EC202 Seminar Week 5
Covering Materials from Week 3

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Recap

- Pure exchange economy
- Utility function
- Edgeworth Box
  - Convex indifference curves
  - Pareto dominating set
  - Pareto optimal set
- Marginal Rate of Substitution (MRS)
- Marginal utility
- Budget set
- Allocation

Notions

- Excess demand and supply

The excess demand of good j, denoted z_j can be calculated as

$$z_j = \sum_{i \in I} x_{ij} - \sum_{i \in I} e_{ij}$$

For a 2×2 economy, this means:
$z_1 = (x_{A1} + x_{B1}) - (e_{A1} + e_{B1}) \quad z_2 = (x_{A2} + x_{B2}) - (e_{A2} + e_{B2})$

- Utility Maximisation Problem

$$\max_{\mathbf{x}_i \in X} u_i(\mathbf{x}_i) \text{ subject to } \mathbf{p} \cdot \mathbf{x}_i \leq \mathbf{p} \cdot \mathbf{e}_i$$

- Walrasian Equilibrium

Definition A price vector $\mathbf{p}^* = (p_1^*, p_2^*, \ldots, p_J^*) \in \mathbb{R}_{\geq 0}^J$ and an allocation $\mathbf{x}^* = (\mathbf{x}_i^*)_{i \in I} \in \mathbb{R}_{\geq 0}^{I \times J}$ constitute a Walrasian Equilibrium if
i) For each consumer $i \in I$, bundle $\mathbf{x}_i^*$ solves the consumer's (UMP) at prices p^*.
ii) All markets clear: $\sum_{i \in I} \mathbf{x}_i^* = \sum_{i \in I} \mathbf{e}_i$
We call $\mathbf{p}^*$ a Walrasian Equilibrium price vector and $\mathbf{x}^*$ a Walrasian Equilibrium allocation.

- Existence?
- Uniqueness?

More Notions…

- What is an Walrasian Equilibrium allocation, **x?**
- Aggregate demand and supply ***function***

  Definition 3.2. The aggregate demand function $\sum_{i\in I}\mathbf{x}_i(\mathbf{p})\in\mathbb{R}^J_{\geq 0}$ is the total demanded by consumers as a function of prices. This is a J-dimensional vector whose j^{th} element is $\sum_{i\in I}x_{ij}(\mathbf{p})$

- Excess demand and supply ***function***

  Definition 3.3. The excess demand function is $\mathbf{z}(\mathbf{p})=\sum_{i\in I}\mathbf{x}_i(\mathbf{p})-\sum_{i\in I}\mathbf{e}_i\in\mathbb{R}^J$. We can also talk about the excess demand for good j as the j^{th} element of this, which is $z_j(\mathbf{p})=\sum_{i\in I}x_{ij}(\mathbf{p})-\sum_{i\in I}e_{ij}$. When $z_j(\mathbf{p})>0$ we say that there is excess demand of good j and when $z_j(\mathbf{p})<0$ we say there is excess supply of good j.

- What is a price vector, v?
- What (the heck) is, Walras' Law?

  Proposition 3.1. Walras' Law: Assuming all individuals have locally nonsatiated preferences, for every price vector $\mathbf{p}\in\mathbb{R}^J_{\geq 0}$ for which individuals' demand functions are finite, $\mathbf{p}\cdot\mathbf{z}(\mathbf{p})=p_1z_1(\mathbf{p})+\cdots+p_Jz_J(\mathbf{p})=0$. In words the value of the excess demand vector equals 0 . This holds for all prices regardless of whether they are equilibrium prices or not.

- ***p*x = w***

In-class Question

Q3. Consider a 2×2 economy where preferences are represented by $u_A, u_B : \mathbb{R}^2_{\geq 0}$ where $u_A = x_{A1}^{\alpha} x_{A2}^{1-\alpha}$ and $u_B = x_{B1}^{\beta} x_{B2}^{1-\beta}$ for some $\alpha, \beta \in (0, 1)$. Assume both goods are in strctly positive supply.

a) Argue that, for any initial endowment, there are no Walrasian Equilibria where one good has price 0 and hence in an Walrasian Equilibrium we must have $(p_1, p_2) \in \mathbb{R}^2_{>0}$.

b) Let prices be $(p_1, p_2) \in \mathbb{R}^2_{>0}$ and incomes be M_A, M_B. Verify that optimal demands are

$$\mathbf{x}_A(\mathbf{p}, M_A) = \left(\frac{\alpha M_A}{p_1}, \frac{(1-\alpha) M_A}{p_2} \right)$$
$$\mathbf{x}_B(\mathbf{p}, M_B) = \left(\frac{\beta M_B}{p_1}, \frac{(1-\beta) M_B}{p_2} \right)$$

In-class Question

- a) If the price of one good equals 0 then a consumer with positive endowment of the other good would not have a nite optimal demand, since they could get more and more utility by forever increasing their demand. Also, at least one of the two consumers must have a positive endowment of the other good.
- b) By drawing indifference curves and Budget constraint, one can justify that the solution must be where slope of budget constraint (price ratio) equals slope of indifference curve (MRS). Also each agent must exhaust their budget. Hence for Andy we solve:

$$\text{MRS} = \frac{\alpha x_{A2}}{(1-\alpha)x_{A1}} = \frac{p_1}{p_2}$$
$$p_1 x_{A1} + p_2 x_{A2} = M_A$$

Solving these two equations simultaneously gives the optimal demand in the question:

$$\mathbf{x}_A(\mathbf{p}, M_A) = \left(\frac{\alpha M_A}{p_1}, \frac{(1-\alpha)M_A}{p_2} \right)$$

The analysis for Bob is identical.

In-class Question

c) Consider the initial endowment $\mathbf{e}_A = (0, 1), \mathbf{e}_B = (1, 0)$:
i) Find optimal demands $\mathbf{x}_A(\mathbf{p})$ and $\mathbf{x}_B(\mathbf{p})$. Show that these demands satisfy Walras' Law.
ii) Find the Walrasian Equilibrium and illustrate it on an Edgeworth box.
iii) Verify on your Edgeworth box and algebraically that both players prefer the Walrasian Equilibrium to their initial allocation.
d) Repeat b) for initial endowment $\mathbf{e}_A = \left(\frac{1}{2}, \frac{1}{2}\right), \mathbf{e}_B = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\alpha < \beta$

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i) Substituting in the value of the endowment as income we get:

$$\mathbf{x}_A(\mathbf{p}) = \left(\frac{\alpha p_2}{p_1}, \frac{(1-\alpha)p_2}{p_2}\right) = \left(\frac{\alpha p_2}{p_1}, (1-\alpha)\right)$$
$$\mathbf{x}_B(\mathbf{p}) = \left(\frac{\beta p_1}{p_1}, \frac{(1-\beta)p_1}{p_2}\right) = \left(\beta, \frac{(1-\beta)p_1}{p_2}\right)$$

Walras' Law states that the value of excess demand should equal 0 . We will verify this holds:

$$\begin{aligned} p_1(x_{A1} + x_{B1} - 1) + p_2(x_{A2} + x_{B2} - 1) &= p_1\left(\frac{\alpha p_2}{p_1} + \beta - 1\right) + p_2\left((1-\alpha) + \frac{(1-\beta)p_1}{p_2} - 1\right) \\ &= \alpha p_2 + p_1\beta - p_1 + p_2(1-\alpha) + (1-\beta)p_1 - p_2 \\ &= 0 \end{aligned}$$

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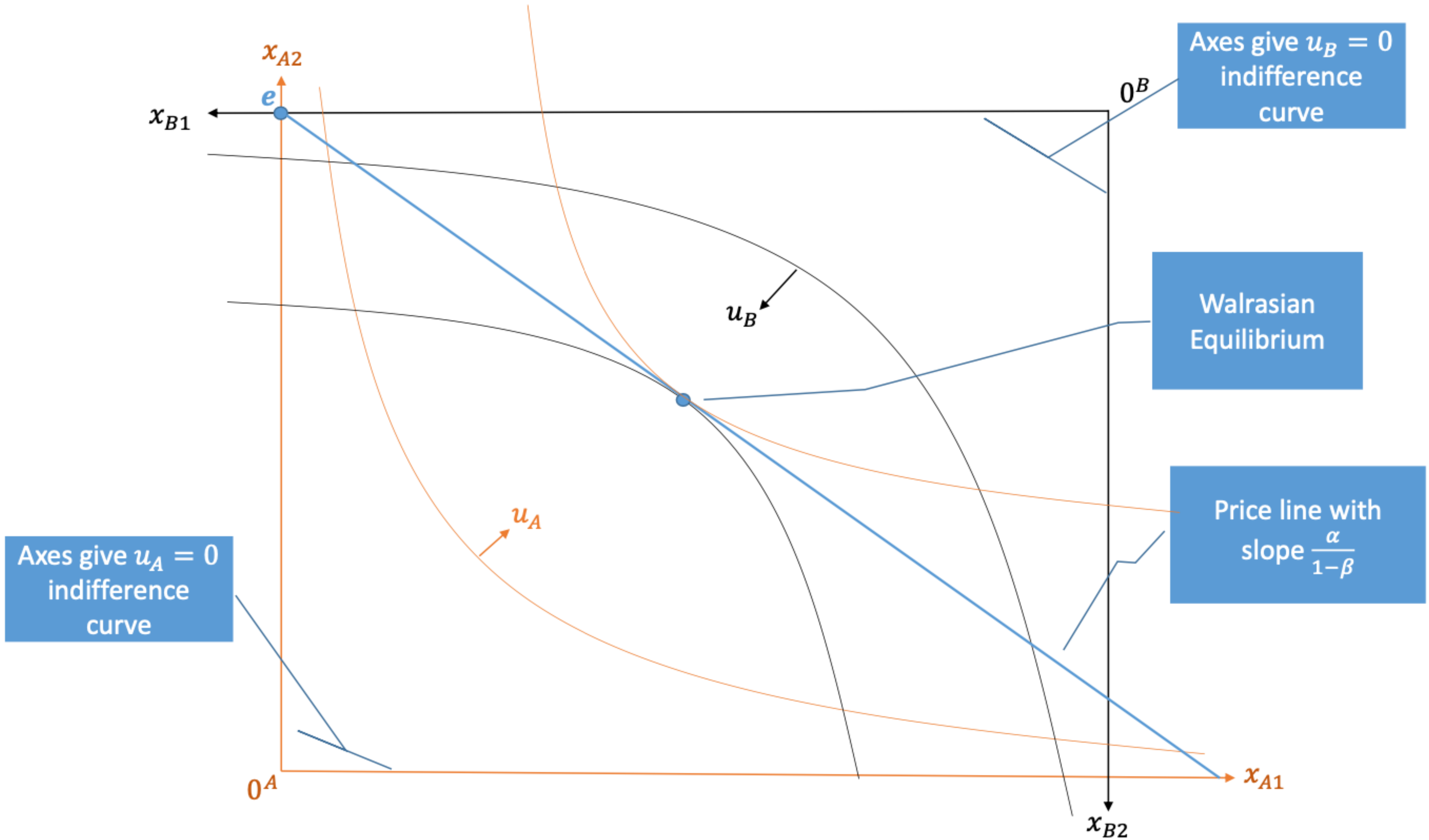
c) ii) In any Walrasian Equilibrium both prices must be positive. For example if $p_1 = 0$ then Andy would demand infinite amount of good 1 and we use the results above: We need markets to clear so:

$$\frac{\alpha p_2}{p_1} + \beta = 1 \quad (\text{ Market for good 1})$$

$$(1-\alpha) + \frac{(1-\beta)p_1}{p_2} = 1 \quad (\text{ Market for good 2})$$

As both prices must be positive, we can normalise $p_2 = 1$ and solve for p_1. When doing this for the market for good 1 we get $p_1 = \frac{\alpha}{1-\beta}$ and as a check, you can verify that this clears the market for good 2 too. Thus our Walrasian Equilibrium is

$$\mathbf{p} = \left(\frac{\alpha}{1-\beta}, 1\right) \mathbf{x}_A = ((1-\beta), (1-\alpha)) \mathbf{x}_B = (\beta, \alpha)$$



x_{A2}
e
x_{B1}
0^B
Axes give $u_B = 0$ indifference curve
u_B
Walrasian Equilibrium
Price line with slope $\frac{\alpha}{1-\beta}$
u_A
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c) iii) From the diagram, we can see that the initial allocation lay on the lowest indifference curve of each agent, while the Walrasian Equilibrium lies on a much higher indifference curve. Algebraically Andy's utility has increased from 0 to $(1-\beta)^\alpha(1-\alpha)^{1-\alpha}$ while Bob's has increased from 0 to $\beta^\beta\alpha^{1-\beta}$.

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d) Substituting in the value of the endowment as income we get:

$$\mathbf{x}_A(\mathbf{p}) = \left(\frac{\alpha\left(\frac{1}{2}p_1 + \frac{1}{2}p_2\right)}{p_1}, \frac{(1-\alpha)\left(\frac{1}{2}p_1 + \frac{1}{2}p_2\right)}{p_2} \right) = \left(\frac{\alpha}{2} + \frac{\alpha p_2}{2p_1}, \frac{(1-\alpha)p_1}{2p_2} + \frac{(1-\alpha)}{2} \right)$$

$$\mathbf{x}_B(\mathbf{p}) = \left(\frac{\beta\left(\frac{1}{2}p_1 + \frac{1}{2}p_2\right)}{p_1}, \frac{(1-\beta)\left(\frac{1}{2}p_1 + \frac{1}{2}p_2\right)}{p_2} \right) = \left(\frac{\beta}{2} + \frac{\beta p_2}{2p_1}, \frac{(1-\beta)p_1}{2p_2} + \frac{(1-\beta)}{2} \right)$$

Walras' Law states that the value of excess demand should equal 0 . We will verify this holds:

$$\begin{aligned} p_1(x_{A1} + x_{B1} - 1) + p_2(x_{A2} + x_{B2} - 1) =& p_1\left(\frac{\alpha}{2} + \frac{\alpha p_2}{2p_1} + \frac{\beta}{2} + \frac{\beta p_2}{2p_1} - 1 \right) \\ &+ p_2\left(\frac{(1-\alpha)p_1}{2p_2} + \frac{(1-\alpha)}{2} + \frac{(1-\beta)p_1}{2p_2} + \frac{(1-\beta)}{2} - 1 \right) \\ =& \, 0 \end{aligned}$$

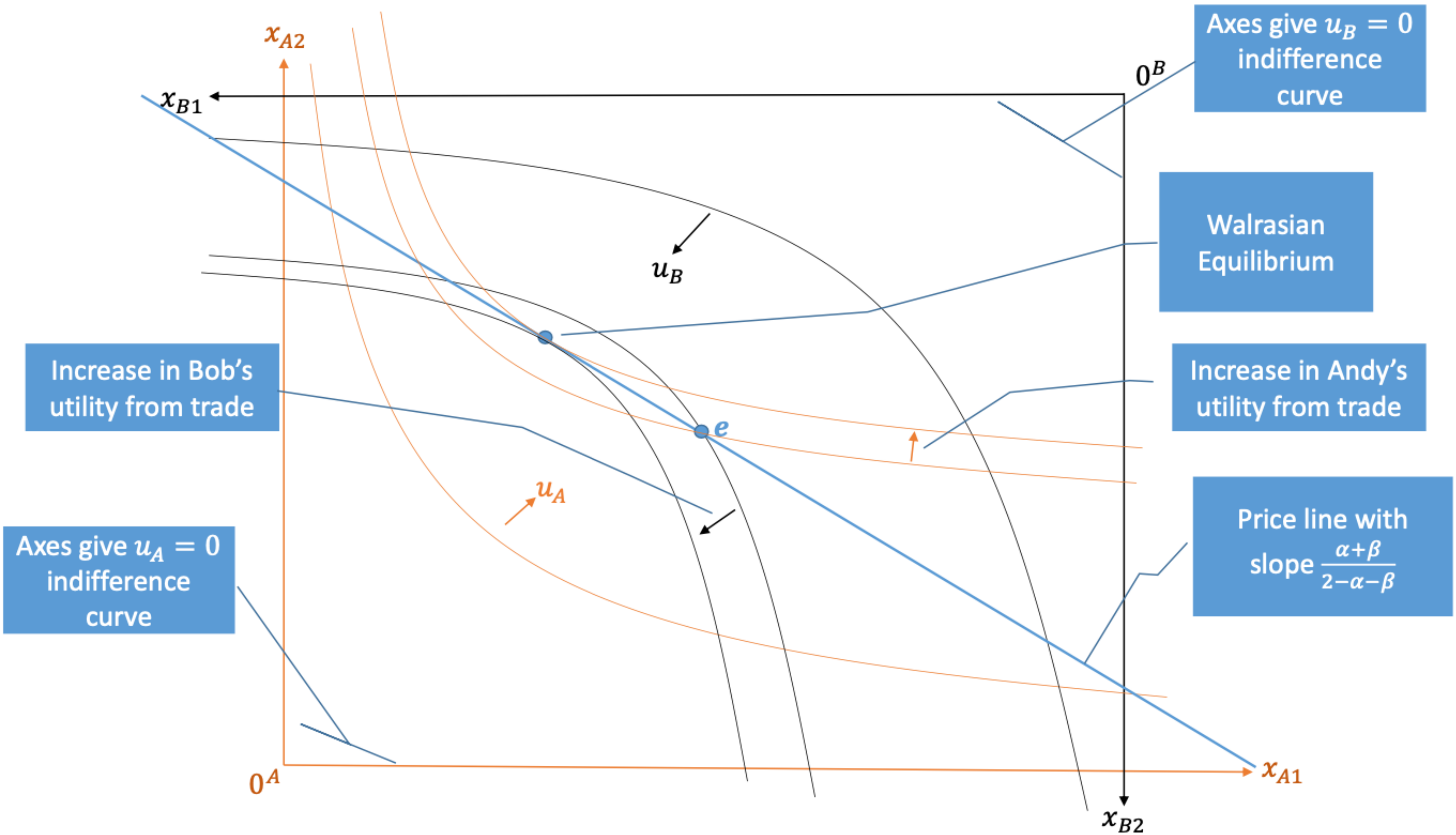
In any Walrasian Equilibrium both prices must be positive for same reason. We need markets to clear so:

$$\frac{\alpha}{2} + \frac{\alpha p_2}{2p_1} + \frac{\beta}{2} + \frac{\beta p_2}{2p_1} = 1 \qquad \text{(Market for good 1)}$$
$$\frac{(1-\alpha)p_1}{2p_2} + \frac{(1-\alpha)}{2} + \frac{(1-\beta)p_1}{2p_2} + \frac{(1-\beta)}{2} = 1 \qquad \text{(Market for good 2)}$$

As both prices must be positive, we can normalise $p_2 = 1$ and solve for p_1. When doing this for the market for good 1 we get $p_1 = \frac{\alpha+\beta}{2-\alpha-\beta}$ (algebra skipped) and as a check, you can verify that this clears the market for good 2 too. Note that as well as $\mathbf{p} = \left(\frac{\alpha+\beta}{2-\alpha-\beta}, 1\right)$ we could write prices as $\mathbf{p} = (\alpha+\beta, 2-\alpha-\beta)$ Thus our Walrasian Equilibrium is

$$\mathbf{p} = \left(\frac{\alpha+\beta}{2-\alpha-\beta}, 1\right)$$
$$\mathbf{x}_A = \left(\frac{\alpha}{2} + \frac{\alpha p_2}{2p_1}, \frac{(1-\alpha)p_1}{2p_2} + \frac{(1-\alpha)}{2}\right) \mid p_1 = \alpha + \beta p_2 = 2 - \alpha - \beta$$
$$\mathbf{x}_B = \left(\frac{\beta}{2} + \frac{\beta p_2}{2p_1}, \frac{(1-\beta)p_1}{2p_2} + \frac{(1-\beta)}{2}\right) \mid p_1 = \alpha + \beta p_2 = 2 - \alpha - \beta$$

As a check, note that $\alpha = \beta = \frac{1}{2}$ gives $\mathbf{p} = (1,1), \mathbf{x}_A = \left(\frac{1}{2}, \frac{1}{2}\right), \mathbf{x}_B = \left(\frac{1}{2}, \frac{1}{2}\right)$ as one would expect. Also note that as α and β increase, meaning agents put more weight on how much of good 1 they get, p_1 also increases. Indeed $p_1 \geq p_2 \iff \alpha + \beta \geq 1$ Also, note that as we are assuming $\alpha < \beta$ we see that Andy will get less of good 1 and more of good 2 than Bob. The diagram below pictures this for some $\alpha + \beta < 1$.


Axes give $u_B = 0$ indifference curve
0^B
x_{A2}
x_{B1}
Walrasian Equilibrium
u_B
Increase in Bob's utility from trade
Increase in Andy's utility from trade
e
u_A
Price line with slope $\frac{\alpha+\beta}{2-\alpha-\beta}$
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