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EC9D31 Advanced Microeconomics
Final Exam 2021-22 - Section A
Questions and Answers

Question 1. Consider the input requirement set

$$V(y) = \{(x_1, x_2, x_3) \mid x_1 + \min\{x_2, x_3\} \geq 3y,\ x_i \geq 0,\ \text{for all } i = 1, 2, 3\}.$$

(a) Does it correspond to a regular (closed and non-empty) input requirement set? **(4 marks)**

(b) Does the technology satisfies free disposal? **(5 marks)**

(c) Is the technology convex? **(5 marks)**

(d) Prove in general that the convexity of the production possibility set Z implies that the production function $f(x)$ is (weakly) concave. **(5 marks)**

(e) State and prove the Constrained Envelope Theorem and Shepard's Lemma. **(6 marks)**

Answers to Q1 We proceed in sequence as follows.

(a) The input requirement set

$$V(y) = \{(x_1, x_2, x_3) \mid x_1 + \min\{x_2, x_3\} \geq 3y,\ x_i \geq 0,\ \text{for all } i = 1, 2, 3\}$$

is closed because all defining inequalities are weak. It is non-empty because the condition $x_1 + \min\{x_2, x_3\} \geq 3y$ is not in conflict with $x_i \geq 0$.

(b) For what it concern free disposal this property is equivalent to the monotonicity of the production function:

$$F(x_1, x_2, x_3) = x_1 + \min\{x_2, x_3\}.$$

Consider an input vector $(x_1', x_2', x_3') \geq (x_1, x_2, x_3)$. By definition of inequality between vectors: $x_i' \geq x_i$ for every $i \in \{1, 2, 3\}$. It then follows that $f(x_1', x_2', x_3') \geq f(x_1, x_2, x_3)$.

(c) As for convexity consider two input vectors, $(x'_1, x'_2, x'_3) \in V(y)$ and $(x_1, x_2, x_3) \in V(y)$, by definition of $V(y)$ we have: $x'_1 + \min\{x'_2, x'_3\} \geq 3y$ and $x_1 + \min\{x_2, x_3\} \geq 3y$. Consider now the input vector $(z_1, z_2, z_3) = \lambda(x'_1, x'_2, x'_3) + (1-\lambda)(x_1, x_2, x_3)$ and $z_1 + \min\{z_2, z_3\}$. Clearly

$$z_1 + \min\{z_2, z_3\} = \lambda x'_1 + (1-\lambda)x_1 + \min\{\lambda x'_2 + (1-\lambda)x_2, \lambda x'_3 + (1-\lambda)x_3\}$$

Consider first the case $\lambda x'_2 + (1-\lambda)x_2 \leq \lambda x'_3 + (1-\lambda)x_3$ then

$$z_1 + \min\{z_2, z_3\} = \lambda x'_1 + (1-\lambda)x_1 + \lambda x'_2 + (1-\lambda)x_2 = \lambda(x'_1 + x'_2) + (1-\lambda)(x_1 + x_2)$$

$$\geq \lambda(x'_1 + \min\{x'_2, x'_3\}) + (1-\lambda)(x_1 + \min\{x_2, x_3\}) \geq 3y$$

A symmetric argument applies for the case $\lambda x'_3 + (1-\lambda)x_3 \leq \lambda x'_2 + (1-\lambda)x_2$.

(d) Consider

$$z = \begin{pmatrix} -x \\ f(x) \end{pmatrix} \in Z, \qquad z' = \begin{pmatrix} -x' \\ f(x') \end{pmatrix} \in Z$$

Convexity of Z implies that for every $0 \leq t \leq 1$

$$t\,z + (1-t)\,z' = \begin{pmatrix} -(t\,x + (1-t)\,x') \\ tf(x) + (1-t)f(x') \end{pmatrix} \in Z$$

By definition of $f(x)$ this means:

$$tf(x) + (1-t)f(x') \leq f(tx + (1-t)x')$$

for every $0 \leq t \leq 1$, the definition of a concave $f(x)$.

(e) Consider the problem:

$$\max_x f(x) \;\text{ s.t. }\; g(x, a) = 0.$$

The Lagrangian is: $\mathcal{L}(x, \lambda, a) = f(x) - \lambda\, g(x, a)$. The Constrained Envelope Theorem states that

$$\frac{d\mathcal{L}(a)}{d\,a} = -\lambda^*(a)\,\frac{\partial g(x^*, a)}{\partial a}$$

To prove it, we proceed as follows. The necessary FOC are:

$$f'(x^*) - \lambda^* \frac{\partial g(x^*, a)}{\partial x} = 0$$

$$g(x^*(a), a) = 0$$

Substituting $x^*(a)$ and $\lambda^*(a)$ in the Lagrangian we get:

$$\mathcal{L}(a) = f(x^*(a)) - \lambda^*(a)\ g(x^*(a), a)$$

Differentiating, we get:

$$\begin{aligned} \frac{d\mathcal{L}(a)}{da} &= \left[f'^*) - \lambda^* \frac{\partial g(x^*, a)}{\partial x}\right] \frac{d\ x^*(a)}{d\ a} \\ &\quad -g(x^*(a), a)\ \frac{d\lambda^*(a)}{da} - \lambda^*(a)\ \frac{\partial g(x^*, a)}{\partial a} \\ &= -\lambda^*(a)\ \frac{\partial g(x^*, a)}{\partial a} \end{aligned}$$

Where the final simplifications follow from the necessary FOC.

Shepard's Lemma states, for every input l:

$$z_l(w, y) = -\frac{\partial c(w, y)}{\partial w_l}.$$

The proof is as follows.

$$c(w, y) = \mathcal{L}(w, y) = wz(w, y) - \lambda\left[f(z(w, y)) - y\right],$$

By the constrained envelope theorem, we obtain:

$$\frac{\partial c(w, y)}{\partial w_l} = -z_l(w, y).$$

Question 2. There are two consumers A and B with the following utility functions and endowments, with $\omega_1 \geq \omega_2$, $\alpha \in [0, 1]$ and $\beta \in [0, 1]$:

$$\begin{aligned} u_A &= \alpha \ln x_{1A} + (1 - \alpha) \ln x_{2A}, \quad \boldsymbol{\omega}_A = (0, \omega_2) \\ u_B &= \beta\sqrt{x_{1B}} + x_{2B}, \quad \boldsymbol{\omega}_B = (\omega_1, 0). \end{aligned}$$

(a) Derive the Marshallian demands $x_i(p, m)$, $i = A, B$. **(5 marks)**

(b) Calculate the market clearing prices and the equilibrium allocations. **(5 marks)**

(c) Explain how the Walrasian equilibrium price of good 1 changes with α, β, ω_1 and ω_2. **(5 marks)**

(d) Explain how consumer A's demand for goods 1 and 2 changes with α, β, ω_1 and ω_2. **(5 marks)**

(e) Explain how consumer B's demand for goods 1 and 2 changes with α, β, ω_1 and ω_2. **(5 marks)**

Answers to Q2 We proceed in sequence as follows.

(a) Let p be the price of good 1 and normalize $p_2 = 1$.

Given price p, consumer A chooses $\mathbf{x}_A$ so that

$$\max\left\{\alpha \ln x_{1A} + (1-\alpha)\ln x_{2A}\right\} \qquad s.t. \qquad px_{1A} + x_{2A} = \omega_2.$$

Hence,

$$\max\left\{\alpha \ln x_{1A} + (1-\alpha)\ln(\omega_2 - px_{1A})\right\},$$

first-order conditions are:

$$\frac{\alpha}{x_{1A}} = p\frac{(1-\alpha)}{\omega_2 - px_{1A}},$$

solving out, $x_{1A} = \alpha\omega_2/p$, substituting back, we obtain: $x_{2A} = \omega_2(1-\alpha)$. Given price p, consumer B chooses $\mathbf{x}_B$ so that

$$\max \beta x_{1B} + (1-\beta)x_{2B} \qquad s.t. \qquad px_{1B} + x_{2B} = p\omega_1.$$

The consumer chooses $x_{1B} = 0$, $x_{2B} = p\omega_1$ for $p > \beta/(1-\beta)$, and $x_{1B} = \omega_1$, $x_{2B} = 0$ for $p < \beta/(1-\beta)$. For $p = \beta/\left(1-\beta\right)$, the consumer chooses any pair x_{1B}, x_{2B} such that $px_{1B} + x_{2B} = p\omega_1$.

(b) Market clearing condition, therefore, is:

$$x_{1A} + x_{1B} = \frac{\alpha\omega_2}{p} + x_{1B} = \omega_1,$$

which is satisfied only for:

$$p = \frac{\beta}{1-\beta},$$

which is the equilibrium price is. So, the equilibrium allocations are

$$x_{1A} = \frac{\alpha\omega_2(1-\beta)}{\beta}, \quad x_{2A} = \omega_2(1-\alpha),$$
$$x_{1B} = \omega_1 - \alpha\omega_2\frac{1-\beta}{\beta}, \; x_{2B} = \alpha\omega_2.$$

(c) The price p of good 1 is:

$$p = \frac{\beta}{1-\beta},$$

differentiating with respect to β, I obtain:

$$\frac{\partial p}{\partial \beta} = \frac{1}{(1-\beta)^2} > 0.$$

The equilibrium price of good 1 is constant in α, ω_1 and ω_2, and increases in β.

(d) Differentiating x_{1A} and x_{2A} with respect to α, β, ω_1 and ω_2, I obtain:

$$\begin{aligned} \frac{\partial x_{1A}}{\partial \alpha} &= \frac{1-\beta}{\beta}\omega_2 > 0, \; \frac{\partial x_{1A}}{\partial \beta} = -\frac{\alpha}{\beta^2}\omega_2 < 0, \; \frac{\partial x_{1A}}{\partial \omega_2} = \frac{\alpha}{\beta}(1-\beta) > 0, \\ \frac{\partial x_{2A}}{\partial \alpha} &= -\omega_2, \; \frac{\partial x_{2A}}{\partial \omega_2} = 1-\alpha. \end{aligned}$$

The demand x_{1A} increases in α, decreases in β and ω_2, and is constant in ω_1. The demand x_{2A} increases in ω_2, decreases in α, and is constant in β and ω_1.

(e) Differentiating x_{1B} and x_{2B} with respect to α, β, ω_1 and ω_2, I obtain:

$$\begin{aligned} \frac{\partial x_{1B}}{\partial \alpha} &= -\frac{1-\beta}{\beta}\omega_2 < 0, \; \frac{\partial x_{1B}}{\partial \beta} = \frac{\alpha}{\beta^2}\omega_2 > 0, \; \frac{\partial x_{1B}}{\partial \omega_1} = 1, \; \frac{\partial x_{1B}}{\partial \omega_2} = -\frac{\alpha}{\beta}(1-\beta) < 0, \\ \frac{\partial x_{2B}}{\partial \alpha} &= \omega_2, \; \frac{\partial x_{2B}}{\partial \omega_2} = \alpha. \end{aligned}$$

The demand x_{1B} increases in β, and ω_1, and decreases in α and ω_2. The demand x_{2B} increases in α and ω_2, and is constant in β and ω_1.