Problem Set 9 – Solutions

Exercise 1

Dividing a cake fairly

a. If player 1 divides the cake unequally then player 2 chooses the larger piece. Thus in any subgame perfect equilibrium player 1 divides the cake into two pieces of equal size.

b. In a subgame perfect equilibrium player 2 chooses P_2 over P_1, so she likes P_2 at least as much as P_1. To show that in fact she is indifferent between P_1 and P_2, suppose to the contrary that she prefers P_2 to P_1. I argue that in this case player 1 can slightly increase the size of P_1 in such a way that player 2 still prefers the now-slightly-smaller P_2. Precisely, by the continuity of player 2's preferences, there is a subset P of P_2, not equal to P_2, that player 2 prefers to its complement $C \setminus P$ (the remainder of the cake). Thus if player 1 makes the division $(C \setminus P, P)$, player 2 chooses P. The piece P_1 is a subset of $C \setminus P$ not equal to $C \setminus P$, so player 1 prefers $C \setminus P$ to P_1. Thus player 1 is better off making the division $(C \setminus P, P)$ than she is making the division (P_1, P_2), contradicting the fact that (P_1, P_2) is a subgame perfect equilibrium division. We conclude that in any subgame perfect equilibrium player 2 is indifferent between the two pieces into which player 1 divides the cake.

I now argue that player 1 likes P_1 as least as much as P_2. Suppose that, to the contrary, she prefers P_2 to P_1. If she deviates and makes a division $(P, C \setminus P)$ in which P is slightly bigger than P_1 but still such that she prefers $C \setminus P$ to P, then player 2, who is indifferent between P_1 and P_2, chooses P, leaving $C \setminus P$ for player 1, who prefers it to P and hence to P_1. Thus in any subgame perfect equilibrium player 1 likes P_1 at least as much as P_2.

To show that player 1 may strictly prefer P_1 to P_2, consider a cake that is perfectly homogeneous except for the presence of a single cherry. Assume that player 2 values a piece of the cherry in exactly the same way that she

Exercise 2

T-period version of Rubinstein alternating offer bargaining game

a) Let x^t be the equilibrium offer after r-1 offers have been rejected, where each x^r is a division of the pie: $x^t = (x_1^t, x_2^t)$, where $x_1^t + x_2^t = 1$.

Solve by backward induction:

Period T

- Even period: Player 2 makes offer x^T
- Player 1 will accept if payoff greater than payoff from rejection (here zero since game ends) ⇒ will accept if $x_1^T \geq 0$
- Player 2 will maximise own share, subject to acceptance: $x^T = (0,1)$

Period T-1

- Odd period: Player 1 makes offer x^{T-1}
- Player 1 will accept if $x_2^{T-1} \geq \delta$ (discounted payoff from continuing to next period)
- Player 2 will maximise own share, subject to acceptance: $x^{T-1} = (1-\delta, \delta)$

Continue argument:

Equilibrium offers will be such that receiver is indifferent between accepting and rejecting. Proposer will offer other player a share equal to discounted payoff she could get in following period:

- $x^{T-2} = \big(\delta(1-\delta), 1-\delta(1-\delta)\big) = (\delta - \delta^2, 1 - \delta + \delta^2)$
- $x^{T-3} = \big(1-\delta(1-\delta+\delta^2), \delta(1-\delta+\delta^2)\big) = (1-\delta+\delta^2-\delta^3, \delta-\delta^2+\delta^3)$
- …
- $x^1 = (1-\delta+\cdots+\delta^{T-2}-\delta^{T-1}, \delta-\delta^2+\cdots-\delta^{T-2}+\delta^{T-1})$
  $= ([1-\delta][1+\delta^2 \ldots + \delta^{T-2}], 1-[1-\delta][1+\delta^2 \ldots + \delta^{T-2}])$

Hence in the subgame perfect equilibrium player 1 will offer x^1 and player 2 will accept in the first period.

b) For $T \to \infty$ the first period offer becomes:

$$x^1 = \left(\frac{1-\delta}{1-\delta^2}, 1 - \frac{1-\delta}{1-\delta^2}\right) = \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$$

Exercise 3

a) Nash Bargaining Problem

- X (outcomes): Any offer $X = (x_1, x_2)$, where $x_1 + x_2 \leq 1$.
- D (failure to agree)
- Utility: $u_i(D) = d_i = 0$, $\quad u_i(x_i) = x_i$
- U (payoff set): $U = \{(v_1, v_2) : v_i = Eu_i(L)$, for some lottery L over $X \cup \{D\}$.

Solution

$\max\,(v_1 - d_1)\,(v_2 - d_2)$ s.t. $v_1 + v_2 \leq 1$ (binding)
and $(v_1, v_2) \geq (0,0)$.

$\mathcal{L} \equiv v_1 v_2 + \lambda(1 - v_1 - v_2)$

F.O.C.:
$\frac{\partial \mathcal{L}}{\partial v_1} = v_2 - \lambda$ =0, $\quad \frac{\partial \mathcal{L}}{\partial v_2} = v_1 - \lambda = 0$, $\quad \frac{\partial \mathcal{L}}{\partial \lambda} = 1 - v_1 - v_2 = 0$
$\Rightarrow v_1 = v_1 = \frac{1}{2}$

Comparison to SPNE (from exercise 2):

$$x^1 = \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$$

$$\lim_{\delta \to 1} \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

For agents who are not impatient the subgame perfect Nash equilibrium is the same as the Nash bargaining solution.

b) A unique bargaining solution satisfies all the axioms:

1. Invariance to equivalent utility representations
2. Symmetry
3. Pareto efficiency
4. Independence of irrelevant alternatives.