EC9A1 Problem Set 1 Solutions

Irina Kholodenko

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Problem 1: Insurance

- There are two states of nature. State 1 occurs with probability $\pi$.

- The consumer’s wealth in state $i$ is $w_i$, and $w_1 > w_2$.

- Suppose the consumer is strictly risk averse.

- Show there exists an insurance policy that makes positive expected profits for the insurer, makes the consumer equally well-off in both states and is preferred to no insurance.
An insurance policy \((p, L)\) costs premium \(p\) in each state of nature and pays \(L\) in case of loss.

Full insurance satisfies \(L = w_1 - w_2\).
Problem 1 cont’d

- We want to find $p$ that satisfies both:

\[ p - (1 - \pi)(w_1 - w_2) > 0 \]  
\[ \pi u(w_1) + (1 - \pi)u(w_2) < u(w_1 - p) \]

- Strictly risk averse means having a strictly concave utility.

\[ \pi u(w_1) + (1 - \pi)u(w_2) < u(\pi w_1 + (1 - \pi)w_2). \]

- Try letting $w_1 - p = \pi w_1 + (1 - \pi)w_2$. Then:

\[ p = (1 - \pi)(w_1 - w_2). \]

- This is fair insurance that makes the company just break even.
Problem 1 method 1 (using continuity of $u$)

- Recall, we want to find $p$ that satisfies both:

  $$p - \left(1 - \pi\right)(w_1 - w_2) > 0 \quad (1)$$
  $$\pi u(w_1) + \left(1 - \pi\right)u(w_2) < u(w_1 - p) \quad (2)$$

- Recall, the consumer is strictly risk averse:

  $$\pi u(w_1) + \left(1 - \pi\right)u(w_2) < u(\pi w_1 + (1 - \pi)w_2).$$

- If $u$ is continuous, then $u(x - \epsilon)$ is close to $u(x)$. There is an $\epsilon > 0$ such that:

  $$\pi u(w_1) + (1 - \pi)u(w_2) < u(\pi w_1 + (1 - \pi)w_2 - \epsilon) < u(\pi w_1 + (1 - \pi)w_2).$$

- Let $w_1 - p = \pi w_1 + (1 - \pi)w_2 - \epsilon$. Then:

  $$p = (1 - \pi)(w_1 - w_2) + \epsilon.$$
Problem 1 method 2 (using the CE)

- Alternatively, we can invoke the certainty equivalent $w^{CE}$ for the consumer’s “gamble.” Because the consumer is strictly risk averse, $w^{CE}$ is less than the expected value of the gamble:

$$\pi u(w_1) + (1-\pi)u(w_2) = u(w^{CE}), \text{ and } w^{CE} < \pi w_1 + (1-\pi)w_2.$$ 

- We will satisfy both conditions we need to satisfy by letting:

$$w_1 - p = \frac{w^{CE} + (\pi w_1 + (1-\pi)w_2)}{2}$$

- This gives us:

$$p = \frac{(1-\pi)(w_1 - w_2) + (w_1 - w^{CE})}{2} > (1-\pi)(w_1 - w_2).$$

- The above inequality follows from, $w^{CE} < \pi w_1 + (1-\pi)w_2$. 
Problem 1 picture

\[ \pi u(w_1) + (1 - \pi) u(w_2) \]

\[ w_1 - p \text{ can take values between here} \]

\[ w_{CE} \]

Utility

Wealth
Problem 2: Market for used cars

- There is a continuum of sellers whose car quality $\theta$ is uniformly distributed on $[0, 1]$. Sellers know their quality, but buyers do not. There are not enough cars for all potential buyers.
- In a trade, the buyer gets $u_b = \theta - p$, and the seller gets $u_s$ to be specified below. A player who does not trade gets 0.
- A). Show that $E(\theta|p) = p$, in every competitive equilibrium of this market.
- B). Let $u_s = p - \theta/2$. Show every $p \in [0, 1/2]$ is an equilibrium price.
- C). Let $u_s = p - \sqrt{\theta}$. Find the equilibrium $p$. Which cars are traded?
- D). Let $u_s = p - \theta^3$. Find the equilibrium $p$. How many equilibria are there?
- E). Which outcomes above are Pareto efficient? Give Pareto improvements where possible.
Problem 2, Part A

- All buyers have the same expected utility:

\[ u_b = \theta - p \implies E(u_b) = E(\theta|p) - p. \]

- Could the equilibrium price give buyers positive expected utility, \( E(u_b) > 0 \)?

- Suppose the price \( p \) makes \( E(u_b) > 0 \). Recall, there are not enough cars for all potential buyers. So, there are buyers who will not get to buy a car, and who want to pay \( p \) or more for a car, because \( E(u_b) > 0 \). This is positive **excess demand**, and it means \( p \) did not clear the market.

- If \( E(u_b) < 0 \), then no buyer agrees to trade, and there is positive **excess supply**, (assuming \( u_s(0, p) > 0 \), if \( p > 0 \)).

- This shows \( E(u_b) = 0 \), in every competitive equilibrium:

\[ E(u_b) = E(\theta|p) - p = 0 \implies E(\theta|p) = p. \]
Problem 2, Part B

- The seller’s utility, \( u_s = p - \theta/2 \), is decreasing in quality. The highest quality seller who agrees to trade at price \( p \) is:

\[
u_s = p - \theta/2 = 0 \implies \theta = \min(2p, 1).
\]

- Thus the average quality seller who agrees to trade is:

\[
E(\theta \mid p) = E(\theta \mid 0 \leq \theta \leq \min(2p, 1)) = \min(p, 1/2).
\]

- This is consistent with \( E(\theta \mid p) = p \) (which we derived in Part A), if \( p \in [0, 1/2] \).

- But, if \( p > 1/2 \), then \( p > \min(p, 1/2) \).

- So, \( p \) is an equilibrium price \( \iff p \in [0, 1/2] \).

- We will see the equilibrium with \( p = 1/2 \) is Pareto efficient (all cars are sold), but the others are not.
Problem 2, Part C

The seller’s utility, \( u_s = p - \sqrt{\theta} \), is decreasing in quality. The highest quality seller who agrees to trade at price \( p \) is:

\[
u_s = p - \sqrt{\theta} = 0 \implies \theta = \min(p^2, 1).
\]

Thus the average quality seller who agrees to trade is:

\[
E(\theta \mid p) = E(\theta \mid 0 \leq \theta \leq \min(p^2, 1)) = \min(\frac{p^2}{2}, \frac{1}{2}).
\]

This is consistent with \( E(\theta \mid p) = p \) (which we derived in Part A), if and only if \( p = 0 \).

When \( p = 0 \), only the seller with \( \theta = 0 \) would agree to trade. This is a measure zero subset of all sellers, and if the trade happens, it generates no surplus for the buyer or seller. We could think of \( p = 0 \) as total market failure, in general . . . but we will see this market has no Pareto improvements anyway.
Problem 2, Part D

► The seller’s utility, \( u_s = p - \theta^3 \), is decreasing in quality. The highest quality seller who agrees to trade at price \( p \) is:

\[
 u_s = p - \theta^3 = 0 \implies \theta = \min(p^{1/3}, 1). 
\]

► Thus the average quality seller who agrees to trade is:

\[
 E(\theta | p) = E(\theta | 0 \leq \theta \leq \min(p^{1/3}, 1)) = \min(0.5p^{1/3}, 1/2). 
\]

► This is consistent with \( E(\theta|p) = p \), if and only if:

\[
 2p = p^{1/3}. 
\]

► So, \( p = 0 \), or \( p = 1/\sqrt{8} \).

► We found 2 equilibria; \( p = 0 \) is the “bad” equilibrium and \( p = 1/\sqrt{8} \) is better but we will see it is still Pareto inefficient.
Problem 2, Part E

- A Pareto improvement is a reallocation of cars and money to the buyers and sellers that makes no one worse off and somebody better off, (relative to the post-trading allocation).
- Recall, there are not enough cars for all buyers, and $u_b = \theta - p$. So, an equilibrium is Pareto inefficient, if exchanging some unsold car of type $\theta$ for a fee of $\theta$ generates positive seller surplus, $u_s(\theta, \theta) > 0$.
- In Part B, the equilibrium $p = 1/2$ is Pareto efficient because all cars are sold. What about $p < 1/2$? Show such an equilibrium is Pareto inefficient, e.g. by exchanging the car of quality 1 for a fee of 1. Does this generate positive surplus without making anybody worse off?
- In Part C, $u_s = \theta - \sqrt{\theta} \leq 0$ for $\theta \in [0, 1]$, so there are no Pareto improvements. It is efficient for nobody to trade.
- Note, some buyers learn $\theta < p$, after they trade, and have ex-post regret. Reversing these trades is not a Pareto improvement, as the sellers who benefited will be worse off.