Course Syllabus: Elections

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Models of Elections

. Elections are modelled as non-cooperative games.
. There may be 2 or more office motivated candidates, possibly with different ideology or valence.
. Candidates’ strategic decisions may include whether and when to run in the election, policy platform, campaign spending amount, ...
. Voters are ideologically differentiated.
. Their decisions may include whether and who to vote, and whether to support a candidate through activism or lobbyism.
. Different electoral rules may be considered.
. Repetition and private information may play a role.
Downsian elections

. Two candidates $i = A, B$ care only about winning the election.
. Candidates $i$ simultaneously commit to policies $x_i \in \mathbb{R}$ if elected.
. There is a continuum of voters.
. The payoff of a voter with ideology $b$ if policy $x$ is implemented is $u(x, b) = L(|x - b|)$, with $L' < 0$.
. Ideologies are distributed according to (continuous and strictly increasing) empirical cumulative distribution $F$, of median $m$.
. After candidates choose platforms, each citizen votes, and the candidate with the most votes wins.
. If $x_A = x_B$, then the election is tied.
Office motivated politicians converge on median positions.

**Theorem (Median Voter Theorem)** The unique Nash Equilibrium of the Downsian election is such that candidates $i = A, B$ choose $x_i = m$, and tie the election.

**Proof.** We calculate candidate payoffs as function of $(x_A, x_B)$.

. Fix any $(x_A, x_B)$ such that $x_A \neq x_B$.

. Because $L' < 0$, each voter with ideology $b$ votes for the candidate $i$ that minimizes $|x_i - b|$.

. Hence, when $x_i < x_j$, candidate $i$’s vote share is $F\left(\frac{x_A + x_B}{2}\right)$, and candidate $j$’s is $1 - F\left(\frac{x_A + x_B}{2}\right)$.

. Now, consider any profile $(x_A, x_B)$ such that $x_i \neq m$ for at least one candidate $i = 1, 2$. 
j’s best response is $BR_j = \{x_j : |x_j - m| < |x_i - m|\}$, by playing a best response, candidate $j$ wins the election.

But if $j$ plays $x_j$ such that $|x_j - m| < |x_i - m|$, $i$’s best response cannot be $x_i$, as $i$ can at least tie the election by playing $m$.

Hence, there cannot be any Nash equilibrium where either candidate $i$ plays $x_i \neq m$.

Suppose now that both candidates play $x_A = x_B = m$.

All voters are indifferent between $x_A$ and $x_B$: the election is tied.

If either candidate $i$ deviates and plays $x_i \neq m$, then she loses the election.

Hence, there is a unique Nash equilibrium: $x_A = x_B = m$. 
Median voter theorem corresponds to equilibrium of the “Hotelling” model of monopolistic competition.

Producers choose to make identical products, in a model of monopolistic competition with horizontal differentiation.

But lack of product differentiation hurts aggregate consumer welfare in Hotelling model, whereas convergence to the median benefits voters in Downsian model.

E.g., if $F$ is uniform on $[0, 1]$, then consumer welfare is maximal in the Hotelling model with $x_A^* = 1/4$, and $x_B^* = 3/4$.

And for general $F$, the optimal products $x_A^*$ and $x_B^*$ are similarly differentiated.

Matters are very different in the Downsian model.
Proposition If voters are risk averse, then the median platforms \( x_A = x_B = m \) are preferred by a majority to any pair \( x'_A, x'_B \).

If \( x'_A, x'_B \) is ‘competitive’, i.e. \( |x'_A - m| = |x'_B - m| \), then \( x_A \) and \( x_B \) are unanimously preferred to \( x'_A, x'_B \).

Proof. Each platform \( x'_i \) in any competitive pair \( x'_A, x'_B \), is voted by 1/2 of voters.

. The pair \( x'_A, x'_B \) is a ‘bet’ with expected value equal to \( m \).

. If voters are risk averse, \( L'' < 0 \), then they all prefer the sure outcome \( x_A = x_B = m \).

. Consider now any distribution \( F \) and platform \( x'_A, x'_B \): the election selects the platform \( x'_i \) closest to \( m \).

. Thus, a majority of voters prefers \( x_A = x_B = m \) to \( x'_A, x'_B \).
**Proposition** If the ideology distribution $F$ is symmetric, $F(b) = 1 - F(2m - b)$ for all $b$, and the loss function $L$ is a power function, $L(|x - b|) = |x - b|^n$ for some integer $n$, then convergence to the median, $x_A = x_B = m$, maximizes “utilitarian” voter welfare $W(x) = - \int_{-\infty}^{+\infty} L(|b - x|) dF(b)$.

**Proof.** If $F$ is symmetric around $m$, $F(b) = 1 - F(2m - b)$ for all $b$, and $L$ is a power function, then all central moments of $F$ coincide with the median $m$ (the zero-th moment).

. Solving $x^* = \arg\max_x \{ W(x) = - \int_{-\infty}^{+\infty} |x - b|^n dF(b) \}$, we obtain that $x^* = m$.

. When $F$ is symmetric, there are also fairness considerations that make median convergence appealing.

. But when $F$ is not symmetric, median convergence does not maximize utilitarian welfare $W$ unless $L$ is a linear function.
Ordinal preferences

. Consider a compact policy space $X$ and a set of voters $N = \{1, \ldots, n\}$, with $n$ odd.

. Preferences are single-peaked on space $X$ with linear order $\succ$, if for each voter $j$ there is a policy $b_j$ such that for all $x, y \in X$,
  . if $b_j \succeq y \succ x$, then $y \succ_j x$,
  . if $x \succ y \succeq b_j$, then $x \succ_j y$.

. Preferences are single-crossing on space $X$ with linear order $\succ$, for voter index permutation $p : N \to N$, whenever
  
  if $x \succ y$ and $p(j) \succ p(i)$, or if $x \prec y$ and $p(j) \prec p(i)$,
  then $x \succ_{p(i)} y$ implies $x \succ_{p(j)} y$.

. A policy $x$ that defeats any other policy $y$ is a Condorcet winner.
**Theorem** Say that an odd number of voters vote among two candidates. If policy $x$ is the Condorcet winner, then both candidates choose $x$ in equilibrium.

**Theorem** (Black, 1948; Gans and Smart, 1996) If an odd number of voters have single-peaked or single-crossing preferences, then the Condorcet winner is the ideal point of the median voter $m$.

There are preference profiles with no Condorcet winners.

1: $x \succ y \succ z$
2: $y \succ z \succ x$
3: $z \succ x \succ y$

The two results are independent: single-crossing condition does not imply single-peakedness, nor vice-versa.
Preferences may be single crossing but not single peaked.

1: $x \succ y \succ z$
2: $x \succ z \succ y$
3: $z \succ y \succ x$

are single crossing on order $x < y < z$ but not single peaked:

$z \succ_2 y \Rightarrow z \succ_3 y$, $x \succ_2 z \Rightarrow x \succ_1 z$, $x \succ_2 y \Rightarrow x \succ_1 y$.

(Not single peaked for any $\succ$ as each $x$, $y$, $z$ is the worst for a voter.)

Preferences may be single peaked but not single crossing.

1: $w \succ x \succ y \succ z$
2: $x \succ y \succ z \succ w$
3: $y \succ x \succ w \succ z$

are single peaked on $w < x < y < z$, but not single crossing:

for $2 < 3$, $z \succ_2 w$ but $z \not\succ_3 w$; for $3 < 2$, $y \succ_3 x$ but $y \not\succ_2 x$. 
Multi-dimensional policy spaces

. Policy platforms are usually multi-dimensional.
. But often multidimensional policy can be projected on a left-right unidimensional space on which voters can be ordered.
. Consider a compact policy space $X \subset \mathbb{R}^d$ and set of voters $N$.
. The voters in $j \in N$ have “intermediate preferences” if every $j$’s payoff can be written as $L_j(x) = J(x) + K(p_j)H(x)$ for some voter index permutation $p$, where $K$ is monotonic, whereas $H(x)$ and $J(x)$ are common to all voters.

**Proposition** Say that an odd number of voters with intermediate preferences vote among two candidates. Then both candidates choose policy $x(p_m)$, the ideal point of the voter $i$ with median $p_m$. 
Suppose agents preferences can be represented by \( L(||x - b_i||) \), where \( b_i \) is vector describing \( i \)'s bliss point in this policy space.

\( L \) decreasing and concave in the Euclidean distance \( ||x - b_i|| \).

**Theorem** (Plott, 1967) There exists a Condorcet winner policy in the interior of a multidimensional policy space \( X \) if and only if there is a policy \( m \) median in all directions.

The existence of a median in all direction requires strong symmetry assumptions on the distribution of individual ideal points.

The ‘top cycle’ of \( X \) is the set of all alternatives \( x \in X \) such that for each \( y \neq x \), there are \( c_1, ..., c_K \) such that \( x = c_1 \succ c_2 \succ ... \succ c_K = y \), where \( \succ \) represents a preference by a majority.

**Theorem** (McKelvey 1976) In a multi-dimensional policy space, if there is no Condorcet winner, then the top cycle is the whole set of alternatives.
Example Consider the divide the dollar game with 3 voters.

. Set of alternatives is $X = \{(x_1, x_2, x_3) \geq 0 : x_1 + x_2 + x_3 = 1\}$.

. Each voter $i$’s payoff is increasing in $x_i$.

. The top cycle is $TC = X \setminus \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

. In fact, every $x \in X$ is defeated by at least one among $(1/2, 0, 1/2)$, $(1/2, 1/2, 0)$ and $(0, 1/2, 1/2)$.

. If $x > 0$, then $x \succ (0, \epsilon, 1 - \epsilon) \succ (1/2, 0, 1/2)$ for some small $\epsilon > 0$ and similarly for $(1/2, 1/2, 0)$ and $(0, 1/2, 1/2)$.

. If exactly two entries of $x$ are positive, then $x$ beats some $x' > 0$, which then indirectly beats all other alternatives.
Agenda setting

. Suppose there are no candidates.
. Voters choose among a finite set of fixed alternatives $X$.
. The choice is made by sequential pairwise elimination.
  E.g., voters choose $x$ vs. $y$, winner is matched to $z$, and so on.
. The ‘agenda’ is the sequence in which alternatives are voted.
. If there is a Condorcet winner, it is selected for all agenda.
. If voters vote sincerely on each alternative, then for every policy $x$ in the top cycle set, there exist agenda that select $x$.
. By McKelvey theorem, the top cycle is $X$: the agenda-setter can determine the outcome.
If voters are strategic and know the agenda, the game is solved by backward induction.

The Banks set includes all alternatives in \( X \) that survive successive elimination by strategic voters for some agenda.

If there is a “status quo” \( \bar{x} \) in \( X \), it is voted last against the penultimate surviving alternative in the agenda.

The inclusion of status quo further restricts the set of alternatives “available” to the agenda setter.
Probabilistic voting

. In Downsian elections, winning probabilities jump discontinuously because voters’ preferences are known.

. Probabilistic voting models smooth out discontinuities by adding “noise” to voters’ preferences.

. If candidates maximize probability to win, then platforms converge to the expected median platform.

. If candidates maximize vote share, then platforms converge to an weighted average platform.

. Platforms may converge also in multi-dimensional policy spaces.
Aggregate uncertainty (Calvert 1985)

. Candidates maximize the probability of winning majority.
. Voters’ preferences do not vary independently.
. Median platform depends on a random common state.
. Each voter $j$ with bliss point $b_j \in \mathbb{R}$ has utility $L(|x - b_j|)$, with $L' < 0$, $L'' < 0$, and $\lim_{z \downarrow 0} L'(z) = 0$, $\lim_{z \uparrow \infty} L'(z) = -\infty$.
. Each ideal point $b_j$ is decomposed as: $b_j = m + \delta_j + e_j$:
  . $\delta_j$ is the fixed $j$’s bias relative to the median platform $m$,
    the empirical distribution of $\delta_j$ across $j$ has median zero;
  . $m$ is the random median platform, with c.d.f. $F$ and median $\mu$;
  . $e_j$ is noise, i.i.d. over $j$, with symmetric density and $E[e_j] = 0$. 

As in the Downsian model there are two candidates, \( i = A, B \) who care only about winning the election.

Candidates \( i \) simultaneously commit to policies \( x_i \in \mathbb{R} \) if elected.

After candidates choose platforms, each voter votes, and the candidate with the most votes wins.

If \( x_A = x_B \), then the election is tied.

**Proposition** In the unique Nash equilibrium of the probabilistic model with aggregate uncertainty, the candidates \( i = 1, 2 \) choose \( x_i \) equal to the median \( \mu \) of the distribution of the median policy \( m \) and tie the election.

**Proof:** Suppose that \( x_i < x_j \), then candidate \( i \) wins the election if \( m < (x_A + x_B)/2 \) and \( j \) wins if \( m > (x_A + x_B)/2 \).
The probability $q_i(x_i, x_j)$ that $i$ wins the election is

$$q_i(x_i, x_j) = \begin{cases} 
\frac{F(x_A+x_B)}{2} & \text{if } x_i < x_j, \\
1/2 & \text{if } x_i = x_j, \\
1 - \frac{F(x_A+x_B)}{2} & \text{if } x_i = x_j.
\end{cases}$$

Given $x_j$, candidate $i$ chooses $x_i$ to maximize $q_i(x_i, x_j)$.

Suppose that $x_j < \mu$. Then, $q_i(x_i, x_j) > 1/2$ and strictly decreasing in $x_i$ for $x_i > x_j$. $i$’s best response is empty.

Likewise, if $x_j > \mu$, then $i$’s best response is empty.

If $x_j = \mu$, then $q_i(x_i, x_j) < 1/2$ and strictly increasing in $x_i$ for $x_i < x_j$, $q(\mu, x_j) = 1/2$, and $q_i(x_i, x_j) < 1/2$ and strictly decreasing in $x_i$ for $x_i > x_j$. $i$’s best response is $x_i = \mu$.

Hence, there is a unique equilibrium: $x_A = x_B = \mu$. 
Vote share maximization (Lyndbeck and Weibull 1993)

. There are $G$ groups of voters $g$ with $s_g$ share of voters in each $g$.
. Candidates $i = A, B$ simultaneously announce platforms $x_i$ in $\mathbb{R}^d$.
. The payoff of voter $k$ in group $g$ is: $u_k(x, i) = L_g(x) + \eta_{ki}$
. $L_g$ is a continuously differentiable loss function, strictly decreasing in the distance $\|x - b_g\|$ from a bliss point $b_g$ in $\mathbb{R}^d$.
. $\eta_{ki}$ are non-policy benefits for $k$ if $i$ is in power.
. Let $\epsilon_k = \eta_{kB} - \eta_{kA}$, drawn independently across individuals, with cumulative distribution $H_g$ on $\mathbb{R}$ and density $h_g$.
. Let $q_{gi}$ be fraction of voters in $g$ that vote candidate $i = A, B$.
. Candidate $i$ picks $x_i$ to maximize vote share $q_i = \sum_{g=1}^{G} s_g q_{gi}$. 
Results

. Each voter \( k \) in group \( g \) votes for \( A \) if \( L_g(x_A) - L_g(x_B) > \varepsilon_k \).
. Vote share for \( A \) in group \( g \) is \( q_{gA} = H_g(L_g(x_A) - L_g(x_B)) \).
. Suppose that
  . \( q_A = \sum_{g=1}^G s_g H_g(L_q(x_A) - L_q(x_B)) \) is strictly concave in \( x_A \)
  . \( q_B = \sum_{g=1}^G s_g [1 - H_g(L_q(x_A) - L_q(x_B))] \) str. concave in \( x_B \).
. Then the equilibrium \( (x_A, x_B) \) solves the FOC:

\[
\sum_{g=1}^G s_g h_g(L_q(x_A) - L_q(x_B)) DL_g(x_A) = 0 \\
\sum_{g=1}^G s_g h_g(L_q(x_A) - L_q(x_B)) DL_g(x_B) = 0,
\]

where \( DL_g(x_i) = (\frac{\partial L_g}{\partial x_{i1}}, \ldots, \frac{\partial L_g}{\partial x_{in}})^T \).
**Proposition** If a pure strategy equilibrium \((x_A, x_B)\) of probabilistic voting model exists, then \(x_A = x_B = x\) such that

\[
\sum_{g=1}^{G} s_g h_g(0) DL_g(x) = 0.
\]

Nash-equilibrium corresponds to solution to maximization of weighted utilitarian social welfare function:

\[
\sum_{g=1}^{G} s_g w_g DL_g(x) = 0,
\]

with group weights \(w_g = h_g(0)\).

Group weight corresponds to group size and responsiveness to policy changes \(h_g(0)\), i.e. share of unbiased voters/swing voters.

When do pure strategy equilibria exist?

Strict concavity of \(q_i\) in \(x_i\) for \(i = A, B\) is hard to check.

A sufficient condition is that for each group \(g\),

\(H_g(L_g(x_A) - L_g(x_B))\) is strictly concave in \(x_A\) and \(x_B\).
Summary

. We have reviewed Downsian and probabilistic elections.
. Two office-motivates candidates credibly commit to platforms.
. Then, voters vote for the preferred platform candidate.
. If policies are uni-dimensional, candidates’ platforms “converge” to the policy preferred by the median voter.
. If the policy space is multi-dimensional, anything goes.
. If there are no candidates and alternatives are voted sequentially, the agenda setter is a dictator unless voters are strategic.
. Equilibrium exist in multi-dimensional policy spaces, if candidates maximize vote shares and voters’ preferences are uncertain.
. This equilibrium is Pareto efficient for the electorate.
Next lecture

1. I will introduce policy motivation in spatial models of elections.
2. Suppose candidates have policy preferences in the aggregate uncertainty probabilistic model.
3. Because of uncertainty, equilibrium platforms diverge.
4. If voters’ preferences may change during campaigns, then platform divergence improves electorate welfare.
5. Suppose candidates have policy preferences, cannot credibly commit to platforms, and choose whether to run or not.
6. There exist equilibria where platforms “diverge” from the median.
7. Candidate may enter elections in the expectation of losing, only to steal votes from perspective winner.