i_{1,t}(\tau_1|x)/d\tau_1|_{t=\tau_1} < 0$. Since for all $\tau_1 > 0$,

$$\mathbb{E}_{\tau_1,\tau_1}\big[\rho_i(0|\lambda)|x, \underline{x}^*_j-\big] < \mathbb{E}_{\tau_1,\tau_1}\big[\rho_i(0|\lambda)|x, \underline{x}^*_j\big] < \mathbb{E}_{\tau_1,\tau_1}\big[\rho_i(0|\lambda)|x, \underline{x}^*_j+\big],$$

it follows that $\sigma^*_{2,i}(x,0) = 0$ by Proposition 3, and hence $\lim_{t\downarrow 0} V^i_{2,t}(\sigma^*_{2,i}(x,0)|x, \underline{x}^*_j-, 0) = 0$. So player i optimally chooses to follow the equilibrium prescription $\sigma^*_{1,i}(x) = 0$.

Second, suppose that x is such that $b_i E_{0,0}[\lambda|x, \underline{x}^*_j+] > c_i$; the player will comply with the equilibrium prescription $\sigma^*_{1,i}(x) > 0$ because $\lim_{t\downarrow 0} V^i_{2,t}(\sigma^*_{2,i}(x,0)|x, \underline{x}^*_j-, 0) \geqslant 0$ and for any τ_1 small enough $\lim_{\tau_1\downarrow 0} dV^i_{1,t}(\tau_1|x)/d\tau_1|_{t=\tau_1} > 0$. □

Proof of Proposition 6. To ease notation, for all $t \geqslant 0$, x, y, $i = A, B$, we introduce the function

$$\mu_i(t,x,y) := E^i_{t,t}\big[\rho_i(t|\lambda)|x, y+\big] - \frac{c_i}{b_i}.$$

The existence of equilibrium strategies $\sigma^*_{2,i}(x,\tau)$ is not an issue. We now show that there exist strategies $\sigma^*_{1,i}(x) \geqslant 0$, $i = A, B$, that satisfy Eq. (3), i.e. such that

$$\mu_i\big(\sigma^*_{1,i}(x), x, {\sigma^*_{1,j}}^{-1}\big(\sigma^*_{1,i}(x)\big)\big) \leqslant 0$$

for $i \neq j = A, B$, with equality if the equation has a positive solution $\sigma^*_{1,i}(x) > 0$. Let the Q operator be the best-response correspondence: if player i plays $\sigma_{1,i}$, player j replies with $\tau_{1,j}$:

$$\begin{pmatrix}\tau_{1,A}\\ \tau_{1,B}\end{pmatrix} = Q\begin{pmatrix}\sigma_{1,A}\\ \sigma_{1,B}\end{pmatrix}.$$

An equilibrium is a fixed point of Q. We show it exists.

Let $K_{\overline{G}}$ be the set of differentiable functions mapping R_+ into itself, that are uniformly bounded below and above and have derivative that are uniformly bounded below by zero and above by $\overline{G} > 0$. Given these properties, by the Arzelà–Ascoli theorem $K_{\overline{G}}$ is compact. Because convex combinations preserve the properties that define $K_{\overline{G}}$ (i.e., differentiability, uniform boundedness below and above, and uniform boundedness below and above of the derivative), $K_{\overline{G}}$ is also convex. A representative member of $K_{\overline{G}}$ is an inverse strategy g_j, with the interpretation that if player j quits first at time t his private signal realization is $g_j(t)$. We aim to show that the best response operator Q maps strategies that are the inverse of functions in $K_{\overline{G}}$ into the same set.

First, a first-order condition for a best response to $g_j(\cdot)$ is $t = \tau_{1,i}(x)$ where

$$\mu_i\big(t, x, g_j(t)\big) = 0. \tag{7}$$

Let $\mu_i^{(k)}$, $k = 1, 2, 3$, denote the partial derivative of μ_i w.r. to its k-th argument, so that Assumption 3 is equivalent to

$$\mu_i^{(3)}\big(t, x, g_j(t)\big) < \overline{G}'\mu_i^{(1)}(t, x, y) < \mu_i^{(2)}(t, x, y)$$

for some $\overline{G}' < 0$. Let $\overline{G} = -1/\overline{G}' > 0$. For every $g_j \in K_{\overline{G}}$, which has $g_j'(t) < \overline{G}$, the following inequality:

$$\mu_i^{(1)}\big(t, x, g_j(t)\big) + \mu_i^{(3)}\big(t, x, g_j(t)\big)g_j'(t) < 0 \tag{8}$$

is always satisfied given Assumption 3 and $\mu_i^{(3)} > 0$. So μ_i is decreasing in t near every solution $t = \tau_{1,i}(x)$ to Eq. (7). By Assumptions 3 and 4, Eq. (7) then has one and only one solution, which is both a local and a global maximum of the value $V_{1,t}$ of being the first to quit. This implies that there exists a unique best response $\tau_{1,i}(x) \geqslant 0$, which is a continuous function of x, and that Q is a continuous functional: a perturbation of g_j in the sup norm can only change τ_i continuously in the same norm.

By the implicit function theorem, the properties of μ_i, and the strict inequality in Eq. (8), the best response $\tau_{1,i}(\cdot)$ is also differentiable and strictly increasing. We showed that the inverse of the best response is uniformly bounded below by the lowest signal realization that makes any player ever enter the race, and above by the upper bound to the signal space.

To establish that the inverse of $\tau_{1,i}$ is in the set $K_{\overline{G}}$ we need to show that this inverse best response has derivative bounded above by $\overline{G}$. Differentiating the first order condition, we require

$$\overline{G} > \frac{d\tau_{1,i}^{-1}(t)}{dt} = \frac{1}{\tau_{1,i}'(\tau_{1,i}^{-1}(t))} = -\frac{\mu_i^{(1)}(t, \tau_{1,i}^{-1}(t), g_j(t)) + \mu_i^{(3)}(t, \tau_{1,i}^{-1}(t), g_j(t))g_j'(t)}{\mu_i^{(2)}(t, \tau_{1,i}^{-1}(t), g_j(t))}.$$

As $\mu_i^{(2)}(t, \tau_{1,i}^{-1}(t), g_j(t)) > 0$, this chain of (in)equalities is equivalent to

$$\overline{G}\mu_i^{(2)}\big(t, \tau_{1,i}^{-1}(t), g_j(t)\big) > -\mu_i^{(1)}\big(t, \tau_{1,i}^{-1}(t), g_j(t)\big) - \mu_i^{(3)}\big(t, \tau_{1,i}^{-1}(t), g_j(t)\big)g_j'(t).$$

As $g_j'(t) > 0$ and $\mu_i^{(3)} > 0$, this is true under the second part of Assumption 3.

Finally, we have to show that the map Q preserves convexity of $K_{\overline{G}}$. Namely, for all $\alpha \in [0, 1]$ the inverse of the function $\tau_{1,i}^{\alpha}(x)$ uniquely defined by

$$\mu_i\big(\tau_{1,i}^{\alpha}(x), x, \alpha g_j\big(\tau_{1,i}^{\alpha}(x)\big) + (1-\alpha)n_j\big(\tau_{1,i}^{\alpha}(x)\big)\big) = 0$$

is also in $K_{\overline{G}}$: differentiable (obvious), increasing (obvious), bounded below and above (obvious), and with derivative bounded above by $\overline{G}$, that is

$$\begin{aligned}\overline{G} > \frac{d(\tau_i^{\alpha})^{-1}(t)}{dt} &= \frac{1}{\tau_{1,i}^{\alpha\prime}((\tau_{1,i}^{\alpha})^{-1}(t))} \\ &= -\frac{\mu_i^{(1)}(t, (\tau_{1,i}^{\alpha})^{-1}(t), \alpha g_j(t) + (1-\alpha)n_j(t))}{\mu_i^{(2)}(t, (\tau_{1,i}^{\alpha})^{-1}, \alpha g_j(t) + (1-\alpha)n_j(t))} \\ &\quad - \frac{\mu_i^{(3)}(t, (\tau_{1,i}^{\alpha})^{-1}, \alpha g_j(t) + (1-\alpha)n_j(t))[\alpha g_j'(t) + (1-\alpha)n_j'(t)]}{\mu_i^{(2)}(t, (\tau_{1,i}^{\alpha})^{-1}, \alpha g_j(t) + (1-\alpha)n_j(t))}.\end{aligned}$$

Rearranging, this is true again under Assumption 3.

Hence $Q : K_{\overline{G}} \to K_{\overline{G}}$ where Q is a continuous map and $K_{\overline{G}}$ is a compact, convex subset of the complete metric space of continuous functions on a bounded set endowed with the sup norm. By Schauder's fixed point theorem an equilibrium exists and the inverse strategies are in $K_{\overline{G}}$. □

Proof of Proposition 7. We want to determine the equilibrium strategy $T_i(x, y)$ of player i. Suppose that the opponent, player j stops at time T. If player i quits at any time $T_1 \leqslant T$, the conditional value evaluated at any time $t > 0$, is

$$\begin{aligned}U_t^i(T_1|\lambda) = {} & \int_t^{T_1} \frac{f_i(s|\lambda)}{1-F_i(t|\lambda)} \frac{1-F_j(s|\lambda)}{1-F_j(t|\lambda)} \left[\int_t^s -c_i e^{-r(v-t)}\,dv + e^{-r(s-t)}b_i\right] ds \\ & + \int_t^{T_1} \frac{f_j(s|\lambda)}{1-F_j(t|\lambda)} \frac{1-F_i(s|\lambda)}{1-F_i(t|\lambda)} \left[\int_t^s -c_i e^{-r(v-t)}\,dv\right] ds \\ & + \left(\frac{1-F_i(T_1|\lambda)}{1-F_i(t|\lambda)}\right)\left(\frac{1-F_j(T_1|\lambda)}{1-F_j(t|\lambda)}\right) \int_t^{T_1} -c_i e^{-r(v-t)}\,dv.\end{aligned}$$

Taking an expectation w.r. to the posterior beliefs $\pi_{t,t}(\lambda|x, y)$ that neither prize has arrived by time T_1 and signals are x, y, we study the derivative with respect to T_1, as long as $T_1 \leqslant T$, we obtain:

$$V_t^i(T_1|x, y) = \int_{\Lambda} U_{i,t}(T_1|\lambda)\pi_{t,t}(\lambda|x, y)\,d\lambda.$$

Hence the first-order condition is:

$$\frac{dV_t^i(T_1|x,y)}{dT_1} = \int_\Lambda \Bigg[\frac{f_i(T_1|\lambda)}{1-F_i(t|\lambda)} \frac{1-F_j(T_1|\lambda)}{1-F_j(t|\lambda)} \Bigg[\int_t^{\tau_1} -c_i e^{-r(v-t)}\,dv + e^{-r(T_1-t)} b_i \Bigg]$$
$$- \frac{f_j(T_1|\lambda)}{1-F_j(t|\lambda)} \left(\frac{1-F_i(T_1|\lambda)}{1-F_i(t|\lambda)} \right) \int_t^{\tau_1} -c_i e^{-r(v-t)}\,dv$$
$$- \left(\frac{1-F_i(T_1|\lambda)}{1-F_i(t|\lambda)} \right) \left(\frac{1-F_j(T_1|\lambda)}{1-F_j(t|\lambda)} \right) c_i e^{-r(T_1-t)} \Bigg] \pi_{t,t}(\lambda|x,y)\,d\lambda.$$

Time consistency requires that we set $t = T_1$ at the maximum. Hence,

$$\left.\frac{dV_t^i(T_1|x,y)}{dT_1}\right|_{t=T_1} = \int_\Lambda \left[\frac{f_i(T_1|\lambda)}{1-F_i(T_1|\lambda)} b_i - c_i e^{-r(T_1-t)} \right] \pi_{T_1,T_1}(\lambda|x,y)\,d\lambda$$
$$= b_i \mathbb{E}_{T_1,T_1}\big[\rho_i(T_1|\lambda)|x,y\big] - c_i. \tag{9}$$

By Assumption 2, this quantity is strictly decreasing at T_1. Hence, first and second order conditions imply that the above equation identifies the unique maximum over the range $T_1 \in [0, T]$.

The value of stopping at any time $T_1 > T$, evaluated at time t is:

$$U_t^i(T_1|\lambda) = \int_t^T \frac{f_i(s|\lambda)}{1-F_i(t|\lambda)} \frac{1-F_j(s|\lambda)}{1-F_j(t|\lambda)} \Bigg[\int_t^s -c_i e^{-r(v-t)}\,dv + e^{-r(s-t)} b_i \Bigg] ds$$
$$+ \int_t^T \frac{f_i(s|\lambda)}{1-F_i(t|\lambda)} \frac{1-F_j(s|\lambda)}{1-F_j(t|\lambda)} \Bigg[\int_t^s -c_i e^{-r(v-t)}\,dv \Bigg] ds$$
$$+ \int_\tau^{T_1} \frac{f_i(s|\lambda)}{1-F_i(t|\lambda)} \frac{1-F_j(\tau|\lambda)}{1-F_j(t|\lambda)} \Bigg[\int_t^s -c_i e^{-r(v-t)}\,dv + e^{-r(s-t)} b_i \Bigg] ds$$
$$+ \frac{1-F_i(T|\lambda)}{1-F_i(t|\lambda)} \frac{1-F_j(T_1|\lambda)}{1-F_j(t|\lambda)} \int_t^{T_1} -c_i e^{-r(v-t)}\,dv.$$

Taking an expectation w.r. to the posterior beliefs $\pi_{t,t}(\lambda|x,y)$, and differentiating with respect to T_1, we obtain:

$$\frac{dV_t^i(T_1|x,y)}{dT_1} = \int_\Lambda \Bigg[\frac{f_i(T_1|\lambda)}{1-F_i(t|\lambda)} \frac{1-F_j(T|\lambda)}{1-F_j(t|\lambda)} \Bigg[\int_t^{T_1} -c_i e^{-r(v-t)}\,dv + e^{-r(\tau_1-t)} b_i \Bigg]$$
$$- \frac{f_i(T_1|\lambda)}{1-F_i(t|\lambda)} \frac{1-F_j(T|\lambda)}{1-F_j(t|\lambda)} \int_t^{T_1} -c_i e^{-r(v-t)}\,dv$$
$$- \frac{1-F_i(T_1|\lambda)}{1-F_i(t|\lambda)} \frac{1-F_j(T|\lambda)}{1-F_j(t|\lambda)} c_i e^{-r(T_1-t)} \Bigg] \pi_{t,t}(\lambda|x,y)\,d\lambda.$$

Simplifying $[1 - F_j(T|\lambda)]/[1 - F_j(t|\lambda)]$ with the expression for $\pi_{t,t}(\lambda|x, y)$, we obtain:

$$\frac{dV_t^i(T_1|x, y)}{dT_1} = \int_\Lambda \left[\frac{f_i(T_1|\lambda)}{1 - F_i(t|\lambda)} \left[\int_t^{T_1} -c_i e^{-r(v-t)}\, dv + e^{-r(T_1-t)} b_i \right] - \frac{f_i(T_1|\lambda)}{1 - F_i(t|\lambda)} \times \int_t^{T_1} -c_i e^{-r(v-t)}\, dv - \frac{1 - F_i(T_1|\lambda)}{1 - F_i(t|\lambda)} c_i e^{-r(T_1-t)} \right] \pi_{t,T}(\lambda|x, y)\, d\lambda;$$

substituting t with T_1, we obtain:

$$\left. \frac{dV_t^i(T_1|x, y)}{dT_1} \right|_{t=T_1} = \int_\Lambda \left[\frac{f_i(T_1|\lambda)}{1 - F_i(T_1|\lambda)} b_i - c_i \right] \pi_{T_1,T}(\lambda|x, y)\, d\lambda = b_i \mathbb{E}_{T_1,T}\big[\rho_i(T_1|\lambda)|x, y\big] - c_i. \tag{10}$$

By Assumption 2, this quantity is strictly decreasing T_1. Hence, first and second order conditions imply that the above equation identifies the unique maximum over the range $T_1 \in [T, +\infty)$. □

Proof of Proposition 8. By contradiction, suppose $T_A(x, y) \geqslant T_B(x, y)$. Then, omitting the arguments of T_A, T_B,

$$E_{T_B,T_B}\big[\rho_B(T_B|\lambda)|x, y\big] \geqslant E_{T_A,T_B}\big[\rho_B(T_A|\lambda)|x, y\big] > E_{T_A,T_B}\big[\rho_A(T_A|\lambda)|x, y\big] = c_A/b_A > c_B/b_B$$

where the first inequality uses $T_A \geqslant T_B$ and $E_{t,T_B}[\rho_B(t|\lambda)|x, y]$ decreasing in t by Assumption 2, the second uses the assumption $\rho_A(\tau|\lambda) < \rho_B(\tau|\lambda)$, the equality uses the optimality of T_A and $T_B > 0$ to avoid the trivial case of no entry, the last one uses the assumption $c_A/b_A > c_B/b_B$. But $E_{T_B,T_B}[\rho_B(T_A|\lambda)|x, y] > c_B/b_B$ contradicts the optimality of T_B. □

Appendix B. Supplementary material

The online version of this article contains additional supplementary material.
Please visit doi:10.1016/j.jet.2009.12.001.

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