Problem 1

Find all the Nash equilibria of the following normal form game:

\[
\begin{array}{ccc}
  & b_1 & b_2 & b_3 \\
 a_1 & 5,6 & 7,2 & 0,8 \\
 a_2 & 3,4 & 5,6 & 1,2 \\
 a_3 & 2,7 & 0,2 & 0,0 \\
\end{array}
\]

Answer

Let’s start by eliminating the strictly dominated strategies. If a strategy is a best reply to some action of the opponent, do not eliminate it. Keep \( a_1 \), since it is the row player’s best reply to \( b_1 \) (and \( b_2 \)). Also, \( a_2 \) is her best reply to \( b_3 \). But, \( a_3 \) is strictly dominated by \( a_2 \). (Why is \( a_3 \) not strictly dominated by \( a_1 \)? Hint: what happens if the column player plays \( b_3 \)? Is \( a_3 \) weakly dominated by \( a_1 \)?)

Note that we do not lose any Nash equilibria of the original game, by eliminating \( a_3 \).

Now, does the column player have a strictly dominated strategy? Keep \( b_3 \) because it is his best reply to \( a_1 \). Also, \( b_2 \) is his best reply to \( a_2 \). Is \( b_1 \) strictly dominated by the mixed strategy \( \sigma_2 = (p b_1, (1-p) b_2) \) for some \( p \in [0,1] \)? It can be shown there is no \( p \) such that \( a_1 \) is dominated. (Alternatively, one can observe that \( b_1 \) is a best reply to \( \sigma_1 = (1/2 a_1, 1/2 a_2) \)). So, we cannot eliminate more strategies.

Second, find all the pure strategy Nash equilibria, in the reduced game. (You can use the method of underlining payoffs corresponding to best replies—you have an equilibrium in pure strategies whenever you underline both numbers in the same cell.) In this game, there are no equilibria in pure strategies:

\[
\begin{array}{ccc}
  & b_1 & b_2 & b_3 \\
 a_1 & 5,6 & 7,2 & 0,8 \\
 a_2 & 3,4 & 5,6 & 1,2 \\
\end{array}
\]

Third, look for all mixed strategy equilibria. If the row player plays a pure strategy, the column player has a unique best reply. This rules out mixing by the column player. Thus, the row player must mix between \( a_1 \) and \( a_2 \) in equilibrium. This implies she is indifferent, i.e. \( u_1(a_1, \sigma_2) = u_1(a_2, \sigma_2) \), if \( \sigma_2 \) is the column player’s equilibrium strategy.

We will use the row player’s indifference condition to learn about the column player’s equilibrium strategy, \( \sigma_2 = (q_1 b_1, q_2 b_2, (1-q_1-q_2) b_3) \):

\[
u_1(a_1, \sigma_2) = 5q_1 + 7q_2 = 3q_1 + 5q_2 + 1 - q_1 - q_2 = u_1(a_2, \sigma_2) \quad \Rightarrow \quad q_1 + q_2 = 1/3.
\]
So far, we showed that the column player puts positive probability on at least one of $b_1$ and $b_2$, and he puts probability $2/3$ on $b_3$. Thus, at least one of the following indifference conditions for the column player must hold: $u_2(\sigma_1, b_1) = u_2(\sigma_1, b_3)$ and/or $u_2(\sigma_1, b_2) = u_2(\sigma_1, b_3)$, if $\sigma_1 = (pa_1, (1-p)a_2)$ is the row player’s equilibrium strategy.

Consider what the column player’s first indifference condition implies for the row player’s equilibrium strategy:

$$u_2(\sigma_1, b_1) = 6p + 4(1-p) = 8p + 2(1-p) = u_2(\sigma_1, b_3) \implies p = 1/2.$$ 

Now, when $\sigma_1 = (1/2a_1, 1/2a_2)$, the column player’s best replies are $b_1$ and $b_3$. They give him an expected payoff of $5$, while $b_2$ yields $4$. So, in this equilibrium, the column player will not put positive probability on $b_2$, and $\sigma_2 = (1/3b_1, 0b_2, 2/3b_3)$.

We found one equilibrium so far. Is there another Nash equilibrium coming from the second possible indifference condition of the column player?

$$u_2(\sigma_1, b_2) = 2p + 6(1-p) = 8p + 2(1-p) = u_2(\sigma_1, b_3) \implies p = 2/5.$$ 

This shows that the column player is indifferent between $b_2$ and $b_3$, if $\sigma_1 = (2/5a_1, 3/5a_2)$. But, then his expected payoff, $22/5$, is less than his expected payoff from playing $b_1$:

$$u_2(\sigma_1, b_1) = 6p + 4(1-p) = 24/5.$$ 

Making the column player indifferent between $b_2$ and $b_3$ is necessary, but not enough to establish a second equilibrium. We also need to consider if the column player has another pure strategy that gives a higher payoff, against the row player’s strategy, $(2/5a_1, 3/5a_2)$. We showed the column player has a better pure strategy, $b_1$, so there is no second equilibrium.
Problem 2

Find all the Nash equilibria of the following normal form game:

<table>
<thead>
<tr>
<th></th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>0,7</td>
<td>2,5</td>
<td>7,0</td>
<td>0,1</td>
</tr>
<tr>
<td>$a_2$</td>
<td>5,2</td>
<td>3,3</td>
<td>5,2</td>
<td>0,1</td>
</tr>
<tr>
<td>$a_3$</td>
<td>7,0</td>
<td>2,5</td>
<td>0,7</td>
<td>0,1</td>
</tr>
<tr>
<td>$a_4$</td>
<td>0,0</td>
<td>0,−2</td>
<td>0,0</td>
<td>10,−1</td>
</tr>
</tbody>
</table>

Answer

Let’s start by eliminating the *strictly* dominated strategies. First, $b_4$ is strictly dominated by the mixed strategy $\sigma_2 = (1/2b_1, 0b_2, 1/2b_3)$. After eliminating $b_4$, $a_4$ is strictly dominated, in the reduced game, by $a_2$. No more strategies can be eliminated, because each is a best reply to some opponent’s strategy.

Second, find all the *pure* strategy Nash equilibria, in the reduced game. This game has exactly one Nash equilibrium in pure strategies, $(s_1, s_2) = (a_2, b_2)$:

<table>
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<th>$b_1$</th>
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<th>$b_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>0,7</td>
<td>2,5</td>
<td>7,0</td>
</tr>
<tr>
<td>$a_2$</td>
<td>5,2</td>
<td>3,3</td>
<td>5,2</td>
</tr>
<tr>
<td>$a_3$</td>
<td>7,0</td>
<td>2,5</td>
<td>0,7</td>
</tr>
</tbody>
</table>

Third, we will show there are no other equilibria in mixed strategies. Let Case 1 be the possibility that the column player is mixing between $b_1$ and $b_2$ only. Such a Nash equilibrium must also be a Nash equilibrium of the game, $\Gamma'$, in which the column player is missing strategy $b_3$:

<table>
<thead>
<tr>
<th></th>
<th>$b_1$</th>
<th>$b_2$</th>
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<tbody>
<tr>
<td>$a_1$</td>
<td>0,7</td>
<td>2,5</td>
</tr>
<tr>
<td>$a_2$</td>
<td>5,2</td>
<td>3,3</td>
</tr>
<tr>
<td>$a_3$</td>
<td>7,0</td>
<td>2,5</td>
</tr>
</tbody>
</table>

In $\Gamma'$, $a_1$ is strictly dominated by $a_2$. After $a_1$ is removed, $b_1$ is dominated by $b_2$. Then, $a_3$ is dominated by $a_2$. Thus, the iterated elimination of strictly dominated strategies shows that $(s_1, s_2) = (a_2, b_2)$ is the only Nash equilibrium of $\Gamma'$. This implies there is no other equilibrium of the *original game*, such that the column player mixes only between $b_1$ and $b_2$.

Let Case 2 be the possibility that the column player is mixing between $b_2$ and $b_3$ only. Similar reasoning will demonstrate there is no other such equilibrium. (Such an NE is also an NE of the game $\Gamma''$, in which the column player is missing strategy $b_1$. But, the only NE of $\Gamma''$ is $(a_2, b_2)$, by the iterated elimination of strictly dominated strategies.)

<table>
<thead>
<tr>
<th></th>
<th>$b_3$</th>
<th>$b_2$</th>
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</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>2,5</td>
<td>7,0</td>
</tr>
<tr>
<td>$a_2$</td>
<td>3,3</td>
<td>5,2</td>
</tr>
<tr>
<td>$a_3$</td>
<td>2,5</td>
<td>0,7</td>
</tr>
</tbody>
</table>
Let Case 3 be the possibility that the column player is putting positive probability on both $b_1$ and $b_3$, (and putting the remaining probability, which may or may not be 0, on $b_2$). We will show there is no such equilibrium. The indifference condition of the column player would pin down the row player’s strategy, $\sigma_1 = (p_1 a_1, (1 - p_1 - p_3) a_2, p_3 a_3)$:

$$u_2(\sigma_1, b_1) = 7p_1 + 2(1 - p_1 - p_3) = 2(1 - p_1 - p_3) + 7p_3 = u_2(\sigma_1, b_3) \implies p_1 = p_3.$$ 

But, when $p_1 = p_3 = p$, the column player’s unique best reply is $b_2$, since:

$$u_2(\sigma_1, b_2) = 5p + 3(1 - 2p) + 5p > 7p + 2(1 - 2p) = u_2(\sigma_1, b_1) = u_2(\sigma_1, b_3).$$

So, the column player cannot put positive probability $p > 0$ on $b_1$ or $b_3$.

We have shown the only equilibrium is $(a_2, b_2)$, because Cases 1-3 are exhaustive.
**Problem 3** (based on Fudenberg & Tirole Exercise 1.4, p. 38)

Two identical firms produce a homogeneous good, whose demand is \( q = 100 - p \). The firms simultaneously choose prices, in \([0, 100]\). Both firms have the same marginal cost 10 to produce a unit of the good. Neither firm can produce more than \( K \) units, (the capacity constraint).

The firms will share the market equally, if both choose the same price \( p \):

\[
q_1(p, p) = q_2(p, p) = \min(50 - p/2, K).
\]

If firm \( i \) chooses a strictly lower price \( p_i < p_j \), then it sells:

\[
q_i(p_i, p_j) = \min(100 - p_i, K).
\]

If \( q_i < K \), there is no demand left for firm \( j \), so \( q_j = 0 \). If firm \( i \) hits its capacity constraint, \( q_i = K \), assume firm \( j \) faces the residual demand, \( \max(100 - K - p_j, 0) \). It sells no more than its capacity constraint:

\[
q_j(p_i, p_j) = \min(\max(100 - K - p_j, 0), K).
\]

**Part A**

Derive the firms’ payoff functions.

**Answer**

Firm \( i \)’s payoff is \( \pi_i(p_i, p_j) = (p_i - 10)q_i(p_i, p_j) \).

**Part B**

Show the game has no pure strategy Nash equilibria, if \( K = 40 \).

**Answer**

We will conclude the result by proving two claims below.

**Claim 1:** No pair of different prices \( p_1 \neq p_2 \) is a Nash equilibrium, if \( K = 40 \).

We need to show that any two prices \( p_i < p_j \) are not best replies to each other. First, the high-price firm \( j \) gets 0 residual demand, if \( p_j > 60 = 100 - K \). So, it can profit by setting a lower price, such as 59, to attract positive demand. Second, if \( p_j \leq 60 \), then the low-price firm \( i \) can profit by setting a higher price, such as \( (p_i + p_j)/2 \). It will still sell the same quantity, \( K = 40 \), at the higher price. This shows there is no equilibrium, with \( p_i < p_j \).
Claim 2: There is no price $p$ such that $p_1 = p_2 = p$ is a Nash equilibrium, if $K = 40$.

We need to show that no price $p$ is a best reply to itself. If $p < 21$, then firm $i$’s profit is:

$$\pi_i(p,p) < (21 - 10)40 = 440.$$  

But, firm $i$ can set a higher price of 30, and get a higher profit, $(30 - 10)(100 - 40 - 30) = 600$. So there is no equilibrium, with $p < 21$.

If $p \geq 21$, then each firm sells $50 - p/2 < 40$. Firm $i$’s profit is then:

$$\pi_i(p,p) = (p - 10)(50 - p/2).$$

It is maximized at $p = 55$. If $p > 55$, firm $i$ can set a price of 55, and increase its profit to $(55 - 10)40$. So there is no equilibrium, with $p > 55$.

Suppose $21 \leq p \leq 55$. Then, $i$ will become the low-price firm and sell $K$ units, if it lowers its price to, $p - 1/10$. The change in $i$’s profit will be:

$$(p - 10 - 1/10)40 - \pi_i(p,p) = (p - 10)(p/2 - 10) - 4 > 0.$$  

Since $p \geq 21$, the change is positive, and $i$’s deviation is profitable. Thus, no $p$ produces an equilibrium.

Part C

Show the game has a pure strategy Nash equilibrium, if $K = 30$.

Answer

Both firms can sell at capacity, $K = 30$, by setting $p = 40$:

$$q_1(40, 40) = q_2(40, 40) = 50 - 40/2 = 30.$$  

Let’s show $p_1 = p_2 = 40$ is a Nash equilibrium. We need to show that $p_i = 40$ is a best reply to the opponent setting $p_j = 40$. Firm $i$’s profit in the candidate equilibrium is:

$$\pi_i(40, 40) = (40 - 10)30 = 900.$$  

Deviating to a lower price clearly decreases the firm’s profit, because the firm is already selling at capacity, $K = 30$. What about deviating to a higher price? We are told that, if firm $i$ sets $p_i > 40$ (but less than 70), it will face the demand, $100 - 30 - p_i = 70 - p_i$. But, then the firm’s profit $\pi_i(p_i, 40) = (p_i - 10)(70 - p_i)$ is less than $(40 - 10)(70 - 40) = 900$, since $\partial \pi_i/\partial p_i = 80 - 2p_i < 0$, for all $p_i > 40$. Thus, deviating to any price higher than 40 is unprofitable. This shows that 40 is the best reply to 40, for each firm. So, $p_1 = p_2 = 40$ is a Nash equilibrium.
Problem 4

One hundred people simultaneously write down an integer between 1 and 100, inclusive. The person who guesses closest to one-third of the average number will win a prize. The prize is divided equally in case of ties.

Part A

Show that no pure strategy strictly dominates any other.

Answer

We will show that every pure strategy $s_i \in \{1, 2, \ldots, 100\}$ can be a losing strategy. Suppose all of player $i$’s opponents guess 1. Then, $i$ will lose the prize, if she plays any strategy in $\{2, \ldots, 100\}$. This shows none of the pure strategies $s_i \geq 2$ strictly dominate any other strategy. Now, suppose that 98 of $i$’s opponents guess 100, and the 99th opponent guesses 30. Then, $i$ will lose the prize if she guesses 1, because 30 is closer than 1 to the target $(9,800 + 30 + 1)/300 = 98.31/3$. This shows each pure strategy sometimes loses, and therefore none strictly dominates any other strategy.

Part B

Find a mixed strategy that strictly dominates 100.

Answer

Note that 100 is weakly dominated by every other pure strategy, but we know from Part A, it is not strictly dominated by any of them. We will show there are mixed strategies that strictly dominate 100.

Consider the mixed strategy $\sigma_i$ that puts equal probability on every number less than 100, i.e. $\sigma_i(s_i) = 1/99$, for every $s_i \in \{1, \ldots, 99\}$. Let player $i$’s opponents play any profile of strategies, $\sigma_{-i}$. The joint distribution $\sigma_{-i}$ assigns some probability, in $[0, 1]$, to the event that all of $i$’s opponents guess 100. In this event, player $i$ will tie for the prize by also playing 100. But, she will do strictly better and win the prize outright by playing the strategy $\sigma_i$. In the complementary event, in which at least one of $i$’s opponents does not guess 100, player $i$ is sure to lose by playing 100. Again, she will do strictly better and win the prize with a positive probability by playing $\sigma_i$. So, $\sigma_i$ strictly dominates the pure strategy 100.

You might have come up with a different mixed strategy, instead of the one above. In fact, every mixed strategy that puts a positive probability on each number between 1 and 33, inclusive, strictly dominates 100.
Part C

Show only one strategy profile survives the iterated elimination of strictly dominated strategies.

Answer

We can use the mixed strategy from Part B to eliminate 100, from every player’s strategy set. In the reduced game, 99 is strictly dominated, by mixing equally among the smaller numbers, \{1, \ldots, 98\}. So, we can eliminate 99, from every player’s strategy set, in the second round of elimination. We continue eliminating strictly dominated strategies, until each player’s strategy set is reduced to \{1\}, after 99 steps of elimination. Thus, each player playing 1 is the only strategy profile to survive. Note, it is the unique Nash equilibrium, in pure or mixed strategies, because every equilibrium must survive this procedure.
**Problem 5**

Two parties are running for office. They simultaneously announce party positions in a policy space that is modeled as the closed unit interval, \([0, 1]\). Each party’s payoff is its vote share. The voters’ ideal points are uniformly distributed on the interval. Each voter votes for the party whose position is closest to her ideal point. If she is equidistant from both parties, she votes for each party with probability 1/2.

**Part A**

Show the game has a unique Nash equilibrium in pure strategies.

**Answer**

First, let’s show that 1/2 is the only best reply to 1/2. Party \(i\)’s vote share will be strictly less than 1/2, if \(p_i \neq 1/2\) but \(p_j = 1/2\). If party \(i\) instead locates at 1/2 with \(j\), then \(i\) will get an expected vote share equal to 1/2. So, \(i\)’s only best reply to \(p_j = 1/2\) is to locate at 1/2, too. This means (1/2, 1/2) is a Nash equilibrium, and there is no other equilibrium in which one party picks 1/2, for sure, while the other locates elsewhere.

Now, suppose the parties choose policy positions, \(p_i \leq p_j\), not equal to 1/2. Then, if \(p_i < 1/2 < p_j\), or \(p_i \leq p_j < 1/2\), party \(i\) can increase its vote share by locating at 1/2, instead of at \(p_i\). And, if \(1/2 < p_i \leq p_j\), party \(j\) can increase its vote share by picking 1/2, instead of \(p_j\). This shows there are no other pure strategy equilibria.

**Remark:** Does this game have other mixed strategy equilibria? Let’s show it does not. Suppose party \(i\) picks 1/2, for sure, and party \(j\) plays any other pure or mixed strategy \(\sigma_j\). Then, \(\sigma_j\) puts positive probability on the event that \(j\) locates somewhere different from 1/2. In this event, \(i\)’s expected payoff is strictly greater than 1/2. So, \(i\)’s overall expected payoff, against \(\sigma_j\), is strictly greater than 1/2. Party \(i\) will do at least as well as this, in any equilibrium in which \(j\) plays \(\sigma_j\). But, party \(j\)’s expected payoff will be less than 1/2, because vote shares add up to 1. So, party \(j\) will have a profitable deviation to 1/2, instead of playing \(\sigma_j\). Thus, there is no equilibrium in which a party plays something other than the pure strategy 1/2.

**Part B**

Show there is no pure strategy Nash equilibrium, with three political parties.

**Answer**

Each party gets an expected vote share of 1/3, if all three pick the same policy position. But, party \(i\) can move to one side and get an expected vote share close to 1/2. Now, suppose at least 2 parties locate at different points. Then, \(p_i < p_j \leq p_k\), or \(p_i \leq p_j < p_k\). In the first case, party \(i\) can raise its vote share by moving closer to \(p_j\). In the second case, party \(k\) can get more votes by moving closer to \(p_j\). This shows there is no Nash equilibrium in pure strategies.