University of Warwick, Department of Economics — Spring 2014
EC941: Game Theory — Prof. Francesco Squintani

# Final Exam

*Answer TWO questions. All questions carry equal weight. Time allowed 2 hours.*

**1.** Each of three people announces an integer from 1 to $K$. If the three integers are different, the person whose integer is closest to 2/3 of the average of the three integers wins \$1. If two or more integers are the same, \$1 is split equally between the people whose integer is closest to 2/3 of the average integer.

a. [15 point] Is there any integer $k$ such that the action profile $(k, k, k)$, in which every person announces the same integer $k$, is a pure strategy Nash equilibrium?

b. [35 points] Are there any other pure strategy Nash equilibria?

**2.** Consider the following Bayesian game. Nature chooses between states $\omega$ and $\omega'$, where $\omega$ is chosen with probability $\pi$. Player 1 observes the state, while player 2 has no information regarding nature's choice. The two players then play a simultaneous move game with payoffs as given below, where player 1 chooses between T and B, and 2 chooses between L and R:

| $\omega$ | L | R |
|---|---|---|
| T | 2,2 | 1,3 |
| B | 3,1 | 0,0 |

| $\omega'$ | L | R |
|---|---|---|
| T | 5,5 | 0,0 |
| B | 0,0 | 2,2 |

a. [15 points] Solve for all the pure strategy Bayesian Nash equilibria of this game as a function of $\pi$.

b. [20 points] Find all the mixed strategy equilibria as a function of $\pi$.

c. [15 points] Consider a different game where neither player observes the realized state. Solve for the pure and mixed strategy Nash equilibria of this game as a function of $\pi$. (Exclude the cases in which $p = 2/3$ or $\pi = 5/6$).

**3.** An exclusive club has the following formal procedure for membership. At each stage of the game, the 'newest member' to have been admitted into the club can either declare the membership-game over or nominate a new candidate to become a member. If a candidate is nominated, the existing members of the club vote whether to admit or reject. If the candidate is rejected, then the membership game is over. If the candidate is admitted, then the game continues with the now-admitted candidate becoming the 'newest member' choosing whether to nominate someone or end the game. Whenever votes occur, the voting is sequential, starting with the newest member (the nomination is his vote) and ending with the first member. The candidate does not get a vote. All votes are observed by everyone. If the candidate gets a half or more of the votes, she is admitted. That is, if there is a tie, then the candidate is admitted. There are no abstentions. Suppose that there are 4 players, A, B, C, and D. Suppose that A is the initial member of the club, and that the players' preferences are expressed as follows:

| A | B | C | D |
|---|---|---|---|
| ac | abcd | acd | abd |
| ab | ab | ac | ad |
| ad | abd | abcd | acd |
| a | abc | abc | abcd |
| abcd | a | a | a |
| abc | ac | ab | ab |
| acd | ad | ad | ac |
| abd | acd | abd | abc |

Thus, for example, B's most preferred outcome would have everyone in the club. Her second preference would be just A and herself. Her third preference would be A, D and herself. And her fourth preference would be AC and herself. All other memberships rank lower in her preferences.

a. [10 points] Suppose three candidates have been admitted. How will the players play?

b. [40 points] Suppose that only A belongs to the club. Find the unique subgame perfect equilibrium

**4.** Consider two firms (players 1 and 2) who are working on a joint project and a bank (player 3) who is a potential investor in the project. First, the entrepreneurs simultaneously decide whether to devote high or low effort to research on the project. They then make a presentation to the bank. If both firms choose high effort in their preliminary research, then the presentation goes well, and otherwise it goes poorly. The bank only observes the outcome of the presentation and not the firms' effort levels. After the presentation, the bank decides whether to invest in the project. Each firm receives a payoff of 5 if the bank invests and 0 otherwise. In addition, choosing high effort costs a firm 1, while choosing low effort is free. Investing costs the bank 2 and brings a return of 3 for each firm who chose high effort (i.e., a return of 6 if both chose high effort, 3 if only one did, and 0 if neither did). If the bank does not invest, his payoff is 0. All players are risk neutral.

a. [15 points] Draw an extensive form representation of this game.

b. [35 points] Find all perfect Bayesian equilibria in which all players choose pure strategies.

**5.** Alex is deciding whether or not to make a loan to Brian who is very poor and who has a bad credit history. Simultaneous to Alex making this decision, Brian must decide whether or not to buy gifts for his grandkids. If he buys gifts, he will be unable to repay the loan. If he does not buy gifts, he will repay the loan. If Alex refuses to give Brian a loan, then Brian will have to go to a loan shark. The payoffs in this game are as follows: if Alex refuses to make a loan to Brian and Brian buys gifts, then both Alex and Brian get 0. If Alex refuses to make a loan to Brian and Brian does not buy gifts, then Alex gets 0 and Brian gets -1. If Alex makes a loan to Brian and Brian buys gift,s then Alex gets -2 and Brian gets 7. If Alex makes a loan to Brian and does not buy gifts, then Alex gets 3 and Brian gets 5.

a. [10 points] Suppose this game is played just once. Find the equilibria of the game.

Now suppose that the game is repeated. Suppose that for both individuals, a dollar tomorrow is worth 2/3 of a dollar today. In addition, suppose that, after each period (and regardless of what happened in the period), Brian has a 1/2 chance of escaping poverty, so that the game ends.

b. [20 points] Is there a subgame perfect equilibrium in which Alex makes the loan to Brian, and Brian repays the loan?

c. [10 points] Suppose that the government introduces regulation of loan sharks, so that Brian's payoff in each period in which he still needs a loan but does not get it from Alex is 1 if he does not buys gifts and 2 if he buys gifts. Is this policy beneficial to Brian in subgame perfect equilibrium, or not?

d. [10 points] Suppose that the government abandons its loan-shark policy and replaces it with a job scheme that increases the probability after each period of Brian escaping poverty to 2/3. Explain the consequences of this policy for the business relationship between Alex and Brian, in subgame perfect equilibrium.

# Answers

**1a** If all three players announce the same integer $k \geq 2$ then any one of them can deviate to $k-1$ and obtain \$1 (since her number is now closer to 2/3 of the average than the other two) rather than \$1/3. Thus no such action profile is a Nash equilibrium. If all three players announce 1, then no player can deviate and increase her payoff; thus (1, 1, 1) is a Nash equilibrium.

**1b** Now consider an action profile in which not all three integers are the same; denote the highest by $k^*$.

Suppose only one player names $k^*$; denote the other integers named by $k_1$ and $k_2$, with $k_1 \geq k_2$. The average of the three integers is $1/3(k^* + k_1 + k_2)$, so that 2/3 of the average is $2/9(k^* + k_1 + k_2)$. If $k_1 \geq 2/9(k^* + k_1 + k_2)$ then $k^*$ is further from 2/3 of the average than is $k_1$, and hence does not win. If $k_1 < 2/9(k^* + k_1 + k_2)$ then the difference between $k^*$ and 2/3 of the average is $k^* - 2/9(k^* + k_1 + k_2) = 7/9k^* - 2/9k_1 - 2/9k_2$, while the difference between $k_1$ and 2/3 of the average is $2/9(k^* + k_1 + k_2) - k_1 = 2/9k^* - 7/9k_1 + 2/9k_2$. The difference between the former and the latter is $5/9k^* + 5/9k_1 - 4/9k_2 > 0$, so $k_1$ is closer to 2/3 of the average than is $k^*$. Hence the player who names $k^*$ does not win, and is better off naming $k_2$, in which case she obtains a share of the prize. Thus no such action profile is a Nash equilibrium.

Suppose two players name $k^*$, and the third player names $k < k^*$. The average of the three integers is then $1/3(2k^* + k)$, so that 2/3 of the average is $4/9k^* + 2/9k$. We have $4/9k^* + 2/9k < 1/2\,(k^* + k)$, so that the player who names k is the sole winner. Thus either of the other players can switch to naming $k$ and obtain a share of the prize rather obtaining nothing. Thus no such action profile is a Nash equilibrium. We conclude that there is only one Nash equilibrium of this game, in which all three players announce the number 1.

**2a.** Player 1 of type $\omega$ chooses $B$ if 2 plays $L$ and $T$ if 2 plays $R$; whereas player 1 of type $\omega'$ chooses $T$ if 2 plays $L$ and $B$ if 2 plays $R$. Hence, possible equilibria are $(BT, L)$ and $(TB, R)$. The first profile is a BNE for all $\pi$, because 2's payoff for playing $L$ is $\pi + 5\,(1-\pi)$ and her payoff for playing $R$ is zero. The second profile is a BNE for all $\pi$ because because 2's payoff for playing $R$ is $3\pi + 2\,(1-\pi)$ and her payoff for playing $L$ is $2\pi$.

**2b.** To find completely mixed strategies, we use the indifference principle. Player 1 of type $\omega$ is indifferent between $B$ and $T$ if and only if

$$2\sigma_L + (1 - \sigma_L) = 3\sigma_L; \quad \text{i.e., } \sigma_L = 1/2.$$

For $\sigma_L = 1/2$, player 1 of type $\omega'$ chooses $T$.

Player 2 of type $\omega'$ is indifferent between $B$ and $T$ if and only if

$$5\sigma_L = 2(1 - \sigma_L); \quad \text{i.e., } \sigma_L = 2/7.$$

For $\sigma_L = 2/7$, player 1 of type $\omega$ chooses $T$.

Hence, it cannot be that both types of player 1 randomize.

Suppose that $\omega$ randomizes, and call her strategy $\sigma_T$. Then, player 2 is indifferent between $L$ and $R$ if and only if:

$$2\pi\sigma_T + \pi\left(1 - \sigma_T\right) + 5\left(1 - \pi\right) = 3\pi\sigma_T; \text{ i.e., } \sigma_T = \frac{5 - 4\pi}{2\pi}.$$

Because $\frac{5-4\pi}{2\pi} \in [0, 1]$ requires $\pi \geq 5/6$, a mixed strategy equilibrium exists only for that range of $\pi$.

Suppose that $\omega'$ randomizes, call her strategy $\sigma'_T$. Then, player 2 is indifferent between $L$ and $R$ if and only if:

$$2\pi + 5\left(1 - \pi\right)\sigma'_T = \pi + 2\left(1 - \pi\right)\left(1 - \sigma'_T\right); \text{ i.e. } \sigma_T = \frac{3\pi - 2}{7\pi - 7}.$$

Because $\frac{3\pi-2}{7\pi-7} \in [0, 1]$ requires $\pi \geq 2/3$, a mixed strategy equilibrium exists only for that range of $\pi$.

| $\omega$ | L | R |
|---|---|---|
| T | 2,2 | 1,3 |
| B | 3,1 | 0,0 |

| $\omega'$ | L | R |
|---|---|---|
| T | 5,5 | 0,0 |
| B | 0,0 | 2,2 |

**2c** This case is equivalent to one in which the players play the 2X2 following game:

| $\omega$ | L | R |
|---|---|---|
| T | $2\pi + 5\left(1 - \pi\right),\ 2\pi + 5\left(1 - \pi\right)$ | $\pi,\ 3\pi$ |
| B | $3\pi,\ \pi$ | $2\left(1 - \pi\right),\ 2\left(1 - \pi\right)$ |

So, when $\pi < 2/3$, then $2\pi + 5\left(1 - \pi\right) > 3\pi$ and $\pi < 2\left(1 - \pi\right)$. Thus, the game has three Nash equilibria: $(T, T)$, $(B, B)$, and $\sigma_T = \sigma_L = \frac{3\pi-2}{9\pi-7}$, where the latter is given by the indifference condition $\left(2\pi + 5\left(1 - \pi\right)\right)\sigma_L + \pi\left(1 - \sigma_L\right) = 3\pi\sigma_L + 2\left(1 - \pi\right)\left(1 - \sigma_L\right)$. When

$2/3 < \pi < 5/6$, then $2\pi + 5(1-\pi) > 3\pi$ and $\pi > 2(1-\pi)$, so there is a unique Nash equilibrium, $(T,T)$. Finally, when $\pi > 5/3$, then $2\pi + 5(1-\pi) < 3\pi$ and $\pi > 2(1-\pi)$, so that the game has 3 Nash equilibria, $(T,R)$, $(B,L)$, and $2\pi + 5(1-\pi)$, and, again, $\sigma_T = \sigma_L = \frac{3\pi-2}{9\pi-7}$.

**3a.** If the club is acd, then neither C nor D will nominate B, as they both prefer acd to abcd. If the club is abd, then D will not nominate C, but B would nominate C, if he were the last player. If the club is abc, then both B and C would nominate D.

**3b.** Suppose A nominates C so C finds herself in a club of ac. The only outcome that player C prefers to ac is acd, so the only nomination she would consider is d or "stop". If she chooses d, then D finds himself in a club of acd. The only memberships D prefers to this are no longer feasible (since he cannot kick anyone out). So he stops. This is good for C so she will indeed nominate D; but this is bad for A since he preferred a to acd. So A will not nominate C.

Next suppose A nominates B. Then B finds himself in a club of ab. He would prefer to get to abcd. If B nominates D then D finds himself in a club of abd which is his most preferred club. So D would stop. This is bad for B so B won't nominate D. But if B nominates C, then C finds herself in a club of abc. C would prefer abcd so she nominates D. This is good for B (who got to abcd) so B would nominate C. But this is bad for A who preferred a to abcd so A won't nominate B.

Next suppose that A nominates D. Then D finds himself in a club of ad. The only club D would prefer would be abd. But if D nominates B, B will nominate C to get to abcd. But D prefers ad to abcd, so D won't nominate B. Instead, D will stop. But this outcome, ad, is better for A than a. So A nominates D (who stops).

**4a** The game can be standardly represented with a game tree.

**4b** Suppose that the bank invests if and only if the presentation is good. Then the firms may either both exert effort, or neither. Both these possibilities occur in equilibrium. There is no PBE in which the bank invests if the presentation is bad. Then, neither firms will invest, and the presentation will be bad, so the firm should not invest in equilibrium. Finally, there is no PBE in which the bank does not invest regardless of the presentation outcome, because if the presentation is good, the bank must believe that both firms invested.

**5a** The game played once is a 2X2 game. The matrix is shown below.

| | repay | gifts |
|---|---|---|
| loan | 3,5 | -2,7 |
| refuse | 0,-1 | 0,0 |

There is only one Nash Equilibrium (refuse; gifts). Because "gifts" is dominant for Brian, there is no other Nash equilibrium.

**5b** Consider the following strategy profile. In period one, Alex makes Brian a loan. Thereafter, Alex continues to make Brian loans (if he is still poor) as long as Brian and has always got a loan and repaid it in the past. But if Brian ever does not repay (or does not get a loan) then Alex never makes a loan to Brian again. In period one, Brian does not buy gifts (and hence repays the loan if he gets one). Thereafter (as long as he is still poor), Brian does not buy gifts (and hence repays the loan if he gets one) as long as he has always got a loan and repaid it in the past. But if Brian ever does not repay (or does not get a loan) then he will return to buying gifts and hence never repay a loan again.

This strategy profile is a subgame perfect equilibrium of the repeated game, because of the following argument. Regardless of the history, Alex is always playing a stage-game best response to Brian's equilibrium action, and no change in Alex's choices ever makes Brian's equilibrium future actions improve Alex's payoff. Hence Alex has no incentive to deviate. Similarly, where the supposed equilibrium instructs Alex to refuse to make loans to Brian for ever, it instructs Brian not to repay. Since Brian is playing a stage-game best response to Alex's equilibrium action and since no change in Brian's choices induces any change in Alex's actions, Brian has no incentive to deviate from this.

However, where the equilibrium specifies that Brian is supposed to repay, the stage-game best response is that he buy gifts and not repay the loan. By the one-shot deviation principle, Brian will not deviate whenever:

$$5 \geq 7\,(1-\delta) + 0\delta; \text{ or } \delta \geq 2/7.$$

Indeed, the discount factor is $1/3 = 1/2 \cdot 2/3$, the probability that the game continues, times the monetary discount factor. Because $\delta = 1/3 > 2/7$, the above strategy profile is a subgame perfect equilibrium.

**5c** The policy undermines Brian's incentive to repay in the proposed equilibrium above. The one-shot deviation inequality now reads

$$5 \geq 7\,(1-\delta) + 2\delta; \text{ or } \delta \geq 2/5;$$

thus the above strategy profile is not a subgame perfect equilibrium any longer. Because the above grim-trigger Nash reversion strategy employs the most powerful punishments for failing to cooperate in the game, and is no longer a subgame perfect equilibrium, it follows that cooperation is not available in subgame perfect equilibrium anymore. The unique subgame perfect equilibrium is the repetition of the stage game unique Nash equilibrium, which yields average discounted payoff of 2 to Brian, instead of the payoff of 5 of cooperation. Hence, the policy is detrimental to Brian.

**5d** Now, the effective discount factor for the loan game is $1/3 \cdot 2/3 = 2/9$. Because $2/9 < 2/7$, the above one-shot deviation inequality is violated, so that, again, cooperation is not sustainable in a subgame perfect equilibrium.