BIASED SOCIAL LEARNING

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Abstract. This paper examines social learning when only one of the two types of decisions is observable. Because agents arrive randomly over time, and only those who invest are observed, later agents face a more complicated inference problem than in the standard model, as the absence of investment might reflect either a choice not to invest, or a lack of arrivals. We show that, as in the standard model, learning is complete if and only if signals are unbounded. If signals are bounded, cascades may occur, and whether they are more or less likely than in the standard model depends on a property of the signal distribution. If the hazard ratio of the distributions increases in the signal, it is more likely that no one invests in the standard model than in this one, and welfare is higher. Conclusions are reversed if the hazard ratio is decreasing.

Keywords: Informational herds, Cascades, Selection bias.

1. Introduction

The process of learning in social contexts confronts the same difficulties as any other statistical analysis. The data available to an individual may be subject to selection bias. Consider the leading example used by Bikchandani, Hirshleifer and Welch [2] (henceforth BHW), for instance. Upon learning that a paper has been previously rejected, a referee at a second journal tilts toward rejection. But what if, as is usually the case, he did not learn about this rejection? Surely, he would nevertheless wonder about the paper's journey onto his desk, and speculate about rejections the paper might have gone through. While publications are by definition observable, rejections are not.

To wit, there are far more significant instances of such bias in academia. How easy is it to publish a paper that finds inconclusive empirical evidence? In medical and social sciences, studies

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whose findings are statistically insignificant get filed away, biasing the published papers toward positive results.[1]

The difficulty in interpreting the absence of negatives is encountered everywhere. Is no one waiting in this line because cabs come by all the time, or because this isn't actually a cab line? Do the low figures of tax evasion reflect the success of deterrent policies, or the success of tax evaders? Why do 90% of mutual funds truthfully claim to have performance in the first quartile of their peers? (The other three quarters of funds have closed. See Elton, Gruber and Blake [7].)

This paper develops a model of biased social learning and revisits the findings of the literature. In this model, individuals arrive randomly over time. As in Smith and Sørensen [15], each agent has some private, conditionally independent information about the relevance of taking some decision immediately upon arrival—say, making an investment. As often in the social learning literature, we assume that the payoff from investing depends on the state of the world, but not on what earlier or later individuals decide. Therefore, values are common, and externalities are purely informational.[2] As is standard as well, signal distributions satisfy the strict monotone likelihood ration property (MLRP).

What sets this model apart from standard models is the following informational assumption. While the decision to invest (but not the payoff from investing) is observable to all future individuals, the failure to do so, and in fact, the mere arrival of individuals (who do not invest), remains hidden. Individuals arriving later will observe "positives" (if and when earlier individuals invested), but not "negatives" (if and when earlier individuals chose to abstain).

Therefore, every individual faces a complex problem of statistical inference: given the observed history, and the randomness in the arrival of individuals, how likely is it that some individuals had the opportunity to invest, but chose not to? And if so, what were their private signals? Note that, in this problem, time plays a key role, as it becomes increasingly more likely, as time passes by, that some individuals must have had the opportunity to invest.

[1]This phenomenon is known as the "file-drawer problem," or the "publication bias." See Scargle [13]. As a result, prominent medical journals no longer publish results of drug research sponsored by pharmaceutical companies unless that research was registered in a public database from the start. Some of them also encourage publication of study protocols in their journal.

[2]We shall also discuss at length a version in which there is only one investment opportunity, in which case there is an obvious payoff externality.

In this context, we ask whether biased social learning exacerbates or mitigates herding. Could it be the case that some investment opportunities, or lucrative projects, remain unexploited because agents considering making it suspect that others must have thought of it, or even tried it before them? How many entrepreneurs, or scientists, stumbling across a new idea, chose not to follow through this idea because of the rational belief that they were unlikely to be the first to think of it?

Our first main result shows that, qualitatively, the absence of negatives does not alter the conditions under which cascades can, or cannot occur. If the informativeness of signals is bounded, wrong herds can occur (that is, they will occur for some prior and payoff parameters). On the other hand, if signals are unbounded, learning is necessarily complete; whether the state of the world is such that investment is profitable or not, agents will eventually learn it.

On the other hand, our second main result shows how, quantitatively, the absence of negatives affects the probability of a wrong herd. Consider the case of bounded signals (so that cascades may occur). What is the probability that no agent ever invests, while agents should, in the case of biased learning, relative to this probability in the benchmark model of BHW, in which all decisions, to invest or not, are observed? As it turns out, the comparison of these probabilities hinges upon a simple statistical property of the signal distribution, the increasing hazard ratio property.[3] If signals satisfy the increasing hazard ratio property (IHRP), that is, if the ratio of the hazard rates increases in the signal, then the probability of no one ever investing is lower under biased learning, *independently* of the state of the world. Conversely, if the hazard ratio is decreasing, then this probability is lower in the benchmark model. While biased learning always leads to higher investment (relative to the benchmark model) under IHRP, it nevertheless leads to lower welfare, at least in the version of our model in which there is only one investment opportunity.[4]

[3]Properties of IHRP in the standard ("BHW") learning model are derived in Herrera and Hörner (2011). Namely, IHRP is the necessary and sufficient condition under for the absence of informational cascades, namely provided the decision of the first individual depends on his signal, the decision of all later individuals will do as well. That is, it ensures that the posterior public belief necessarily stays in the learning region provided that the prior lies in it. More precisely, IHRP guarantees that this is the case after "good news," that is, after an observed investment decision. There is a corresponding property for the case of "bad news."

[4]This version allows us to focus on the history of no observed investment.

The first models of sequential decisions and observational learning by Banerjee [1] and Bickchandani, Hirshleifer and Welch [2], and their subsequent generalization by Smith and Sørensen [15] all assume that all actions are observed by later individuals. Namely, agents could observe the precise sequence of decisions made by all the predecessors. Later work, notably Çelen and Kariv [4], Callander and Hörner [3] and Smith and Sørensen [14] relaxes this assumption and considers the case in which either a subsample of the sequence, or a statistic thereof is observable. As these authors show, the asymptotic properties of social learning may radically change. For instance, Çelen and Kariv [4] show that, when agents only observe the action of their immediate predecessor, beliefs do not converge. Therefore, complete learning never occurs, as beliefs and actions end up cycling. Hence, limiting the information available to agents may alter the qualitative properties of learning in general, although it turns out not to do so in the case of biased learning. Callander and Hörner [3] show that if agents can only observe the fraction of agents having taken each action, rather than the entire sequence, then it might be optimal to take the action that was adopted by the minority of predecessors. A similar observational assumption is made in Hendricks, Sorensen and Wiseman [9].

Guarino, Harmgart and Huck [8] also analyze a framework in which one of the two actions is not observable, assuming simpler and more essential public information: the only public state variable is the aggregate number of people that took the observable action. Interestingly, they obtain that only cascades on the observable action can occur, never on the non observable action.

Chari and Kehoe [6] develop an observational learning model with a similar investment bias, in the sense that more information is revealed after observing an investment than after observing a decision not to investment. Each investment amount is an observable continuous variable, so the investor's private signal can be fully inferred (when this investment amount is positive). As in usual models, in case of a non-investment is observed then only a truncation on the investor's private signal can be inferred. In a sense, their model adds information in a biased fashion to the standard model (investment decisions become more informative), while we suppress information in a biased fashion, by assuming that decisions not to invest are not observable.

Our model is also related to models of endogenous timing such as the elegant paper of Chamley and Gale [5]. In their model as in ours, whether an agent has an opportunity to invest or not is a random variable. In their model, there is a finite number of agents who are all present from the start, and may choose to wait before investing, if they wish to. Inefficiently low investment occurs because agents might decide to wait, in the hope that others will act first, and thereby reveal valuable information. In our model, agents arrive at random times, and the total number of agents having the opportunity to invest is almost surely countably infinite, so that the collective information of the agents reveals the state. In both their model and ours, agents must be careful in interpreting an observed absence of investment. In Chamley and Gale [5], this absence might also reflect strategic delay, rather than bad news, while in ours, it might simply be because no agent happened to face this choice. As we show, in the version of our model in which the game stops after the first investment opportunity, agents have no incentive to engage in strategic delay.

Section 2 introduces the set-up. Section 3 develops the analysis and parametric examples. The qualitative results are stated in Section 4. Section 5 focuses on the quantitative results regarding the probability that no one ever invests. Most proofs are relegated to an appendix.

2. Set-up

2.1. **Information.** Imagine a situation in which there are two states of the world. The state of the world is denoted $\theta \in \{0,1\}$. We refer to state 0 as the bad state, and to state 1 as the good state. The *ex ante* probability of the good state is denoted

$$p_0 := \mathbb{P}[\theta = 1] \in (0,1).$$

There is a countable infinity of agents (or individuals, or players). Each agent receives a private signal (or type) $x \in X := [0,1]$. Private signals are conditionally independent across agents. The conditional distribution of this signal is identical across agents. Conditional on state θ, the distribution (c.d.f) is denoted F_θ, and assumed twice differentiable on $(0,1)$, with density f_θ that is strictly positive on $(0,1)$.

Private signals provide valuable information about the state to the agents. The distributions are assumed throughout to satisfy the strict monotone likelihood ratio property (MLRP henceforth), and this assumption will be implicit in all formal statements. That is, defining the likelihood ratio

$$l(x) := \frac{f_1(x)}{f_0(x)},$$

on the interval $(0,1)$, we assume that l is strictly increasing. This guarantees that higher values of the signal lead to higher posterior probabilities that the state is good, for all priors. See

Milgrom [11]. In some instances, without loss of generality, we will set $l(1/2)$ to 1, so that the signal $x = 1/2$ leaves any given prior probability belief unchanged.

2.2. **Actions and Payoffs.** Each agent i faces a binary choice. He may either invest or not. The decision to invest is denoted I, while the decision not to is denoted N. An action, therefore, is an element $a_i \in \{N, I\}$. Investing is costly: the action I entails a cost $c \in (0,1)$. The return from investment is random, and depends on the state of the world. We normalize its expectation to 0 in the bad state, and to 1 in the good state. The payoff from not investing is set to 0, so that, under complete information, an agent would invest if and only if the state were good. To summarize, the payoff of an agent is given by

$$\begin{aligned} u(N) &:= 0, \\ u(I) &:= -c + \begin{cases} 0 & \text{if } \theta = 0, \\ 1 & \text{if } \theta = 1. \end{cases} \end{aligned}$$

Note that it is optimal for an agent to invest if and only if he assigns a probability of at least c to the good state. Throughout, we refer to this probability as the agent's belief.

2.3. **Timing and Histories.** We are now ready to describe the extensive-form game. Time is continuous and the horizon is infinite. There is Poisson arrival process defined over dates $t \in \mathbb{R}_+$, with associated random point process $\{T_i\}_{i\geq 0}$, with $T_0 := 0$, and $T_i \leq T_{i+1}$ for all $i \geq 0$. The intensity of the Poisson process, λ, is constant and independent of the state of the world.

The random variable T_i determines agent i's *arrival time*. That is, agent i must take an action j at the date of the realization t_i of T_i. Because agents cannot delay their decision, any discounting is irrelevant and ignored.

Arrival times are not observed, and neither are decisions not to invest. Private signals are not observed either. Further, agents do not know their index, i.e. agent i does not know that, by definition of his index, $i-1$ agents had the opportunity to invest before him.[5] However, decisions to invest are observed (of course, the corresponding arrival time is then inferred). A (complete) history up to date t, then, specifies the state of the world, the infinite sequence of

[5]Their belief about their rank is the improper uniform prior, so that they are *a priori* equally likely to be anywhere in the sequence.

private signals of players, the date t and the sequence $\{(t_i, a_i)\}_i$, with $t_i \le t$, for all i, of arrival times and actions taken by the corresponding agent. This sequence is (almost surely) finite. Agents, however, only observe a subset of these arrival times. The relevant history, then, is the public history $h_t := (t, \{(t_i, I)\}_i)$, which is the subset of the complete history that includes all times at which an agent decided to invest, as well as the current date t. Note that the public history does not include the identity i of the agents that decided to invest, so that it is not possible to infer from such a history how many agents actually had the opportunity to invest up to time t. Denote the set of public histories up to time t by H_t, and let $H := \cup_{t \ge 0} H_t$ denote the set of all histories. Set $H_0 := (0, \{\varnothing\})$. A history $h_t = (t, \{\varnothing\})$ indicates that no investment has taken place up to date t.

2.4. **Strategies and Equilibrium.** A strategy for agent i specifies, for each possible signal x, arrival time t_i and public history h_t, $t = t_i$, an action choice (possibly mixed). That is, a behavior strategy for agent i is a measurable mapping

$$\sigma_i : X \times H \to \triangle\{N, I\},$$

where $\triangle\{N, I\}$ denotes the set of lotteries over $\{N, I\}$. Since a player only knows his own type, this is a game of incomplete information, and one should therefore specify each player's assessment over the complete history, for each possible (x_i, h_t), $t = t_i$. Given that agents cannot delay their decision, and given the payoff specification, it is clear that the only relevant probability that affects their choice is their belief p_i over the state of the world, given (x, h_t). We build this directly in the definition of an equilibrium.

A *Perfect Bayesian Equilibrium* consists of a strategy profile $\sigma := \{\sigma_i\}_i$ and a profile of beliefs $p_i : X \times H \to [0, 1]$, all i, such that (i) each player's strategy is a best-reply at every information set, and (ii) the beliefs p_i are consistent with Bayes' rule at every information set that is reached with positive probability, given σ.

Note that, given MLRP, the optimal strategy of an agent must be a pure strategy, and more precisely a cut-off strategy. That is, he should invest if and only if his signal is high enough. We focus on symmetric equilibria. This implies that an equilibrium will be uniquely determined by a measurable function x_t that specifies the cut-off type above which an agent i invests at time t. Along with the prior belief p_0, this determines, in particular, the *public belief* $p_t := \mathbb{P}[\theta = 1 | h_t]$

(henceforth, belief) about the state given the observed history h_t. Note also that p_t is a summary statistic for h_t. In the sequel, it we shall also use the likelihood ratio of p_t, denoted L_t, and defined by

$$L_t := \frac{p_t}{1-p_t} = \frac{\mathbb{P}\left[\theta = 1|h_t\right]}{\mathbb{P}\left[\theta = 0|h_t\right]}.$$

Because L is strictly increasing in p, we shall sometimes, with an abuse of terminology, refer to this ratio as the public belief as well.[6]

3. Analysis

3.1. **Threshold Signal and Public Belief.** As mentioned, an equilibrium can be summarized by two functions of the public history h_t: the belief p_t that the state is good, and the cut-off x_t such that an agent invests at date t if and only if his signal exceeds x_t.

Bayes' rule provides one relationship between x_t and the belief p_t. Namely, p_t determines x_t since, given p_t, an agent of type x_t must be indifferent between investing or not, at least when x_t is in $(0,1)$:[7]

$$x_t := x \text{ solves } \mathbb{E}\left[\theta|x, h_t\right] = \mathbb{P}\left[\theta = 1|x, h_t\right] = c.$$

Using Bayes' rule, this means that the threshold x_t solves

$$\frac{f_1(x)\, p_t}{f_1(x)\, p_t + f_0(x)\,(1-p_t)} = c.$$

Using the likelihood ratio, the relationship between the threshold x and the public belief L takes the simple product form, for any given c,

$$l(x)\, L = \frac{c}{1-c}, \tag{1}$$

which highlights the inverse relationship between the public belief and the signal cut-off: if agents are more optimistic, that is, if the public belief L is higher, a lower threshold signal x is required for an investment opportunity to be deemed profitable.

There is a second relationship between x_t and p_t. Namely, the threshold x_t determines the evolution of the belief p_t, along with the initial value p_0. The evolution of p_t over some small

[6]Here and in what follows we omit the time subscripts when the relation holds in general.

[7]If, given p_t, it is optimal to invest independently of the signal, set x_t equal to 0. Similarly, set it to 1 if it is always optimal not to invest.

interval of time $(t, t+dt]$, $dt > 0$, depends on whether an investment occurs or not during that interval. Accordingly, we divide the analysis into two cases.

3.2. **Evolution without Investment.** Consider first the case in which there has been no investment up to time t. Note that, in this case, p_t is a continuous function, because the distribution over arrival times is continuous. It follows that x_t is continuous as well. Let

$$G_{\theta,t} := \mathbb{P}\left[h_t = (t, \{\varnothing\})|\theta\right]$$

be the probability of this event, conditional on state θ. Since investments arrive at rate $\lambda F_\theta(x_t)$, this probability is given by

$$G_{\theta,t} = e^{-\lambda \int_0^t (1-F_\theta(x_s))ds}.$$

To see this, note that, from time t to time $t+dt$, this probability evolves as follows (neglecting terms of order dt^2 or higher):

$$G_{\theta,t+dt} = G_{\theta,t} \cdot \left(F_\theta\left(x_t\right) \cdot \lambda dt + 1 \cdot (1 - \lambda dt)\right).$$

Indeed, the probability that no one invests up to time $t+dt$ is the probability that no one invests up to time t, multiplied by the probability that no one invests in the time interval $(t, t+dt]$. This latter probability is the sum of two terms. With probability $1 - \lambda dt$, no agent arrives during this time interval. With probability $F_\theta\left(x_t\right) \cdot \lambda dt$, some agent arrives in the time interval, but his signal is below the threshold. Because x_t is continuous, it follows that $G_{\theta,t}$ is differentiable and solves

$$\frac{G'_{\theta,t}}{G_{\theta,t}} = -\lambda\left(1 - F_\theta\left(x_t\right)\right),$$

along with $G_{\theta,0} = 1$. This integrates out to the formula above.

After a no-investment history $h_t = (t, \{\varnothing\})$, the public belief evolves according to

$$p_t = \mathbb{P}[\theta = 1|h_t] = \frac{\mathbb{P}[h_t|\theta = 1] \cdot \mathbb{P}[\theta = 1]}{\mathbb{P}[h_t]} = \frac{G_{1,t}p_0}{\mathbb{P}[h_t]}.$$

Since $G_{1,t}$ is a function of $(x_s)_{s\leq t}$, this provides a second relationship between the belief p_t and the threshold x_t.

We now combine the two relationships. As pointed out, the threshold x_t solves

$$\frac{f_1\left(x\right)p_t}{f_1\left(x\right)p_t + f_0\left(x\right)\left(1-p_t\right)} = \frac{f_1\left(x\right)G_{1,t}p_0}{f_1\left(x\right)G_{1,t}p_0 + f_0\left(x\right)G_{0,t}\left(1-p_0\right)} = c.$$

Using the formula for $G_{\theta,t}$, and setting

$$\gamma := \frac{1-p_0}{p_0}\frac{c}{1-c},$$

it follows that x_t solves

$$l(x_t) = \gamma \frac{G_{0,t}}{G_{1,t}} = \gamma e^{\lambda \int_0^t (F_0(x_s) - F_1(x_s)) ds}. \tag{2}$$

Since the right-hand side is differentiable in t, and l is differentiable, the function x_t must be differentiable as well. By the implicit function theorem, the function x_t solves

$$\frac{l'(x_t)\, x_t'}{l(x_t)} = \lambda \left(F_0(x_t) - F_1(x_t)\right),$$

with initial condition $x_0 = l^{-1}(\gamma)$. Integrating, we obtain the following implicit characterization of the threshold x_t:

$$t = g(x_t) := \frac{1}{\lambda}\int_{x_0}^{x_t} \frac{l'(x)}{l(x)} \left(F_0(x) - F_1(x)\right)^{-1} dx.$$

While the right-hand side admits no closed-form solution in general, we shall provide a few examples in which we can solve for x_t explicitly. Note that this also gives us a characterization of p_t, or equivalently, L_t, since

$$L_t = \frac{c}{1-c}\frac{1}{l(x_t)}.$$

While we have focused so far on the history in which there is no investment at all, observe that the same analysis applies to the evolution of the belief for arbitrary histories, over any interval of time over which no investment takes place, provided the initial condition is accordingly modified. If s is the last date at which an investment is observed, and the belief immediately after this investment is p_s, we simply replace p_0 by p_s in the definition of γ.

Finally, observe that a simple change of variables yields

$$\int_0^t (1 - F_\theta(x_s))\, ds = \int_{x_0}^{x_t} (1 - F_\theta(x))\, g'(x)\, dx = \frac{1}{\lambda}\int_{x_0}^{x_t} \frac{l'(x)(1-F_\theta(x))}{l(x)(F_0(x) - F_1(x))} dx.$$

This gives us the following formula for the probability of no investment in state θ, namely

$$G_{\theta,t} = e^{-\int_{x_0}^{x_t} \frac{l'(x)(1-F_\theta(x))}{l(x)(F_0(x)-F_1(x))} dx}.$$

3.3. **Evolution after an Investment.** If an investment occurs at date t, the evolution of the belief is discontinuous at date t. The belief p_t jumps up, since agents become suddenly more optimistic (by MLRP). Simultaneously, the threshold x_t jumps down. More precisely, let (L_t, x_t) denote the belief and threshold immediately before the investment, and $\left(L_t^+, x_t^+\right)$ these values immediately after the investment. By Bayes' rule, the public belief jumps up to

$$L_t^+ = \frac{1 - F_1(x_t)}{1 - F_0(x_t)} L_t > L_t.$$

Therefore, the threshold x_t jumps down to the solution to

$$l\left(x_t^+\right) L_t^+ = \frac{c}{1-c}.$$

Taken together, the formulas derived in the last two subsections allow us to solve recursively for the threshold x_t after any arbitrary history h_t.

3.4. **Examples.** We provide here a pair of parametric examples. Consider first the case in which the distributions are given by, for all $x \in [0,1]$,

$$F_1(x) = \frac{e^{ax} - 1}{e^a - 1}, \text{ and } F_0(x) = \frac{e^a - e^{a(1-x)}}{e^a - 1},$$

for some $a > 0$. Note that the range of the likelihood ratio is $l(x) \in \left[e^{-a}, e^a\right]$. The parameter a is a measure of the informativeness of the private signals, as a larger a implies a larger range of possible likelihood ratio values.

For the initial condition $\gamma = 1$ (which obtains, for instance, for $p_0 = 1/2$ and $c = 1/2$), the evolution of the belief and of the threshold up to the first investment are given by

$$x_t = \frac{1}{a} \ln \left(\frac{e^{\frac{\lambda t + a}{2}} + 1}{e^{\frac{\lambda t - a}{2}} + 1} \right), \qquad p_t = \frac{e^a}{e^a + \left(\frac{e^{\frac{\lambda t + a}{2}} + 1}{e^{\frac{\lambda t - a}{2}} + 1} \right)^2}.$$

The details are in the appendix. As discussed, if an investment takes place at some date, both the public belief and the threshold jump. The evolution of the cut-off and the public belief is shown below for the case in which $\lambda = a = 1$, and an investment takes place at date $t = 5$.

The *ex ante* probability of no investment ever in the good state is given by

$$G_{1,t} = \left(\frac{e^{-\frac{a}{2}} + e^{-\frac{\lambda t}{2}}}{e^{-\frac{a}{2}} + 1} \right)^2.$$

Since the limit of this probability as $t \to \infty$ is bounded away from zero, there is a positive probability that no one ever invests, although the state is good. This probability decreases as the informativeness of the signals a increases.

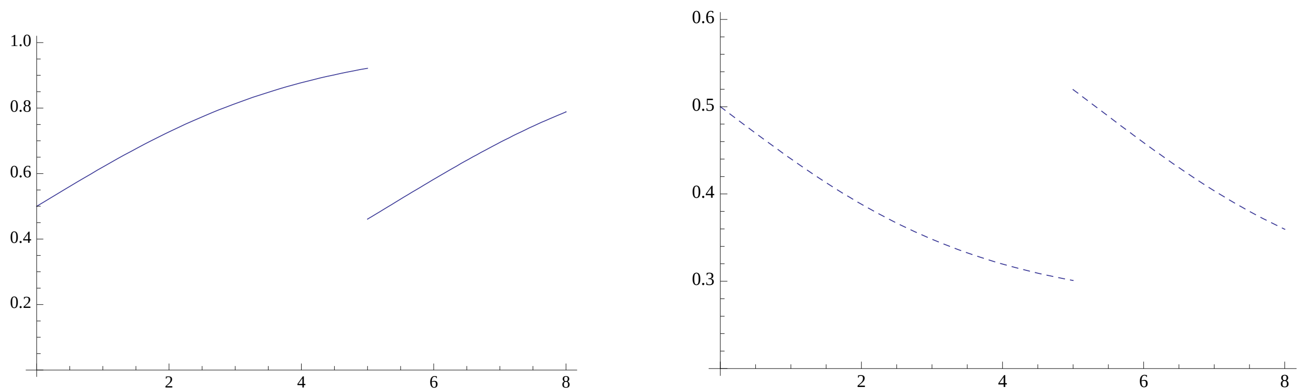

FIGURE 1. Cut-off (left) and belief (right) over time with an investment at $t = 5$.

As a second example, take the power distributions given by, for all $x \in [0, 1]$,

$$F_1 = x^2, \quad F_0 = 1 - (1 - x)^2 .$$

The probability that no one ever invests while the state is good is given by

$$G_{1,t} = \frac{1 - x_t}{x_t} e^{\frac{1}{2x_t} - 1},$$

which tends to 0 as $t \to \infty$, because, along such a history, the cut-off x_t tends to one. Here as well, details can be found in the appendix. This means that, in this example, almost surely, investments will eventually take place when the state is good. Since this is true for the first investment, it is also true for later investments, so that the total number of investments is unbounded.

As is easy to check, the likelihood ratio of the signal distribution is a bounded function in the first example, while it is not in the second. We shall prove in the next section that this distinction explains the different asymptotic properties of these two examples.

4. Asymptotic Properties

Although the learning process is biased, its asymptotic properties are the same as in traditional models of social learning. As is standard, we define private signals to be *unbounded* if

$\lim_{x\to 0} l(x) = 0$ and $\lim_{x\to 1} l(x) = \infty$. This means that extreme signals are arbitrarily informative. Signals are *bounded* if the first limit is strictly positive, and the second is finite. Note that this does not partition the set of all distributions (for instance, it could be that $f_1(0)$ is equal to zero, but $f_0(1)$ is not).

From equation (1), it follows that an agent with the highest possible private signal, signal $x = 1$, will be indifferent between investing or not (if ever) if $l(1)L_t = c/(1-c)$, where L_t is the likelihood ratio of the public belief resulting from the public history up to date t. Let $\underline{L}$ denote the highest likelihood ratio for which, given that the public history leads to this likelihood ratio, it is optimal for such an agent not to invest (with the convention that $\underline{L} = 0$ if he always does). It follows that, if the signals are unbounded, $\underline{L} = 0$, while with bounded signals,

$$\underline{L} := \frac{c}{1-c}\frac{1}{l\left(1\right)} < \frac{c}{1-c}.$$

Similarly, define $\overline{L}$ to be the lowest likelihood ratio for which an agent with the lowest possible signal, signal $x = 0$, finds it optimal to invest. With unbounded signals, $\overline{L} = +\infty$, while with bounded signals,

$$\overline{L} := \frac{c}{1-c}\frac{1}{l\left(0\right)} > \frac{c}{1-c}.$$

In terms of beliefs, this means that, defining

$$\underline{p} := \frac{\underline{L}}{1+\underline{L}}, \text{ and } \overline{p} := \frac{\overline{L}}{1+\overline{L}},$$

the probabilities $\underline{p}, \overline{p}$ are in $(0,1)$ if signals are bounded, while if signals are unbounded we have $\underline{p} = 0$ and $\overline{p} = 1$. If signals are unbounded, then, independently of the history up to t, an agent arriving at date t will follow his signal if this signal is extreme enough. While it is not hard to see how this implies complete learning if the state is good, this is only slightly subtler if the state is bad: although later agents do not observe the informative actions of the agents with sufficiently low signals, they will infer as much from their absence over the long-run. We see here the key role of two assumptions: the arrival rate is common knowledge (so that the absence of negatives can be correctly interpreted), and there is only one action that is hidden (so that agents can infer it from its absence). When signals are bounded, cascades can happen, just as in the traditional model, and for the same reason: histories leading to beliefs above $\overline{p}$, or below $\underline{p}$ have positive probability under either state. In both cases, eventually, almost all agents take the same action. This discussion is summarized in the following set of results.

Proposition 1. *Beliefs converge a.s., with limit:*

$$p_\infty := \lim_t p_t \in [0, \underline{p}] \cup [\overline{p}, 1].$$

The threshold signal x_t converges a.s. to

$$\lim_t x_t = x_\infty \in \{0, 1\}.$$

Proof. See Appendix. □

This does not yet say that beliefs converge to the correct value if signals are unbounded, but simply that they converge to either 0 or 1. Turning to investments, we have the following.

Proposition 2. *If signals are unbounded, the probability that there is no investment ever in the good state is zero. It is positive if they are bounded.*

Proof. See Appendix. □

With unbounded signals if the number of investments is finite, then, after the last investment, the public belief would decrease to the lower bound $\underline{p} = 0$. On the other hand, if there was an infinite number of investments, the belief could not converge to the lower bound $\underline{p} = 0$, because, after each investment, the public belief must exceed: $p_t \geq c$. Hence, in that case, beliefs must converge to the upper bound $\overline{p} = 1$. This is summed up in the following lemma.

Lemma 3. *With unbounded signals, the total number of investments is finite (resp., infinite) if and only if the belief converges to the lower bound $\underline{p} = 0$ (upper bound $\overline{p} = 1$).*

As expected, complete learning occurs if and only if signals are unbounded.

Proposition 4. *The belief converges to the correct value almost surely if and only if signals are unbounded.*

Proof. See Appendix. □

Together with Proposition 1, this means that, if signals are unbounded, the number of investments is a.s. infinite in the good state, and finite in the bad state.

5. One-Investment Game

In both the traditional, BHW model, and in this model, learning is complete when signals are unbounded. Therefore, we turn our attention to the case of bounded signals. To compare the likelihood of cascades under biased learning, relative to traditional learning, one should compare, in particular, the probability that there is a total of exactly k investments over the infinite horizon under both scenarios, for all integers k. We start with a simpler comparison, by focusing on the probability that at least one investment is ever made. We may, and will interpret this as a game in which there is a single investment opportunity; obviously, this interpretation does not affect the agents' behavior, since agents cannot choose the timing of their decision. In fact, assuming that agents cannot wait before taking their action is not restrictive. If other agents do not delay either, an agent cannot gain by waiting: either nothing happens, or the game ends. This result does not require agents to discount future payoffs. More formally,

Proposition 5. *If all other agents follow the cut-off strategy x_t described in Section 3, delaying investment is not optimal, for any discount rate.*[8]

Proof. See Appendix. □

Observe that this version of the game admits independent economic interpretations, in which the winner takes all. It is of no use to discover a product that has been already patented, or to prove a result that has been already published.

5.1. **The BHW Model.** The BHW model differs from ours in two respects: all actions are observed, and arrivals are not random. But given that actions are observed, whether arrivals are random or not is irrelevant to the decisions of the agents, and we might, for concreteness, keep on viewing arrivals as random. The public history up to date t is summarized by the sequence of individuals who arrived up this to date, and what their decisions where. Because we are assuming here that a single investment ends the game, this further reduces to the number n of agents who arrived and chose not to invest. Between each arrival, the threshold that characterizes the optimal strategy is now constant. Let x_k denote this cut-off when there have been k decisions not to invest so far.

[8]While it seems plausible that acting immediately is also the unique equilibrium strategy in this game, addressing this question would require defining the continuous-time game which is beyond the scope of this paper.

Therefore, the probability that no agent invests, among the first n agents, conditional on the state θ, is given by

$$B_{\theta,n} := \prod_{k=0}^{n-1} F_\theta\left(x_k\right).$$

The thresholds x_k can then be solved recursively. Clearly, $x_0 = l^{-1}(\gamma)$, and, from Bayes' rule, x_n, $n \geq 1$, solves

$$l\left(x_n\right)\frac{B_{1,n}}{B_{0,n}} = \gamma,$$

since the right-hand side is the likelihood ratio of an agent with private signal x_n, given the public history. Thus, the thresholds solve the first-order difference equation

$$l\left(x_{n+1}\right) = \frac{F_0\left(x_n\right)}{F_1\left(x_n\right)} l\left(x_n\right).$$

Our objective is then to compare the limit of this probability,

$$B_{\theta,\infty} := \prod_{k=0}^{\infty} F_\theta\left(x_k\right),$$

with the analogous probability under biased learning derived in subsection (3.2), namely,

$$G_{\theta,\infty} := e^{-\int_{x_0}^{1} \frac{1-F_\theta(x)}{F_0(x)-F_1(x)} \frac{l'(x)}{l(x)} dx}.$$

While those two expressions bear little in common, we shall see that the comparison hinges upon a simple statistical property, the increasing hazard ratio property (IHRP), defined as follows. The *hazard ratio* at the signal x is the ratio of the hazard rates conditional on the good and the bad state, that is,

$$H\left(x\right) := \frac{1-F_0\left(x\right)}{1-F_1\left(x\right)} l\left(x\right).$$

The (strict) *increasing hazard ratio property* (IHRP) holds if this mapping is strictly increasing (see Herrera and Hörner (2011) for properties of IHRP in the standard herding model).

5.2. **The Result.** We are finally ready to compare the probabilities of no investment ever in both models, conditional on a given state. One might suspect that this comparison depends on the state, but it turns out that this is not the case, because the ratio of these conditional probabilities is the same in both models, as the next lemma establishes.

Lemma 6. *It holds that*

$$\frac{G_{1,\infty}}{G_{0,\infty}} = \frac{B_{1,\infty}}{B_{0,\infty}}.$$

Proof. By definition,

$$G_{1,\infty} = e^{-\int_{x_0}^{1}\left(1+\frac{1-F_0(x)}{F_0(x)-F_1(x)}\right)\frac{l'(x)}{l(x)}dx} = \frac{\gamma}{l(1)} G_{0,\infty},$$

and similarly, by definition,

$$B_{1,\infty} = \frac{\prod_{k=0}^{\infty} F_1(x_k)}{\prod_{k=0}^{\infty} F_0(x_k)} B_{0,\infty} = \frac{\gamma}{l(1)} B_{0,\infty}.$$

The result follows. □

The main result of this section establishes that the probability that no investment ever takes place is higher in the BHW model than in the biased learning model, if IHRP holds. That is, under IHRP, biased learning leads to higher investment, independently of the state. If the hazard ratio is constant, both models lead to the same amount of investment, and biased learning leads to lower investment if the hazard ratio is decreasing.

Proposition 7. *Assume IHRP. Conditional on either state, the probability of investment is always larger in the hidden action model:*

$$G_{\theta,\infty} < B_{\theta,\infty}.$$

This inequality is reversed if the hazard ratio is decreasing.

Proof. See Appendix. □

The intuition behind the equation of motion of the threshold x is that since the news arrival is more likely in state 1 than in state 0 (and vice versa for the lack thereof) then the lack of news makes the public belief drop, and hence the threshold x increase. The key aspect when comparing the overall chance of investment in the G and the B model is understanding by how much the threshold x changes with continued lack of news: as x increases the news arrival rates naturally drop in both states, the question is by how they drop much in each state. With IHRP the relative drop is larger for the arrival rate in state 1 than in state 0, so a news arrival (resp., lack thereof) would be less good news (resp., less bad news) than with the previous threshold x, because it is relatively less likely than before to come from the good state. Hence, under IHRP, in the absence of news the threshold would move relatively less, reducing less the chance of investment in the next instant. The converse happens with DHRP: investment is choked off more as time passes without news and investment becomes less likely overall in the hidden action model. In formulas, the threshold in the absence of investment evolves according to the law integral law (2), which is equivalent to the differential law

$$\frac{l'(x_t)}{l(x_t)}dx_t = (1 - \lambda dt\,(1 - F_0(x_t))) - (1 - \lambda dt\,(1 - F_1(x_t))) > 0$$

The left-hand side represents the private belief: the change in the threshold scaled by the relative change in the likelihood ratio; the right-hand side represents the observational learning: the difference between the investment news arrival rates (or lack thereof) in the two states. Since the investment news arrival rate in each state at time t is $\lambda(1 - F_\theta(x_t))$, the hazard rate $\frac{f_\theta(x_t)}{1-F_\theta(x_t)}$ measures the relative change of this arrival rate as the threshold changes. IHRP implies that the relative change is larger in the good state than in the bad state, and vice versa for DHRP.

5.3. **Welfare Comparison.** Since the probabilities of cascades are not the same in both models, it is natural to wonder which one aggregates information better. In the good state, it is optimal for someone to invest, while in the bad state, it is optimal for all to abstain. Therefore, we define welfare by the expectation of the utility of this eventual outcome (either someone eventually invests, or no one does). Since players do not internalize the informational externalities, there is no reason to expect *a priori* that having more information is necessarily better. Indeed, whether this is the case depends here again on the hazard ratio, as the next result establishes.

Proposition 8. *Under IHRP, the welfare in the hidden action model is lower than in the benchmark model. It is higher if the hazard ratio is decreasing.*

Proof. The welfare in the hidden action model is, by definition,

$$\begin{aligned} W(G) &:= \mathbb{E}_\theta\left[(1-G_\theta)\,u(I)+(G_\theta)\,u(N)\right] \\ &= p_0(1-G_1)(1-c)+(1-p_0)(1-G_0)(-c) \\ &= p_0(1-c)\left((1-G_1)-\gamma\left(1-\frac{l(1)}{\gamma}G_1\right)\right) \\ &= p_0(1-c)(1-\gamma+(l(1)-1)\,G_1), \end{aligned}$$

and likewise for $W(B)$. Since $l(1)>1$, as 1 is the highest signal, this expression is increasing in G_1, and therefore

$$W(G)<W(B),$$

whenever the IHRP is satisfied. The inequality is obviously reversed if the hazard ratio is decreasing. □

Therefore, even though the model with biased learning performs better in the good state, as it always leads to a higher probability of investment, it achieves a lower welfare than the BHW model, at least under IHRP.

5.4. **Many Investments.** So far, the comparison between the model with biased learning and the benchmark BHW model has been performed in the game with one investment. Comparing the two models more generally is difficult (in the IHRP case). For example, computing the probability of exactly two investments involves a summation (or, in the biased learning model, an integration) over the times at which one agent first invested. Even in the simple example of an exponential distribution, that satisfies IHRP (see Section 3.4.), closed-form solutions appear elusive. Nevertheless, numerical computations can be performed in this case, which we briefly present here.

Consider first the model with biased learning. Let $G_\theta^0(x_0,x_t)$ denote the probability of no investment in state θ during the time interval required for the threshold to go from an initial value x_0 to the value x_t, and let $G_\theta^n(x_0,1)$ denote the probability of exactly n eventual investments in state θ (i.e., over the time interval required for the threshold to go from the initial value x_0 to 1). This probability must satisfy the recursion

$$G_\theta^{n+1}(x_0,1)=\int_{x_0}^{1} G_\theta^0(x_0,x_t)\,\lambda\,(1-F_\theta(x_t))\,G_\theta^n\left(x_t^+,1\right)dx_t,$$

as the probability that there are $n+1$ investments overall can be decomposed as the sum of the probabilities, as t varies, that the first investment occurs exactly at time t, and that exactly n investments overall are made afterwards (given the resulting new initial value).

In the exponential example, considering the state $\theta = 1$ for instance, we obtain

$$G_1^0(x_0, x_t) = e^{-\lambda \int_0^t (1-F_1)dt} = e^{-\int_{x_0}^{x_t} \frac{l'}{l} \frac{1-F_1}{F_0-F_1} dx} = \left(\frac{e^{ax_0}-1}{e^{ax_t}-1} \right)^2,$$

and the recursion becomes

$$G_1^{n+1}(x_0, 1) = \int_{x_0}^1 \left(\frac{e^{ax_0}-1}{e^{ax}-1} \right)^2 \left(2a \frac{e^{ax}}{e^{ax}-1} \right) G_1^n(x/2, 1)\, dt.$$

This integration can only be performed explicitly in the case $n = 0$.

A similar decomposition can be used in the BHW model: the probability of n overall investments is the sum, over the index k of the first agent to invest, of the probability that the first agent to invest is the k-th agent, times the probability that there are exactly $n-1$ agents investing in the game (where, for the latter probability, we use as initial belief the public belief resulting from a first investment by the k-th agent).

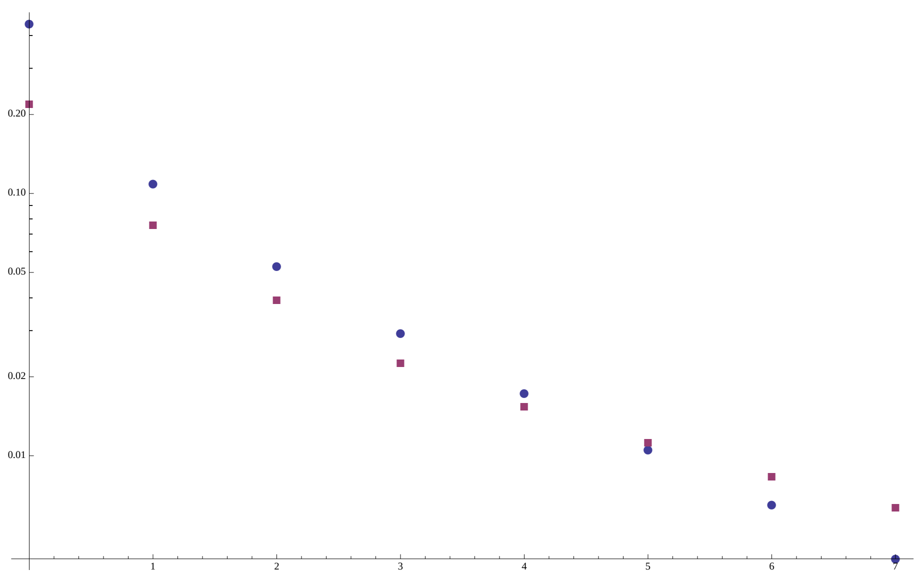

FIGURE 2. Log plot of the probabilities of n eventual investments for $a = 1/4$ (dots: BHW benchmark; squares: biased learning model).

These recursions allow numerical computation of these probabilities. Figure 2 depicts the (log plot of) the probabilities of n eventual investments in both models, in the exponential example with parameter $a = 1/4$ (the same pattern arises for all values of a). These probabilities cross exactly once. That is, there exists an integer n^* such that, for $n < n^*$, the probability of n eventual investments is higher in the BHW model, while the opposite is true for $n \geq n^*$. It can be numerically verified that the probability distribution of n or less eventual investments of the BHW model first-order stochastically dominates the one of the biased learning model. This suggests that, at least in this example, restricting attention to the one-investment game does not seem too misleading.

6. Conclusions

At least as far as convergence is concerned, we have shown that relaxing the assumptions that all types of decisions are observable does not change significantly the asymptotic learning properties of the model. Beliefs always converge and will converge to the true value for sure if and only if private beliefs are unbounded. So, even if one action is hidden and can only be inferred, the market aggregates the information correctly anyway when beliefs are unbounded.

With bounded signals, whichever model delivers a higher probability of investment depends on a property of the signal distribution. If the hazard ratio is increasing in the signal, then investment is more likely in the model with biased learning, and welfare is lower. These conclusions are reversed if the hazard ratio is decreasing.

7. Appendix

7.1. **Proofs.**

Proof of Proposition 1. Since the public belief p_t is a bounded martingale, the martingale convergence theorem implies almost sure convergence to a value $p_\infty \in [0, 1]$.

Beliefs cannot converge to any interior value

$$p_\infty \in \left(\underline{p}, \overline{p}\right),$$

namely, beliefs cannot settle within any $\varepsilon > 0$ of any interior value $p_\infty \in \left(\underline{p}, \overline{p}\right)$, as beliefs would jump discretely after any investment, or would decrease to $\underline{p}$ otherwise. The threshold signal x_t must converge because it is a continuous function of the belief. It must converge to its boundaries

$x_\infty \in \{0,1\}$, as the limit threshold cannot be any interior value $x_\infty \in (0,1)$: otherwise, a later investment would almost surely occur, and the belief and the threshold would jump. The result follows. □

Proof of Proposition 2. In a history without any investment, the belief would converge to its lower bound $\underline{p}$ and the threshold to one: $x_\infty = 1$. We now show that with unbounded beliefs this is a zero probability history, and conversely:

$$l(1) = +\infty \quad \Longleftrightarrow \quad \lim_t G_{1,t} = 0.$$

Observe that the probability of no investment ever under state 1 converges, as it is an increasing and bounded function. Since

$$F_0(x_t) - F_1(x_t) \leq 1 - F_1(x_t),$$

it follows that from (2) that

$$G_{1,t} = e^{-\int_0^t \lambda(1-F_1(x_s))ds} \leq \frac{c}{1-c}\frac{1-p_0}{p_0}\frac{1}{l(x_t)},$$

so

$$\lim_t G_{1,t} = 0 \quad \text{if } l(1) = +\infty.$$

Conversely, if beliefs are bounded we have that $l(1) < \infty$. Then

$$h(x) := \frac{l(x)}{l'(x)}\lambda(F_0(x) - F_1(x))$$

converges to 0, and, so for all x sufficiently close to 1,

$$x_t' = h(x) > -h'(1)(1-x_t) - M(1-x_t)^2,$$

with $h'(1) < 0$ as $h(x) > 0$, for some $M > 0$. This implies that, for all t sufficiently large,

$$1 - x_t \leq \frac{-h'(1)}{M + C_1 e^{-h'(1)t}},$$

for some constant C_1. Since

$$1 - F_1(x_s) \leq f_1(1)(1-x_s) + C_2(1-x_s)^2,$$

and

$$\int^t \frac{-h'(1)\,ds}{M + C_1 e^{-h'(1)s}} = \ln\left(C_1 + Me^{-h'(1)t}\right)/M < \ln(C_1 + M)/M,$$

it follows that $G_{1,t}$ is bounded below, so that

$$l(1) < +\infty \Rightarrow \lim_t G_{1,t} > 0.$$

$\square$

Proof of Proposition 4. Assume signals are unbounded. By Proposition 1, the public belief must either converge to 0 or 1. Assume that the state is good. The belief cannot converge to 0, because that would imply that the number of investments is finite, contradicting Proposition 2 (recall that, after an investment, the public belief exceeds c).

If the state is bad, then the public belief converges to zero. By the martingale property, if the belief converges to 1 in the good state it must converge to zero in the bad state, namely

$$p_t = \mathbb{E}_t[p_\infty] = \mathbb{P}_\infty[\theta = 1]\, p_t + \mathbb{P}_\infty[\theta = 0](1 - p_t), \text{ and so } \mathbb{P}_\infty[\theta = 1] = 1 \Rightarrow \mathbb{P}_\infty[\theta = 0] = 0.$$

Assume that signals are bounded. In the good state, beliefs can converge to the wrong value $\underline{p}$ because there is a positive probability of zero investment history. In the bad state, beliefs converge to $\overline{p}$ with positive probability. Indeed, if they did not, then it would be the case that $\overline{p} = 1$ (by definition of $\overline{p}$), and this is impossible with bounded beliefs. $\square$

Proof of Proposition 5. Let $\tau(x, t)$ denote the stopping time of a player arriving at instant t with signal x, and let $F(s; t, x)$, $s \geq t$, denote the corresponding c.d.f. Fix an equilibrium and suppose for the sake of contradiction that $0 < F(s; t, x) < 1$ for some x, t and $s > t$, for some finite s. Let q_τ denote the private belief of this agent at time $\tau \geq t$, given his signal x and the equilibrium strategies (conditional on the event E_τ that no one invested up to τ). Observe that q_τ is non-increasing, and constant over some interval of time $[t', t'')$ if and only if $F(t''; s, x) = F(t'; s, x)$ for all $s \leq t'$ and $x \in [0, 1]$. Assume that it is strictly profitable to invest at time t with signal x. Then, because the payoff from investing is strictly increasing in q_t, $F(s; t, x) < 1$ is only possible if $q_s = q_t$, and player i assigns probability one to no one investing before (or at the same) time than he does. In particular, any other player arriving in the interval of time (t, s) must invest with probability zero in that interval of time. Consider the event that some player arrives in this interval of time with a signal $x \geq x_t$, i.e. a player whose payoff from investing immediately is strictly positive. (x_t here depends obviously upon the equilibrium strategies.) This event has strictly positive probability, and thus, given the equilibrium strategies, there exists a player arriving at some time $t' \in (t, s)$ whose probability of investing first (after s) is strictly less than

1. This player would profitably gain from investing immediately at time t'. Assume now that it is strictly unprofitable to invest at time t with signal x. Plainly it remains unprofitable to invest at any later time, and so it cannot be that $0 < F(s;t,x)$ for some finite s. Therefore, a player is unwilling to delay unless $x = x_t$, i.e. he is indifferent between investing immediately or never. This event has zero probability, and so does not affect the analysis, in particular the determination of x_t. □

Proof of Proposition 7. IHRP means that the inverse of H, denoted m, i.e.

$$m(x) := \frac{f_0(x)/(1-F_0(x))}{f_1(x)/(1-F_1(x))},$$

is strictly decreasing, so that, for $m = m(x_k)$,

$$\frac{1-F_1(x)}{1-F_0(x)} \leq ml(x),$$

for all $x \in [x_k, x_{k+1}]$, with equality for $x = x_k$. That is, we have

$$\frac{F_0(x)-F_1(x)}{1-F_0(x)} \leq ml(x) - 1,$$

which implies that the right-hand side is positive, as the left-hand side is.

Given Lemma 5, it is enough to show $B_{\theta,\infty} > G_{\theta,\infty}$ for $\theta = 0$. We have

$$\prod_{k=0}^{\infty} F_0\left(x_k\right) > e^{-\int_{x_0}^{1} \frac{1-F_0(x)}{F_0(x)-F_1(x)} \frac{l'(x)}{l(x)} dx} = \prod_{k=0}^{\infty} e^{-\int_{x_k}^{x_{k+1}} \frac{1-F_0(x)}{F_0(x)-F_1(x)} \frac{l'(x)}{l(x)} dx},$$

so it suffices to show that, for all k,

$$\ln F_0(x_k) + \int_{x_k}^{x_{k+1}} \frac{l'(x)}{l(x)} \frac{1-F_0(x)}{F_0(x)-F_1(x)} dx \geq 0.$$

We have

$$\int_{x_k}^{x_{k+1}} \frac{l'(x)}{l(x)} \frac{1-F_0(x)}{F_0(x)-F_1(x)} dx \geq \int_{x_k}^{x_{k+1}} \frac{l'(x)}{l(x)(ml(x)-1)} dx$$
$$= \ln \frac{ml(x_{k+1})-1}{l(x_{k+1})} - \ln \frac{ml(x_k)-1}{l(x_k)} = \ln \frac{ml(x_k) - \frac{F_1(x_k)}{F_0(x_k)}}{ml(x_k)-1},$$

using that $l(x_{k+1}) = \frac{F_0(x_k)}{F_1(x_k)} l(x_k)$. Therefore, the inequality

$$\ln F_0(x_k) + \int_{x_k}^{x_{k+1}} \frac{l'(x)}{l(x)} \frac{1-F_0(x)}{F_0(x)-F_1(x)} dx \geq 0,$$

would be implied by

$$\ln F_0(x_k) + \ln \frac{ml(x_k) - \frac{F_1(x_k)}{F_0(x_k)}}{ml(x_k) - 1} \geq 0,$$

Rearranging, this is equivalent to

$$\frac{1 - F_1(x_k)}{1 - F_0(x_k)} \geq ml(x_k),$$

but this is the case, since in fact both sides are equal by definition of m. By immediate inspection, this chain of inequalities is tight if the hazard ratio is constant, and reversed if it is decreasing. $\square$

7.2. **Exponential Example.** Take the following c.d.f.

$$F_1 = \frac{e^{ax} - 1}{e^a - 1}, \quad F_0 = \frac{e^a - e^{a(1-x)}}{e^a - 1}.$$

The threshold can be expressed as

$$\begin{aligned}
\lambda t &= \int_{x_0}^{x} 2a \int \frac{e^a - 1}{1 + e^a - e^{as} - e^{a(1-s)}} ds = \left[2 \ln \left(\frac{e^{as} - 1}{e^a - e^{as}} \right) \right]_{x_0}^{x} \quad \text{or} \\
\lambda t &= 2 \ln \left(\frac{e^{ax} - 1}{e^a - e^{ax}} \right) - A_0 \quad \Longrightarrow \quad x_t = \frac{1}{a} \ln \left(\frac{e^{\frac{\lambda t + A_0}{2} + a} + 1}{e^{\frac{\lambda t + A_0}{2}} + 1} \right).
\end{aligned}$$

With a uniform prior, we have

$$\left(x_0 = \frac{1}{2} \quad \Longrightarrow \quad A_0 = -a \right) \quad \Longleftrightarrow \quad x_t = \frac{1}{a} \ln \left(\frac{e^{\frac{\lambda t + a}{2}} + 1}{e^{\frac{\lambda t - a}{2}} + 1} \right).$$

The likelihood ratio and the public belief are, respectively,

$$\begin{aligned}
l(x_t) &= e^{a(2x_t - 1)} = e^{-a} \left(\frac{e^{\frac{\lambda t + a}{2}} + 1}{e^{\frac{\lambda t - a}{2}} + 1} \right)^2 \to e^a, \\
L_t &= e^a \left(\frac{e^{\frac{\lambda t - a}{2}} + 1}{e^{\frac{\lambda t + a}{2}} + 1} \right)^2 \to e^{-a} \quad \Longrightarrow \quad p_t = \frac{L_t}{1 + L_t} \to \frac{1}{1 + e^a}.
\end{aligned}$$

Because

$$e^{ax_t} = \frac{e^{\frac{\lambda t + a}{2}} + 1}{e^{\frac{\lambda t - a}{2}} + 1}, \quad \text{and} \quad \int_0^t \frac{1}{e^{\frac{\lambda t - a}{2}} + 1} dt = t - \frac{2}{\lambda} \ln \left(\frac{e^{\frac{\lambda t - a}{2}} + 1}{e^{-\frac{a}{2}} + 1} \right),$$

the probability of no investment in the good state is

$$\begin{aligned} G_{1,t} &= e^{-\int_0^t \lambda(1-F_1)dt} = e^{-\lambda\int_0^t\left(1-\frac{e^{ax}-1}{e^a-1}\right)dt} = e^{-\lambda\int_0^t \frac{1}{e^{\frac{\lambda t-a}{2}}+1}dt} \\ &= e^{-\lambda t}\left(\frac{e^{\frac{\lambda t-a}{2}}+1}{e^{-\frac{a}{2}}+1}\right)^2 = \left(\frac{e^{-\frac{a}{2}}+e^{-\frac{\lambda t}{2}}}{e^{-\frac{a}{2}}+1}\right)^2 \to \left(\frac{e^{-\frac{a}{2}}}{e^{-\frac{a}{2}}+1}\right)^2. \end{aligned}$$

Given the symmetry

$$F_0(x;a) = F_1(x;-a),$$

the probability of no investment in the bad state is

$$G_{0,t} = \left(\frac{e^{\frac{a}{2}}+e^{-\frac{\lambda}{2}t}}{e^{\frac{a}{2}}+1}\right)^2 \to \left(\frac{e^{\frac{a}{2}}}{e^{\frac{a}{2}}+1}\right)^2.$$

After an investment, the threshold $x_{\tau+}$ decreases discontinuously (e.g. take $a=1$)

$$\begin{aligned} l(x_{\tau+}) &= \frac{1-F_0(x_\tau)}{1-F_1(x_\tau)} l(x_\tau) = \left(e^{-x_\tau}\right) l(x_\tau) \\ e^{2x_{\tau+}-1} &= e^{x_\tau-1} \quad \Longrightarrow \quad x_{\tau+} = \frac{x_\tau}{2}, \end{aligned}$$

and the belief increases discontinuously to

$$p_{\tau+} = \frac{\left(e^{x_\tau}\right)p_\tau}{(1-p_\tau)+\left(e^{x_\tau}\right)p_\tau} > p_\tau.$$

7.3. **Polynomial Example.** Take the following c.d.f.

$$F_1 = x^2, \quad F_0 = 1-(1-x)^2.$$

In this example beliefs are unbounded, namely: $l(x) \in [0,\infty]$. For $\lambda = 1$, $c = 1/2$ ($\Longrightarrow$ $x_0 = 1/2$), the threshold is determined implicitly by

$$t = \int_{0.5}^{x_t}\left(\frac{1}{2x^2(1-x)^2}\right)dx = \frac{1}{2}\left(\frac{1}{1-x_t}-\frac{1}{x_t}\right)+\ln\left(\frac{x_t}{1-x_t}\right).$$

The change of variables

$$t = \frac{1}{2}\left(\frac{1}{1-x}-\frac{1}{x}\right)+\ln\left(\frac{x}{1-x}\right) := g(x)$$

gives

$$\int_0^t (1-F_1(x_s))\,ds = \int_{0.5}^{x_t}(1-F_1(x))\,g'(x)\,dx = \ln\frac{x_t}{1-x_t}-\frac{1}{2x_t}+1,$$

so the probability that nobody invests in the good state is

$$G_{1,t} = e^{-\lambda \int_0^t (1-F_1(x_s))ds} = \frac{1-x_t}{x_t} e^{\frac{1}{2x_t}-1} \to 0,$$

and the probability that nobody invests in the bad state is

$$G_{0,t} = e^{\frac{1}{2x_t}-1} \to e^{-\frac{1}{2}} \simeq 61.$$

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