Information Transmission in Political Networks*

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Abstract

Motivated by political economy applications such as networks of policy-makers, interest groups, or judges, I formulate and study a model of information transmission in networks of ideologically differentiated agents. When all agents' ideologies are sufficiently diverse, the optimal network is the line in which the agents are ordered according to their ideologies. When agents are partitioned in ideologically diverse clusters, each composed of agents with similar views, it is optimal for all agents that the clusters organize as factions: stars whose only links are with ideologically close clusters through star centers (the faction leaders). Such optimal networks obtain as Nash equilibria of a game in which each link requires sponsorship by both connected agents, and are the unique strongly pairwise stable networks. These results suggest positive and normative rationales for "horizontal" links between like-minded agents in political networks, as opposed to hierarchical networks such as the star, that have been shown to prevail in organizations where agents' preferences are more closely aligned.

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1 Introduction

Social connections and networks are understood to be an important, if not fundamental, feature of political economy. Several empirical studies investigate networks of political agents (e.g., cooperative webs of policy makers, ties among interest groups and policy stakeholders, and personal connections among judges).[1] Yet, theoretical analysis of political networks is still underdeveloped, despite ample network research often motivated by the study of firms and other organizations. One main established function of political networks is to facilitate the flow of information (Victor, Montgomery and Lubell, 2017).[2] I study information transmission in networks of political agents, and investigate political network optimality and endogeneous stability. My results are in line with stylized facts, and suggest a normative rationale for often observed political network patterns.

The information transmission model I study comprises a set of agents linked in a network. One agent is called at random to make a policy decision, identified as a point on the real line (the left-right spectrum). All agents care about every policy decisions, possibly to a different extent. Each would like a decision closer to her ideal policy, which depends on an unknown state of the world and on an individual ideal preference. Hence, all agents prefer policies informed about the state. But given information, each would like to push policy towards her ideological preference.

Agents may hold information correlated with the state of the world, and hence relevant for the policy choice, that they can transmit through the links in the network. At least some of this information is verifiable. An agent's choice is whether to relay it to her linked peers, or withhold it from some or all of them. With small probability, information may decay in transmission. There are as many rounds of communication as needed for all information to have the possibility to travel to every possible decision maker in the network. Then the policy choice is made. The solution concept is perfect Bayesian equilibrium.

Network links are costly, and formed ex-ante (i.e., before agents receive any information from nature). I calculate both the networks that maximize the sum of the agents' ex-ante equilibrium payoffs, and the networks that form endogenously when the agents sponsor links ex-ante.

To make this model concrete, we may think about policy makers in different jurisdictions. One of them faces a policy challenge on a specific public administration issue. She would gain from the experience gathered by her peers, who may have a specific expertise in the area, or have faced similar policy challenges in the past. The most valuable information often takes the form of pilot studies or other policy experiments, whose results cannot be falsified, but can be easily concealed

[1] An exhaustive review is provided by the handbook edited by Victor, Montgomery and Lubell (2017). The importance of netwoks in politics has been recognized as early as by Routt (1938).

[2] Information transmission is one of the main themes of theoretical political economy (see, e.g., the literature on expert advice to Congress started with Gilligan and Krehbiel, 1987).

in the abundance of bureaucratic documents. In order to make better decisions, the policy maker consults her network of peers, but these connections are costly to maintain.[3]

Other examples include networks of lobbyists, interest groups, political experts, advocates, consultants, investigative journalists, bloggers, or academics.[4] One of them may be called to campaign or provide advice on a specific matter. In order to deliver more effectively, she will try and consult her peers to collect information, but again, these connections are costly to maintain. An intriguing example concerns informal networks of judges. Jurists may seek advice from their peers when confronted with legal issues (although they ought to abstain from delving into the facts of specific cases). And there is ample evidence that informal networks of judges play an important role in judicial sentences.[5]

The first and most important question I pose within my model is how to efficiently organize these political networks ex-ante. When the agents' ideological preferences are sufficiently aligned, I recover results of the early network literature (e.g., Jackson and Wolinsky, 1996, and Bala and Goyal, 2000). The optimal network is complete when the link cost is small enough, and it is empty when the cost is too large. In the case of interest with "intermediate" link cost, every ex-ante optimal network is a star (i.e., there is a unique "center" agent to which all other agents are linked, and such "periphery" agents are not linked with each other).

Results change dramatically when agents' ideologies are sufficiently diverse within every pair of agents. Then, the agents only relay information that conforms to their individual ideological biases relative to the decision maker. As a result, in the case of interest of intermediate link cost, the optimal network is the line in which the agents are ordered according to their ideal preferences. It is optimal that links only form between the ideologically closest agents.[6] But I prove that this

[3]The empirical/descriptive literature on networks of policy makers includes work by Zafonte and Sabatier (1998) on the cooperative links among San Francisco Bay area government agencies, and the study by Gerber, Henry and Lubell (2013) on the connections among planning and management agencies in five main California regions.

[4]Among studies on interest group networks, Laumann and Knoke (1987) and Carpenter, Esterling and Lazer (2004) trace the connections among lobbyists, government agencies, and congressional staff in the 1970's health and energy policy domains, König and Bräuninger (1998) investigate the ties among interest groups, trade unions, governmental agencies and legislators in the events that shaped 1980's German labour policies, Koger, Masket and Noel (2009) study the network of US party candidates, activists, interest groups and media outlets using donor and subscriber names data, and Box-Steffensmeier and Christenson (2014) build a network of interest groups coalitions using *amicus curiae* briefs to the US Supreme Court.

[5]Several studies demonstrate the influence on sentences of judicial networks constructed with various methodologies, e.g., by tracking precedent citation across Courts (Caldera, 1985), or by following the flow of clerks across judges (Katz and Stafford, 2010). Weiser (2015) interviewed some judges in Manhattan's Federal District Court and reports that almost all have consulted colleagues when puzzled by legal or other issues.

[6]There is overwhelming evidence in all sorts of domains that ideological and policy preference proximity is a main driver of political connections (e.g., Zafonte and Zapatier, 1998, and Gerber, Henry and Lubell, 2013, on policy makers networks; Laumann and Knoke, 1987, König and Bräuninger, 1998, Carpenter, Esterling and Lazer, 2004, and Box-Steffensmeier and Christenson, 2014, on networks of interest groups; and Caldera, 1985, on networks of judges).

does not lead to communication breakdown. And when agents are very likely to be informed, any agent's choice in the ordered line is almost as informed as if all information was shared.

Of special interest is the case in which agents are partitioned in ideologically diverse clusters, each composed of agents with similar views. I show that it is optimal *for all agents* that the clusters organize as factions, whose leaders communicate only with the leaders of the ideologically nearest factions. Formally, each cluster organizes as a star, and the only links across such stars connect the centers of ideological neighbor stars. Factionalization is a common organizational structure in political networks.[7] My analysis provides a rationalization based on optimal information transmission.

It is interesting that these results are derived in a model where link costs among agents are unrelated to ideological differences. In applications such as the ones discussed earlier, link costs are likely smaller within ideological clusters than across. When they are sufficiently small, the clusters optimally organizes as complete networks, instead of stars. Heterogeneous within-cluster costs would lead to the optimality of intermediate structures with a center agent to which all the other agents are linked, and each pair of such periphery agents are linked if and only if their link costs are sufficiently small.

I then turn to the question of whether these optimal networks form endogenously and whether they are stable. I focus on the case in which agents sponsor links primarily to access other agents' information, as they do not care much about their decisions. In the case of interest of intermediate link cost, I show that the earlier characterized optimal networks obtain as Nash equilibria of a network formation game, in which the agents invest ex-ante in the links they wish to form, and each link requires sponsorship by both connected agents. Further, the optimal networks are pairwise stable in the sense of Jackson and Wolinsky (1996), and they are the unique "strongly" pairwise stable networks. That is to say, the optimal networks include all links that are beneficial to both connected agents individually, and are the unique Nash equilibrium networks that include all links whose cost is smaller than their aggregate value to the connected agents.

To summarize: A possibly quintessential feature of politics is that agents hold diverse ideal views, and my analysis highlights the key importance of ideology for political networks optimality and stability. My study of information transmission in political networks provides a normative and positive rationale for political agents to form "horizontal" links with like-minded peers, beyond

[7]Ideological factionalization is a fact of life in political groups, as early documented by Janda (1980). For example, the study by Gerber, Henry and Lubell (2013) on regional cooperation among policy-makers in California uncovers 5 geographical cliques roughly organized as stars or complete networks. The interest groups network constructed by Box-Steffensmeier and Christenson (2014) "appears to resemble a host of tightly grouped factions" with homogeneous ideology, that are organized similarly to stars. The judicial network constructed by Caldera (1985) can be approximated as a collection of star-shaped subnetworks, in which "Courts cite the most prestigious among the culturally closest Courts."

their possibly intrinsic preference for doing so. In contrast, hierarchical networks such as the star are more likely prevalent, and useful, in organizations such as armies or companies in which the agents' preferences are closely aligned.[8]

It is remarkable that my results are in line with stylized facts established by the vast empirical literature on political networks, and in particular on the role of networks in facilitating the flow of information. This literature documents that preference-based homophily is a main driver of networks of policy makers, judges, and interest groups. Modal network structures can be described as a collection of subnetworks, differentiated by political preferences, and loosely connected among each other. Each subnetwork is tightly connected and hierarchically differentiated between a core of leaders to which less prominent agents are connected. These periphery agents may or may not be connected among each other, ostensibly depending on connection costs.[9]

While this paper's main motivation/application are political networks of ideological agents, my analysis can be applied to other organizations in which preferences differ across agents. These include, for example, companies where distinct divisions have divergent strategic objectives, partnerships of enterpreneurs with different wealth, risk attitude or discount factor, and even possibly academic departments composed of different research groups. The star-shaped factional networks I describe as optimal are not uncommon. In many companies, distinct divisions are organized as separate hierarchies, whose only formal connnections is between heads of divisions within meetings coordinated by higher management. Academic departments are sometimes partitioned according to their members' research interests, and each research group refers to a coordinator who liaises with the other group coordinators.

2 Literature Review

One of the main motivations of theoretical network economics studies is information transmission. Seminal papers such as Jackson and Wolinsky (1996) and Bala and Goyal (2000) identify the maximally centralized architecture of the star as socially optimal, and determine conditions under which this optimal architecture arises in equilibrium, and is a stable network. These results, generalized in a number subsequent studies, suggest that agents with aligned preferences should be organized in a strongly centralized, hierarchical, manner.[10]

[8]Organization design studies model hierarchical organizations as trees, where the (maximal) distance from terminal nodes identifies the agent's level in the hierarchy. Hierarchies with more than one level are regarded as optimal for minimizing information processing costs together with the risk of information decay. I briefly discuss this literature in Section 2.

[9]An extended review of this literature is available upon request.

[10]A detailed review of network economics literature is in the book by Jackson (2010) and the handbook edited by Bramoullé, Galeotti and Rogers (2017), for example.

In organization design and studies on the theory of the firm, hierarchical networks are regarded as the optimal structure for reducing the costs of information processing (Radner, 1993; Bolton and Dewatripont, 1994; Garicano, 2000), and for preventing conflicts between subordinates and their superiors (Friebel and Raith, 2004). Instead, cliques emerge among subgroups of agents who wish to coordinate their actions (Calvó, de Martí and Prat, 2015).[11]

All these papers study network models of communication that rule out the possibility that information is selectively withheld or that agents lie. Agents choose whether and how much information they send before they learn their information. In line with standard concepts of strategic communication (e.g., Milgrom, 1981, and Crawford and Sobel, 1982) instead, I allow agents to selectively withhold verifiable information (and also to lie on unverifiable information).[12]

Unlike the papers above, I study networks of political agents, who may wish to mislead each other because of their ideological preference differences. Factions form and are optimal in my model because agents with close ideologies are less likely to mislead each other, regardless of how much they care about each other actions. By contrast, for example, cliques in Calvó, de Martí and Prat (2015) form among agents who have the highest interest in coordinating their actions.

As earlier pointed out, networks are known to be very important in political economy, but there are few formal models of political networks. Assuming truthful communication of willingness to participate in collective action, Chwe (2000) characterizes minimal networks that lead to coordinated action. Instead, I study strategic communication and decentralized decisions, and determine optimal and stable networks. Canen, Trebbi and Jackson (2017) structurally estimate a model of endogenous network formation and legislative activity. Unlike my paper, where network homophily is micro-founded, legislators are assumed to form links according to ideological proximity. They do not study optimal networks, which is the main point of my paper. While I consider network transmission of multi-agent (verifiable) information, Bloch, Demange and Kranton (2018) study the spreading of possibly false information, originated by a single agent, in a network where agents may either wish correct decisions are made, or have a private agenda in favor of one alternative. Unlike my paper, they do not consider network optimality.[13]

A recent paper by Gieczewski (2016) studies transmission of verifiable information in networks, but does not consider network formation, stability nor optimality. He models information as continuous signals that can be transmitted at zero cost, so that full information transmission may

[11]For a more detailed exposition of the organization design literature, see the handbook by Gibbons and Roberts (2012), for example.

[12]The relevance of strategic communication in organizations is underlined and analyzed, for example, by Dessein (2002), and Alonso, Dessein and Matouscheck (2008).

[13]More distantly related to my work, Battaglini and Patacchini (2018) study how interest groups allocate campaign contributions when congressmen are connected by social ties.

take place in equilibrium. Nevertheless, there is no full learning, unless the network is sufficiently dense. Signals closer to the mean are more likely to propagate, because agents tend to block signals contrary to their bias, as is the case in my model.

In a broad sense, my paper belongs to the recent surge of studies of multi-agent communication in political economy, developed within the framework of Galeotti, Ghiglino and Squintani (2013). Patty (2013) determines the optimal exclusion and inclusion policies to maximize information sharing in meetings. Dewan et al. (2015) investigate the optimal assignment of decision-making power in the executive of a parliamentary democracy. Penn (2016) studies the engagement and association of societal groups and subcultures. Dewan and Squintani (2016) analyze the formation of party factions.

All these papers study transmission of unverifiable information (cheap talk). Instead, this paper assumes that at least some of the information transmitted is verifiable. As is well known, this distinction is crucial for equilibrium communication. As I later show, there is no information transmission whatsoever when ideologies are sufficiently diverse, within the framework of Galeotti et al. (2013). Hence, the optimal network is empty (no agent is linked with any other agent) regardless of link costs. In stark contrast, the optimal network is the ordered line when some information is verifiable, and information transmission is never fully blocked in equilibrium.

3 The model

My analysis of information transmission in political networks is based on the set up that I describe below. After presenting it, I will discuss the motivation and implications of its major assumptions, so as to assess the robustness of my analysis.

A set $\mathcal{N}$ of n agents is connected in a network N, that describes who can transmit information to whom. Formally, N is a $n \times n$ matrix with entries $n_{ij} \in \{0, 1\}$, for $i = 1, ..., n$ and $j = 1, ..., n$. A link connects agent i to j whenever $n_{ij} = 1$, in which case j is called a neighbor of i. The network N is symmetric, $n_{ij} = n_{ji}$ for all i, j: each agent i is linked to j if and only if j is linked to i.

A state of the world θ is uniformly distributed between 0 and 1. There is a random non-empty set $E \subseteq \mathcal{N}$ of expert agents who may be informed about θ. The random set of actually informed agents I is a possibly proper subset of E. The set I can be empty. Every agent $i \in I$ receives from nature a signal $s_i \in \{0, 1\}$ informative of θ according to the probability distribution $\Pr(s_i = 1|\theta) = \theta$. The signals s_i are independent across agents. A random agent $d \in \mathcal{N}$ is called to make a decision $\hat{y}_d \in \mathbb{R}$. Every agent i can be the decision maker and can be an expert agent. I denote by ξ the joint distribution of (d, E, I). I assume that for all $d \in \mathcal{N}$, (i) $\sum_{E:e\in E} \sum_{I \subseteq E} \xi(d, E, I) > 0$, and (ii) $\xi(d, E, I) > 0$ only if $I \subseteq E \neq \varnothing$.

When the state is θ, the decision maker is d, and the decision $\hat{y}_d$, the utility of each agent i is $u_i(\hat{y}_d; \theta) = -\alpha_{id}(\hat{y}_d - \theta - b_i)^2$. For every agent i, the weights α_{id} are all strictly positive and $\sum_{d \in \mathcal{N}} \alpha_{id} = 1$. Every agent i would like that every decision $\hat{y}_j$ is close to her bliss point $\theta + b_i$, which is composed of the "unbiased optimum" θ and the idiosyncratic ideological bias b_i. Yet, agents need not care equally about any agents' decisions. The preference biases b_i are common knowledge. Without loss of genericity I stipulate that $b_1 < ... < b_n$, with the notation $\mathbf{b} = (b_1, ..., b_n)$.

At the beginning of the game, nature selects the state θ, the decision maker d, the sets E and I of expert and informed agents, and the realizations of the signals s_i for the agents $i \in I$. Each agent $i \in I$ is then informed of her private signal s_i, and every agent learns the identity of the decision maker d. Agent d also learns who is in the set E of expert agents. (Whether or not any other agent observes E is irrelevant.) Agents are not informed of whether other (expert) agents hold a signal or not, and of the other agents' signal realizations.

The agents may transmit signals to the decision maker along the network N. There are T rounds of information transmission. I assume that $T \geq n - 1$ so that every signal has the possibility to travel to every agent in any (connected) network N.[14]. If informed of a signal s_i at any round t, any agent j chooses whether to relay s_i or not to some or all of her neighbors k in the network N. Neither the content of signal s_i nor the identity of its "source", agent i, can be falsified.

Formally, any strategy choice of agent j of reporting signal s_i to a neighbor k at a history h^t is denoted by $r_{jkdi}(\omega_{ij}; h^t)$ as a function of the information $\omega_{ij} \in \{0, \phi, 1\}$ held by agent j on signal s_i at history h^t. If j is not informed of s_i then $\omega_{ij} = \phi$, and else $\omega_{ij} = s_i$. The verifiability assumption entails that $r_{jkdi}(\omega_{ij}; h^t) \in \{\omega_{ij}, \phi\}$ for all $\omega_{ij} \in \{0, \phi, 1\}$. If j is informed of s_i, $\omega_{ij} = s_i$, the notation $r_{jkdi}(s_i; h^t) = s_i$ identifies the choice of relaying i's signal s_i to j. Instead, $r_{jkdi}(s_i; h^t) = \phi$ represents the choice of withholding s_i.[15]

As in the network papers cited in the introduction, information can be lost or decay in transmission. If j plays $r_{jkdi}(s_i; h^t) = s_i$, then k observes s_i with probability $\delta \leq 1$. With probability $1 - \delta$, agent k does not receive s_i and observes the report ϕ. I focus on the case of small decay, in which δ is close to one.[16] For future reference, let me define as $q(\tau, \ell)$ the probability that a signal s_i reaches an agent d in network N along a path p of length ℓ in τ periods if it moves one step in p with probability δ in every period. The probability $q(\tau, \ell)$ is defined recursively as

[14] A network N is connected if for every pair of agent i and j, there exists at least a path p that connects i and j. Two agents i and j are linked by the path $p = (i, h_1, ..., h_{\ell-1}, d)$ of length ℓ in network N, if i is linked to h_1, h_k is linked to h_{k+1} for every $k = 1, ..., \ell - 2$, and $h_{\ell-1}$ is linked to d.

[15] Once agent j has observed a signal s_i, further disclosure of s_i is immaterial, because s_i cannot be confused with any other signal.

[16] The possibility of signal decay is not needed for my main results, it is included in the model to allow comparison with early network theoretical results. My model does not include information processing costs, but it can be shown that their addition would reinforce my main results.

$q(\tau, \ell) = \delta q(\tau - 1, \ell - 1) + (1 - \delta) q(\tau - 1, \ell)$ for $\tau \geq \ell$, together with $q(\tau, \ell) = 0$ if $\tau < \ell$, and $q(\tau, 0) = 1$ for all $\tau \geq 0$. For δ close to one, $q(\tau, \ell)$ is approximately equal to δ^ℓ for all $\tau \geq \ell$.

In the final stage of the game, agent d chooses $\hat{y}_d$ and the payoffs $u_i(\hat{y}_d; \theta)$ are realized by every agent i. The solution concept is (pure and mixed strategy) perfect Bayesian equilibrium. For any given network N, there may be multiple equilibria. But I will show that each agent i ranks them in the same order ex-ante (that is, before agents receive any information from nature). As customary in the communication game literature, I focus on the equilibria (that I call "optimal") that maximize each agent's ex-ante payoffs.

Network formation is costly, and each link costs $c > 0$. I am interested in the networks that are ex-ante optimal in the utilitarian sense, that is, networks that maximize the sum of the agents' payoffs ex-ante, including links costs. As I will explain later in details, I also study endogenous equilibrium networks when link formation requires ex-ante sponsorship by both linked agents (they each must pay $c/2$ to sponsor the link), and when connections can be ex-ante unilaterally sponsored by one of the linked agents, who pays the whole cost c.

The model is kept as simple as possible, in order to deliver insights in the clearest fashion. Before proceeding with the analysis, I now discuss motivation and implications of each major assumption, so as to assess the robustness of my results.

It is easy to generalize my analysis to the case in which agents can receive more than one binary signal, and to the case in which each signal s_i may take any finite number of realizations. My main results continue to hold qualitatively. If the signals s_i are drawn from a continuum, then my main results generalize as long as there is a small cost $c_0 > 0$ to transmit information along the network. The assumption of a uniform prior is also immaterial, as my main results hold qualitatively for general beta prior distributions, within the beta-binomial statistical model.[17]

The representation of agents' utilities as quadratic loss functions with diverse bliss-points is predominant in communication games since Crawford and Sobel (1982), and this paper's formulation is a simple generalization of this standard representation. It can be proved that my main results generalize qualitatively to utility representations that satisfy the single-peakedness and single-crossing assumptions standard in the communication games literature.

Decentralized decision making is an integral part of my model. I assume that a random agent is called to make decisions. This assumption is based on my motivating examples. At any moment in time, judges in different Courts face different legal questions, policy makers in different jurisdictions confront with different policy challenges and emergencies, and different interest groups and policy

[17] The beta-binomial model is one of the main statistical models used for Bayesian updating, together with the normal-normal model.

stake-holders campaign or provide advice on different topics. A decision-making problem in my model represents one of these different matters, and it is then natural to assume a random decision maker.[18] And if I allow multiple decision makers, my results generalize qualitatively as long as expert agents are sufficiently likely informed, and every agent can be the unique decision maker with positive probability.

I assume that the decision maker knows who the expert agents are. In my motivating examples, they are the policy-makers, lobbyists and judges known to have expertise in the area of the decision maker's problem, for example because of educational backgound or past experience. While such expert agents need not be actually informed, it is reasonable to assume that their identities would be public. And this assumption is unrestrictive if every agent can be expert, as I stipulate in the second part of the analysis.

My model considers verifiable information transmission. To be clear, I do not claim that political agents do not engage in cheap talk. However, the possibility of cheap talk communication is irrelevant for my main results, as I show later. What I assume is that *some* of the exchanged information (modelled as signals s_i) is verifiable, and I also stipulate that the "source," agent i, of each signal s_i is identifiable.[19] These assumptions are hardly contestable within my motivating examples. The briefings and reports of interest groups and policy advocates are filled with verifiable data and facts, whose source is identifiable. Likewise, policy makers largely base decisions on policy trials data, and judges base rulings on careful interpretation of selected legislation, legal doctrine and precedents.

4 Optimal networks

This section calculates the optimal network N in my model, as a function of the cost and decay parameters c and δ, as well as the bias differences $|b_i - b_j|$ across agents i and j. I define optimal the network N that maximizes the sum of the agents i ex-ante equilibrium payoffs $U_i(N)$ given N, minus $c(N) \equiv c \sum_{i=1}^{n} \sum_{j<i} n_{ij}$, the sum of the costs of the links in N.

In the interest of a smoother presentation, I begin the exposition with the case in which there is a unique random expert agent e, informed with probability $\rho \in (0, 1]$. I assume that every pair of agents (e, d) can be drawn as expert and decision maker with positive probability. Thus, I restrict

[18] As is common in studies of information aggregation, I abstract from the possibility of interaction across decision-making problems and game repetition effects. Their investigation would be a nice extension of my analysis.

[19] My results generalize qualitatively to the case in which the source i of any signal s_i is not identifiable, as long as the realization of signal s_i is verifiable, regardless of whether or not agents can distinguish different signals s_i and s_j. It is also immaterial whether or not any agent j can certify to any agent k that any signal s_i has arrived to her. But it matters that j cannot certify to k that she has *not* received s_i, and I find this assumption realistic.

attention to distributions ξ such that $\xi(d, E, I) > 0$ if and only if E is a singleton set $\{e\}$, that $\xi(d, \{e\}, I) > 0$ for every pair of agents (d, e) and $I = \varnothing, \{e\}$, and that the probability that the expert e is actually informed $\frac{\xi(d,\{e\},\{e\})}{\xi(d,\{e\})}$ is equal to ρ, where $\xi(d, \{e\}) \equiv \xi(d, \{e\}, \{e\}) + \xi(d, \{e\}, \varnothing)$.

The case of a single expert agent: Communication strategies. In order to determine optimal networks, one needs to calculate equilibrium communication strategies given network N, expert e and decision maker d. Suppose that e and d are linked via a single path p of length ℓ. The agents' order of biases along p is arbitrary.

Consider an agent j on the path p and say that $b_j > b_d$. Agent j would like to bias agent d's decision $\hat{y}$ to the right. When the signal s of agent e is equal to 1, agent j has no incentive to withhold s, as it conforms with her bias relative to d. By relaying $s = 1$ to the next agent on the path p, agent j can only (weakly) increase the probability that d receives signal s, and move her decision $\hat{y}$ to the right, closer to j's bliss point $b_j + E[\theta|s]$. When j is informed of $s = 0$, the signal s is contrary to her bias $b_j - b_d > 0$ relative to d. By relaying $s = 0$ to the next agent on p, agent j can only move the decision $\hat{y}$ to the left, farther from her bliss point $b_j + E[\theta|s]$. Unless the bias $b_j - b_d$ is smaller than the threshold calculated in Lemma 1 below, agent j prefers to withhold $s = 0$. And by symmetry, any agent j with $b_j - b_d < 0$ prefers to withhold $s = 1$ in equilibrium unless $|b_j - b_d|$ is smaller than the same threshold.[20]

Lemma 1 *Suppose $\ell + 1$ agents j are linked along a path p from agent e informed of a signal s with probability $\rho \in (0, 1]$ to decision-maker d. In every equilibrium, every such agent j with $b_j \geq b_d$ (respectively, $b_j \leq b_d$) prefers to relay the signal $s = 1$ (resp., $s = 0$) to the next agent on p, but every agent j such that $b_j - b_d > \frac{1}{6[2-\rho q(T,\ell)]}$, (respectively, $b_j - b_d < -\frac{1}{6[2-\rho q(T,\ell)]}$) prefers to withhold signal $s = 0$ (resp., $s = 1$).*

When agents j are sufficiently biased relative to b_d, $|b_j - b_d| > \frac{1}{6[2-\rho q(T,\ell)]}$, they withhold signals contrary to their bias. Because strategic withholding of information is the research subject of this paper, let me momentarily suppose that $|b_j - b_d| > \frac{1}{6[2-\rho q(T,\ell)]}$ for every agent j on the path p from e to d. I now describe a simple necessary and sufficient condition for informed decision making.

If every agent j's ideological bias $b_j - b_d$ has the same sign, then the signal s reaches agent d in equilibrium if and only if it conforms to the agents' biases relative to agent d. (Let me call this: "one-sided information transmission"). As in the simpler case of verifiable information transmission between two agents studied by Milgrom (1981), this leads to informed decision making. Both receiving and not receiving signal s are informative to agent d in equilibrium. If not observing s,

[20]The formal proofs of Lemma 1 and of the subsequent results are in Appendix A.

she believes that s is contrary to the agents' biases with probability $\frac{1}{2-\rho q(T,\ell)} > \frac{1}{2}$.[21]

Instead, information trasmission to agent d fully breaks down when $b_j > b_d$ for some agents j on the path p, and $b_k < b_d$ for some other agents k on p. The former relay the signal $s = 1$ but block $s = 0$, whereas the latter block $s = 1$ and relay $s = 0$. As a result, the signal s never reaches agent d, who acts fully uninformed. Unlike in the simpler case of two agents, equilibrium disclosure of verifiable information need not be informative. For future reference, I call *bias reversal path* any path p from an agent e to an agent d in which there are both agents j with significant positive bias, $b_j - b_d > \frac{1}{6[2-\rho q(T,\ell)]}$, and agents k with negative bias $b_k - b_d < -\frac{1}{6[2-\rho q(T,\ell)]}$.

Completing the equilibrium communication characterization, Proposition 1 calculates the equilibria that maximize the agents' ex-ante payoffs, beyond the case in which $|b_j - b_d| > \frac{1}{6[2-\rho q(T,\ell)]}$ for every agent j on p. Depending on the biases $\mathbf{b}$, the optimal equilibria is either such that both signal realizations $s = 0$ and $s = 1$ are relayed along p (I call this "full communication"), or that all agents relay one signal realization s and some randomize on the "opposite" realization $1 - s$, or that one signal realization is relayed and the opposite one is blocked, or that both signal realizations $s = 0$ and $s = 1$ are blocked.[22]

For brevity, I use the notations $\underline{b} = \min_j b_j$, $\overline{b} = \max_j b_j$, and $\overline{j} = \arg\max_j |b_j - b_d|$, to denote respectively the extreme left and right ideologies b_j on p, and the agent j with the largest absolute bias $|b_j - b_d|$ relative to d.

Proposition 1 *Suppose $\ell + 1$ agents j are linked along a path p from agent e informed of a signal s with probability $\rho \in (0, 1]$ to decision-maker d.*

1. If $|b_j - b_d| \leq \frac{1}{12}$ for every agent j on p, then in the optimal equilibrium every agent relays both $s = 0$ and $s = 1$ to the next agent on p.

2. If $\frac{1}{12} < \overline{b} - b_d \leq \frac{1}{6(2-\rho q(T,\ell))}$ and $\underline{b} \geq \overline{b} - \frac{1}{6}$, then the optimal equilibrium is in mixed strategies: every agent on p relays $s = 1$, and $s = 0$ is relayed by every agent except $\overline{j}$ who randomizes.
A symmetric characterization holds when $-\frac{1}{6[2-\rho q(T,\ell)]} \leq \underline{b} - b_d < -\frac{1}{12}$ and $\overline{b} \leq \frac{1}{6} + \underline{b}$.

3. If there exists agents j, k on p such that $b_j - b_d \geq -\frac{1}{6[2-\rho q(T,\ell)]}$ and $b_k - b_d \leq \frac{1}{6[2-\rho q(T,\ell)]}$, then neither $s = 0$ nor $s = 1$ ever reaches agent d in any equilibrium.

4. Else, information transmission is one-sided in the optimal equilibrium. If $b_j - b_d > -\frac{1}{6[2-\rho q(T,\ell)]}$ (respectively, $b_j - b_d < \frac{1}{6[2-\rho q(T,\ell)]}$) for every agent j on p, then every agent relays $s = 1$ (resp. $s = 0$) along p, but $s = 0$ (resp. $s = 1$) never reaches agent d.[23]

[21] With small decay and ρ close to one, not receiving signal s is interpreted almost certainly as evidence that s is contrary to the agents' biases, because $1/[2 - \rho q(T, \ell)] \approx 1/(2 - \rho\delta^\ell) \to 1$ for $\delta \to 1$ and $\rho \to 1$.

[22] I omit the possibility of equilibrium with randomization on both $s = 0$ and $s = 1$, as I prove in the appendix that this possibility is non-generic in $\mathbf{b}$.

[23] It is immediate that case 4 includes the case in which $|b_j - b_d| > \frac{1}{6[2-\rho q(T,\ell)]}$ for all j on p, and there are no

It is intuitive that when the agents' ideologies are sufficiently close ($|b_j - b_d| < \frac{1}{12}$ for every j on p), there is an an equilibrium in which no agent withholds information. Consider an agent j with $b_j - b_d > 0$ who is informed that $s = 0$, in contrast with her bias. By blocking the transmission of signal s along the path p, agent j increases the probability that s does not reach d, thus moving the equilibrium decision $\hat{y}_d$ to the right from $b_d + E[\theta|s = 0] = b_d + 1/3$ to $b_d + E[\theta|\phi] = b_d + \frac{1}{2}$. (Agent d's expectation about θ if not observing s equals $\frac{1}{2}$ in this equilibrium, because the event that s is lost in transmission is independent of whether $s = 0$ or $s = 1$.) When $2(b_j - b_d) \leq \frac{1}{2} - 1/3 = 1/6$; this movement of $\hat{y}_d$ "leapfrogs" the bliss-point $b_j + E[\theta|s = 0]$ of agent j by so much that it makes j worse off. Agent j prefers to not deviate from equilibrium and to relay $s = 0$ to the next agent on p. A symmetric argument holds for any agent j with $b_j < b_d$ who is informed that $s = 1$.

Because the length of the path p is ℓ, the signal s reaches d with probability $\rho q(T, \ell)$, and I prove in the appendix that each agent i's ex-ante expected equilibrium payoffs are:

$$U_{ied}(p) = -\alpha_{id}(b_i - b_d)^2 - \alpha_{id}\frac{1}{18}(1 + \frac{1 - \rho q(T, \ell)}{2}). \tag{1}$$

Suppose next that the agents are not so ideologically close to establish full communication. But say that their biases relative to the decision maker are still "sufficiently small" (in the sense of the conditions of case 2. of Proposition 1). Then optimal equilibrium communication is in mixed strategy and asymmetric across signal realizations, but retains features of full communication. All agents relay both $s = 0$ and $s = 1$, except the most biased agent $\bar{j}$ who randomizes disclosure of the signal realization s contrary to her bias. There may multiple, outcome equivalent, equilibria. In the simplest one to describe, agent $\bar{j}$ relays s in every history h^t with the same probability:

$$\sigma = \frac{1 - 12|b_{\bar{j}} - b_d| + 6\rho q(T, \ell)|b_{\bar{j}} - b_d|}{\rho q(T, \ell)(1 - 6|b_{\bar{j}} - b_d|)}.$$

I show in the appendix that, regardless of whether signal realization $s = 0$ or $s = 1$ is the one randomly withheld in equilibrium, each agent i's ex-ante expected payoffs are:

$$\begin{aligned} U_{ied}(p) &= -\alpha_{id}(b_i - b_d)^2 - \alpha_{id}\frac{1}{18}(1 + \frac{1 - \rho q(T, \ell)(1 + \sigma) + \rho^2 q(T, \ell)^2 \sigma}{2 - \rho q(T, \ell)(1 + \sigma)}) \\ &= -\alpha_{id}(b_i - b_d)^2 - \alpha_{id}\frac{1}{18}(1 + 6|b_{\bar{j}} - b_d|[1 - \rho q(T, \ell)]). \end{aligned} \tag{2}$$

It is immediate that the payoffs U_{ied} increase in σ, the equilibrium probability that the extreme bias agent $\bar{j}$ relays the signal realization contrary to her bias. And further, U_{ied} decrease in the absolute magnitude $|b_{\bar{j}} - b_d|$ of the bias of the most biased agent, as is intuitive.

As the agents' biases increase further, the mixed strategy equilibrium described above ceases to exist. If there are no bias reversals on the path p (either $b_j - b_d > \frac{1}{6[2 - \rho q(T, \ell)]}$ for all j on

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bias reversals.

p, or $b_j - b_d < -\frac{1}{6[2-\rho q(T,\ell)]}$ for all j on p), then equilibrium communication is one sided: it is asymmetric, but now in pure strategies. All agents j relay the signal realization s that conforms with their biases relative to agent d, and the "opposite" signal realization $1-s$ is blocked from d.[24] Regardless of whether signal realization $s=0$ or $s=1$ is the one withheld in equilibrium, each agent i's ex-ante expected payoffs are:

$$U_{ied}(p) = -\alpha_{id}(b_i - b_d)^2 - \alpha_{id}\frac{1}{18}(1 + \frac{1-\rho q(T,\ell)}{2-\rho q(T,\ell)}). \tag{3}$$

Finally, if the path p includes bias reversals (there are agents j,k such that $b_j - b_d > \frac{1}{6[2-\rho q(T,\ell)]}$ and $b_k - b_d < -\frac{1}{6[2-\rho q(T,\ell)]}$), then communication breaks down and d's decision is not informed. Each agent i's ex-ante payoffs are:

$$U_{ied}(p) = -\alpha_{id}(b_i - b_d)^2 - \alpha_{id}\frac{1}{12}. \tag{4}$$

It is immediate by comparing the expressions (1) - (4) that every agent i's ex-ante payoffs $U_{ied}(p)$ follow the same ranking. They all prefer full communication to agent d to one-sided communication, which they all prefer to communication breakdown.

The case of a single expert agent: Optimal networks. Earlier identified conditions that determine whether bias differences are too small or sufficiently large to entail strategic information withholding, also determine as a result whether the optimal network is a star or the ordered line, for the case of interest with small decay and intermediate link costs. Specifically, if $|b_i - b_j| \leq \frac{1}{6(2-\rho)}$ for all pairs of agents i,j (i.e., $b_n - b_1 \leq \frac{1}{6(2-\rho)}$), then every optimal network N is a star. There is a unique agent i (the center) to which all other agents j are linked, and there are no other links. Instead, if $|b_i - b_j| > \frac{1}{6(2-\rho)}$ for all pairs i,j (i.e., $b_{i+1} - b_i > \frac{1}{6(2-\rho)}$, for all $i < n$), then the optimal network N is the ordered line. Every agent $i = 2, ..., n-1$ is linked to both agents $i-1$ and $i+1$, and there are no other links.

The analysis focuses on intermediate link costs. It is intuitive that if the connection cost c is sufficiently small, then the optimal network N is the complete network, i.e., the network in which every pair of agents i,j is connected. Likewise, if c is too large, then the optimal network N is not connected: there exist agents i and j that are not linked by any path. Let me define $\bar{\bar{c}}$ as the maximum cost for which the optimal network is connected, in the (hypothetical) case that one-sided information transmission with no decay takes place among every pair of agents i and j. For future reference, a network N is minimally connected if every pair of agents i and j is linked by a unique path.

[24] If $|b_j - b_d| < \frac{1}{6[2-\rho q(T,\ell)]}$ for all agents j on p, then both one sided equilibria exist and are optimal: one in which $s=0$ is relayed and $s=1$ blocked, and one in which $s=1$ is relayed and $s=0$ blocked.

Proposition 2 *Suppose there is one randomly drawn expert e, informed with probability $\rho > 0$.*

1. (Close ideologies) When $b_n - b_1 \leq \frac{1}{12}$, there exist thresholds $\bar{\delta} < 1$ and $\underline{c}(\delta) < \bar{c}(\delta)$, with $\lim_{\delta \to 1} \underline{c}(\delta) = 0$ and $\lim_{\delta \to 1} \bar{c}(\delta) = \bar{\bar{c}}$, such that for all $\delta \in (\bar{\delta}, 1)$ and $c \in (\underline{c}, \bar{c})$, every optimal network is a star S.[25]

2. (Divergent ideologies) When $b_{i+1} - b_i > \frac{1}{6(2-\rho)}$ for all $i < n$, there exist thresholds $\bar{\delta} < 1$ and $\underline{c}(\delta) < \bar{c}(\delta)$, with $\lim_{\delta \to 1} \underline{c}(\delta) = 0$ and $\lim_{\delta \to 1} \bar{c}(\delta) = \bar{\bar{c}}$, such that for all $\delta \in (\bar{\delta}, 1]$ and $c \in (\underline{c}, \bar{c})$, the optimal network is the ordered line L.[26]

The first result in Proposition 2 confirms earlier results on the optimality of star networks that date back at least to Jackson and Wolinsky (1996). Ideology differences are small, and so each agent relays all signals, regardless of the identities of the decision maker d and the expert e. The only obstacle to information transmission is decay. The stars are then the optimal networks, because they connect all agents with the shortest sum of paths, and hence minimize decay.

The most important result of Proposition 2 concerns the case in which ideology differences are sufficiently large, according to the threshold calculated in Lemma 1. In the case of interest with intermediate cost and small decay, the optimal network is the ordered line L. It is optimal for the whole group of political agents that each one of them forms connections only with the ideologically closest agents. But this does not lead to information transmission breakdown. Equilibrium communication from the expert agent to the decision maker is one sided. And when the expert agent is very likely to be informed, the decision maker's choice in the ordered line is almost as informed as if all information was shared.

The intuition for this result is as follows. Suppose momentarily that the optimal network is a tree (i.e., it is minimally connected). There is only one path connecting any pair of agents e and d, and the information flow is fully blocked when that path has bias reversals. Evidently, there are no bias reversal paths in ordered line L: If $b_e < b_d$, then $b_k < b_d$ for all agents on the path from e to d, and vice versa. With the ordered line L, information is never fully blocked.

Moving on to consider other trees, I first focus on the case of 4 agents. Interchanging agents' identities, there are two classes of trees: the line, and the star. It can be seen that every star has at least a bias reversal path,[27] and of course, non-ordered lines contain bias reversal paths.

[25] Analogously to $\bar{c}$, the threshold $\bar{\bar{c}}$ is defined as the maximum cost for which the optimal network is connected, in the hypothetical case there is full communication without decay among every pair of agents. When $\delta = 1$, there exists $\bar{c}$ such that if $b_n - b_1 \leq \frac{1}{12}$, then every minimally connected network is optimal for all $c \in (0, \bar{c})$.

[26] The concept of optimality is expressed in terms of the sum of agents i ex-ante payoffs U_i minus the aggregate link costs. The same result holds qualitatively if considering any weighted sum of agents' payoffs and link costs. Fixing any weight distribution, there exist thresholds $\bar{\delta}$, $\underline{c}$, and $\bar{c}$ such that exactly the same stated results hold. However, the specific threshold values depend on the weight distribution considered.

[27] When agent 1 is the center, then the path from agent 3 to 2 has a bias reversal, and so does the path from 4

Then, I note that for any number of agents $n \geq 4$, the only tree that does not contain a star or a non-ordered line as a sub-tree is the n-agent ordered line L. Hence, the unique minimally connected network in which information is never fully blocked is the ordered line. When decay is small (δ close to one), the first concern is to ensure that information is not blocked, as opposed to the length of the communication paths. The ordered line L is thus the (unique) optimal tree.

Let me then consider networks N that are not minimally connected. When the link cost c is not too small, adding links to minimally connected networks is wasteful. When c is not too high, it is optimal that there is a path between each pair of agents, because each agents' information is useful to every agent: the optimal network must be connected. I prove in Appendix A that for small decay, there exist a non-empty "intermediate" range of costs $(\underline{c}, \overline{c})$, function of δ, such that for all $c \in (\underline{c}, \overline{c})$ the unique optimal network is the ordered line.

As the decay factor δ converges to one, the cost range $(\underline{c}(\delta), \overline{c}(\delta))$ grows to cover the whole range $(0, \overline{\overline{c}})$. This last result holds because, as decay becomes negligible, the penalty borne by the ordered line relative to networks with shorter paths (the fact that signals are more likely lost, as the total length of paths is larger) also becomes negligible.

I conclude this part of the analysis with a short digression that considers cheap talk communication (as in Galeotti et al., 2013), instead of transmission of verifiable information. Formally, I lift the verifiability restriction on reporting strategies that $r_{jkdi}(\omega_{ij}; h^t) \in \{\omega_{ij}, \phi\}$ given j's information $\omega_{ij} \in \{0, \phi, 1\}$ on signal s_i, for any agents d, i, j, k, and history h^t. Regardless of the realization of signal s_i, the messages $r_{jkdi}(s_i; h^t)$ may here take any value out of possibly large message sets M^t_{jkdi}, that I assume finite for simplicity. The remainder of the model of Section 3 is unchanged.

I first consider cheap talk from an expert agent e who holds a signal s informative of θ with probability ρ, to a decision maker d along a path p of length ℓ. I focus on the case of small decay, δ close to one. I prove in Appendix B that, if there is any agent j on p with $|b_j - b_d| > \frac{1}{6(2-\rho)}$, then communication fully breaks down in the unique equilibrium that is outcome equivalent to any equilibrium of the game with verifiable information transmission.[28] And if $|b_j - b_d| > \frac{1}{6}$ for all agents j on p, then equilibrium communication fully breaks down also if allowing for equilibria with general cheap talk strategies.

This is in stark contrast with the case of verifiable information transmission, where even if $b_{i+1} - b_i > \frac{1}{6}$ for all agents $i < n$, equilibrium supports (one-sided) information transmission along

to 3 when agent 2 is the center. The cases when 4 and 3 are the star centers are analogous to when the center is 1 and 2, respectively, by interchanging 1 with 4 and 2 with 3.

[28]In every period t, the decision maker d receives the same report $r_{jd}(\omega_j; h^t)$ from the last agent j on p, regardless of the signal realization s and of the history h^t.

any path p without bias reversals (Proposition 1). The implications for network optimality are also stark. In the case of interest of small decay and intermediate link cost, if the agents ideological biases are sufficiently diverse, $b_{i+1} - b_i > \frac{1}{6}$ for all $i < n$, then the optimal network N is empty, when the signal s is not verifiable.[29] Instead, if the signal s is verifiable, then the optimal network is the ordered line (Proposition 2).

The same reasoning shows that adding cheap talk on top of transmission of verifiable information would be irrelevant, if the agents' ideologies are sufficiently diverse. If together with the verifiable signal s, there were also unverifiable signals $\tilde{s} \in \{0, 1\}$ informative of θ with probability $\Pr(\tilde{s} = 1|\theta) = \theta$, then the decision maker would not be informed of any such signals $\tilde{s}$ in equilibrium, when $b_{i+1} - b_i > \frac{1}{6}$ for all agents $i < n$. And for any general distribution of unverifiable signals $\tilde{s}$ informative of θ, it can be shown that the decision maker would not be informed of $\tilde{s}$ in equilibrium, when the biases $b_{i+1} - b_i$ are sufficiently large.

The case of multiple expert agents. The logic behind Proposition 2 extends to the case in which there is more than one expert agent, i.e., the random set E is not a singleton set. To make the exposition simpler, let me assume that $\xi(d, E, I) > 0$ if and only if $E = \mathcal{N}$: every agent i may be informed on θ. Then, the condition that identifies "sufficiently large" ideological differences for the ordered line L to be the optimal network is that $b_{i+1} - b_i > \frac{1}{2(n+2)}$ for all $i < n$. And note that this implies that $b_n - b_1 > \frac{n-1}{2(n+2)} \to \frac{1}{2}$ for $n \to \infty$, a requirement seemingly not very demanding even in the aggregate. As in Proposition 2, I focus on link cost c smaller than the threshold $\bar{\bar{c}}$, the maximum cost for which the optimal network is connected if one-sided information transmission with no decay were to take place between every pair of agents i and j.

Proposition 3 *Suppose that every agent may be informed on θ, and that ideologies are sufficiently divergent, $b_{i+1} - b_i > \frac{1}{2(n+2)}$ for all $i < n$. Then, there exists thresholds $\bar{\delta} < 1$ and $\underline{c}(\delta) < \bar{c}(\delta)$, with $\lim_{\delta \to 1} \underline{c}(\delta) = 0$ and $\lim_{\delta \to 1} \bar{c}(\delta) = \bar{\bar{c}}$, such that for all $\delta \in (\bar{\delta}, 1]$ and $c \in (\underline{c}, \bar{c})$, the optimal network is the ordered line L.*

The proof of this result is an extension of the proof of Proposition 2. Again, I show that the ordered line L is the only tree for which the transmission of none of signals s_i is blocked to any possible decision maker d. Hence, L is the optimal minimally connected network for small decay, δ close to 1; and L is the optimal network unless the link cost c is too small (when a connected network with loops may dominate L) or c is too high (when a not connected network may dominate L).

[29] The empty network N is such that $n_{ij} = 0$ for all i, j.

Now however, there are potentially n different signals, and hence there is an informational loss if the decision maker d does not receive all of them. I prove in the appendix that when $b_{i+1} - b_i > \frac{1}{2(n+2)}$ for all $i < n$, any agent j wants to block a signal s_i contrary to her bias $b_j - b_d$ relative to agent d, upon knowing that none of the other $n-1$ signals is withheld from d in equilibrium. As in Proposition 2, the ordered line L is the only tree that makes such blocking impossible, and hence it dominates all other trees, as long as it is not too much penalized by decay.

The optimality of factionalization. To conclude the analysis of optimal networks, suppose that agents are partitioned into ideologically diverse clusters of agents with similar ideology. These ideological clusters are common in politics.[30] For example, they arise when agents form their views as students of different party schools or political organizations.[31] Divisive issues such as abortion, civil rights, or environmental protection also tend to polarize views and create ideologically coherent clusters.

The results presented so far show that if all agents are ideologically close, then they should optimally organize as a star network, whereas if they are all ideologically diverse, then the optimal network is a line ordered according to their ideologies. These results bear intuitive implications for ideologically clustered groups of agents. In the case of interest of small decay and intermediate link cost, it is optimal for all agents that each cluster organizes as a star, and that the centers of these stars are connected according to the clusters' ideological order.[32]

Proposition 4 *Suppose that every agent may be informed on θ. Say that there are $M \leq N$ ideological clusters of agents $C_1, ..., C_M$ such that for all $m = 1, ..., M$, and all $i \in C_m$, $|b_i - b_j| \leq \frac{1}{2(n+2)(n+1)}$ for all $j \in C_m$, and $b_i - b_k > \frac{1}{2(n+2)}$ for all $k \in C_{m-1}$. Then, there exist $\bar{\delta} < 1$ and $\underline{c}(\delta) < \bar{c}(\delta)$, with $\lim_{\delta \to 1} \underline{c}(\delta) = 0$ and $\lim_{\delta \to 1} \bar{c}(\delta) = \bar{\bar{c}}$, such that for all $\delta \in (\bar{\delta}, 1)$ and $c \in (\underline{c}, \bar{c})$, every optimal network has the following properties:*

1. For every m, the agent(s) in cluster C_m organize as a star S_m (possibly made of a single agent);

2. For every $m = 2, ..., M-1$, the center of star S_m is linked with the centers of S_{m-1} and S_{m+1}, and there are no other links across clusters.

This result provides an efficiency rationale for ideological factionalization, which is shown to

[30] An early study by Rose (1964) identifies these ideological clusters as "stable set[s] of attitudes, rather than stable set[s] of politicians," as opposed to organized party factions, described as groups "self-consciously organized as a body, with a measure of discipline and cohesion."

[31] For example, Argentinian politicy makers and economists often divide themselves among those who studied in national universities and favor socially oriented policy making, and "fresh-water" US university trained purponents of free-market measures. Similar divisions are present in other countries.

[32] For brevity, the next results (Propositions 4 - 6) are presented for the case in which all agents may be informed on θ, analogous results hold for the case of a single expert agent.

optimize information transmission among political agents. Each ideological group C_m forms a faction organized as a star S_m, whose center is naturally interpreted as the faction leader, because every other member of the faction reports only to her. Communication across factions occurs only through the leaders, and each faction leader communicates only with the leaders of the ideologically closest factions.

It is important that Proposition 4 predicts that factionalization is optimal for the whole set of agents, despite the assumption that link costs among agents are unrelated to ideological differences. In many applications such as the ones discussed in the introduction, link costs are likely increasing with ideological differences. A full analysis for the case of heterogeneous link costs is outside the boundaries of this paper.[33] But it is easy to see what would happen in simple cases, such as for example when some links among agents in the same cluster are costless. Then, each cluster optimally organizes as a "generalized star" for small decay. There is a single center to which all the other agents are linked, and each pair of these periphery agents are linked if only if their link is costless.[34] Conversely, it is easily seen that if the cost of forming links across clusters grows sufficiently, it becomes optimal that clusters are not connected.

Proposition 4 does not cover all possible bias profiles. It need not be generally the case that agents are clustered according to ideology. For example, it may be that all pairs of agents are ideologically close, in the sense that $b_{i+1} - b_i \leq \frac{1}{2}\frac{1}{(n+2)(n+1)}$ for all $i < n$, and yet agents are not clustered, because $b_{i+2} - b_i > \frac{1}{2}\frac{1}{(n+2)(n+1)}$ for all $i < n-1$. Beyond the case of clustered ideologies, the optimal network can take forms different from the structure characterized in Proposition 4. This is demonstrated in the following example, that is presented for the case of a single random expert agent, to make matters simpler.

Example 1. Suppose there are $n = 6$ agents. A decision maker d and an expert agent e, informed with probability $\rho \leq 1$, are randomly drawn. The biases are as follows: $\frac{1}{12} < b_2 - b_1 \leq \frac{1}{6(2-\rho)}$, $b_3 - b_2 > \frac{1}{6(2-\rho)}$, $\frac{1}{12} < b_4 - b_3 \leq \frac{1}{6(2-\rho)}$, $b_5 - b_4 > \frac{1}{6(2-\rho)}$, and $\frac{1}{12} < b_6 - b_5 \leq \frac{1}{6(2-\rho)}$.

Let me first consider the following network N:

[33] Galeotti, Goyal and Kamphorst (2006) expand the framework of Bala and Goyal (1996) to consider heterogeneities in the costs and values of connecting agents. Value heterogeneity is the starting point of their analysis, whereas here it is derived from first principles in a model of information transmission. Another important difference is that in their model, the value of connecting a pair of agents is assumed to depend only on the agents connected, and not on the path that connects them. Here, the value of connecting two agents does not depend only on these two agents' preferences, but also on the preferences of all the agents on the connecting paths.

[34] By continuity, this result holds also when these links have sufficiently small cost, holding decay fixed. But holding link costs fixed and making decay small, the optimal within-cluster network morphs into the tree (not necessarily a star) that minimizes the sum of link costs. This distinction complicates the general characterization of the optimal within-cluster network for small heterogeneous link costs and small decay.

$$\begin{array}{ccccc} 1 & — & 3 & — & 5 \\ | & & | & & | \\ 2 & & 4 & & 6 \end{array}$$

In line with the optimal network characterization of Proposition 4, every pair of agents whose ideology difference is smaller than $\frac{1}{6(2-\rho)}$ is linked, and these pairs (stars) are then linked according to their ideology order. Because $b_{i+1} - b_i > \frac{1}{12}$ for all $i < n$, there cannot be any path p along which both signals $s = 0$ and $s = 1$ reach any agent d. By Proposition 4, there is one sided information transmission across all pairs of agents e and d. However, there is significant signal decay: the sum of the lengths of the paths among pairs of agents is 72.

Let me compare network N with the following network N':

$$\begin{array}{ccccc} 1 & — & 3 & — & 5 \\ & & | & & \\ 2 & — & 4 & — & 6 \end{array}$$

While network N' can be represented as two linked stars, $1-3-5$ and $2-4-6$, the connections within these stars are among ideologically distant agents, whereas the connection across the stars is among ideologically close agents. This is the opposite of the optimal networks characterized in Proposition 4.

Nevertheless, one-sided communication takes place among every pair of agents e and d also in network N'. In fact, there are no bias reversals in any paths between $e \leq 3$ and $d \geq 4$ (or $e \geq 4$ and $d \leq 3$). And there is one-sided communication also along any other path with $d = 3$ or $d = 4$. This is because (i) agents 3 and 4 are so ideologically close that they may send each other either $s = 0$ or $s = 1$ in equilibrium (cf. Proposition 1), and (ii) there is a path without bias reversal from any agent e to either $d = 3$ or $d = 4$. An analogous argument shows that there is also one-sided communication along any path with $e = 3$ and $e = 4$.

The case in which $e = 6$ and $d = 5$, or $e = 5$ and $d = 6$, is more subtle. The path p from 6 to 5 goes through 3 and 4. Because $b_4 - b_5 < -\frac{1}{6(2-\rho)}$, the signal $s = 1$ is blocked. Nevertheless, because agent 6 is sufficiently close to 5, $b_6 - b_5 \leq \frac{1}{6(2-\rho)}$, she is willing to relay signal $s = 0$ along p, and this signal is not blocked by either 3 or 4. The converse argument holds when $e = 5$ and $d = 6$. By symmetry, the cases in which $e = 0$ and $d = 1$, or $e = 1$ and $d = 0$, are addressed in the same way.

Network N' dominates network N because it achieves the same information transmission with less decay. The sum of the lengths of the paths in N' is 58, whereas in network N it is 72. It can be further proved that N' is the efficient network for the above profile of biases. This is because it achieves the maximum possible information transmission, with the shape closest to a star. $\Diamond$

Having concluded my analysis of optimal networks, the next section studies endogenous network formation and stability both in the case that links require the sponsorship of both connected agents, and when links can be individually sponsored.

5 Endogenous network formation and stability

Beyond network efficiency, an important question is how networks form endogenously and whether they are stable. This question has been addressed by Jackson and Wolinsky (1996) in terms of "pairwise stability." The idea is that every link needs the cooperation of both linked agents to form and be maintained, and that agents do not forego the opportunity of forming mutually advantageous links. As is the case for optimality, also stability is assessed here "ex-ante", i.e., before the agents are informed by nature.

Pairwise stable networks. Pairwise stability can be described as a refinement of the set of Nash Equilibria of a "bilateral sponsorship" endogenous network formation game, in which the agents simultaneously submit lists of links they sponsor, and each link forms if it is in the lists of both connected agents.

In the context of this paper suppose that, before nature discloses any information, all agents i simultaneously submit a profile $l_i \in \{0,1\}^n$ that identifies by $l_{ij} = 1$ the agents $j \neq i$ with whom i sponsors a link by paying the upfront cost $c/2$. The link between any agents i and j forms, $n_{ij} = 1$, if and only if $l_{ij} = 1 = l_{ji}$, else $n_{ij} = 0$. Each agent i pays $c/2$ for every j such that $l_{ij} = 1$ also in the event that the link with j does not form. Given the endogenous network $N = \{n_{ij}\}_{(i,j)\in\mathcal{N}^2}$, the game proceeds as before, with nature disclosing d and E publicly and s_i privately to each agent $i \in I$, followed by T rounds of information transmission along N, and by the decision $\hat{y}_d$.

As in Jackson and Wolinsky (1996), I consider two different pairwise stability concept. A network N is said (weakly) pairwise stable if two conditions are met. First, N obtains as a Nash equilibrium of the bilateral sponsorship game, i.e., for every link $n_{ij} = 1$, both agents i and j are better off with the network N, than with the network $\hat{N}$ in which $\hat{n}_{ij} = \hat{n}_{ji} = 0$ and $\hat{n}_{hk} = n_{hk}$ for every other pair h, k, and not paying the cost $c/2$ for the severed link between i and j. Second, for every link $n_{ij} = 0$, either agent i or j, or both, are better off with the network N, than with the network $\hat{N}$ in which $\hat{n}_{ij} = \hat{n}_{ji} = 1$ and $\hat{n}_{hk} = n_{hk}$ for every other pair h, k, and paying the cost $c/2$ of the added link between i and j.[35]

In the second pairwise stability concept I consider, the second requirement is strengthened to

[35] Multiple equilibria are unavoidable in the bilateral sponsorship game, as a best response of any agent i to $\ell_{ji} = 0$ is $\ell_{ij} = 0$ regardless of whether the link with j would be beneficial or not.

require that there are no "joint incentives" to form extra links in the network N. I demand that, for every link $n_{ij} = 0$, the sum of the equilibrium payoffs of the agents i and j with the network N is larger than the sum of these agents' payoffs with the network $\hat{N}$ in which $\hat{n}_{ij} = \hat{n}_{ji} = 1$ and $\hat{n}_{hk} = n_{hk}$ for every other pair h, k, including the cost c of the added link between i and j. Under this "strong" pairwise stability concept, a network N is unstable if there is an agent i who is so interested in forming a link with another agent j, to be willing to subsidize j in case the value of the extra link to j falls short of the cost $c/2$.

I assume that for every agent i, the weight α_{ii} is sufficiently close to one that the main concern of agent i is to obtain information from the other agents j for her decision $\hat{y}_i$, rather than improving their information. By repeating the analysis of Section 4, I prove that the optimal networks obtain as Nash equilibria of the bilateral sponsorship game, under conditions that are qualitatively analogous. For small decay and intermediate link costs, the ordered line L obtains as Nash equilibrium when ideological preferences are divergent, in the sense that $b_{i+1} - b_i \geq \frac{1}{2(n+2)}$ for all $i < n$. When the agents are partitioned in ideological clusters, there is a Nash equilibrium in which the clusters organize as stars whose centers are connected along the ideological order, and there are no other links across stars. And most importantly, for low decay and intermediate link cost, the optimal networks are uniquely implemented as strongly pairwise stable networks.[36]

Proposition 5 *Suppose the agents simultaneously choose whether or not to sponsor links with each other in the bilateral sponsorship game. After the network is formed, every agent may receive information on θ.*

1. For diverse ideologies, $b_{i+1} - b_i \geq \frac{1}{2(n+2)}$ for all $i < n$, there exist $\overline{\alpha} < 1$, $\bar{\delta}' < 1$ and $\underline{c}'(\delta) < \overline{c}'(\delta)$, such that when $\alpha_{ii} > \overline{\alpha}$ for all i, $\delta > \bar{\delta}'$ and $c \in (\underline{c}', \overline{c}')$, the ordered line L obtains as Nash equilibrium and the unique strongly pairwise stable network.

2. Suppose there are $M \leq n$ clusters of agents $C_1, ..., C_M$ such that for all $m = 1, ..., M$ and all $i \in C_m$, $|b_i - b_j| \leq \frac{1}{2(n+2)(n+1)}$ for all $j \in C_m$ and $b_i - b_k > \frac{1}{2(n+2)}$ for all $k \in C_{m-1}$. Then, there exist $\overline{\alpha} < 1$, $\bar{\delta}' < 1$ and $\underline{c}'(\delta) < \overline{c}'(\delta)$ such that, when $\alpha_{ii} > \overline{\alpha}$ for all i, $\delta > \bar{\delta}'$ and $c \in (\underline{c}', \overline{c}')$, in every strongly pairwise stable network, (i) the agent(s) in every cluster C_m are organized as a star S_m; (ii) the center of every star S_m, $m = 2, ..., M-1$, is linked with the centers of the stars S_{m-1} and S_{m+1}, and there are no other links across clusters.

These results are intuitive. For small decay and intermediate costs, the ordered line L obtains as Nash equilibrium of the bilateral sponsorship game because no agent would profit by breaking any link, and cannot unilaterally establish alternative links. And the same reasons that make L

[36] The thresholds δ', $\underline{c}'$ and $\overline{c}'$ of Proposition 5 below need not coincide with the analogous thresholds of Propositions 2 - 4.

the optimal network imply that L is also the unique strongly pairwise stable network. In fact, L is the unique minimally connected network in which the path between every pair of agents has no bias reversals. It is then the unique strongly pairwise stable network, when the link cost is not so low that agents wish to pay for duplicate paths, nor it is so high that pairs of agents would forego the opportunity to exchange information by co-sponsoring a link.

However, I also show in the appendix that the ordered line L is not uniquely selected by weak pairwise stability, even when every pair of agents is ideologically distant. Other trees such as all stars are weakly pairwise stable networks. While in every star there exists pairs of agents i and j such that the path from i and j has a bias reversal, it is never the case that also the path from j to i has bias reversals. The star is then weakly pairwise stable because, although agent i would benefit by forming a link with j, agent j would not and prefers not sponsor the link. Nevertheless, not all networks are weakly pairwise stable. I show in the appendix that weak pairwise stability selects minimally connected networks and may rule out non-ordered lines.

The same reasoning demonstrates that when the agents are partitioned in ideological clusters, every strongly pairwise stable network is such that the clusters organize as stars, the centers are connected along the ideological order and there are no other links across the clusters.

Having concluded that the optimal networks of Propositions 2 - 4 obtain as Nash equilibria of a bilateral network sponsorship game, and are the unique strongly pairwise stable networks, I now turn to consider endogenous network formation in a game of unilateral sponsorship.

Unilateral sponsorship game. Unlike in the game of bilateral sponsorship, suppose that each link is formed and maintained by one of the two connected agents (as in Bala and Goyal, 2000). Again, before nature discloses any information, all agents i simultaneously submit a link proposal profile $l_i \in \{0,1\}^n$. Here, a link between any pair of agents i and j forms as long as one of them pays for the link: $n_{ij} = 1$ if and only if $l_{ij} + l_{ji} > 0$. For brevity, I study this game focusing on the case of diverse ideologies, $b_{i+1} - b_i \geq \frac{1}{2(n+2)}$ for all $i < n$. Again, I focus on the case that α_{ii} is close to one for every agent i.

I show that the ordered line L obtains as Nash equilibrium of the unilateral sponsorship game, with small decay and intermediate link cost, if and only if there are at most five agents. In the equilibrium implementing L, I further establish that every link is sponsored by the most ideologically moderate agent, i.e., by the agent closer to the median agent(s) in $\mathcal{N}$.

Proposition 6 *Suppose that n ideologically diverse agents, $b_{i+1} - b_i \geq \frac{1}{2(n+2)}$ for all $i < n$, simultaneously choose whether or not to unilaterally sponsor links with each other. After the network is formed, every agent may receive information on θ.*

1. For $n \leq 5$, there exist $\overline{\alpha} < 1$, $\overline{\delta}'' < 1$ and $\underline{c}''(\delta) < \overline{c}''(\delta)$ such that when $\alpha_{ii} > \overline{\alpha}$ for all i, $\delta > \overline{\delta}''$ and $c \in (\underline{c}'', \overline{c}'')$, the ordered line L obtains as Nash equilibrium of unilateral sponsorship game. For $n = 4, 5$ and $\delta < 1$, every link $n_{ij} = 1$ is sponsored by the most moderate among i and j.[37]

2. For $n \geq 6$, the ordered line L cannot obtain as Nash equilibrium of the unilateral sponsorship game, when $\delta < 1$ and α_{ii} is sufficiently close to one for all i.

I prove this result first by showing that the ordered line L obtains as an equilibrium for $\delta < 1$ only if all links are paid by the most moderate agent. This is best explained taking the case of 5 agents as an example. The link between agents 2 and 3 in L cannot be sponsored by 2 in equilibrium, because she would prefer to deviate and connect to 4, instead. Through agent 4, agent 2 has access to the same set of agents, 3, 4 and 5, but with shorter paths (and none of these paths has bias reversals, as is the case when linking to 3). Instead, agent 3 has no incentive to deviate when sponsoring the link to 2, because switching to 1 gives her exactly the same payoffs. This "shortcut argument" holds for any n and for any pair of ideological neighbors, to conclude that L obtains as a Nash equilibrium only if all links are paid by the most moderate agent.

As a consequence, whether the ordered line L is implemented as an equilibrium depends on whether or not the most moderate agent of every pair of ideological neighbors would benefit by bypassing the link with her neighbor, instead of sponsoring it. In turn, this depends on the length of the line of agents she reaches through the link with her neighbor. As long as $n \leq 5$, there are not enough agents for any shortcut to be a profitable deviation, and L obtains as a Nash equilibrium. But for any $n \geq 6$, the ordered line L cannot be an equilibrium. In this case, neither agent 3 nor 4 are willing to sponsor a link between each other, as each of them prefers to bypass that link and find a shortcut.

Having established that the optimal network L cannot obtain as a Nash equilibrium of the unilateral sponsorship game for $n \geq 6$ agents, I conclude this section by explicitly constructing the optimal Nash equilibrium for $n = 6$.

Example 2. Suppose there are 6 agents with diverse ideologies, $b_{i+1} - b_i > \frac{1}{2(n+2)}$ for all $i < n$, who simultaneously choose whether or not to unilaterally sponsor links of cost c. After the network N is formed, a decision-maker d is randomly drawn, and every agent may receive information on θ. For every agent i, the weight α_{ii} is so close to one that i form links just to maximize the information she receives from the other agents. The probability of decay is small. The link cost c is sufficiently low that each agent i would rather pay c than having signals blocked from her, but c is not so low that i wants to form duplicate links to avoid signal decay.

[37] In the case of $n = 3$, every link in L can be sponsored by either of the two linked agents.

In light of Proposition 6, there is no Nash equilibrium that implements the ordered line L. Consider the strategy profile l such that $l_{21} = 1$, $l_{32} = 1$, $l_{35} = 1$, $l_{42} = 1$, $l_{45} = 1$, $l_{56} = 1$, and $l_{ij} = 0$ for all other i, j. Agent 2 sponsors a link with 1, agent 3 sponsors a link with 2 and 5, and symmetrically agent 5 sponsors a link with 6, and agent 4 sponsors a link with 5 and 2.

The profile l yields a network N not minimally connected. As in the ordered line L, no signal is blocked from any agent. To establish that l is a Nash equilibrium, I check that no agent prefers to deviate. Evidently, this is not the case for the agents 1 and 6 who do not pay for any link. Agents 2 and 5 do not gain by severing their links to 1 and 6 as they would lose information; nor do 3 and 4 by severing the links with 2 and 5 and sponsoring links with 1 and 6 instead. Finally, the link preferred by 3 to reach the agents 4, 5 and 6 is the link with 5; and symmetrically, the best link for 4 to access 1, 2 and 3 is the link with 2. Hence, l is a Nash equilibrium. Because the network N induced by l is as close as possible to the optimal network L, I conclude that l is the optimal Nash equilibrium. $\diamondsuit$

6 Conclusion

Motivated by political economy applications such as networks of policy-makers, interest groups, or judges, I have formulated and solved a model of information transmission in networks of ideological agents. In line with early theoretical network analyses, every optimal and stable network is a star, when agents' preferences are sufficiently close. But when all agents' ideologies are sufficiently diverse, I have found that the optimal network is the line in which agents are ordered according to their ideologies. This network obtains as Nash equilibrium of a game in which each link requires sponsorship by both connected agents, and it is the unique strongly pairwise stable network.

In the case agents are partitioned in ideologically diverse clusters, each composed of agents with similar views, I have proved that it is optimal for all agents that the clusters organize as factions, whose leaders communicate with the leaders of the ideologically nearest factions. Formally, each cluster organizes as a star, and the only link across such stars connect the centers of ideological neighbor stars. Again, these optimal networks obtain as Nash equilibria of a bilateral link sponsorship game, and are uniquely strongly pairwise stable.

My results suggest positive and normative rationales for why factionalization and connections between ideologically close agents are so common in political networks. Instead, such 'horizontal' connections are less likely useful in organizations where the agents' preferences are more closely aligned, such as the army or some commercial companies.

This paper may be useful for future theoretical and empirical research on political networks. The analysis is based on a simple model and functional forms, it should be easy to test results in lab experiments. By and large, my findings confirm and explain empirical stylized facts. Political network structures can be described as a collection of subnetworks, differentiated by political preferences, and loosely connected among each other. Each subnetwork is tightly connected and hierarchically differentiated between a core of leaders to which less prominent agents are connected. As well as stable, my analysis suggests that these network structures are efficient for the purposes of information transmission.

I have kept my model as simple as possible, in order to deliver insights in the clearest fashion. This paper's analysis can be generalized and modified in several directions. For example, it would be possible to study the robustness of a political network when an antagonist attempts to remove agents from the network. This could happen within a party's network of elected officials when politicians are up for re-election. The opposing parties may try to disrupt the party's network by strategically allocating campaign resources. This question could then be modelled as a variation of the Colonel Blotto game (Borel, 1921).

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Appendix A: Omitted Proofs

Proof of Lemma 1. In every equilibrium, the final decision $\hat{y}_d$ of agent d maximizes $E[u_d(\hat{y}_d;\theta)|\omega_d] = -E[(\hat{y}_d - \theta - b_d)^2|\omega_d]$, where $\omega_d \in \{0, 1, \phi\}$ denotes agent d's equilibrium information. Hence the strategy of agent d is $y_d = b_d + E[\theta|\omega_d]$. If the signal s reaches agent d, she plays

$$y_d = b_d + E[\theta|s] = \begin{cases} b_d + 1/3 & \text{if } s = 0 \\ b_d + 2/3 & \text{if } s = 1. \end{cases}$$

If s does not reach agent d, she plays $y_d = b_d + E[\theta|\phi]$.

Consider an arbitrary agent j informed of signal s at an arbitrary history h^t. Agent j chooses whether or not to relay s to the next agent k on the path p. Omitting its dependence on h^t for brevity, let $Q_{jd}(s)$ be the equilibrium probability that s reaches agent d if agent j plays $r_{jkdi}(s; h^t) = s$, thus relaying s to k. With probability $1 - Q_{jd}(s)$, agent d's information is $\omega_d = \phi$. Likewise, let $Q_{jd}(\phi)$ be the equilibrium probability that the signal s reaches d if agent j plays $r_{jkdi}(s; h^t) = \phi$ and withholds s from k.

Agent j's expected payoffs if relaying signal s to the next agent k on the path p at history h^t are:

$$u_{jd}(s; h^t) = -\alpha_{jd}E[(b_d + E[\theta|s] - b_j - \theta)^2|s]Q_{jd}(s) - \alpha_{jd}E[(b_d + E[\theta|\phi] - b_j - \theta)^2|s][1 - Q_{jd}(s)],$$

whereas agent j's payoffs if not relaying s are:

$$u_{jd}(\phi; h^t) = -\alpha_{jd}E[(b_d + E[\theta|s] - b_j - \theta)^2|s]Q_{jd}(\phi) - \alpha_{jd}E[(b_d + E[\theta|\phi] - b_j - \theta)^2|s][1 - Q_{jd}(\phi)].$$

Hence, agent j strictly prefers to relay s if and only if:

$$\begin{aligned} u_{jd}(s; h^t) - u_{jd}(\phi; h^t) &= -\alpha_{jd}[Q_{jd}(s) - Q_{jd}(\phi)] \left\{E[(b_d + E[\theta|s] - b_j - \theta)^2|s] - E[(b_d + E[\theta|\phi] - b_j - \theta)^2|s]\right\} \\ &\equiv -\alpha_{jd}\Delta Q_{jd}(s; h^t)\Delta\mathcal{L}_{jd}(s) > 0. \qquad (5) \end{aligned}$$

At time t, there are $T - t + 1$ periods left to deliver s to d, including the current period. Suppose that agent j is ℓ_j steps away from d on the path p. Say that each successor of j on p relays signal s to the next agent on p at every history. Then, if j relays s to the next agent k on p, the signal s reaches d with probability $q(T - t + 1, \ell_j)$, whereas if j withholds it, s reaches d with probability at most $q(T - t, \ell_j)$. Because $q(T - t + 1, \ell_j) > q(T - t, \ell_j)$, it follows that $\Delta Q_{jd}(s; h^t) > 0$. If $\Delta\mathcal{L}_{jd}(s) < 0$, then j relays s to her next agent k on p, whereas if $\Delta\mathcal{L}_{jd}(s) > 0$, then j withholds s. Because $\Delta\mathcal{L}_{jd}(s)$ is independent of t and h^t, I have concluded that if every j's successor on the path p relays s to her next agent on p at every history, then either j relays s to her next agent on p at every history h^t, or j withholds s at every history h^t. I also note that the argument (trivially) covers the last agent on the path p before d, as she does not have any successors other than d.

By induction, I then conclude that in every equilibrium:

1. if $\Delta\mathcal{L}_{jd}(s) < 0$ for all agents j on the path p, then each agent j relays s to the next agent k on p at every history h^t, so that s reaches d with probability $q(T, \ell)$.

2. if there exists an agent j such that $\Delta\mathcal{L}_{jd}(s) > 0$, then signal s does not ever reach d, as on every path it is blocked by at least one such agent.

Next, I simplify $\Delta\mathcal{L}_{jd}(s)$ as follows:

$$\begin{aligned}
\Delta\mathcal{L}_{jd}(s) &= E[(b_d + E[\theta|s] - \theta - b_j)^2 - (b_d + E[\theta|\phi] - \theta - b_j)^2|s] \\
&= E[(b_d - b_j - \theta)^2|s] + E[\theta|s]^2 + 2E[b_d - b_j - \theta|s]E[\theta|s] \\
&\quad - E[(b_d - b_j - \theta)^2|s] - E[\theta|\phi]^2 - 2E[b_d - b_j - \theta|s]E[\theta|\phi] \\
&= E[\theta|s][E[\theta|s] + 2(b_d - b_j) - 2E[\theta|s]] + E[\theta|s]E[\theta|\phi] \\
&\quad - E[\theta|\phi][E[\theta|\phi] + 2(b_d - b_j) - 2E[\theta|s]] - E[\theta|s]E[\theta|\phi] \\
&= E[\theta|s][E[\theta|\phi] + 2(b_d - b_j) - E[\theta|s]] - E[\theta|\phi][E[\theta|\phi] + 2(b_d - b_j) - E[\theta|s]] \\
&= [E[\theta|\phi] - E[\theta|s]][2(b_j - b_d) - (E[\theta|\phi] - E[\theta|s])] \\
&\propto \begin{cases} 2(b_j - b_d) - (E[\theta|\phi] - E[\theta|s]) & \text{if } s = 0 \\ -2(b_j - b_d) + (E[\theta|\phi] - E[\theta|s]) & \text{if } s = 1. \end{cases}
\end{aligned} \tag{6}$$

Hence, there are two cases in which $\Delta\mathcal{L}_{jd}(s) < 0$ and agent j strictly prefers to relay signal s (unless it is blocked by her successors in equilibrium):

$$\begin{aligned}
s = 0 \text{ and } b_j - b_d &< \frac{E[\theta|\phi] - E[\theta|s]}{2} = \frac{E[\theta|\phi] - 1/3}{2}, \\
s = 1 \text{ and } b_j - b_d &> \frac{E[\theta|\phi] - E[\theta|s]}{2} = \frac{E[\theta|\phi] - 2/3}{2},
\end{aligned} \tag{7}$$

whereas $\Delta\mathcal{L}_{jd}(s) > 0$ in the converse, complementary cases, and agent k strictly prefers to withhold signal s.

As a consequence, I show that every agent j on the path p such that $b_j > b_d$ (respectively, $b_j < b_d$) always relays the signal $s = 1$ (resp., $s = 0$) to the next agent k on p in equilibrium, unless s is blocked by j's successors.

To prove it, I first note that the event that s does not reach agent d is on the path of play of every possible equilibrium, because when an agent j relays s to the next agent k on p, the signal s is lost with probability $1 - \delta > 0$. As a result, agent d's equilibrium beliefs $E[\theta|\phi]$ are pinned down by the Bayes rule, and are therefore bounded strictly between $E[\theta|s = 0] = 1/3$ and $E[\theta|s = 1] = 2/3$. When $s = 0$, the inequality $E[\theta|\phi] > E[\theta|s = 0]$ implies that conditions (7) holds when $b_j < b_d$; and conversely when $s = 1$, the inequality $E[\theta|\phi] < E[\theta|s = 1]$ implies that conditions (7) holds when $b_j > b_d$. In these two cases, agent j prefers to relay s to the next agent k on p at every history h^t and in every equilibrium.

Conversely, every agent j on the path p such that $b_j - b_d > \frac{1}{6[2-\rho q(T,\ell)]}$ (resp., $b_j - b_d < -\frac{1}{6[2-\rho q(T,\ell)]}$) withholds signal $s = 0$ (resp. $s = 1$) from the next agent k on p at every history h^t in every equilibrium, unless s is blocked by j's successors.

In every equilibrium in fact, the expected value $E[\theta|\phi]$ is weakly smaller than in the equilibrium in which every agent j on p relays signal $s = 0$ with probability one, and signal $s = 1$ is blocked from agent d. In this "boundary" equilibrium,

$$E[\theta|\phi] = \int_0^1 \theta f(\theta|\phi) d\theta = \int_0^1 \theta \frac{\theta + (1-\theta)[1 - \rho + \rho(1 - q(T,\ell))]}{\int_0^1 [\theta + (1-\theta)[1 - \rho + \rho(1 - q(T,\ell))]] d\theta} d\theta = \frac{1 - \rho q(T,\ell)/3}{2 - \rho q(T,\ell)}, \tag{8}$$

because with probability θ, it is the case that $s = 1$, and then s does not ever reach d; whereas with probability $1 - \theta$, it is the case that $s = 0$, so that s does not reach d either when not observed by the expert agent e, with probability $1-\rho$, or when lost in transmission, with probability $1-q(T,\ell)$. The inequality $E[\theta|\phi] \leq \frac{1-\rho q(T,\ell)/3}{2-\rho q(T,\ell)}$ implies that when $s=0$, $E[\theta|\phi]-E[\theta|s] = \frac{1-\rho q(T,\ell)/3}{2-\rho q(T,\ell)} - 1/3 \leq \frac{1}{3[2-\rho q(T,\ell)]}$, so that conditions (7) cannot hold when $b_j - b_d > \frac{1}{6[2-\rho q(T,\ell)]}$, and as a result $s=0$ is blocked in equilibrium. Conversely, when $b_j - b_d < -\frac{1}{6(2-\rho q(T,\ell))}$ and $s=1$, conditions (7) cannot hold and $s=1$ is blocked. ■

Proof of Proposition 1: Full communication. I prove that there exists an equilibrium in which every agent j on the path p relays both $s=0$ and $s=1$ at every history h^t if and only if $|b_j - b_d| \leq \frac{1}{12}$ for all j. I note that, when the agents conform to these strategies, it is the case that $E[\theta|\phi] = \frac{1}{2}$, because the event that signal s does not reach agent d is independent of whether $s=0$ or $s=1$. Hence, conditions (7) hold if and only if either $s=0$ and $b_j - b_d < \frac{E[\theta|\phi]-1/3}{2} = \frac{\frac{1}{2}-1/3}{2} = \frac{1}{12}$, or if $s=1$ and $b_j - b_d > \frac{\frac{1}{2}-2/3}{2} = -\frac{1}{12}$. As a result, each agent j has no incentive to deviate from the equilibrium strategies if and only if $|b_j - b_d| \leq \frac{1}{12}$.

Each agent's i expected payoffs $U_{ied}(p)$ are as calculated follows. Because neither $s=0$ nor $s=1$ is ever withheld by any agent j on p, they both reach agent d with probability $\rho q(T,\ell)$. In either case, d chooses $\hat{y}_d = E[\theta|s] + b_d$, and each agent i's expected payoffs are, for both $s=0$ and $s=1$,

$$\begin{aligned}
&-\alpha_{id} E[(E[\theta|s] + b_d - \theta - b_i)^2|s] = -\alpha_{id} E[(E[\theta|s]-\theta)^2 + (b_d - b_i)^2 + (E[\theta|s]-\theta)(b_d-b_i)|s] \\
&\quad = -\alpha_{id}(b_d - b_i)^2 - \alpha_{id} E[(E[\theta|s]-\theta)^2|s] \\
&\quad = -\alpha_{id}(b_d - b_i)^2 - \alpha_{id}\frac{1}{18}.
\end{aligned}$$

With complementary probability $1-\rho q(T,\ell)$, agent d acts uninformed and chooses $y_d = \frac{1}{2} + b_d$. Each agent i's ex-ante payoffs are:

$$\begin{aligned}
&-\alpha_{id} E[(\frac{1}{2} + b_d - \theta - b_i)^2] = -\alpha_{id} E[(\frac{1}{2}-\theta)^2 + (b_d - b_i)^2 + (\frac{1}{2}-\theta)(b_d - b_i)] \\
&\quad = -\alpha_{id}(b_d - b_i)^2 - \alpha_{id} E[(\frac{1}{2}-\theta)^2] \\
&\quad = -\alpha_{id}(b_d - b_i)^2 - \alpha_{id}\frac{1}{12}.
\end{aligned}$$

Because the signals $s=0$ and $s=1$ are equally likely ex-ante, I can wrap up as follows:

$$\begin{aligned}
U_{ied}(p) &= -\alpha_{id}(b_d - b_i)^2 - \alpha_{id}\rho q(T,\ell)\frac{1}{18} - \alpha_{id}[1-\rho q(T,\ell)]\frac{1}{12} \\
&= -\alpha_{id}(b_i - b_d)^2 - \alpha_{id}\frac{1}{18}(1 + \frac{1-\rho q(T,\ell)}{2}).
\end{aligned}$$

Proof of Proposition 1: Communication breakdown. Suppose that there exists agents j,k on the path p such that $b_j - b_d > \frac{1}{6[2-\rho q(T,\ell)]}$ and $b_k - b_d < -\frac{1}{6[2-\rho q(T,\ell)]}$. By Lemma 1, both signals $s=0$ and $s=1$ are blocked in equilibrium. Agent d acts uninformed and chooses $y_d = \frac{1}{2} + b_d$. Each agent i's ex-ante payoffs are:

$$\begin{aligned}
U_{ied}(p) &= -\alpha_{id} E[(\frac{1}{2} + b_d - \theta - b_i)^2] \\
&= -\alpha_{id}(b_d - b_i)^2 - \alpha_{id}\frac{1}{12}.
\end{aligned}$$

Proof of Proposition 1: One sided equilibrium. I first consider equilibrium in which every agent j on the path p relays $s = 0$ and at least one j blocks $s = 1$. With these strategies, agent d's expectation of θ upon not observing s is $E[\theta|\phi] = \frac{1-\rho q(T,\ell)/3}{2-\rho q(T,\ell)}$, cf. expression (8). Hence, each agent j on p prefers not to deviate from relaying $s = 0$ if and only if $b_j - b_d \leq \frac{E[\theta|\phi]-1/3}{2} = \frac{1-\rho q(T,\ell)/3}{2[2-\rho q(T,\ell)]} - \frac{1}{6} = \frac{1}{6[2-\rho q(T,\ell)]}$. Further, $s = 1$ is blocked if and only if $b_j - b_d < \frac{[E[\theta|\phi]-2/3]}{2} = \frac{1-\rho q(T,\ell)/3}{2[2-\rho q(T,\ell)]} - \frac{1}{3} = -\frac{1-\rho q(T,\ell)}{6[2-\rho q(T,\ell)]}$ for some j on p.

In sum, the one sided equilibrium in which $s = 0$ is relayed and $s = 1$ is blocked exists if and only if $b_j - b_d \leq \frac{1}{6[2-\rho q(T,\ell)]}$ for all j on p, and $b_j - b_d < -\frac{1-\rho q(T,\ell)}{6[2-\rho q(T,\ell)]}$ for some j on p. A symmetric argument shows that the one sided equilibrium in which $s = 1$ is relayed and $s = 0$ is blocked exists if and only if $b_j - b_d \geq -\frac{1}{6[2-\rho q(T,\ell)]}$ for all j on p, and $b_j - b_d > \frac{1-\rho q(T,\ell)}{6[2-\rho q(T,\ell)]}$ for some j on p.

Because $\rho q(T,\ell) \in (0,1]$, it is the case that $\frac{1-\rho q(T,\ell)}{6[2-\rho q(T,\ell)]} < \frac{1}{12} < \frac{1}{6[2-\rho q(T,\ell)]}$. Thus the area of existence of the one sided equilibria overlaps with the area of existence of the full communication equilibrium ($|b_j - b_d| \leq \frac{1}{12}$ for all j on p), when $|b_j - b_d| \leq \frac{1}{6[2-\rho q(T,\ell)]}$ for all j on p, and $b_j - b_d < -\frac{1-\rho q(T,\ell)}{6[2-\rho q(T,\ell)]}$ or $b_j - b_d > \frac{1-\rho q(T,\ell)}{6[2-\rho q(T,\ell)]}$ for some j on p. If instead $|b_j - b_d| > \frac{1}{12}$ for some j on p, then the full communication equilibrium does not exist and the one-sided equilibria is optimal. And note that the analysis conducted so far covers the whole parameter space. If the path p has bias reversals (there exist j, k such that $b_j - b_d < -\frac{1}{6[2-\rho q(T,\ell)]}$ and $b_j - b_d > \frac{1}{6[2-\rho q(T,\ell)]}$), then communication breaks down in equilibrium. Else, if $|b_j - b_d| \leq \frac{1-\rho q(T,\ell)}{6[2-\rho q(T,\ell)]}$ for all j on p, so that one sided equilibria do not exist, then $|b_j - b_d| < \frac{1}{12}$ for all j on p, so that full communication equilibrium exists. And for all other bias profiles $\mathbf{b}$, there exist one sided communication equilibria.

The calculation of the agents' ex-ante payoffs in one-sided equilibria is postponed to the next part of the proof, as a special case of the ex-ante payoffs in the optimal mixed strategy equilibria.

Proof of Proposition 1: Optimal mixed strategy equilibrium. Let me first note that it is enough that one agent j on p blocks a signal realization s for it not to reach d. Second, conditions (7) imply that if any agent j such that $b_j > b_d$ is indifferent on whether or not $s = 0$ reaches the decision maker d, then any agent k such that $b_k > b_j$ strictly prefers that $s = 0$ is blocked. And a converse argument holds for $s = 1$ and agents j, k such that $b_k < b_j < b_d$. Third I note that, generically, there may be at most one agent who is indifferent on whether or not the signal contrary to her bias reaches d. Hence, I do not consider the possibility of randomization on both signal realizations $s = 0$ and $s = 1$. Fourth, I note that for the parameter range in which the one-sided (pure strategy) equilibrium characterized above does not exist, there is full communication breakdown in every equilibrium.

These considerations imply that the only fighting chance for any mixed strategy equilibrium to maximize the agents' payoffs are that the most biased agent $\bar{j}$, the agent j whose $|b_j - b_d|$ is maximal, randomizes on the signal realization s contrary to her bias, and all other agents relay both $s = 0$ and $s = 1$ at every history. This is the only case in which more information is released than with one-sided information transmission.

Suppose that the most biased agent $\bar{j}$ is such that $b_{\bar{j}} > b_d$. Agent $\bar{j}$ relays $s = 1$ at every history, and randomizes (at least) at some histories when $s = 0$. For this to be the case, $\bar{j}$ must be indifferent between relaying and withholding $s = 0$. Hence it must be the case that $b_{\bar{j}} - b_d = \frac{E[\theta|\phi]-1/3}{2}$, cf. condition (7). As a consequence, agent $\bar{j}$ is indifferent between relaying and withholding $s = 0$ in *every* history h^t, and I may focus on the mixed strategy equilibrium in which $\bar{j}$ always randomizes and always with the same probability σ of relaying $s = 0$.

For this strategy profile, agent d's expectation of θ if not receiving s is:

$$E[\theta|\phi] = \int_0^1 \theta \frac{\theta[1-\rho+\rho(1-\sigma)+\rho\sigma(1-q(T,\ell))] + (1-\theta)[1-\rho+\rho(1-q(T,\ell))]}{\int_0^1 [\theta[1-\rho+\rho(1-\sigma)+\rho\sigma(1-q(T,\ell))] + (1-\theta)[1-\rho+\rho(1-q(T,\ell))]]d\theta} d\theta$$
$$= \frac{1-\rho q(T,\ell)(1+2\sigma)/3}{2-\rho q(T,\ell)(1+\sigma)}.$$

For agent $\bar{j}$ to be indifferent between relaying and withholding $s=0$, the indifference condition simplifies to:

$$b_{\bar{j}} - b_d = \frac{E[\theta|\phi]-1/3}{2} = \frac{\frac{1-\rho q(T,\ell)(1+2\sigma)/3}{2-\rho q(T,\ell)(1+\sigma)} - 1/3}{2} = \frac{1-\sigma\rho q(T,\ell)}{6[2-\rho q(T,\ell)(1+\sigma)]}.$$

Hence, the mixed strategy σ takes the form:

$$\sigma = \frac{1-12(b_{\bar{j}}-b_d)+6\rho q(T,\ell)(b_{\bar{j}}-b_d)}{\rho q(T,\ell)[1-6(b_{\bar{j}}-b_d)]} \tag{9}$$

which is easily shown to be admissible, i.e., $\sigma \in [0,1]$, if and only if $\frac{1}{12} \leq b_{\bar{j}} - b_d \leq \frac{1}{6[2-\rho q(T,\ell)]}$.

Further, note that for all agents $k \neq \bar{j}$ such that $b_k > b_d$, it is the case that $b_k - b_d < \frac{1-\sigma\rho q(T,\ell)}{6[2-\rho q(T,\ell)(1+\sigma)]}$. They all strictly prefer to relay signal realization $s=0$ despite it being contrary to their biases.

Now, consider all agents k such that $b_k < b_d$. They relay $s=0$ at every history h^t, and they relay $s=1$ if and only if

$$b_k - b_d > \frac{E[\theta|\phi]-2/3}{2} = \frac{\frac{1-\rho q(T,\ell)(1+2\sigma)/3}{2-\rho q(T,\ell)(1+\sigma)} - 2/3}{2} = -\frac{1-\rho q(T,\ell)}{6[2-\rho q(T,\ell)(1+\sigma)]} = -\frac{1}{6} + b_{\bar{j}} - b_d,$$

where the last equality is obtained by plugging in (9). And I note that this condition implies that $b_k - b_d \geq -\frac{1}{12}$ when $b_{\bar{j}} - b_d \geq \frac{1}{12}$, which implies that $|b_k - b_d| < |b_{\bar{j}} - b_d|$ for all agents k with $b_k < b_d$.

In sum, I have proved that when $\frac{1}{12} < \bar{b} - b_k \leq \frac{1}{6(2-\rho q(T,\ell))}$ and $\underline{b} \geq \bar{b} - \frac{1}{6}$, then the optimal equilibrium is in mixed strategies: Every agent j relays $s=1$, and $s=0$ is relayed by every agent except $\bar{j}$ who relays $s=0$ with probability σ at every history h^t. A symmetric characterization holds when $-\frac{1}{6(2-p\rho)} \leq \underline{b} - b_k < -1/12$ and $\bar{b} \leq \frac{1}{6} + \underline{b}$.

I now calculate the agents' ex-ante payoffs. Without loss of generality, I again consider the mixed-strategy equilibrium in which the disclosure of $s=0$ is randomized. Hence, $s=1$ reaches d with probability $\rho q(T,\ell)$, whereas $s=0$ reaches d with probability $\sigma\rho q(T,\ell)$. Because $s=0$ and $s=1$ are equally likely ex-ante, each agent i's expected payoffs ex-ante can be written as:

$$U_{ied}(p) = -\alpha_{id}(b_d-b_i)^2 - \alpha_{id}\frac{\sigma\rho q(T,\ell)}{2}E[(1/3-\theta)^2|s=0] - \alpha_{id}\frac{\rho q(T,\ell)}{2}E[(2/3-\theta)^2|s=1]$$
$$-\alpha_{id}(1-\frac{1+\sigma}{2}\rho q(T,\ell))E[(E[\theta|\phi]-\theta)^2|\phi].$$

The last term is calculated as follows:

$$E[(E[\theta|\phi]-\theta)^2|\phi] = E[\theta^2|\phi] - (E[\theta|\phi])^2 = \frac{4-\rho q(T,\ell)(1+\sigma)-2\sigma\rho q(T,\ell)}{12-6\rho q(T,\ell)(1+\sigma)} - \left[\frac{1-\rho q(T,\ell)(1+2\sigma)/3}{2-\rho q(T,\ell)(1+\sigma)}\right]^2$$
$$= \frac{\rho^2 q(T,\ell)^2[(1+\sigma)^2+2\sigma]+6[1-\rho q(T,\ell)(1+\sigma)]}{18[2-\rho q(T,\ell)(1+\sigma)]^2}.$$

Plugging this term into the formula for U_{ied} and simplifying, I obtain:

$$\begin{aligned}U_{ied}(p) =& -\alpha_{id}(b_d - b_i)^2 - \alpha_{id}\frac{\sigma\rho q(T,\ell)}{2}\frac{1}{18} - \alpha_{id}\frac{\rho q(T,\ell)}{2}\frac{1}{18} \\ & -\alpha_{id}[1 - \frac{1+\sigma}{2}\rho q(T,\ell)][\frac{\rho^2 q(T,\ell)^2[(1+\sigma)^2 + 2\sigma] + 6[1-\rho q(T,\ell)(1+\sigma)]}{18[2-\rho q(T,\ell)(1+\sigma)]^2}] \\ =& -\alpha_{id}(b_d - b_i)^2 - \alpha_{id}\frac{1}{18}[1 + \frac{1-\rho q(T,\ell)(1+\sigma) + \sigma\rho^2 q(T,\ell)^2}{2-\rho q(T,\ell)(1+\sigma)}] \\ =& -\alpha_{id}(b_i - b_d)^2 - \alpha_{id}\frac{1}{18}(1 + 6|b_{\bar{j}} - b_d|[1-\rho q(T,\ell)]).\end{aligned}$$

One sided communication equilibria can be understood as the extreme case of the above mixed strategy equilibria in which $\sigma = 0$. The most extreme agent always withholds the signal realiazation contrary to her bias. Hence, each agent i's ex-ante payoffs in the one sided communication equilibria are:

$$U_{ied} = -\alpha_{id}(b_d - b_i)^2 - \alpha_{id}\frac{1}{18}[1 + \frac{1-\rho q(T,\ell)}{2-\rho q(T,\ell)}].$$

■

Proof of Proposition 2. I start by proving the second statement. Suppose that ideologies are divergent, i.e., that $b_{i+1} - b_i > \frac{1}{6(2-\rho)}$ for all $i < n$, so that $b_{i+1} - b_i > \frac{1}{6[2-\rho q(T,\ell)]}$ for all $\rho > 0$, $T \geq \ell > 0$, $\delta \leq 1$, and all $i < n$.

Proceeding in two steps, I first restrict attention to minimally connected networks (i.e., to trees), and show that the optimal tree is the ordered line L.

To prove this result, I start by noting that for any n, the ordered line L has a path p without bias reversals between every pair of agents e and d. If it is the case that $e < d$, then every agent $k = e, ..., d-1$ on the path p is such that $b_k < b_d$; whereas if $e > d$, then every agent $k = d+1, ..., e$ on p is such that $b_k > b_d$. By Proposition 1, this implies that whenever $e < d$, signal $s = 0$ reaches the decision maker d with probability $\rho q(T, |d-e|)$, and $s = 1$ does not reach d. When $e > d$, signal $s = 1$ reaches agent d with probability $\rho q(T, |d-e|)$, and $s = 0$ does not. Thus, simplifying the notation $\xi(d, \{e\})$, each agent i's payoffs are:

$$\begin{aligned}U_i(L) =& -\sum_{(e,d)\in\mathcal{N}^2} \xi_{de}\alpha_{id}\left\{(b_d - b_i)^2 + \frac{1}{18}[1 + \frac{1-\rho q(T,|d-e|)}{2-\rho q(T,|d-e|)}]\right\} \\ \to& -\sum_{(e,d)\in\mathcal{N}^2} \xi_{de}\alpha_{id}\left\{(b_d - b_i)^2 + \frac{1}{18}(1 + \frac{1-\rho}{2-\rho})\right\} \text{ as } \delta \to 1.\end{aligned}$$

Consider any tree N with at least a pair of agents e and d linked by a bias reversal path p. Because N is a tree, there are no other paths connecting e to d. By Proposition 1, both signals $s_e = 0$ and $s_e = 1$ are blocked from agent d along the path p. Let $P(N) = \{(e,d) : (b_k - b_d)(b_j - b_d) > 0$ for all k, j on the path between e and $d\}$ be the set of agent pairs (e,d) linked by a path p without bias reversals (note that $(e,d) \in P(N)$ if $e = d$, trivially). I let $\ell_N(e,d)$ be the length of the path between any pair e and d

of agents in the tree N. Then, each agent i's payoffs are:

$$U_i(N) = -\sum_{(e,d)\in P(N)} \xi_{de}\alpha_{id}\left\{(b_d-b_i)^2+\frac{1}{18}[1+\frac{1-\rho q(T,\ell_N(e,d))}{2-\rho q(T,\ell_N(e,d))}]\right\} - \sum_{(e,d)\notin P(N)} \xi_{de}\alpha_{id}\left\{(b_d-b_i)^2+\frac{1}{12}\right\}$$

$$\to -\sum_{(e,d)\in P(N)} \xi_{de}\alpha_{id}\left\{(b_d-b_i)^2+\frac{1}{18}(1+\frac{1-\rho}{2-\rho})\right\} - \sum_{(e,d)\notin P(N)} \xi_{de}\alpha_{id}\left\{(b_d-b_i)^2+\frac{1}{12}\right\} \text{ as } \delta\to 1.$$

Hence, for δ close to one, the payoff difference between the ordered line L and tree N is:

$$U_i(L)-U_i(N) \approx \sum_{(d,e)\notin P(N)} \xi_{de}\alpha_{id}[\frac{1}{12}-\frac{1}{18}(1+\frac{1-\rho}{2-\rho})] = \frac{1}{36}\frac{\rho}{2-\rho}\sum_{(d,e)\notin P(N)} \xi_{de}\alpha_{id} > 0.$$

Because every tree N has the same number of links, $n-1$, and hence the same aggregate link cost $c(N)=(n-1)c$, it then follows that for δ close to 1, any tree N with a bias reversal path between any pair of agents e and d is dominated by the ordered line L.

It is obvious that every line network N that does not conform with the agents' bias order includes pairs of agents e and d connected only by a bias reversal path.

Turning to consider trees that are not lines, I first notice that every such a tree N must contain a 4-agent star S as a sub-tree. Hence, if I show that every 4-agent star S includes at least a bias reversal path, then I have concluded that every tree N that is not a line has at least a bias reversal path, and the proof that L is the optimal tree is complete. In fact, if the "extreme-bias" agent 1 is the center of S, then there is a bias reversal path, for example, from agent $e=3$ to agent $d=2$. When the "moderate" agent 2 is the center of S, there is a bias reversal path from $e=4$ to $d=3$. The cases when 4 or 3 are the centers of S are analogous to when the centers are 1 or 2 respectively, by interchanging 1 with 4 and 2 with 3.

I have proved that the optimal minimally connected network is the ordered line L, for δ close to one.

The second step of the proof consists in showing the existence of thresholds $\underline{c}(\delta)$ and $\bar{c}(\delta)$ such that $\lim_{\delta\to1}\underline{c}(\delta)=0$ and $\lim_{\delta\to1}\bar{c}(\delta)=\bar{\bar{c}}$, and that for δ close to one and $c\in(\underline{c},\bar{c})$, the sum of the agents' ex-ante payoffs $U_i(L)$ with the ordered line L minus the sum of link costs $c(L)=(n-1)c$ is larger than with any connected network N with more links than L, and than with any network N that is not connected.

Consider any connected network N that is not minimally connected. Because $b_{i+1}-b_i > \frac{1}{6[2-\rho q(T,\ell)]}$ for all $\ell>0$, all $i=1...,n-1$, and all $\delta\le 1$, full communication and mixed strategy communication are impossible in equilibrium between any pair of agents e and d. Consider the hypothetical (and counterfactual) worst case in which there is one sided communication along every path in network N (including bias reversal paths). Even in this case, each agent i's ex-ante payoffs $U_i(N)$ are approximately the same as with the ordered line L, for δ close to one. In fact, let $q_N(T;e,d)$ be the probability that one signal realization s reaches d along one of the paths from e. Note that $q_N(T;e,d)\ge q(T,\underline{\ell}(e,d))$, where $\underline{\ell}(e,d)$ is the length of the shortest path from e to d. Then, because $\lim_{\delta\to1} q_N(T;e,d)\ge \lim_{\delta\to1} q(T,\underline{\ell}(e,d))=1$,

$$U_i(N) = -\sum_{(e,d)\in\mathcal{N}^2} \xi_{de}\alpha_{id}\left\{(b_d-b_i)^2+\frac{1}{18}[1+\frac{1-\rho q_N(T;e,d)}{2-\rho q_N(T;e,d)}]\right\}$$

$$\to -\sum_{(e,d)\in\mathcal{N}^2} \xi_{de}\alpha_{id}\left\{(b_d-b_i)^2+\frac{1}{18}(1+\frac{1-\rho}{2-\rho})\right\} = U_i(L) \text{ as } \delta\to 1.$$

By construction, network N has more than the $n-1$ links of L. Hence for $\delta = 1$ and $c > 0$, the ordered line L dominates every connected network N with loops, in terms of the sum of agents' ex-ante payoffs $U_i(N)$ minus the sum of link costs $c(N)$. By continuity, for δ close to one, there exists a threshold $\underline{c}$ function of δ with $\lim_{\delta\to 1} \underline{c}(\delta) = 0$, such that L dominates every connected network N with loops when $c > \underline{c}(\delta)$.

Now I consider networks N that are not connected, again taking the hypothetical worst case in which there is one sided information transmission even along bias reversal paths in N. Letting the set of pairs of agents linked via a path be $\bar{P}(N)$, the ex-ante payoffs of each agent i are then:

$$U_i(N) = -\sum_{(e,d)\in \bar{P}(N)} \xi_{de}\alpha_{id} \left\{ (b_d - b_i)^2 + \frac{1}{18}[1 + \frac{1-\rho q_N(T; e, d)}{2-\rho q_N(T; e, d)}] \right\} - \sum_{(e,d)\notin \bar{P}(N)} \xi_{de}\alpha_{id} \left\{ (b_d - b_i)^2 + \frac{1}{12} \right\}$$

$$\to -\sum_{(e,d)\in \mathcal{N}^2} (b_d - b_i)^2 - \frac{1}{18}(1 + \frac{1-\rho}{2-\rho}) \sum_{(e,d)\in \bar{P}(N)} \xi_{de}\alpha_{id} - \frac{1}{12} \sum_{(e,d)\notin \bar{P}(N)} \xi_{de}\alpha_{id} \text{ as } \delta \to 1.$$

Hence, for $\delta = 1$,

$$\sum_{i\in\mathcal{N}} U_i(L) - \sum_{i\in\mathcal{N}} U_i(N) = \frac{1}{36}\frac{\rho}{2-\rho} \sum_{(e,d)\notin \bar{P}(N)} \xi_{de} \sum_{i\in\mathcal{N}} \alpha_{id} > 0,$$

and any not connected network N that dominates L must have sum of link costs $c(N) < c(L) = (n-1)c$. Hence for $\delta = 1$, the ordered line L may be dominated by a not connected network N in terms of the sum of agents' ex-ante payoffs $U_i(N)$ minus the sum of link costs $c(N)$ only if N has less than $n-1$ links. By definition, the individual link cost c cannot be below the threshold $\bar{\bar{c}}$. By continuity, for δ close to one, there exists a threshold $\bar{c}$ function of δ with $\lim_{\delta\to 1} \bar{c}(\delta) = \bar{\bar{c}}$, such that for $c < \bar{c}$, L dominates every network N that is not connected.

I have proved the second statement of Proposition 2. When δ is sufficiently close to one, there exist thresholds $\underline{c}(\delta) < \bar{c}(\delta)$, with $\lim_{\delta\to 1} \underline{c}(\delta) = 0$ and $\lim_{\delta\to 1} \bar{c}(\delta) = \bar{\bar{c}}$, such that the ordered line L is the optimal network for $c \in (\underline{c}, \bar{c})$.

Turning to prove the first statement, suppose that $b_n - b_1 \leq \frac{1}{12}$. Consider any minimally connected network N, and any pair of agents e and d. By Lemma 1, there is full communication along any path from any informed agent e to any decision maker d. Hence, each agent i's ex-ante payoffs are:

$$U_{ied} = -\alpha_{id}(b_i - b_d)^2 - \alpha_{id}\frac{1}{18}[1 + \frac{1-\rho q(T, \ell_N(e, d))}{2}].$$

which for δ close to one can be approximated with the Taylor expansion:

$$U_{ied} \approx -\alpha_{id}(b_i - b_d)^2 - \alpha_{id}\frac{1}{18}[1 + \frac{1-\rho}{2} + (1-\delta)\rho\ell_N(e, d)\frac{1}{2}],$$

because for $\delta \approx 1$, it is the case that $q(T, \ell) \approx \delta^\ell$.

The above functional form is covered by Proposition 1 of Jackson and Wolinsky (1996). It thus follows as a corollary that every optimal minimally connected network is a star S.

The demonstration that for δ sufficiently close to one, there exist thresholds $\underline{c}(\delta) < \bar{c}(\delta)$ such that also every optimal network is a star for $c \in (\underline{c}, \bar{c})$ is analogous to the first part of the proof. ■

Proof of Proposition 3. The first part of the proof extends the analysis of Lemma 1, that determined conditions for equilibrium disclosure of a unique signal s to a decision maker d, to the case in which every agent i may hold a signal s_i informative on θ. I note that here, because the decision maker d can receive up to n signals, information trasmission is inefficient even if only one of them is blocked.

Let me consider a path p from an agent i to agent d, and an agent j on p who holds i's signal s_i at a history h^t. Say that $b_j > b_d$. Arguments analogous to the ones leading to conditions (7) in the proof of Lemma 1 show that, unless s_i is blocked by j's successors on p, agent j relays $s_i = 1$ at every history h^t in equilibrium, and relays $s_i = 0$ if and only if:

$$b_k - b_d \leq E_{l,r}[\frac{E_\theta[\theta|\omega_{id} = \phi; l, r] - E_\theta[\theta|\omega_{id} = 0; l, r]}{2}|\Omega_j(h^t)], \tag{10}$$

where ω_{id} is agent d's information on s_i at period T, r is the random number of signals s_k other than s_i received by d in equilibrium, l is the random number of signals s_k equal to 1, and $\Omega_j(h^t)$ is the information held by agent j at history h^t used to make equilibrium inferences on r and l. The right-hand side of inequality (10) can be bounded as follows:

$$E[\frac{E[\theta|\omega_{id} = \phi; l, r] - E[\theta|\omega_{id} = 0; l, r]}{2}|\Omega_j(h^t)] \leq E[\frac{E[\theta|\omega_{id} = 1; l, r] - E[\theta|\omega_{id} = 0; l, r]}{2}|\Omega_j(h^t)]$$
$$= E[\frac{1}{2}(\frac{l+2}{r+3} - \frac{l+1}{r+3})|\Omega_j(h^t)]. \tag{11}$$

In any efficient network and equilibrium, there is no signal blocking. At least one of the realizations $s_k = 0$ or $s_k = 1$ of the signal s_k of every agent k is relayed to any decision maker d. The event $\mathcal{E}$ that agent j is informed of all $n-1$ signals s_k different from s_i has positive probability. In this event, for δ close to one, agent j expects also agent d to be informed of all these $n-1$ signals with probability close to one, in the efficient equilibrium. As a consequence, the right-hand side of inequality (11) can be approximated as $\frac{1}{2(n+2)}$, for $\Omega_j(h^t) = \mathcal{E}$.

Hence, if $b_j - b_d > \frac{1}{2(n+2)}$, then agent j blocks $s_i = 0$ at least when $\Omega_j(h^t) = \mathcal{E}$, unless $s_i = 0$ is blocked from d by j's successors on p. Likewise, if $b_j - b_d < -\frac{1}{2(n+2)}$, then j blocks $s_i = 1$ at least when $\Omega_j(h^t) = \mathcal{E}$.

Moving on to show Proposition 3, as in the proof of Proposition 2, the first step consists in restricting attention to minimally connected networks, and in showing that the optimal tree is the ordered line L, for δ sufficiently close to one.

I begin by noting that although $b_{i+1} - b_i \geq \frac{1}{2(n+2)}$ for all $i < n$, none of the signals is fully blocked along the ordered line L, regardless of the decision maker d. The first part of this proof implies that given decision maker d, each agent $j < d$ relays every signal $s_i = 0$ to her neighbor $j+1$ along the path to d, and each agent $j > d$ relays every signal $s_i = 1$ to her neighbor $j-1$ along the path to d, in equilibrium.

Instead, every other tree N entails the blocking of some signal s_i to some possible decision maker d with positive probability. Proceeding as in the proof of Proposition 2, I see that if N is a tree that contains a 4-agent star S, then there is a decision maker d in S from whom some signals s_i originating in S are blocked in equilibrium with positive probability. Likewise, I see that if N is a tree that contains a 4-agent line that is not ordered, there are decision makers d in the line from whom some signals s_i originating in the line are blocked with positive probability.

The second step of the proof consists in showing the existence of thresholds $\underline{c}(\delta) < \overline{c}(\delta)$ for δ close to one, such that $\lim_{\delta \to 1} \underline{c}(\delta) = 0$ and $\lim_{\delta \to 1} \overline{c}(\delta) = \overline{\overline{c}}$, and that for $c \in (\underline{c}, \overline{c})$, the sum of the agents' ex-ante payoffs $U_i(L)$ with the ordered line L minus the sum of link costs $c(L)$ is larger than with any connected network N with more links than L, and than with any network N that is not connected. The proof of this step is omitted as it is an extension of the second step of the proof of Proposition 2 analogous to the above extension of the first step. ■

Proof of Proposition 4. The first part of the proof extends the analysis of part 1 of Proposition 1, that determined conditions for full communication between a unique informed agent e and a decision maker d, to the case in which every agent i may hold a signal s_i informative of θ.

Proceeding as in the proof of Proposition 3, consider an agent j such that $b_j > b_d$ who is informed of a signal s_i. Unless s_i is blocked by j's successors on the path p to agent d, agent j relays $s_i = 1$ to the next agent on p at every history h^t in equilibrium, and relays $s_i = 0$ if and only if condition (10) is satisfied. In the full communication equilibrium, agent d's expectation of θ if receiving r signals of which l equal to one is $E[\theta|l, r] = \frac{l+1}{r+2}$. Hence, inequality (10) simplifies to:

$$b_j - b_d \leq E_{l,r}[\frac{E_\theta[\theta|\omega_{id} = \phi; l, r] - E_\theta[\theta|\omega_{id} = 0; l, r]}{2}|\Omega_j(h^t)] = E[\frac{1}{2}(\frac{l+1}{r+1} - \frac{l+1}{r+2})|\Omega_j(h^t)]$$
$$= E[\frac{1}{2}\frac{l+1}{(r+2)(r+1)}|\Omega_j(h^t)] \leq \frac{1}{2(n+2)(n+1)},$$

where the latter bound is approximately achieved when $\Omega_j(h^t)$ is a vector of $n-1$ signals all equal to zero, so that $r = n$ and $l = 0$. This is an event with positive probability in the full communication equilibrium.

Conversely, any agent j such that $b_j < b_d$ relays the signal $s_i = 1$ contrary to her bias if and only if:

$$b_j - b_d \geq E_{l,r}[\frac{E_\theta[\theta|\omega_d = \phi; l, r] - E_\theta[\theta|s_i = 1; l, r]}{2}|\Omega_j(h^t)] = E[\frac{1}{2}(\frac{l+1}{r+1} - \frac{l+2}{r+2})|\Omega_j(h^t)]$$
$$= -E[\frac{1}{2}\frac{r+1-l}{(r+2)(r+1)}|\Omega_k] \geq -\frac{1}{2(n+2)(n+1)},$$

where the latter bound is approximately achieved when $\Omega_j(h^t)$ is a vector of $n-1$ signals all equal to one, so that $r = l = n$.

Wrapping up, there is equilibrium full communication on path p if and only if $|b_j - b_d| \leq \frac{1}{2(n+2)(n+1)}$ for all agents j on p.

Let me now turn to prove Proposition 4. Again, the first step is to restrict attention to minimally connected networks, to then consider general networks.

For any cluster C_m, $m = 1, ..., M$, and any possible decision maker $d \in C_m$, consider any signal s_i such that $i \in C_m$. The first part of this proof implies that neither $s_i = 0$ and $s_i = 1$ is blocked from d along any path of agents j in C_m, because $|b_j - b_d| \leq \frac{1}{2}\frac{1}{(n+2)(n+1)}$ for all such agents. Hence, the same arguments that prove the first part of Proposition 2 apply to each cluster C_m. For δ close to one, there exist thresholds $\underline{c}(\delta) < \overline{c}(\delta)$ such that $\lim_{\delta\to1} \underline{c}(\delta) = 0$ and $\lim_{\delta\to1} \overline{c}(\delta) = \overline{\overline{c}}$, and that if $c \in (\underline{c}, \overline{c})$, then each cluster C_m is optimally organized as a star S_m, as stated in the hypothesis.

I now consider communication across clusters. Take any signal s_i such that $i \in C_{m'}$, with $m' < m$. Let me show that in any tree with the properties stated in the hypothesis, the signal realization $s_i = 0$ reaches every decision maker $d \in C_m$, whereas $s_i = 1$ is blocked. In fact, for every agent j on the path p between i and d, it must be either the case that $b_j < b_d$ or that $|b_j - b_d| \leq \frac{1}{2(n+1)(n+2)}$. Hence, every j on p has bias $b_j - b_d < \frac{1}{2(n+1)}$, and the results in the first part of the proof of Proposition 3 imply that

j relays $s_i = 0$ along the path p in equilibrium. Instead, because $d \in C_m$ and $i \in C_{m'}$ with $m' < m$, it follows that $b_i - b_d < -\frac{1}{2(n+1)}$ and signal $s_i = 1$ is blocked from d. The case in which $m' > m$ is handled symmetrically.

Consider any tree N in which the clusters C_m are organized as stars S_m. If the stars S_m are not connected along the ideological order, then arguments analogous to those in the second part of the proof of Proposition 3 conclude that some signal s_i is blocked in equilibrium from some decision maker d. Finally, I note that connecting every pair stars S_m and S_{m+1} for $m < M$ in the tree N through the star centers minimizes the sum of the paths among pairs of agents i and d with $i \in S_m$ and $d \in S_{m'}$ and $m \neq m'$.

For δ close to one, I have concluded that the set of optimal minimally connected networks consists of the trees N with the characteristics stated in the hypothesis. The proof of Proposition 4 is concluded by showing that for δ close to one, there exist thresholds $\underline{c}(\delta) < \overline{c}(\delta)$ such that $\lim_{\delta\to 1} \underline{c}(\delta) = 0$ and $\lim_{\delta\to 1} \overline{c}(\delta) = \overline{\overline{c}}$, and that such trees N are also the optimal networks, for $c \in (\underline{c}, \overline{c})$. Again, the proof of this step is analogous to the proof of the same step in Proposition 2, and hence is omitted. ■

Proof of Proposition 5. The proof of the first part builds on the proof of Proposition 3.

I begin by proving that there are thresholds $\overline{\alpha} < 1$, $\overline{\delta}' < 1$ and $\overline{c}'(\delta)$ such that $\lim_{\delta\to 1} \overline{c}'(\delta) = \overline{\overline{c}}$, and that when $\alpha_{ii} > \overline{\alpha}$ for all i, $\delta > \overline{\delta}'$ and $c < \overline{c}'$, the ordered line L obtains as Nash equilibrium.

Consider a profile of link strategies l, such that $l_{i,i+1} = 1 = l_{i+1,i}$ and $l_{i,i-1} = 1 = l_{i-1,i}$ for every $i = 2, ..., n-1$, and all other links l_{ij} equal zero. The resulting network N is the ordered line L. For any j such that $l_{ij} = 0$, any agent i pays $c/2$ without changing N by deviating to $l_{ij} = 1$. This deviation is clearly not profitable. By deviating from $l_{i,i+1} = 1$ to $l_{i,i+1} = 0$, agent i deletes her link with $i+1$ and loses all signals s_e from agents $e > i$, instead of receiving them with positive probability. And by deviating from $l_{i,i-1} = 1$, she loses all signals s_e from agents $e < i$. Hence, for the same reasons that a connected network is optimal in the proof of Proposition 3, there exist thresholds $\overline{\alpha} < 1$, $\overline{\delta}' < 1$ and $\overline{c}'_i > 0$ such that for all $\alpha_{ii} > \overline{\alpha}$, $\delta > \overline{\delta}'$ and $c < \overline{c}_i$, agent i does not want to delete neither link $l_{i,i+1} = 1$, nor $l_{i,i-1} = 1$. The same arguments also imply that $\lim_{\delta\to 1} \overline{c}'_i(\delta) = \overline{\overline{c}}$. Hence, for $c < \overline{c}' \equiv \min\{\overline{c}'_i : i \in \mathcal{N}\}$, the ordered line L obtains as Nash equilibrium, and $\lim_{\delta\to 1} \overline{c}'(\delta) = \overline{\overline{c}}$.

Let me now turn to pairwise stability. I begin by showing that there exist thresholds $\overline{\alpha} < 1$, $\overline{\delta}' < 1$ and $\underline{c}'(\delta)$ such that $\lim_{\delta\to 1} \underline{c}'(\delta) = 0$, and that when $\alpha_{ii} > \overline{\alpha}$ for all i, $\delta > \overline{\delta}'$ and $c > \underline{c}'$, networks N with loops are not weakly pairwise stable. I consider a loop O of agents in network N and prove that the agent $\underline{i}$ with the smallest index i (and ideology b_i) in O gains by severing one of its two links in O, for δ close to 1, unless the link cost c is too small. In fact, every agent $i \in O$ has (at least) two paths without bias reversals to $\underline{i}$, by construction. Further, every agent $i \notin O$ connected to $\underline{i}$ only through agents in O has either (at least) two paths without bias reversals to $\underline{i}$ (when $b_i < b_{\underline{i}}$), or none at all (when $b_i > b_{\underline{i}}$). Hence, by severing one of its two links in the loop O, agent $\underline{i}$ does not block the transmission of any signal s_i to her. For $\alpha_{\underline{ii}}$ and δ close to one and when the link cost c is not too small, agent $\underline{i}$ gains by severing one of its two links in O.

I have concluded that there exist thresholds $\overline{\alpha} < 1$, $\overline{\delta}' < 1$, $\underline{c}'(\delta) < \overline{c}'(\delta)$ such that for when $\alpha_{ii} > \overline{\alpha}$ for all i, $\delta > \overline{\delta}'$ and $c \in (\underline{c}', \overline{c}')$, every weakly pairwise stable network needs to be minimally connected. Because strong pairwise stability is more demanding, also every strongly pairwise stable network needs to be minimally connected.

It is easy to show that the ordered line L is weakly pairwise stable under the parametric restrictions stated in the hypothesis. In L, no signal is blocked to any agent, and there are no duplicate links. So, for δ close to one, each agent loses by forming a duplicate link, unless the link cost c is too low.

To prove that all stars are weakly pairwise stable under analogous parametric restrictions, I note that in any star S there does not exist any pair of agents i, j such that both the path from i to j and the path from j to i have bias reversals. In fact, suppose that i and j are periphery agents. Letting $b_i > b_j$ without loss of generality, there is a bias reversal path connecting i and j if and only if the biases of the center agent c of S relative to i and j have the same signs, $(b_c - b_i)(b_c - b_j) > 0$. Hence, when both biases are positive (negative), there is a bias reversal from j to i (respectively, from j to i) but not from i to j (respectively, not from j to i). And of course, there cannot be bias reversal paths if either i or j is the center agent, because all these paths are composed of only two agents. Because there does not exist any pair of agents i, j such that both the path from i to j and the path from j to i have bias reversals, there does not exist any pair of agent i, j that both gain by forming a link with each other, when δ, α_{ii} and α_{jj} are close to 1, and the link cost c is not too small.

To show that there are trees that are not weakly pairwise stable under the parametric restrictions stated in the hypothesis, I consider the 4-agent line $2-1-4-3$. Agents 2 and 3 would gain by forming a link, unless the link cost c is too high, because the signal s_2 is blocked from 3 and vice-versa. Indeed, the presence of 4 in the path from 2 to 3 implies a bias reversal, and the presence of 1 in the path from 3 to 2 implies a bias reversal. And if the link cost c is too high for 2 and 3 to want to form a direct link, then it is also too high for 1 to maintain a link with 2. In either case, the line $2-1-4-3$ cannot be weakly pairwise stable.

To show that the ordered line L is the unique strongly pairwise stable network, first recall that every strongly pairwise stable network needs to be minimally connected, for δ and α_{ii} close to one for all i, and intermediate link cost c. Then, I note that there exists thresholds $\bar{\delta}' < 1$, $\overline{\alpha}' < 1$ and $\overline{c}'(\delta)$, such that $\lim_{\delta \to 1} \overline{c}'(\delta) = \overline{\overline{c}}$ and that when $\alpha_{ii} > \overline{\alpha}$ for all i, $\delta > \bar{\delta}'$, and $c < \overline{c}'$, there is a joint incentive to form a link by a pair of agents i and j if and only if i and j are not connected by a path without bias reversal. In the proof of Proposition 2, I have shown that L is the unique minimally connected network for which there is a path without bias reversals across every pair of agents i, j. Hence, the only minimally connected network that can be strongly pairwise stable is L, for δ close to one, unless c is too small.

The ordered line L is indeed strongly pairwise stable unless c is too large. Formally, there exists thresholds $\bar{\delta}' < 1$, $\overline{\alpha}' < 1$ and $\underline{c}(\delta)'$ such that $\lim_{\delta \to 1} \underline{c}(\delta)' = 0$, and that when $\alpha_{ii} > \overline{\alpha}$ for all i, $\delta > \bar{\delta}'$, and $c < \overline{c}$, L is strongly pairwise stable. Under these conditions in fact, no agent i has any incentive to sever any of her links $n_{ij} = 1$ in L and lose the signals s_e gathered through that link.

I have concluded that there exist thresholds $\overline{\alpha} < 1$, $\bar{\delta}' < 1$, $\underline{c}'(\delta) < \overline{c}'(\delta)$ such that when $\alpha_{ii} > \overline{\alpha}$ for all i, $\delta > \bar{\delta}'$ and $c \in (\underline{c}', \overline{c}')$, the unique strongly pairwise stable network is the ordered line L.

The proof of the second part of the proposition is built on the proof of Proposition 4 in the same way as the proof of the first part is built on the proof of Proposition 3. It is omitted and available upon request. ■

Proof of Proposition 6. I first show that there exist a threshold $\overline{\alpha} < 1$ such that if $\delta < 1$ and $\alpha_{ii} > \overline{\alpha}$ for all i, then the only network formation strategy profile l' that implements the ordered line L as a Nash equilibrium is such that every agent i with $1 < i \leq \lfloor (n+1)/2 \rfloor$ plays $l'_{i,i-1} = 1$, that every i with $\lfloor (n+1)/2 \rfloor \leq i < n$ plays $l'_{i,i+1} = 1$, and that $l'_{i,j} = 0$ for all other pairs of agents i, j.

Suppose that a strategy profile $l \neq l'$ yields the ordered line L. Hence, there must exist an agent i such that $1 \leq i < \lfloor (n+1)/2 \rfloor$ with $l_{i,i+1} = 1$, or an agent i such that $\lfloor (n+1)/2 \rfloor < i \leq n$ with $l_{i,i-1} = 1$. I consider the former, as the arguments for the latter are symmetric. The link with agent $i+1$ allows agent i to access the information of the agents $i+1$, $i+2$, ..., n. Agent i receives signal s_j of the agent $j = i+k$ with probability $q(T,k)$ for $k = 1$, ..., $n-i$. Suppose that agent i deviates, and plays $l_{i,i+1} = 0$ and $l_{i,\lceil (n+i+1)/2 \rceil} = 1$. In words, i bypasses $i+1$ by forming a link with the agent $\lceil (n+i+1)/2 \rceil$ who is in the middle of the ordered line between $i+1$ and n. This way, agent i has still access to the information of agents $j = i+1$, $1+2$, ..., n, but now with shorter paths, and hence smaller decay. In fact, she receives one of each signals s_j with probabilities $q(T,1), q(T,2), ..., q(T, \lfloor (n-i-1)/2 \rfloor + 1)$, then again one signal s_j with probabilities $q(T,2), ..., q(T, \lfloor (n-i-1)/2 \rfloor + 1)$, and when $n-i-1$ is odd, also one signal with probability $q(T, \lceil (n-i-1)/2 \rceil)$. Instead, by maintaining her link with agent $i+1$, agent i receives one of each signals s_j with probabilities $q(T,1), q(T,2), ..., q(T, \lfloor (n-i-1)/2 \rfloor + 1)$, and then one signal s_j with probabilities $q(T, \lfloor (n-i-1)/2 \rfloor + 2), ..., q(T, n-i)$. As long as $n-i \geq 3$, agent i is then more likely better informed by connecting to agent $\lceil (n+i+1)/2 \rceil$ than by connecting to $i+1$. Because $i < \lfloor (n+1)/2 \rfloor$, the condition $n-i \geq 3$ is satisfied if $n \geq 3 + (\lfloor (n+1)/2 \rfloor - 1)$, and this holds when $n \geq 4$.

I have established that every agent i such that $1 \leq i < \lfloor (n+1)/2 \rfloor$ with $l_{i,i+1} = 1$ strictly prefers to deviate and play $l_{i,i+1} = 0$ and $l_{i,\lceil (n+i+1)/2 \rceil} = 1$. A symmetric argument holds for every agent i such that $\lfloor (n+1)/2 \rfloor < i \leq n$ with $l_{i,i-1} = 1$.

This concludes that any strategy profile $l \neq l'$ that yields the ordered line L cannot be implemented as a Nash equilibrium of the unilateral sponsorship game, when $\delta < 1$ and α_{ii} is sufficiently close to one for all i.

Now, I show that for $n \leq 5$, there exists thresholds $\overline{\alpha} < 1$, $\overline{\delta}'' < 1$, and $\underline{c}''(\delta) < \overline{c}''(\delta)$ such that if $\alpha_{ii} > \overline{\alpha}$ for all i, $\delta < \overline{\delta}''$ and $c \in (\underline{c}'', \overline{c}'')$, the above strategy profile l' is a Nash equilibrium of the unilateral sponsorship game, and hence the ordered line L obtains as a Nash equilibrium.

To prove this, I check whether any agent i has any profitable deviation. Again, for δ close to 1, no agent i would benefit by sponsoring extra links, i.e., by playing $l_i > l'_i$, if the link cost c is not too small. And unless the link cost is too high, it is not profitable to deviate from l'_i by adopting a strategy $l_i < l'_i$ for α close to one, because agent i foregoes valuable information by severing her link(s).

I now show that for any agent i such that $1 < i \leq \lfloor (n+1)/2 \rfloor$, it would not be profitable to deviate by setting $l_{i,i-1} = 0$, together with $l_{i,j} = 1$ for some $j < i-1$. The cases $n = 3$ and $n = 4$ are trivial, and so is the case with $n = 5$ for any agent i other than 3. When $n = 5$, agent 3 may deviate from l'_3, by setting $l_{3,2} = 0$ together with $l_{3,1} = 1$. But if doing so, her payoffs would not change: instead of receiving $s_2 = 0$ with probability $q(T,1)$ and $s_1 = 0$ with probability $q(T,2)$, agent 3 would receive $s_1 = 0$ with probability $q(T,1)$ and $s_2 = 0$ with probability $q(T,2)$. Hence, agent 3 has no reasons to deviate from the strategy l'_3. The analysis for i such that $\lfloor (n+1)/2 \rfloor \leq i < n$ is symmetric. I have shown that for $n \leq 5$, the strategy profile l' is a Nash equilibrium.

I now turn to the case in which $n > 5$. Consider the central agent $i = \lfloor (n+1)/2 \rfloor$. The strategy profile l' is not a Nash equilibrium for any $\delta < 1$ and α_{ii} close to one for all i, because agent i has an incentive to deviate from the strategy l'_i in which $l'_{i,i+1} = 1$ to a strategy l such that $l_{i,\lceil (n+i+1)/2 \rceil} = 1$ and $l_{ij} = 0$ for all other agent $j > i$, as long as $n \geq 6$. In fact, the same calculations as in the first part of the proof prove that, for $n \geq 6$, agent $i = \lfloor (n+1)/2 \rfloor$ has access to the signals of the same agents $i+1, ..., n$ by deviating from l' to l, but with shorter paths and hence higher probability. ■

Appendix B: Cheap talk

The cheap talk version of the game introduced in Section 3 is defined as follows. I lift the restriction on reporting strategies that $r_{jkdi}(\omega_{ij}; h^t) \in \{\omega_{ij}, \phi\}$ given j's information $\omega_{ij} \in \{0, \phi, 1\}$ on signal s_i, for any agents d, i, j and k and history h^t. Regardless of the realization of signal s_i, the reports $r_{jkdi}(s_i; h^t)$ may here take any value out of possibly large message sets M^t_{jkdi}, that I assume finite for ease of exposition. The remainder of the model of Section 3 is unchanged.

The first part of the analysis considers equilibria of the game of cheap talk that are outcome equivalent to the equilibria of the game of verifiable information transmission of Section 3. Suppose $\ell + 1$ agents j are linked along a path p from an expert e, informed of an unverifiable signal s with probability ρ, to a decision-maker d. There are T rounds of communication. In each round t, any agent j on the path p may communicate to his successor k. All communication takes the form of cheap talk.

I begin by considering the one-sided information equilibria of the verifiable information transmission game, specifically those in which every agent j relays only signal realization $s = 0$ to the next agent k on the path p. Regardless of the specific report strategies r adopted, any such equilibrium is strategically and outcome equivalent to the equilibrium of the cheap talk game in which $M^t_{ikdi} = \{0, \phi, 1\}$, $r_{jkdi}(0; h^t) = 0$ and $r_{jkdi}(\phi; h^t) = r_{jkdi}(1; h^t) = \phi$, for all t, i, d, j and k. The possibility that the message r_{jkdi} is lost with probability $1-\delta$ is again represented by the assumption that when $r_{jkdi} \neq \phi$, agent k receives report ϕ with probability δ.

Suppose momentarily that there is no decay, $\delta = 1$. Then in the one-sided equilibrium, if receiving the report $r_{jkdi} = \phi$ from her predecessor j on p in every period t, then the expectation about θ of any agent k (including the decision maker d) is $E[\theta|\phi] = \frac{1-\rho/3}{2-\rho}$ —cf. expression (8). If instead k receives the report $r_{jkdi} = 0$ in any period t, her expectation about θ is $E[\theta|\phi] = \frac{1}{3}$ henceforth. On the equilibrium path, agent k cannot ever receive report $r_{jkdi} = 1$, nor she can receive report $r_{jkdi} = \phi$ after receiving $r_{jkdi} = 0$. If these events, her equilibrium beliefs are not pinned down by the Bayes rule. I want to show sufficient conditions under which the one-sided equilibrium does not exist. Hence, I assume off path beliefs that do not make off path deviations from equilibrium more profitable than deviations on the equilibrium path.

Suppose that the information $\omega_j(h^t)$ about s at history h^t of an agent j on the path p with $b_j > b_d$ is $\omega_j = 0$. Then, as in the proof of Proposition 1, $u_{jd}(0; h^t) \geq u_{jd}(\phi; h^t)$ and j sends the equilibrium message $r_{jkdi} = 0$ to k instead of deviating to $r_{jkdi} = \phi$ if and only if $2(b_j - b_d) \leq E[\theta|\phi] - E[\theta|0] = \frac{1}{3(2-\rho)}$, or $b_j - b_d \leq \frac{1}{6(2-\rho)}$. Now however, also any agent j on the path p with $b_j < b_d$ has the opportunity to deviate from equilibrium and send $r_{jkdi} = 0$ instead of $r_{jkdi} = \phi$, when her information about s is $\omega_j(h^t) \in \{\phi, 1\}$. She does not deviate if and only if $u_{jd}(0; h^t) \leq u_{jd}(\phi; h^t)$, that is, $2(b_j - b_d) \geq E[\theta|0] - E[\theta|\phi] = -\frac{1}{3(2-\rho)}$, or $b_j - b_d \geq -\frac{1}{6(2-\rho)}$.

Wrapping up, for $\delta = 1$ the one-sided communication equilibrium in which agents reveal $s = 0$ exists if and only if $|b_j - b_d| \leq \frac{1}{6(2-\rho)}$ for all agents j on the path p. To extend the result to δ close to one, I note that the payoff functions $u_{jd}(s; h^t)$ and $u_{jd}(\phi; h^t)$ are continuous in δ—cf. expression (5). Hence, for δ close to one, it is still the case that $u_{jd}(0; h^t) > u_{jd}(\phi; h^t)$ for any agent j on the path p with $b_j - b_d < -\frac{1}{6(2-\rho)}$ and $\omega_j(h^t) \in \{\phi, 1\}$, if the opponents abide to the one sided equilibrium strategies. Likewise, it is still the case that $u_{jd}(0; h^t) < u_{jd}(\phi; h^t$ for any agent j on p with $b_j - b_d > \frac{1}{6(2-\rho)}$, even if $\omega_j(h^t) = 0$. In other terms, also if δ is close to one it is the case that the one-sided communication equilibrium in which the agents reveal $s = 0$ exists if and only if $|b_j - b_d| \leq \frac{1}{6(2-\rho)}$ for all j on p. A symmetric argument leads to the same conclusion for the one-sided communication equilibrium in which the agents reveal $s = 1$.

Now consider full communication equilibria. The agents reveal both signal realizations $s = 0$ and $s = 1$. Any such equilibrium is outcome and strategic equivalent to the equilibrium of the cheap talk game in which $M^t_{jkdi} = \{0, \phi, 1\}$ and $r_{jkdi}(\omega_j; h^t) = \omega_j$ for all t, i, d, j and k. The existence of this equilibrium requires the truth-telling conditions $2(b_j - b_d) \leq E[\theta|\phi] - E[\theta|0] = \frac{1}{6}$ and $2(b_j - b_d) \leq E[\theta|1] - E[\theta|\phi] = \frac{1}{6}$ for $b_j > b_d$, and $2(b_j - b_d) \geq E[\theta|\phi] - E[\theta|0] = -\frac{1}{6}$ and $2(b_j - b_d) \geq E[\theta|1] - E[\theta|\phi] = -\frac{1}{6}$ for $b_j > b_d$. These existence conditions are more demanding than the above conditions for the existence of the one-sided communication equilibrium.

The same is true for any mixed strategy equilibrium that is outcome equivalent to any equilibrium of the verifiable information transmission game. Any mixing across the messages $r_{jkdi} = 0$ and $r_{jkdi} = \phi$ of the one sided communication equilibrium makes $E[\theta|0]$ larger than $\frac{1}{3}$ and $E[\theta|\phi]$ smaller than $\frac{1-\rho/3}{2-\rho}$, thus making the equilibrium conditions, $2(b_j - b_d) \leq E[\theta|\phi] - E[\theta|0]$ for $b_j > b_d$, and $2(b_j - b_d) \geq E[\theta|0] - E[\theta|\phi]$ for $b_j < b_d$, harder to be satisfied. And conversely, any mixing across the messages $r_{jkdi} = \phi$ and $r_{jkdi} = 1$ makes $E[\theta|1]$ smaller and $E[\theta|\phi]$ larger, thus making the equilibrium conditions harder to be satisfied. And of course, any mixing across the messages $r_{jkdi} = 0$, $r_{jkdi} = \phi$ and $r_{jkdi} = 1$ of the full communication equilibrium makes equilibrium conditions even harder to satisfy.

For δ close to one, I conclude that if $|b_j - b_d| > \frac{1}{6(2-\rho)}$ for some agents j on the path p, then the babbling equilibrium is the only equilibrium of the cheap talk game that is outcome equivalent to an equilibrium of the verifiable information transmission game.

I now show that if $b_{i+1} - b_i > \frac{1}{6}$ for all $i < n$, then there do not exist any perfect Bayesian equilibria other than the babbling equilibrium, regardless of whether they are equivalent to equilibria of the verifiable information transmission game, or take more complicated forms.

I study cheap talk games with message spaces M^t_{jkdi} constructed as follows. In period $t = 1$, let $M^t_{jkdi} = \{0, \phi, 1\}$ for all agents i, d, j, k; and for any time $t > 1$, let $M^t_{jkdi} = \times_{h \neq j,k} M^{t-1}_{hjdi} \times M^{t-1}_{jkdi}$. This is because I want to allow for the possibility that in each period t, each agent j can report to any one neighbor k not only the messages r_{hj} she just learnt from her neighbors $h \neq j$ but also what she could have learnt and said at any earlier period τ. In the message spaces constructed above, for all agents j and k, the space M^t_{jkdi} includes M^{t-1}_{jkdi} at every period t, and then by concatenation M^t_{jkdi} also includes M^{t-2}_{jkdi}, ..., M^1_{jkdi}. The possibility that j's report r_{jkdi} is lost in transmission at any time t is represented by the assumption that with probability $1 - \delta$, agent k receives a report vector whose components are all equal to ϕ, when j sent a different report r_{jkdi}.

Again, I begin by considering the case of no decay, $\delta = 1$. For each neighbor j of the decision maker d, let me introduce the following collection of cheap talk games between agents j and d. Nature draws θ from the uniform distribution on $[0, 1]$, and then $s \in \{0, \phi, 1\}$ according to $\Pr(s = 1|\theta) = \rho\theta$, $\Pr(s = 0|\theta) = \rho(1 - \theta)$. Then, there are T periods of communication. In each period $t = 1, ..., T$, nature sends agent j a report r_{njt} out of the message set $M^t_{nj} = \times_{h \neq j} M^t_{hjdi}$. That is, nature can send any vector of reports r_{hj} that j would have received from her neighbors h. The report r_{njt} depends on the signal s according to the probability distribution $\zeta(\mathbf{r}_{nj}|s)$, where $\mathbf{r}_{nj}$ is a profile of T reports r_{njt}. In each period $t = 1, ..., T$, agent j also sends a message r_{jdt} to agent d function of the profile $\mathbf{r}_{njt}$ of t reports $r_{nj\tau}$ for $\tau = 1, ..., t$, possibly adopting a mixed strategy σ_{jd}. After these T rounds of communication, agent d chooses $\hat{y}_d$ as function of the profile of messages $\mathbf{r}_{jd}$ received from j.

Any given nature signal disclosure distribution ζ identifies a different game between agent d and one of her neighbors j. If no information is revealed to d in any Bayesian Nash equilibrium of any of these games, then no information can be revealed to d also in any Bayesian Nash (and hence a fortiori any perfect Bayesian) equilibrium of the game in which agent e is informed of s with probability ρ and agents may communicate for T rounds before d's decision. Hence, I want to prove that every Bayesian Nash equilibrium of every game between agent d and one of his neighbors j is a babbling equilibrium. The message $r_{jdt}(\mathbf{r}_{njt})$ of agent j to d is the same for every profile of nature messages $\mathbf{r}_{njt}$ at every time t.

I first note that, because there is no decay, $\delta = 1$, it is without loss of generality to restrict attention to agent j's strategies σ_{jd} in which j does not transmit any information to d until the last period T. (I.e., at every time $t < T$, agent j sends the same message r_{jdt} in every history h^t.) After collecting all the reports $\mathbf{r}_{nj}$ from nature, j decides what to communicate to agent d, and sends a message $r_{jdT} \in M^T_{jd}$. This restriction is without loss of generality because by construction, every vector of nature's reports $\mathbf{r}_{nj}$ is identified by at least a message in the space M^T_{jd}. Hence, I just want to show that if $|b_j - b_d| > \frac{1}{6}$, then agent j sends the same message r_{jdT} at every history h^T in every equilibrium.

Suppose by contradiction that there exists an equilibrium in which $\sigma_{jd}(\mathbf{r}_{nj}) \neq \sigma_{jd}(\mathbf{r}'_{nj})$ for (at least) two report vectors $\mathbf{r}_{nj}$ and $\mathbf{r}'_{nj}$ such that $E[\theta|\mathbf{r}_{nj}] \neq E[\theta|\mathbf{r}'_{nj}]$. Then, there would exist messages r_{jdT} and r'_{jdT} such that $y_d(r_{jdT}) \neq y_d(r'_{jdT})$. For the strategy σ_{jd} to be an equilibrium strategy, it would then need to be the case that

$$u_{jd}(r_{jdT}; \mathbf{r}_{nj}) = -E[(y_d(r_{jdT}) - \theta - b_j)^2|\mathbf{r}_{nj}] \geq u_{jd}(r'_{jdT}; \mathbf{r}_{nj}) = -E[(y_d(r'_{jdT}) - \theta - b_j)^2|\mathbf{r}_{nj}]$$
$$u_{jd}(r'_{jdT}; \mathbf{r}'_{nj}) = -E[(y_d(r'_{jdT}) - \theta - b_j)^2|\mathbf{r}'_{nj}] \geq u_{jd}(r_{jdT}; \mathbf{r}'_{nj}) = -E[(y_d(r_{jdT}) - \theta - b_j)^2|\mathbf{r}'_{nj}]$$

Because agent d is sequentially rational, it must be that $y_d(r_{jdT}) = b_d + E[\theta|r_{jdT}]$ and $y_d(r'_{jdT}) = b_d + E[\theta|r'_{jdT}]$, and I can rewrite the above conditions as:

$$E[(b_d + E[\theta|r_{jdT}] - \theta - b_j)^2|\mathbf{r}_{nj}] \leq E[(b_d + E[\theta|r'_{jdT}] - \theta - b_j)^2|\mathbf{r}_{nj}]$$
$$E[(b_d + E[\theta|r'_{jdT}] - \theta - b_j)^2|\mathbf{r}'_{nj}] \leq E[b_d + E[\theta|r_{jdT}] - \theta - b_j)^2|\mathbf{r}'_{nj}].$$

Simplifying these inequalities as in (6), I obtain:

$$\left[2(b_j - b_d) + 2E[\theta|\mathbf{r}_{nj}] - E[\theta|r_{jdT}] - E[\theta|r'_{jdT}]\right]\left[E[\theta|r'_{jdT}] - E[\theta|r_{jdT}]\right] \leq 0$$
$$\left[2(b_j - b_d) + 2E[\theta|\mathbf{r}'_{nj}] - E[\theta|r_{jdT}] - E[\theta|r'_{jdT}]\right]\left[E[\theta|r'_{jdT}] - E[\theta|r_{jdT}]\right] \geq 0. \quad (12)$$

I note that if agent j's strategy σ_{jd} identifies $\mathbf{r}_{nj}$ and $\mathbf{r}'_{nj}$ perfectly, i.e. both the support of $\sigma_{jd}(\mathbf{r}_{nj})$ and of $\sigma_{jd}(\mathbf{r}'_{nj})$ are disjoint from the supports of $\sigma_{jd}(\mathbf{r}''_{nj})$ for every other report vector $\mathbf{r}''_{nj}$, then $E[\theta|\mathbf{r}_{nj}] = E[\theta|r_{jdT}]$ and $E[\theta|\mathbf{r}'_{nj}] = E[\theta|r'_{jdT}]$. The decision maker identifies $\mathbf{r}_{nj}$ from the message r_{jdT} and $\mathbf{r}'_{nj}$ from the message r'_{jdT}, and the conditions (12) take the same form as in (6):

$$2(b_j - b_d) \leq E[\theta|r'_{jdT}] - E[\theta|\mathbf{r}_{nj}], \quad 2(b_j - b_d) \geq E[\theta|r_{jdT}] - E[\theta|\mathbf{r}'_{nj}]$$

Instead, if agent j's strategy σ_{jd} pools or randomizes information, it determines a Blackwell scrambling of nature's report distribution ζ. As a consequence, agent j must have more extreme beliefs about θ than the decision maker d, for some realizations $\mathbf{r}_{nj}$, $\mathbf{r}'_{nj}$ of ζ and r_{jdT} and r'_{jdT} of σ_{jd}. For such values of $\mathbf{r}_{nj}$, $\mathbf{r}'_{nj}$, r_{jdT} and r'_{jdT}, it is then either the case that $E[\theta|r_{jdT}] < E[\theta|\mathbf{r}_{nj}]$ and $E[\theta|r'_{jdT}] > E[\theta|\mathbf{r}'_{nj}]$ when $E[\theta|r'_{jdT}] > E[\theta|r_{jdT}]$, or that $E[\theta|r_{jdT}] > E[\theta|\mathbf{r}_{nj}]$ and $E[\theta|r'_{jdT}] < E[\theta|\mathbf{r}'_{nj}]$ when $E[\theta|r'_{jdT}] < E[\theta|r_{jdT}]$.

Suppose without loss of generality that $E[\theta|r'_{jdT}] > E[\theta|r_{jdT}]$. Say that $b_j > b_d$. Then, the first equilibrium condition (12) fails if:

$$2(b_j - b_d) > E[\theta|r_{jdT}] + E[\theta|r'_{jdT}] - 2E[\theta|\mathbf{r}_{nj}].$$

Because $E[\theta|r_{jdT}] \leq E[\theta|\mathbf{r}_{nj}]$, the latter condition simplifies to:

$$2(b_j - b_d) > E[\theta|r'_{jdT}] - E[\theta|\mathbf{r}_{nj}],$$

and because $E[\theta|r'_{jdT}] \leq 2/3$ and $E[\theta|\mathbf{r}_{nj}] \geq 1/3$, this simplifies to: $2(b_j - 2b_d) > 1/3$, i.e., $b_j - b_d > 1/6$.

Hence, if $b_j - b_d > 1/6$, then the hypothesized (possibly partially) separating strategy σ_{jd} cannot be part of an equilibrium. A symmetric argument shows that if $b_j - b_d < -1/6$, then the second equilibrium condition (12) fails, and again σ_{jd} cannot be part of an equilibrium.

I have concluded that there is no equilibrium information transmission in the game in which agent e is informed of s with probability ρ and agents may communicate with no decay, $\delta = 1$, for T rounds before d's decision. The extension of this result to when δ is close to one is based on the same arguments used in the first part of the analysis of cheap talk. Because the payoff functions $u_{jd}(r_{jdT}; \mathbf{r}_{nj})$ are continuous in δ for all r_{jdT} and $\mathbf{r}_{nj}$, if one or both inequalities (12) fail for $\delta = 1$, then they must still fail for δ close to 1 for the same strategy profiles. As a result, there there still cannot be any equilibrium information transmission.

I have concluded that if $b_{i+1} - b_i > \frac{1}{6}$ for all $i < n$, then the babbling equilibrium is the only equilibrium of the cheap talk game in which an expert agent e is informed of signal s such that $\Pr(s = 1|\theta) = \theta$. The result can be generalized to any unverifiable signal s informative of θ, that is, any signal s such that $E[\theta|s]$ is not the same for all realizations of s. For any such signal s, there exist a threshold $\beta(s)$ such that if $b_{i+1} - b_i > \beta(s)$ for all $i < n$, then the unique equilibrium of the cheap talk game is the babbling equilibrium.

It is obvious that these results hold also if agents have verifiable information to communicate, together with these unverifiable signals. Hence, if together with the verifiable signal s, there were also unverifiable signals $\tilde{s} \in \{0, 1\}$ informative of θ with probability $\Pr(\tilde{s} = 1|\theta) = \theta$, any possible decision maker d would not be informed of any such signals $\tilde{s}$ in equilibrium, when $b_{i+1} - b_i > \frac{1}{6}$ for all agents $i < n$. And for any general distribution of unverifiable signals $\tilde{s}$ informative of θ, it can be shown that d would not be informed of $\tilde{s}$ in equilibrium, if the biases $b_{i+1} - b_i$ are sufficiently large.