EC202 Seminar
Week 3
(on materials of week 1)

Junxi Liu

17 Oct 2022

Welcome!

- My name is Junxi Liu, a second-year MRes/PhD student in the economics department
- Classes and office hours: week 3 to week 9; another one in term 3
- Office hours:
  - Tuesdays 4pm-6pm, Social Science 1.128b
  - Wednesdays 2pm-4pm, Social Science 0.86
  - Booking: junxiliu.com
- Contact:
  - junxi.liu.1@warwick.ac.uk
  - For technical questions, please try to book an office hour
- Format:
  - Fridays: Google form link to collect questions and general thoughts
  - Sundays: slides sent by email and on website
  - Mondays: go through in-class questions and other material based on the poll

Brief Introduction

- UK system
- Academic career?
- Don’t be shy about anything and try your best to engage
- Please give feedbacks
- General advice: treat it as a math class
- Make absolutely sure that you understand the intuition, definition, logic, thought process, and methods
- It’s the class to hone your thinking abilities
- Absence

In-class Question

Q5. Consider a student deciding on their housing choices. We model this as having a budget of M to split between two goods: the first being accommodation, where the more luxurious a place the student rents, the more they have to pay, and the second being the composite good (ie money to spend on all other goods). We let h denote units of housing quality. A basic model could measure this in square metres or a more sophisticated measurement would include things like condition of the house, location, amenities etc. Let g be units of the composite good, that is money to spend on all other things. Let the price of housing be p_h per unit and the price of the composite good be 1.

We let our consumption set be $X = \mathbb{R}^2_{\geq 0}$. For each of the following utility functions $u : X \to \mathbb{R}$, draw indifference curves and budget constraint, write down the utility maximisation problem and solve it.

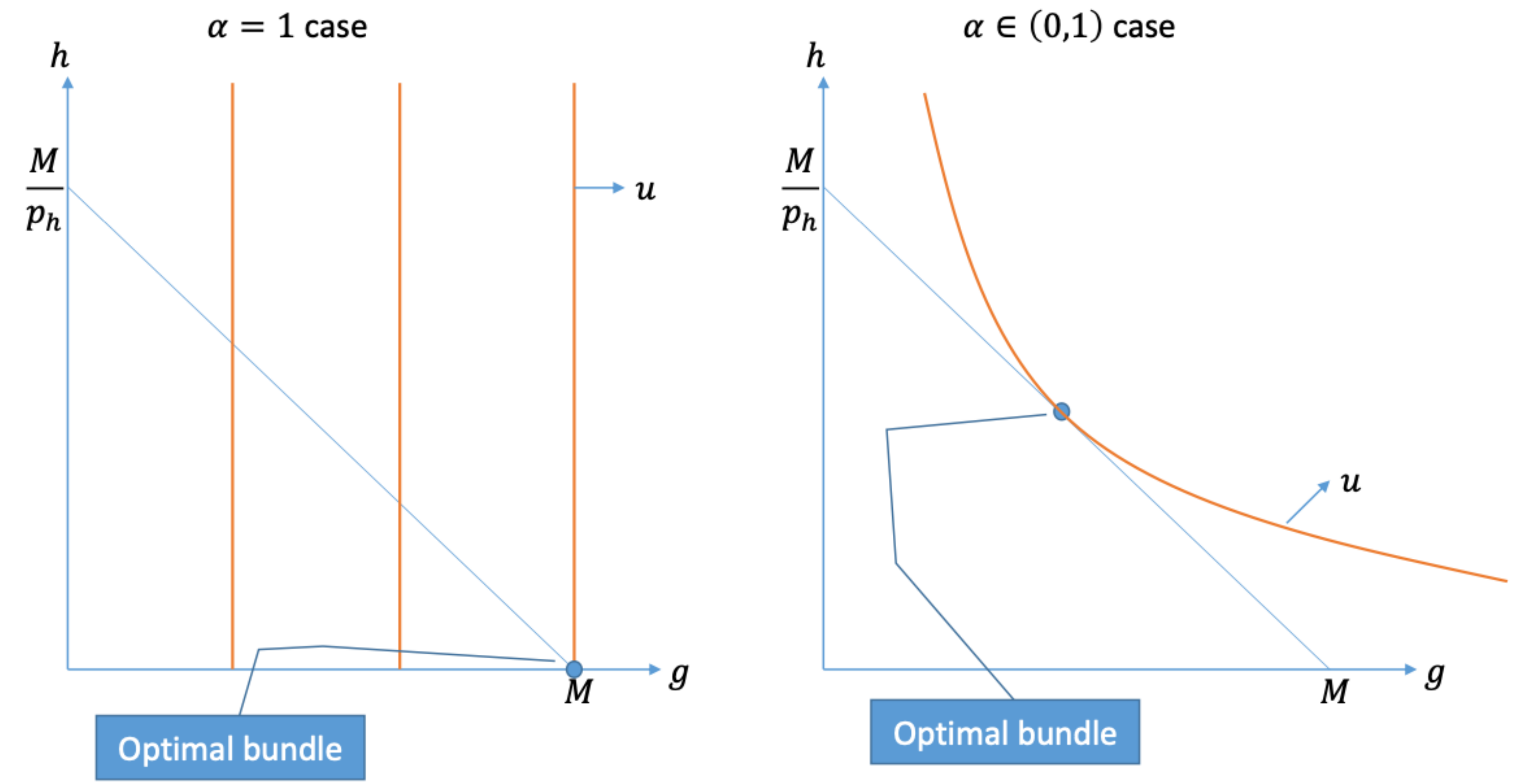
Define the problem

- We want to maximum utility
- We have a constraint
- A few notes:
  - Dimensions
  - Domain
  - Corner cases
  - Derivative technics

$$\max_{(g,h)\in\mathbb{R}^2_{\geq 0}} u(g,h) \quad \text{s.t. } p_h h + g \leq M$$

a) (quasi) Cobb-Douglas

$$u(g,h) = g^{\alpha} h^{1-\alpha} \text{ for some exogeneous } \alpha \in [0,1]$$



Student only values the composite good so spends all income on that.

Student solves UMP where all income expended and bang per buck of good 1 equals bang per buck of good 2.

a) (quasi) Cobb-Douglas

$u(g,h) = g^\alpha h^{1-\alpha}$ for some exogeneous $\alpha \in [0,1]$

The $\alpha = 1$ case can be seen from the diagram. The optimal bundle is $(g,h)^* = (M,0)$.

The $\alpha = 0$ case is conceptually similar: now the studegnt only cares about housing and so indifference curves are horizontal and so the optimal bundle is $(g,h)^* = \left(0, \frac{M}{p_h}\right)$.

For $\alpha \in (0,1)$ we set

$$\frac{MU_h}{p_h} = \frac{MU_g}{p_g} \iff \frac{(1-\alpha)g^\alpha h^{-\alpha}}{p_h} = \frac{\alpha g^{\alpha-1}h^{1-\alpha}}{1}$$
$$\iff (1-\alpha)g = \alpha h p_h$$

We can substitute this back into $p_h h + g = M$ to get the optimal bundle is

$$(g,h)^* = \left(M\alpha, \frac{M(1-\alpha)}{p_h}\right)$$

- Convexity
- Other methods: Lagrangian, MRS equals slope

b) (log-form quasi) Cobb-Douglas

$$u(g,h) = \begin{cases} \alpha \ln g + (1-\alpha)\ln h & g,h > 0 \\ -\infty & g = 0 \text{ or } h = 0 \end{cases} \quad \text{for some exogeneous } \alpha \in (0,1)$$

- Same as a): increasing monotonic transformation

c) Complementary goods

$$u(g, h) = \min\{\alpha g, h\} \text{ for some exogeneous } \alpha > 0.$$

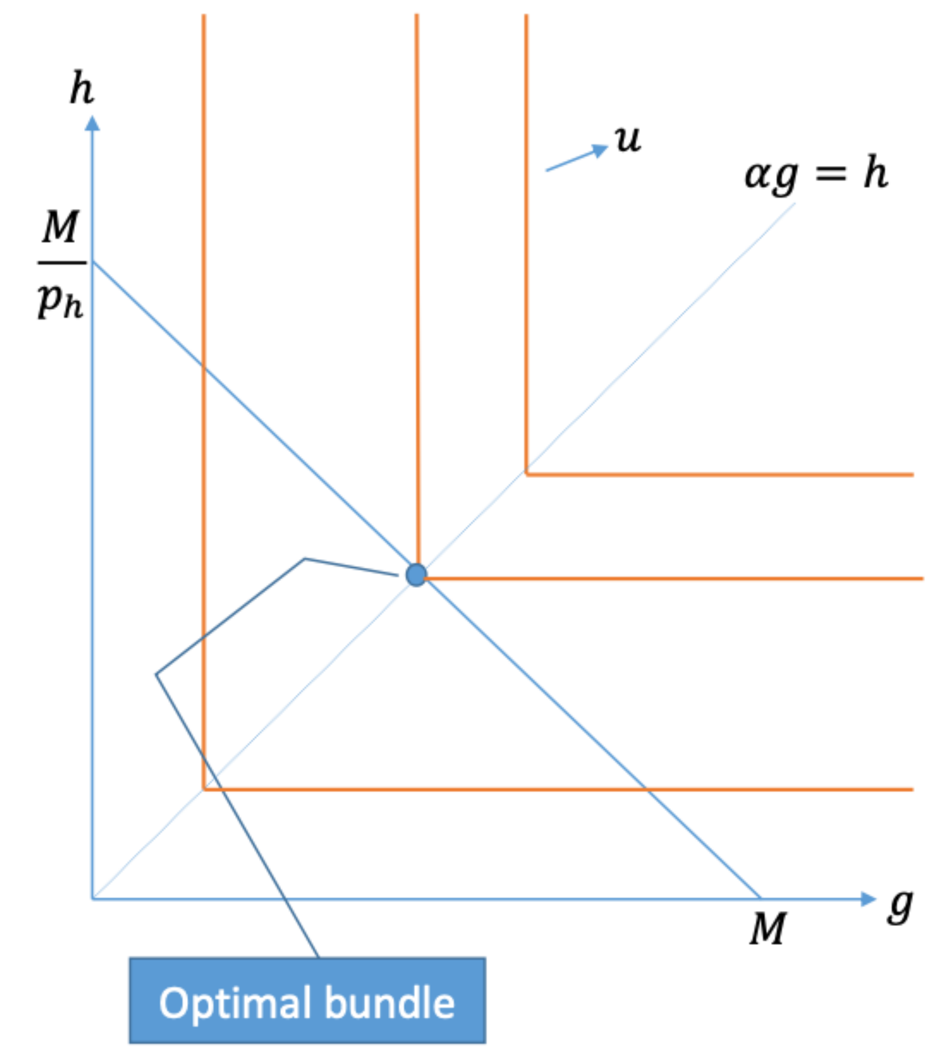
As seen in the diagram below the optimal bundle satisfies

$$\alpha g = h \;\text{ and }\; p_h h + g = M$$

Solving these two equations simultaneously gives

$$(g, h)^* = \left(\frac{M}{1 + \alpha p_h}, \frac{\alpha M}{1 + \alpha p_h}\right)$$

c) Perfect complements



Optimal bundle lies at kink of indifference curve, on budget line.

d) Substitute goods

$$u(g,h) = \alpha g + h \text{ for some exogeneous } \alpha > 0$$

We get 3 different cases depending on whether the slope of the indifference curve is steeper, shallower or equal to the slope of the budget constraint. The diagram above shows the case when $\alpha > \frac{1}{p_h}$ and the optimal bundle is $(g,h)^* = (M,0)$.

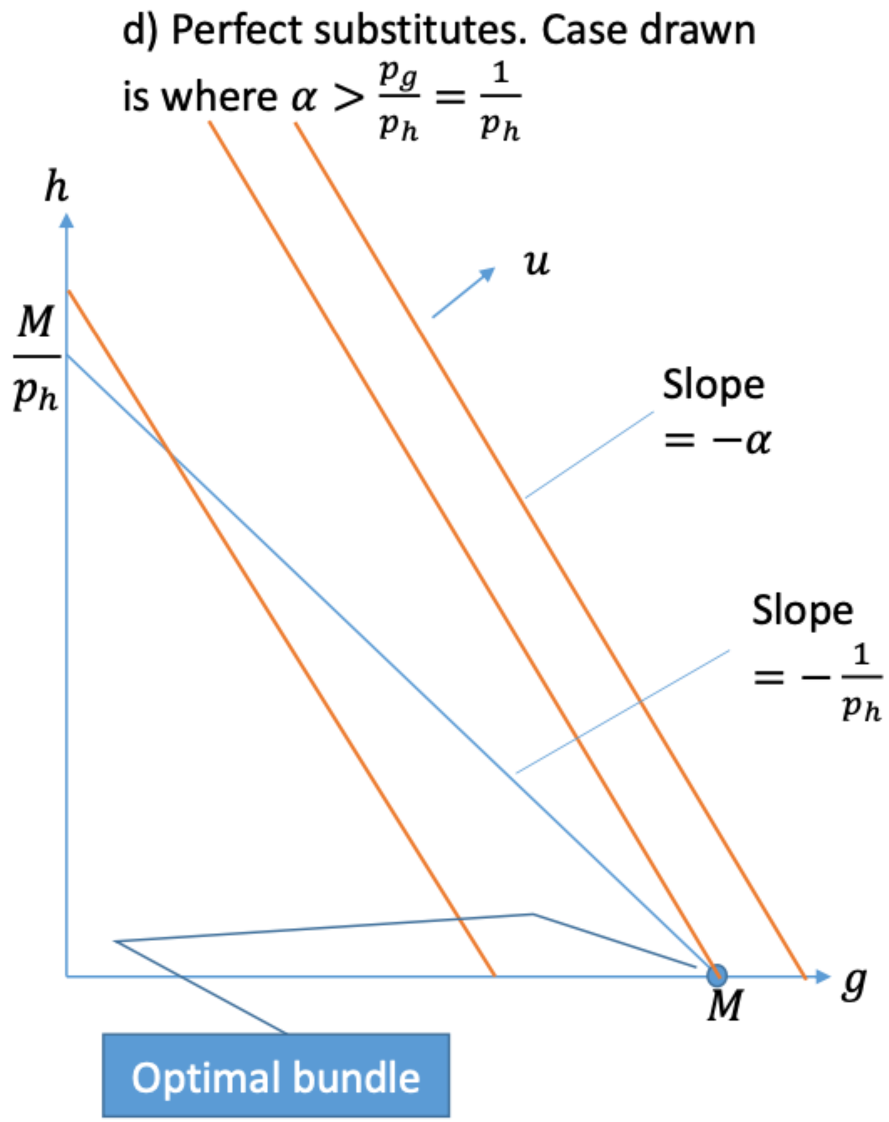
The case when the indifference curve is shallower than the budget line is conceptually similar and would lead us to the optimal bundle at $(g,h)^* = \left(0, \frac{M}{p_h}\right)$.

Finally, the indifference curve could be the same slope as the budget line. Here the budget line and the highest chievable indifference curve exactly coincide and so utility is maximised anywhere along the indifference curve.

Another way to approach this is to compare the bang per buck the consumer gets from spending on each good.

$$\frac{MU_g}{p_g} \geq \frac{MU_h}{p_h} \iff \alpha \geq \frac{1}{p_h}$$

So when $\alpha > \frac{1}{p_h}$, the bang per buck on the composite good is higher and so this is why the consumer spends all income on that good. For $\alpha < \frac{1}{p_h}$ the bang per buck on housing is greater and so the consumer spends all income on that. When $\alpha = \frac{1}{p_h}$ the bang per buck on each good is the same and so the consumer can spend all their money on either one or a mixture of the two.

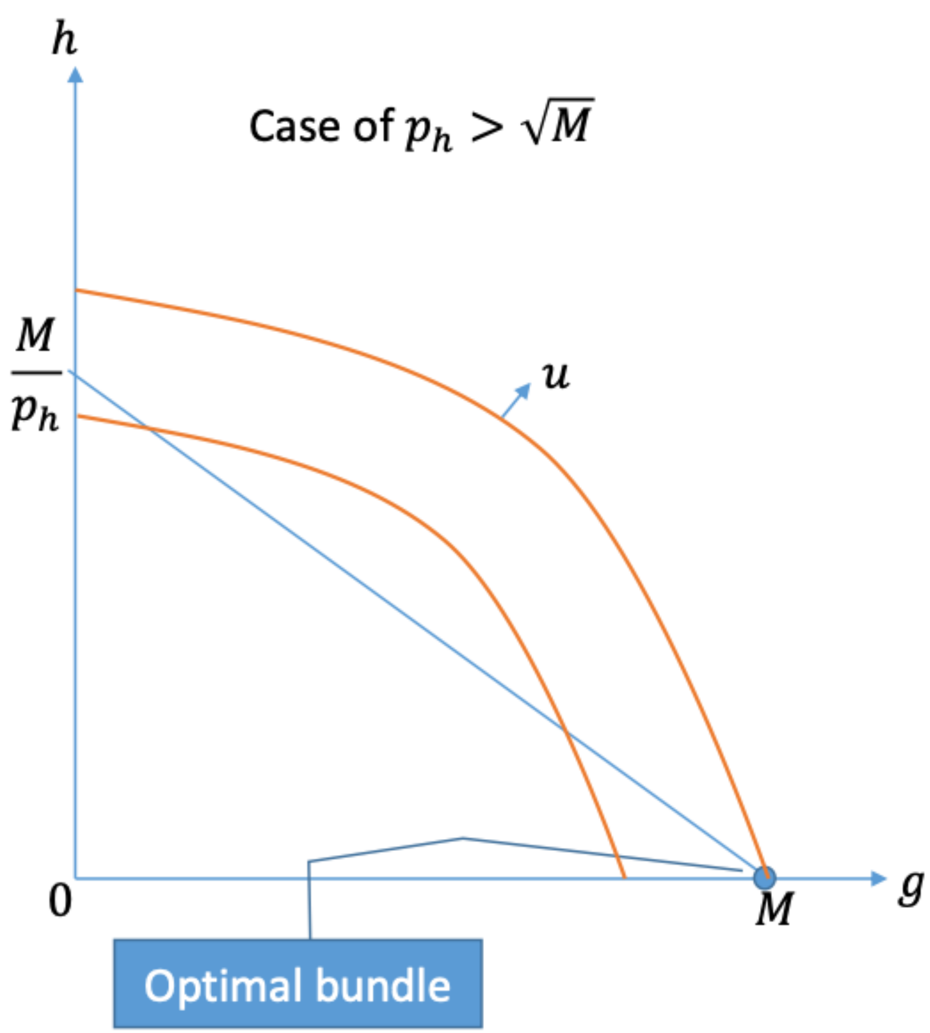


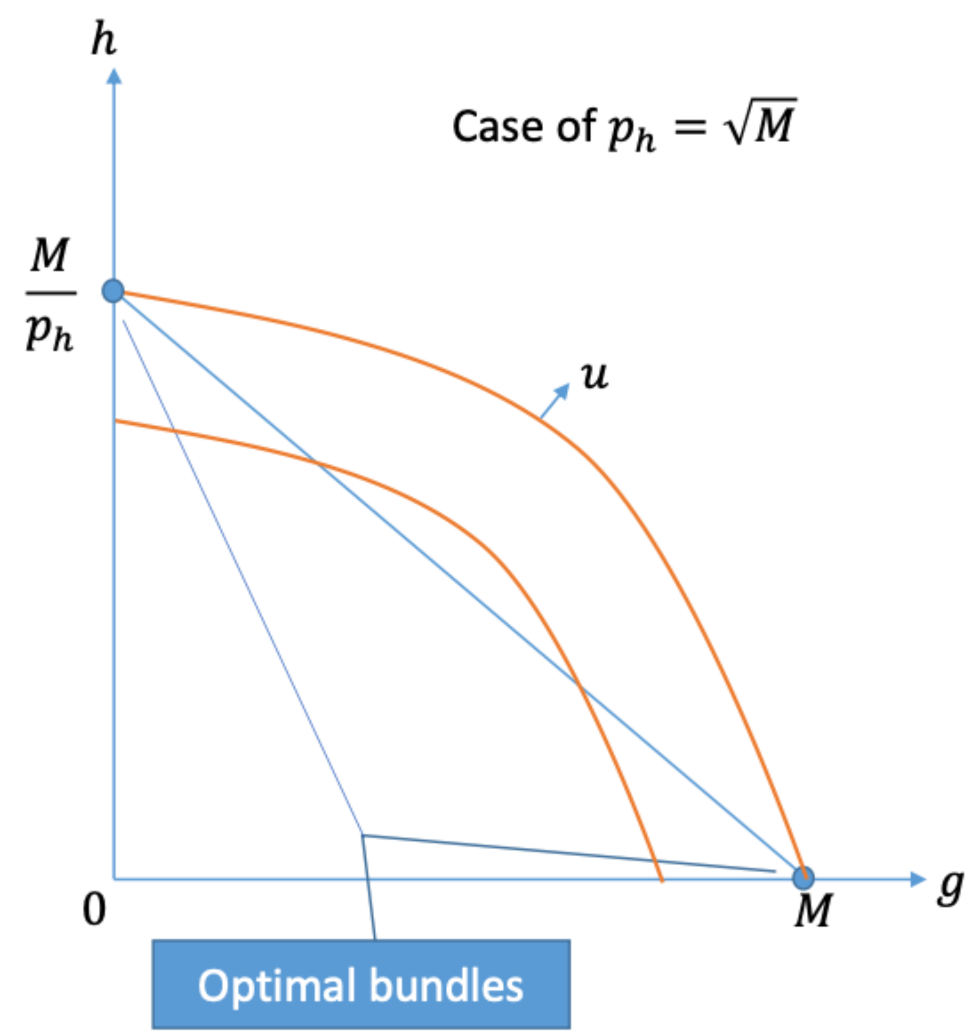
Maximise utility by spending all income on the good with higher bang per buck.

e) Power function

$$u(g,h) = g + h^{\alpha} \text{ where } \alpha \in \left\{\frac{1}{2}, 2\right\}$$

e) $\alpha = 2$





Indifference curve is curved the opposite way from normal. The two corners of budget set are the candidates for optimal bundles. You should check utility at each and compare.

It is also possible for both corners to give the same utility as each other and so for both to be optimal bundles. Although this is unlikely and only happens at one very specific price.

e) Power function

$$u(g,h) = g + h^{\alpha} \text{ where } \alpha \in \left\{\frac{1}{2}, 2\right\}$$

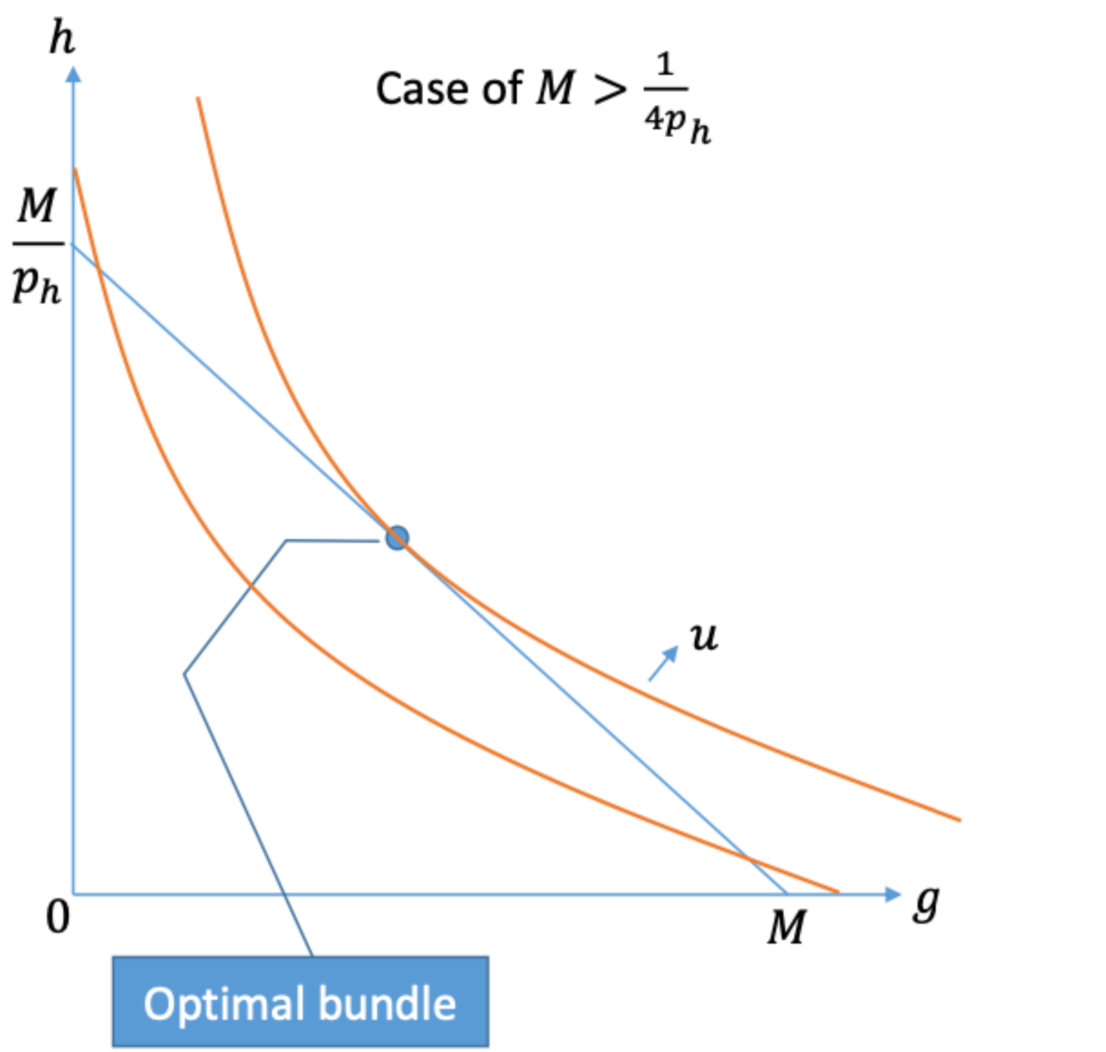
- Here the indifference curves are curved in the opposite direction to normal and so equating slope of indifference curve to slope of budget line or equivalent method would be solving a utility minimisation instead of a maximisation problem.

Observe that only the corners of the budget set, $(g,h) = (M,0)$ or $\left(0, \frac{M}{p_h}\right)$ can be optimal bundles. So we simply compare their utilities: $u(M,0) = M$, while $u\left(0, \frac{M}{p_h}\right) = \left(\frac{M}{p_h}\right)^2$. This brings us to the following results:
If $p_h > \sqrt{M}$ then $(g,h)^* = (M,0)$.
If $p_h < \sqrt{M}$ then $(g,h)^* = \left(0, \frac{M}{p_h}\right)$.
If $p_h = \sqrt{M}$ then both $\left(0, \frac{M}{p_h}\right)$ and $(M,0)$ are optimal bundles. Two of these three cases are depicted below. Students should be able to use these diagrams to draw the diagram for the $p_h < \sqrt{M}$ case.

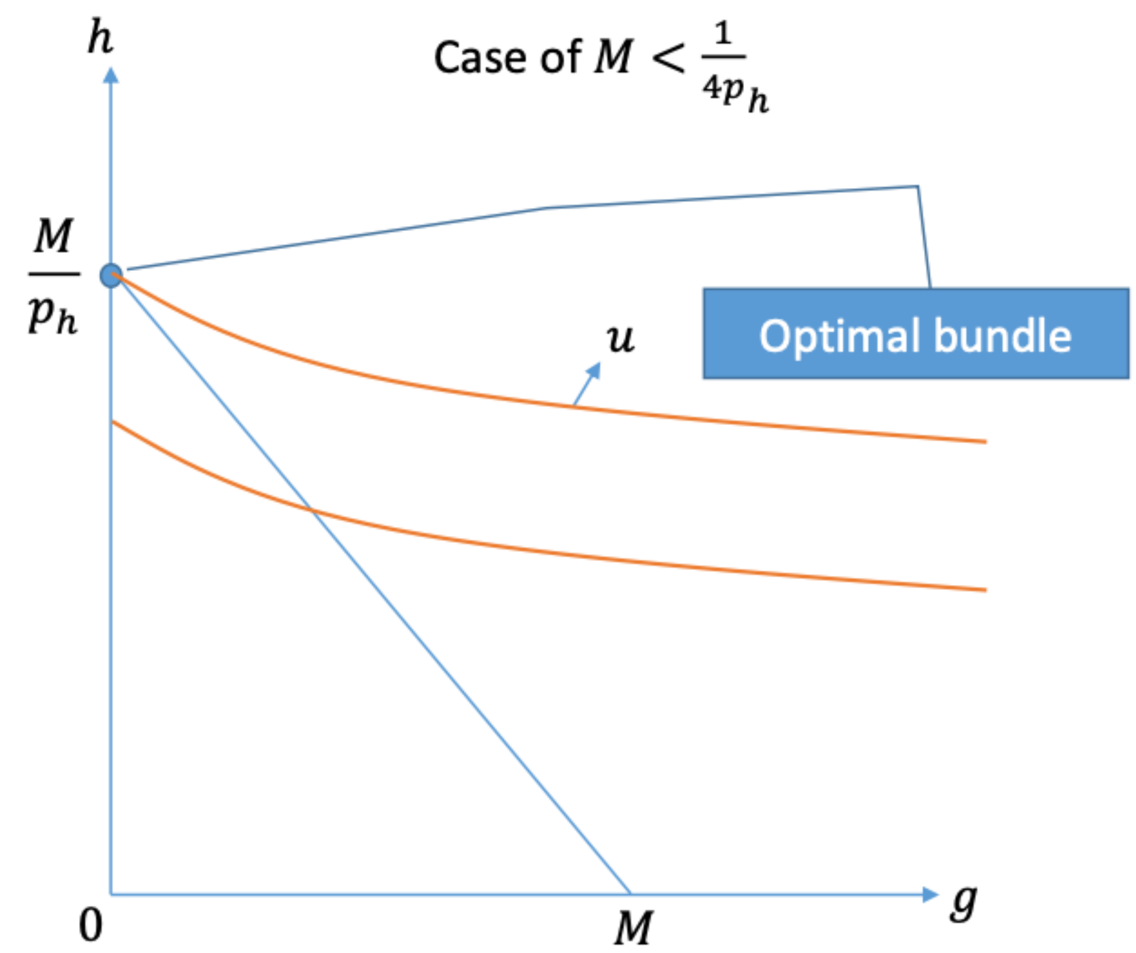
e) Power function

$$u(g,h) = g + h^{\alpha} \text{ where } \alpha \in \left\{\frac{1}{2}, 2\right\}$$

e) $\alpha = 0.5$



Indifference curve is curved the right way for convex preferences. There exists point along budget line where slope of budget line equals that of indifference curve so this is optimal bundle.



It is also possible that everywhere along the budget line, the slope of the indifference curve is shallower than the slope of budget line. In this instance we have a corner solution. Note we cannot have the opposite – that slope of IC is steeper than slope of budget line everywhere since slope of IC tends towards 0 as we approach $(M, 0)$.

e) Power function

$$u(g,h) = g + h^\alpha \text{ where } \alpha \in \left\{\frac{1}{2}, 2\right\}$$

- Here preferences are convex
- However, it is possible that the slope of the indifference curve could always be shallower than the slope of the budget constraint, leading to a corner solution.

$$\frac{MU_g}{p_g} \geq \frac{MU_h}{p_h} \iff h \geq \frac{1}{4p_h^2}$$

$$\frac{MU_g}{p_g} < \frac{MU_h}{p_h} \iff h < \frac{1}{4p_h^2}$$

If $M < \frac{1}{4p_h}$ then the consumer doesn't get any good g and so we have a corner solution.

If $M > \frac{1}{4p_h}$, the consumer will get both goods and we have an interior solution where slope of indifference curve equals slope of budget constraint.

If $M > \frac{1}{4p_h}$:

$$h = \frac{1}{4p_h^2} \text{ and } p_h h + g = M$$

$$(g,h)^* = \left(M - \frac{1}{4p_h}, \frac{1}{4p_h^2}\right).$$

If $M < \dfrac{1}{4p_h}$, then $(g,h)^* = \left(0, \dfrac{M}{p_h}\right)$.

Q6

Let $X = \mathbb{R}^3_{\geq 0}$ Consider perfect substitutes in the 3 good case: $u : X \to \mathbb{R}$ is defined by $u(x_1, x_2, x_3) = \alpha x_1 + \beta x_2 + x_3$ for some exogeneous $\alpha > 0, \beta > 0$. Find the optimal bundle as a function of income and prices.

Preferences are convex (just about as linear) so we get correct answer by considering bang per buck of each good:

$$\frac{MU_1}{p_1} = \frac{\alpha}{p_1} \quad \frac{MU_2}{p_2} = \frac{\beta}{p_2} \quad \frac{MU_3}{p_3} = \frac{1}{p_3}$$

Our consumer spends all their income on whichever good has the highest bang per buck. If two or more goods are tied for highest then the consumer can do any mixture amongst those goods.

Key Concepts — Consumption

Definition 3.1. The Consumption Set is defined as the set of commodities (or goods) consumers can consume. Let there be J goods then we define the consumption set by $X = \mathbb{R}^J_{\geq 0}$. A typical member of this set is $\mathbf{x} = (x_1, x_2, \ldots, x_J)$ where x_j is the consumer's consumption of good j, for $j \in \{1, 2, \ldots, J\}$

Definition 3.2. The budget constraint equation is: $p_1 x_1 + \ldots + p_J x_J \leq M$.

This leads us onto the next definition:

Definition 3.3. The Walrasian budget set is the set of bundles our consumer can choose between

$$\left\{(x_1, \ldots x_J) \in \mathbb{R}^J_{\geq 0} \mid p_1 x_1 + \ldots + p_J x_J \leq M\right\}$$

Key Concepts — Preferences

Definition 3.5. Given $\succeq$ we define:
The strict preference relation: $\hat{\mathbf{x}} \succ \overline{\mathbf{x}} \iff \hat{\mathbf{x}} \succeq \overline{\mathbf{x}}$ and not $\hat{\mathbf{x}} \preceq \overline{\mathbf{x}}$.
The indifference relation: $\hat{\mathbf{x}} \sim \overline{\mathbf{x}} \iff \hat{\mathbf{x}} \succeq \overline{\mathbf{x}}$ and $\hat{\mathbf{x}} \preceq \overline{\mathbf{x}}$.
Two important axioms on preferences are completeness and transitivity:
Definition 3.6. Completeness: For any two bundles of goods in $X, \hat{\mathbf{x}} = (\hat{x}_1, \ldots \hat{x}_J)$ and $\overline{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_J)$, at least one of $\hat{\mathbf{x}} \succeq \overline{\mathbf{x}}$ or $\overline{\mathbf{x}} \succeq \hat{\mathbf{x}}$ must hold. (If both hold then $\overline{\mathbf{x}} \sim \hat{\mathbf{x}}$)
Definition 3.7. Transitivity: If we prefer bundle 1 to bundle 2 and bundle 2 to bundle 3 , then we should also prefer bundle 1 to bundle 3 . If $\mathbf{x}^{(1)} \succeq \mathbf{x}^{(2)}$ and $\mathbf{x}^{(2)} \succeq \mathbf{x}^{(3)}$ then $\mathbf{x}^{(1)} \succeq \mathbf{x}^{(3)}$

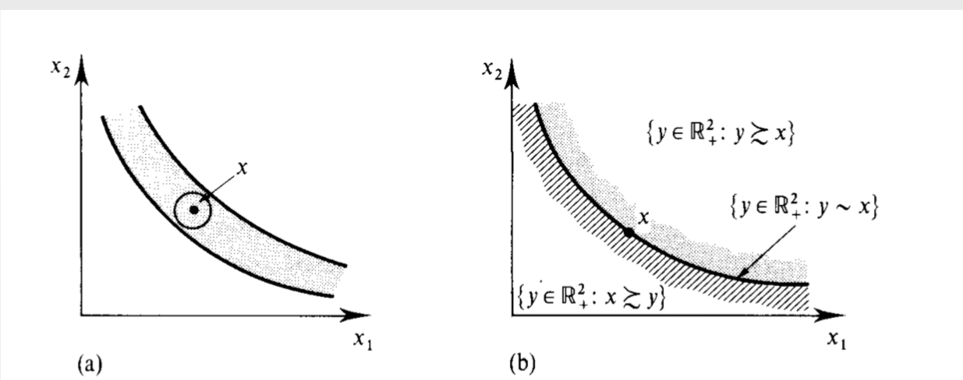
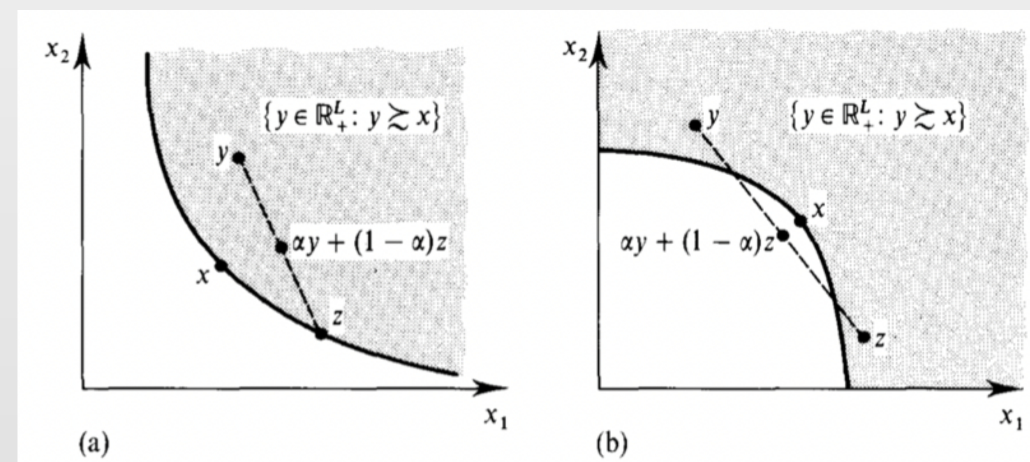


Figure 3.B.2
(a) A thick indifference set violates local nonsatiation.
(b) Preferences compatible with local nonsatiation.



Definition 3.8. Convex preferences: For every $\mathbf{x} \in X$, the upper contour set $\{\hat{\mathbf{x}} \in X \mid \hat{\mathbf{x}} \succeq \mathbf{x}\}$ is a convex set. In other words, if $\hat{\mathbf{x}} \succeq \mathbf{x}$ and $\overline{\mathbf{x}} \succeq \mathbf{x}$ then $\alpha\hat{\mathbf{x}} + (1-\alpha)\overline{\mathbf{x}} \succeq \mathbf{x}$ for any $\alpha \in [0,1]$
Definition 3.11. Preferences are Monotone if for every $\mathbf{x}, \hat{\mathbf{x}} \in X$, if $x_j < \hat{x}_j \forall j \in J$ then $\hat{\mathbf{x}} \succ \mathbf{x}$.
Definition 3.12. Preferences are locally non-satiated if for every $\mathbf{x} \in X, \forall \varepsilon > 0, \exists \hat{\mathbf{x}} \in X$ such that $\|\mathbf{x} - \hat{\mathbf{x}}\| < \varepsilon$[7] and $\hat{\mathbf{x}} \succ \mathbf{x}$

Key Concepts — Preferences

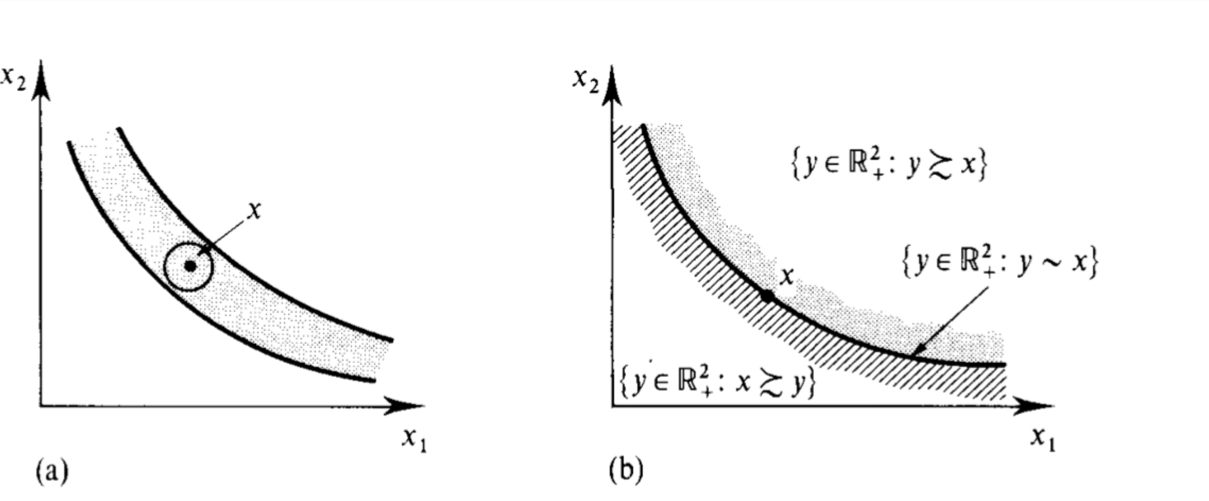
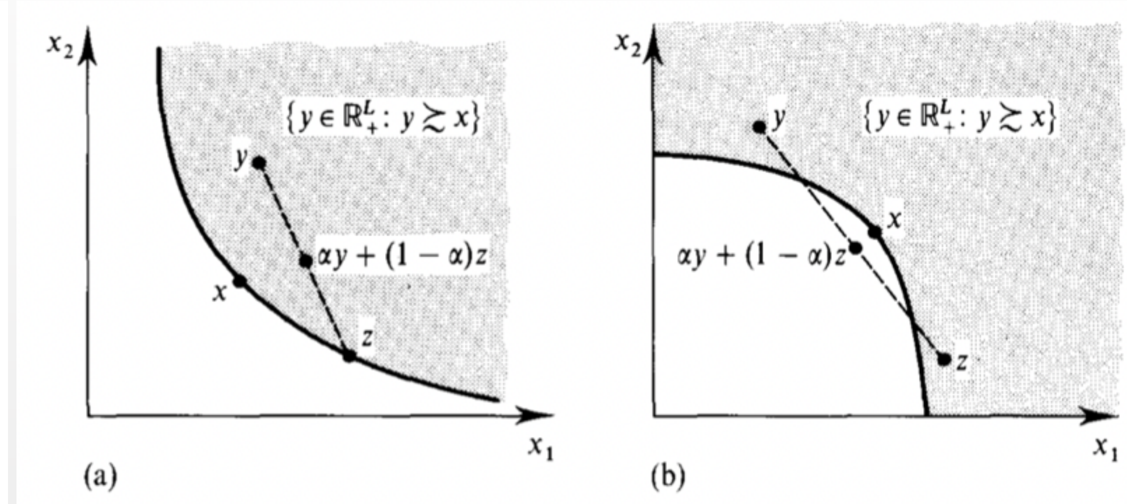


Figure 3.B.2
(a) A thick indifference set violates local nonsatiation.
(b) Preferences compatible with local nonsatiation.



Definition 3.8. Convex preferences: For every $\mathbf{x} \in X$, the upper contour set $\{\hat{\mathbf{x}} \in X \mid \hat{\mathbf{x}} \succeq \mathbf{x}\}$ is a convex set. In other words, if $\hat{\mathbf{x}} \succeq \mathbf{x}$ and $\overline{\mathbf{x}} \succeq \mathbf{x}$ then $\alpha\hat{\mathbf{x}} + (1-\alpha)\overline{\mathbf{x}} \succeq \mathbf{x}$ for any $\alpha \in [0,1]$

Definition 3.9. Strictly convex preferences: For every $\mathbf{x} \in X$, if $\hat{\mathbf{x}} \succeq \mathbf{x}$ and $\overline{\mathbf{x}} \succeq \mathbf{x}$, with $\hat{\mathbf{x}} \neq \overline{\mathbf{x}}$ then for any $\alpha \in (0,1)$, $\alpha\hat{\mathbf{x}} + (1-\alpha)\overline{\mathbf{x}} \succ \mathbf{x}$

Definition 3.10. Preferences are Strongly Monotone if for every $\mathbf{x}, \hat{\mathbf{x}} \in X$, if $x_j \leq \hat{x}_j \forall j \in J$ and $\exists j \in J$ with $x_j < \hat{x}_j$ then $\hat{\mathbf{x}} \succ \mathbf{x}$.

Definition 3.11. Preferences are Monotone if for every $\mathbf{x}, \hat{\mathbf{x}} \in X$, if $x_j < \hat{x}_j \forall j \in J$ then $\hat{\mathbf{x}} \succ \mathbf{x}$.

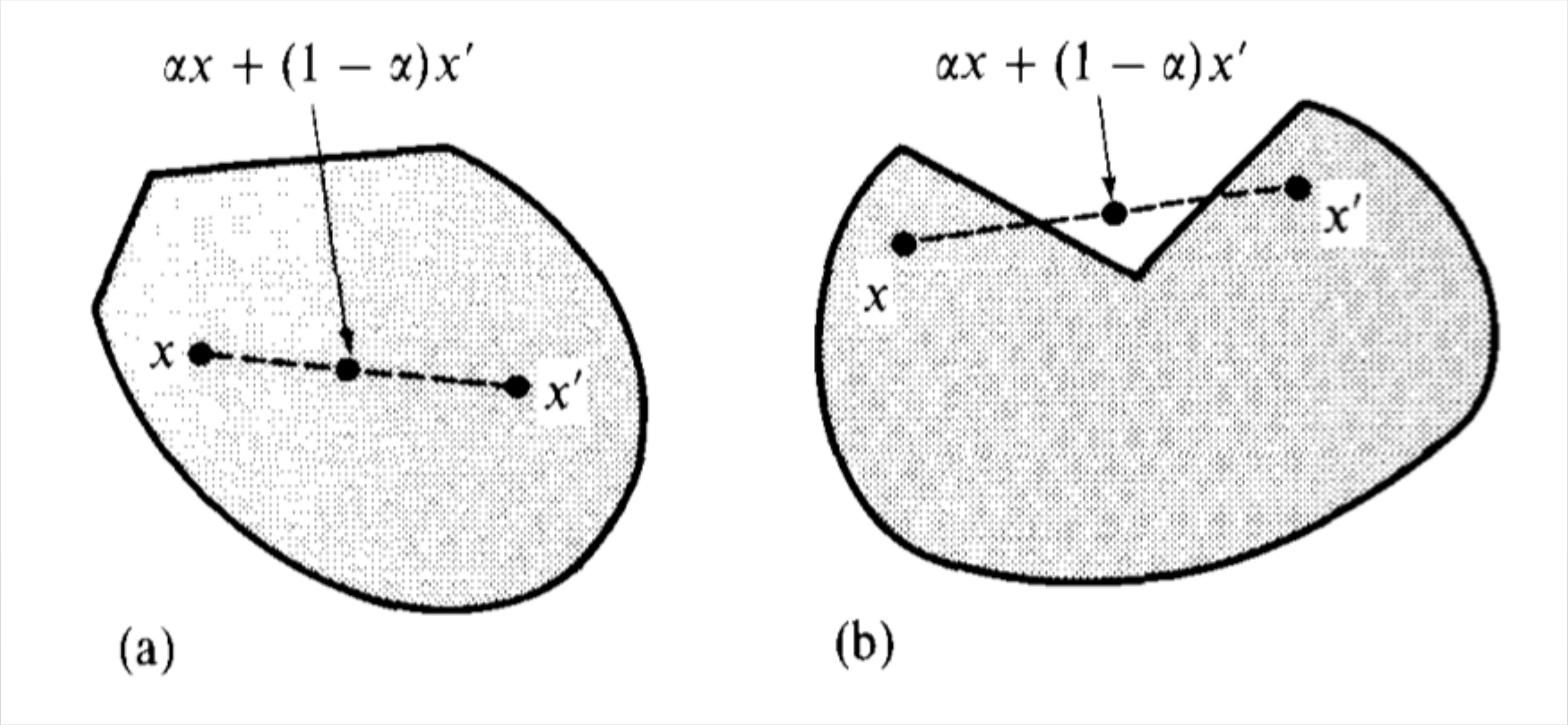
Definition 3.12. Preferences are locally non-satiated if for every $\mathbf{x} \in X$, $\forall \varepsilon > 0$, $\exists \hat{\mathbf{x}} \in X$ such that $\|\mathbf{x} - \hat{\mathbf{x}}\| < \varepsilon^7$ and $\hat{\mathbf{x}} \succ \mathbf{x}$

Key Concepts — Utility Functions and Sets

Definition 3.13. The preference relation $\succeq$ can be represented by a utility function $u : X \to \mathbb{R}$ if for every pair of bundles $\overline{\mathbf{x}}, \hat{\mathbf{x}} \in X, \hat{\mathbf{x}} \succeq \overline{\mathbf{x}} \iff u(\hat{\mathbf{x}}) \geq u(\overline{\mathbf{x}})$

Definition 3.14. A function $u : X \to \mathbb{R}$ is quasi-concave if its upper level sets, $\{x \in X : u(x) \geq c\}$, are convex for every $c \in \mathbb{R}$.

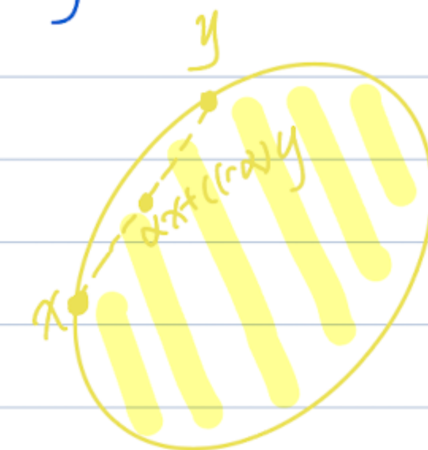
Definition M.G.1: The set $\boldsymbol{A} \subset \mathbb{R}^N$ is convex if $\alpha x + (1-\alpha)x' \in A$ whenever $x, x' \in A$ and $\alpha \in [0,1]$.



Def 3.0.0 Convexity and Concavity

A set is convex

if, for all x and y in X, we have $[\alpha x + (1-\alpha)y] \in X$, for any $\alpha \in [0,1]$. A set that's not convex is called non-convex set.

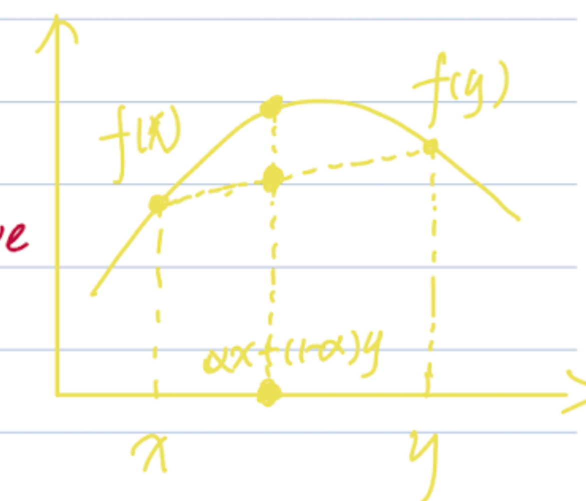


A function is concave

if, $f[\alpha x + (1-\alpha)y] \geq \alpha f(x) + (1-\alpha) f(y)$
for all x, y, $\alpha \in [0,1]$

A function is strictly concave

if, $f[\alpha x + (1-\alpha)y] > \alpha f(x) + (1-\alpha) f(y)$
for all x, y, $x \neq y$, $\alpha \in (0,1)$



A function is quasiconcave

if, $f[\alpha x + (1-\alpha)y] \geq \min\{f(x), f(y)\}$
for all x, y, $\alpha \in [0,1]$

$\leq \max\{f(x), f(y)\}$ quasiconvex

or, the upper contour sets $\{x \in A : f(x) \geq t\}$ are Convex sets,

$\{x : f(x) \leq t\}$ ~~convex~~ quasiconvex

which means, $f(x) \geq t$ and $f(y) \geq t$ implies that $f[\alpha x + (1-\alpha)y] \geq t$.
for any $t \in R$, all x, y, and $\alpha \in [0,1]$