EC202 Week 7
Covering Materials from Week 5

Junxi Liu

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Recap of what we've been through so far…

- Pure exchange economy:
  - Week 2 material: Edgeworth Box, Pareto Efficiency
  - Week 3 material: Walrasian equilibrium
- Economy with production (Robinson Crusoe economy):
  - Week 4 material: Walrasian equilibrium with production

The Welfare Theorems

- The First Fundamental Theorem of Welfare Economics
  - If (markets are complete and) everyone's preferences are locally non-satiated then any Walrasian Equilibrium is Pareto optimal.
  - (Assuming markets are complete.) Given an economy with fundamentals listed in Lecture 4, Definition 2.1, let (**p**, **x**, **y**) be a Walrasian Equilibrium. If all consumers have locally non-satiated preferences then the allocation (**x**,**y**) is Pareto efficient.
- The Second Fundamental Theorem of Welfare Economics
  - (Assuming markets are complete) Let **x** be a Pareto efficient allocation. If all agents have continuous, convex and locally non-satiated preferences then there exists a reallocation of resources such that for some price schedule **p**, (**p**, **x**) is a Walrasian Equilibrium.
  - (Assuming markets are complete.) Given the fundamentals listed in Lecture 4, Definition 2.1, let (**x**, **y**) be a Pareto efficient al location where $x_{ij} > 0$ forall $i \in I$ and $j \in J$. Suppose
    - i) all preferences are convex, continuous and locally non-satiated.
    - ii) all production sets are convex, closed and satisfy free-disposal.

    Then there exists a distribution of resources such that (**x**,**y**) is a Walrasian Equilibrium allocation.

In-class Question

Q3. Crusoe has 10 units of time (good 1) to allocate between work and leisure and 2 units of the consumption good (good 2). If he works for k hours he can produce $2\sqrt{k}$ units of the consumption good and can freely dispose of each good. Crusoe has utility function $u : \mathbb{R}^2_{\geq 0} \rightarrow \mathbb{R}$ where

$$u(x_1, x_2) = x_1^{\frac{1}{10}} x_2^{\frac{2}{10}}$$

a) Find the Pareto efficient bundle(s) and draw a diagram to illustrate them. (Hint: in this case algebra gets messy, so just show that the solution of Crusoe working 4 hours satisfies the first order condition.)
b) Write down the firm's production set.
c) What if anything can we learn about the Walrasian Equilibrium or Equilibria from the First Welfare Theorem?
d) What if anything can we learn about the Walrasian Equilibrium or Equilibria from the Second Welfare Theorem?
e)Find the Walrasian Equilibrium or Equilibria.

Solution: To start with, note that the utility function $u(x_1, x_2) = x_1^{\frac{1}{10}} x_2^{\frac{2}{10}}$ represents the same preferences as $v = x_1 x_2^2$ and so I will use the latter to make the algebra simpler.
a) Crusoe will optimise where he does not freely dispose (waste) either good and so letting $k \geq 0$ be the amount of time Crusoe devotes to labour, we solve

$$\max_{k \geq 0} v = x_1 x_2^2 \text{ s.t. } x_1 = 10 - k, x_2 = 2 + 2\sqrt{k}$$

Normally, I would solve by writing $v(k)$ and taking $\frac{dv}{dk} = 0$. While you can apply that method, the algebra gets a little messy and so it's easier to set slope of production function equal to slope of indifference curve:

$$| \text{ MRS } | = \frac{d}{dk}(2\sqrt{k})$$
$$\iff \frac{x_2^2}{2x_1 x_2} = k^{-\frac{1}{2}}$$
$$\iff \sqrt{k} = \frac{2x_1}{x_2} = \frac{2(10-k)}{2+2\sqrt{k}} = \frac{10-k}{1+\sqrt{k}}$$
$$\text{subbing in } k = 4 \implies 2 = \frac{10-4}{1+2}$$

So we get the solution $k = 4$ meaning our Pareto optimum is

$$(x_1, x_2) = (6, 6)$$

b) The firm has production set

$$Y = \left\{ \mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0, y_2 \leq 2\sqrt{-y_1} \right\}$$

c) To apply the First Welfare Theorem, we need to show that preferences satisfy local non-satiation. We only have one utility function: $u(x_1, x_2) = x_1^{\frac{1}{10}} x_2^{\frac{2}{10}}$. Take any $\mathbf{x} \in \mathbb{R}^2_{\geq 0}$, and consider an $\varepsilon-$ ball around $\mathbf{x}$. For any $\varepsilon > 0$, the $\varepsilon-$ ball contains bundles with slightly more of both goods and any such bundle is strictly preferred to $\mathbf{x}$. Note that this argument actually shows the slightly stricter property of monotonicity. Alternatively, one could draw a diagram with indifference curves and shade in elements of the $\varepsilon-$ ball which are preferred to $\mathbf{x}$, but to be rigorous enough should show this for points on both types of indifference curve: $u = 0$ and $u > 0$.

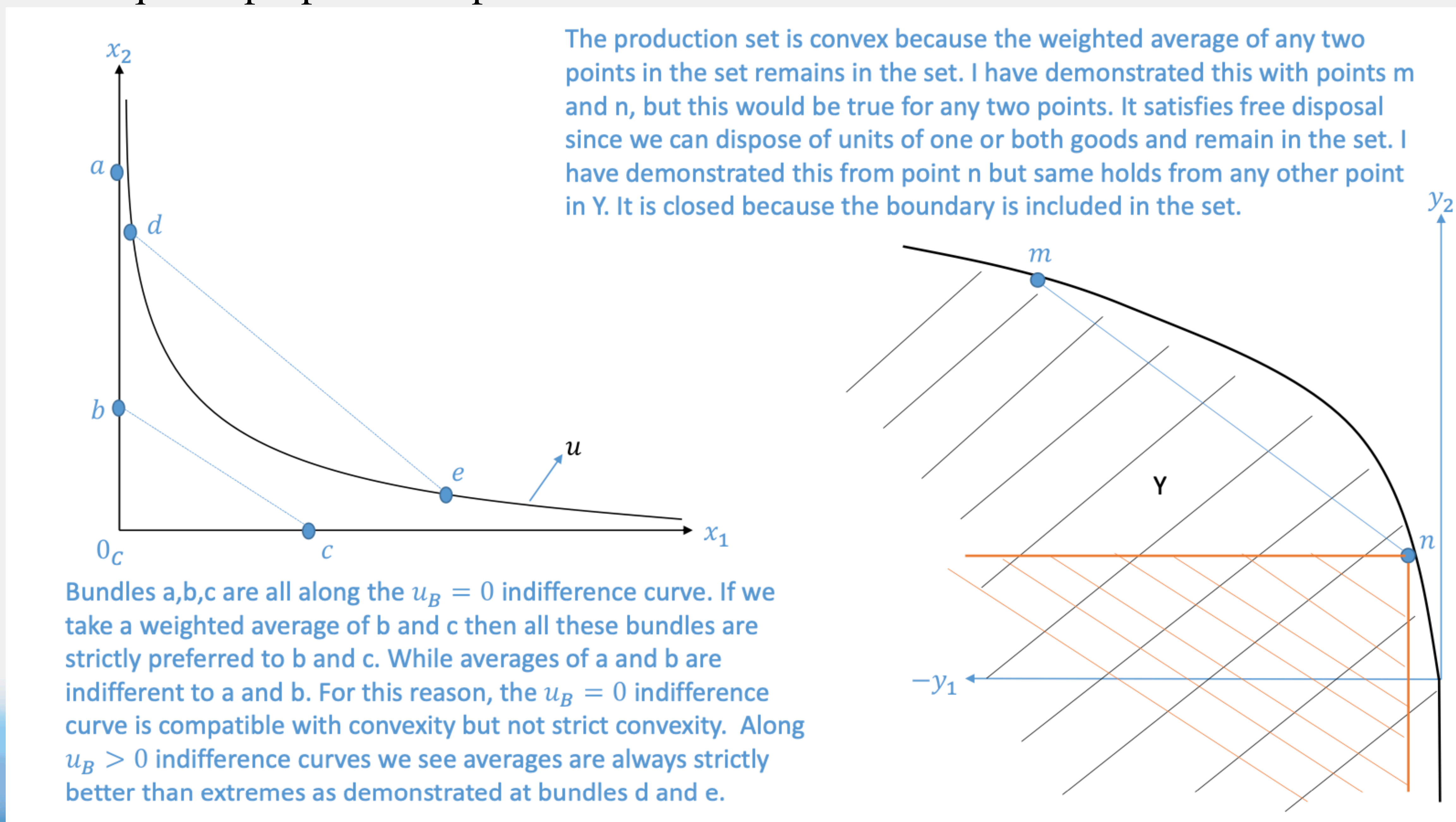
Since we have shown local non-satiation, we can conclude by the First Welfare Theorem, that all Walrasian Equilibria are Pareto efficient. As there is only one Pareto efficient allocation, this means that if a Walrasian Equilibrium exists, it must be at $(x_1, x_2) = (6, 6)$.

d) To apply the 2nd Welfare Theorem we need:

- Preferences are convex, continuous, locally non-satiated.

- Production sets are convex, closed and satisfy free disposal.

Local non-satiation has already been shown. Continuity is immediate as we have a continuous utility function. Convexity can be seen on the diagram below. We can see that the upper level sets are convex, or if we take any two bundles on the same indifference curve, the average of those bundles is weakly preferred to the original bundle. The diagram below also justifies that the three required properties of production sets hold too.



Since we have justified all the necessary assumptions, we can apply the 2nd Welfare Theorem: Every Pareto efficient allocation can be supported as a Walrasian Equilibrium for some reallocation of resources. Here there is only one Pareto efficient allocation - what makes Crusoe best off, which happens when $(x1,x2) = (6,6)$. We also only have one possible allocation of resources - since there is only one consumer, Crusoe must own all the resources and the firm. Therefore starting from this allocation, we must have a Walrasian Equilibrium at the Pareto efficient allocation where $(x1, x2) = (6, 6)$.

e) For Walrasian Equilibrium, we need i) firm profit maximises, ii) Crusoe chooses optimal demand, iii) markets clear.

Firstly consider firm profit maximising: The firm's profit maximisation problem is $\max_{\mathbf{y}\in Y} \mathbf{p}.\,\mathbf{y}$. Letting the amount of input be k and substituting $y_1 = -k$ and $y_2 = 2\sqrt{k}$ we can solve this as follows:

$$\pi = \mathbf{p}\cdot\mathbf{y} = -p_1 k + 2p_2\sqrt{k}$$
$$\frac{d\pi}{dk} = 0 \iff -p_1 + p_2 k^{-\frac{1}{2}} = 0$$
$$\iff k = \left(\frac{p_2}{p_1}\right)^2$$
$$\iff \mathbf{y}(\mathbf{p}) = \left(-\left(\frac{p_2}{p_1}\right)^2, \frac{2p_2}{p_1}\right)$$

(We can argue that the first order condition is sufficient either by π being concave or by drawing a diagram and seeing that our maximum lies where the iso-profit line is tangential to the boundary of the production set.) The profit can be found by subbing $\mathbf{y}(\mathbf{p})$ back into the profit function:

$$\pi = \mathbf{p}\cdot\mathbf{y} = -p_1\left(\frac{p_2}{p_1}\right)^2 + p_2\left(\frac{2p_2}{p_1}\right) = \frac{p_2^2}{p_1}$$

Crusoe maiximises utility subject to budget constraint so solves:

$$\max_{\mathbf{x}\in\mathbb{R}^2_{\geq 0}} u = x_1^{\frac{1}{10}} x_2^{\frac{2}{10}} \text{ subject to } p_1x_1 + p_2x_2 \leq 10p_1 + 2p_2 + \frac{p_2^2}{p_1}$$

Crusoe maximises utility by spending $\frac{1}{3}$ of his income on good 1 and $\frac{2}{3}$ on good 2. This gives

$$\mathbf{x}(\mathbf{p}) = \left(\frac{10p_1 + 2p_2 + \frac{p_2^2}{p_1}}{3p_1}, \frac{2\left(10p_1 + 2p_2 + \frac{p_2^2}{p_1}\right)}{3p_2} \right)$$

As we already know what the Walrasian Equilibrium allocation should be: we can just check that at this allocation, there are prices satisfying conditions i) to iii). We know we need $(x_1, x_2) = (6, 6)$ and so by market clearing we need $(y_1, y_2) = (-4, 4)$. This implies that $\mathbf{p} = (1, 2)$. The last thing to check is that subbing $\mathbf{p} = (1, 2)$ into $\mathbf{x}(\mathbf{p})$ gives $(x_1, x_2) = (6, 6)$

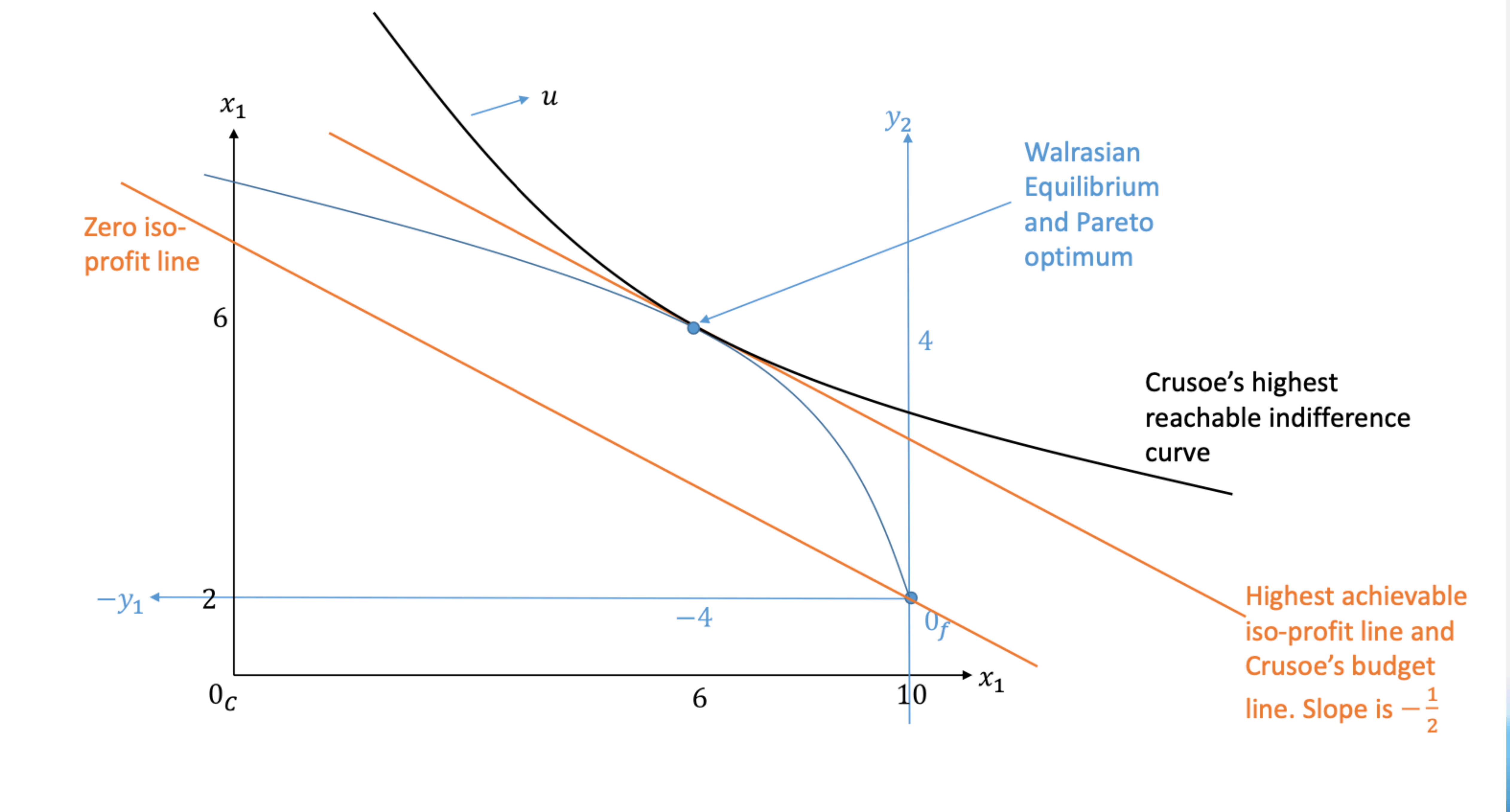
$$\mathbf{x}(\mathbf{p}) = \left(\frac{10p_1 + 2p_2 + \frac{p_2^2}{p_1}}{3p_1}, \frac{2\left(10p_1 + 2p_2 + \frac{p_2^2}{p_1}\right)}{3p_2} \right)$$

$$\implies \mathbf{x}(1, 2) = \left(\frac{10 + 4 + 4}{3}, \frac{2(10 + 4 + 4)}{6} \right) = (6, 6)$$

Thus we have confirmed our Walrasian Equilibrium:

$$\mathbf{p} = (1, 2) \quad \mathbf{x} = (6, 6) \quad \mathbf{y} = (-4, 4)$$

By the First Welfare Theorem, this is the unique Walrasian Equilibrium allocation and therefore also the unique price ratio. Although students could check this by solving for Walrasian Equilibrium the same way as in Lecture 4 and Problem Set 4 - by writing market clearing conditions in terms of x(p) and y (p), normalising p1 = 1 and calculating p2 = 2.



In-class Question

Q4. Repeat Q3 but with changing the preferences and production technology to:
- Let Crusoe have preferences represented by $u(x_1, x_2) = 2x_1 + x_2$.
- Let Crusoe's production technology be the ability to transform k units of good 1 into $2k$ units of good 2 .

Solution:

a) If Crusoe spends k units of time working to produce 2k units of good 2 , relative to the initial endowment $(k=0)$, he loses $2k$ units of utility from good 1 and gains $2k$ units of utility from good 2 and so is indifferent between any such bundle. Thus anywhere along the boundary of the feasible set is Pareto efficient. Diagramatically the boundary of the feaisble set (production frontier) and the indifference curves of Crusoe are both straight lines of slope -2 and thus anywhere along the production frontier is a Pareto optimum. Thus the Pareto Set is

$$\left\{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \mid x_1 = 10 - k, x_2 = 2 + 2k, k \in [0,1]\right\}$$

b) The firm has production set

$$Y = \left\{\mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0, y_2 \leq -2y_1\right\}$$

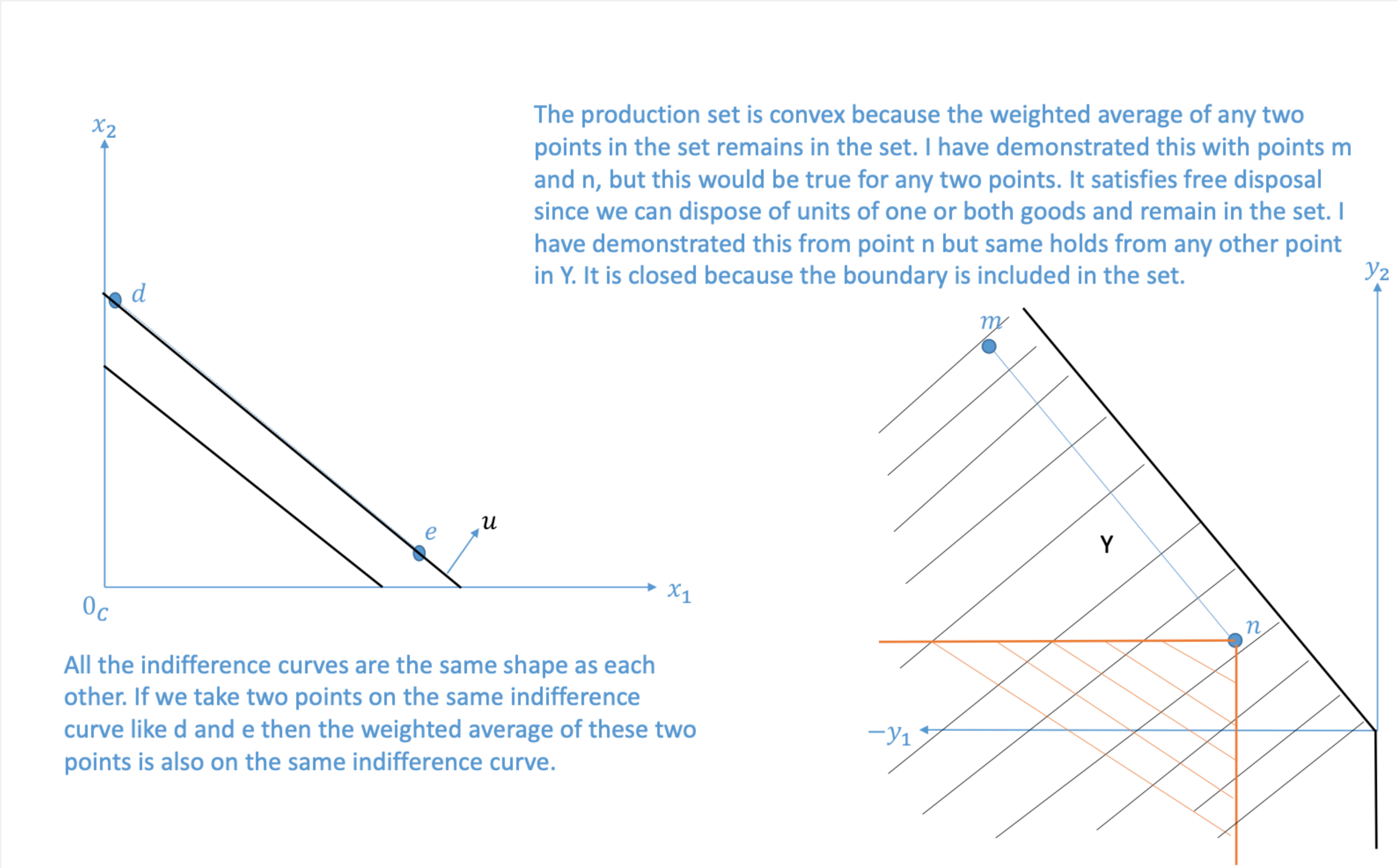
c) To apply the First Welfare Theorem, we need to show that preferences satisfy local non-satiation. We only have one utility function: $u(x_1, x_2) = 2x_1 + x_2$. Take any $\mathbf{x} \in \mathbb{R}^2_{\geq 0}$, and consider an $\varepsilon-$ ball around $\mathbf{x}$. For any $\varepsilon > 0$, the $\varepsilon-$ ball contains bundles with slightly more of both goods and any such bundle is strictly preferred to $\mathbf{x}$. Note that this argument actually shows the slightly stricter property of monotonicity. Alternatively, one could draw a diagram with indifference curves and shade in elements of the $\varepsilon-$ ball which are preferred to $\mathbf{x}$.

Since we have shown local non-satiation, we can conclude by the First Welfare Theorem, that all Walrasian Equilibria are Pareto efficient and so lies in the set identified in a).

d) To apply the 2 nd Welfare Theorem we need:

- Preferences are convex, continuous, locally non-satiated.
- Production sets are convex, closed and satisfy free disposal.

Local non-satiation has already been shown. Continuity is immediate as we have a continuous utility function. Covexity just holds since for any two bundles on the same indifference curve, the averages of those two bundles also lies on that indifference curve. We can also see that the upper level sets are convex. The diagram below also justifies that the three required properties of production sets hold too.



Since we have justied all the necessary assumptions, we can apply the 2nd Welfare Theorem: Every Pareto ecient allocation can be supported as a Walrasian Equilibrium for some reallocation of resources. Here there are innitely many Pareto ecient allocations as found in a). But still only one possible allocation of resources - since there is only one consumer, Crusoe must own all the resources and the rm. Therefore starting from this allocation, we must have the whole set of Pareto ecient allocations as Walrasian Equilibria.

e) In solving the profit maximisation problem, we have 3 cases, depending on the slope of the iso-profit lines compared to the slope of the production frontier: When the iso-profit lines are shallower than the production frontier $\left(\frac{p_1}{p_2} < 2\right)$, there is no solution as profits keep increasing as we increase production. When the iso-profit lines are steeper than the production frontier $\left(\frac{p_1}{p_2} > 2\right)$, the profit maximising output occurs at the firm's origin - ie doing nothing. When they are the same slope as each other, all points along the production frontier lie along the same iso-profit line (the zero iso-profit line) and the firm can choose any one of these. Thus we get:

$$\mathbf{y}(\mathbf{p}) = \begin{cases} \varnothing & \frac{p_1}{p_2} < 2 \\ \{\mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0, y_2 = -2y_1\} & \frac{p_1}{p_2} = 2 \\ \mathbf{0} & \frac{p_1}{p_2} > 2 \end{cases}$$

Crusoe maiximises utility subject to budget constraint so solves:

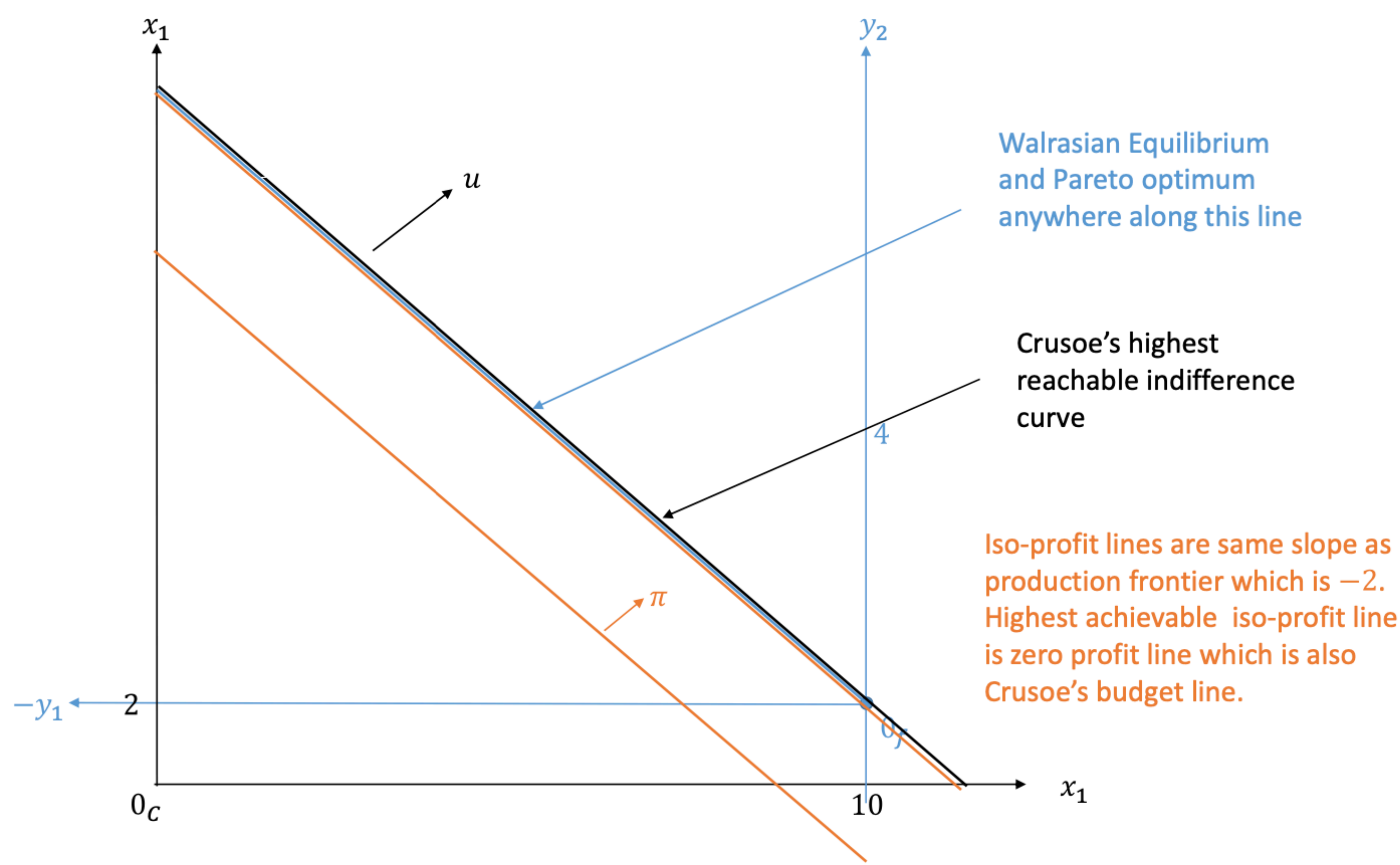
$$\max_{\mathbf{x} \in \mathbb{R}^2_{\geq 0}} u = 2x_1 + x_2 \text{ subject to } p_1 x_1 + p_2 x_2 \leq 10p_1 + 2p_2$$

Crusoe maximises utility by spending his entire income on whichever good gives the higher bang per buck: when $\frac{p_1}{p_2} < 2$ this is good 1 ; when $\frac{p_1}{p_2} > 2$ this is good 2 ; when $\frac{p_1}{p_2} = 2$ the bang per buck of each good is always the same and so any allocation along the budget line is a solution. So we get:

$$\mathbf{x}(\mathbf{p}) = \begin{cases} \left\{ \left(\frac{10p_1+2p_2}{p_1}, 0 \right) \right. & \frac{p_1}{p_2} < 2 \\ \lambda\left(\frac{10p_1+2p_2}{p_1}, 0 \right) + (1-\lambda)\left(0, \frac{10p_1+2p_2}{p_2}\right) \mid \lambda \in [0,1] \Big\} & \frac{p_1}{p_2} = 2 \\ \left(0, \frac{10p_1+2p_2}{p_2}\right) & \frac{p_1}{p_2} > 2 \end{cases}$$

Looking for Walrasian Equilibrium in the three different cases: i) $\frac{p_1}{p_2} < 2$ is impossible due to the firm having no profit maximising output. iii) $\frac{p_1}{p_2} > 2$ is impossible because we then get excess demand of good 2 and excess supply of good 1. In case ii) Both the firm and Crusoe have infinitely many points solving their optimisation problems, but as long as they take compatible actions with each other we get a Walrasian Equilibrium. That is for each $k \in [0, 10]$ there is an Equilibrium of the following form where the firm demands k units of labour and Crusoe demands $10 - k$ units of leisure:

$$\mathbf{p} = (2, 1) \quad \mathbf{x} = (10 - k, 2 + 2k) \quad \mathbf{y} = (-k, 2k)$$



x_1
y_2
u
Walrasian Equilibrium
and Pareto optimum
anywhere along this line
Crusoe's highest
reachable indifference
curve
4
Iso-profit lines are same slope as
production frontier which is -2.
Highest achievable iso-profit line
is zero profit line which is also
Crusoe's budget line.
π
$-y_1$
2
0_C
10
x_1

In-class Question

Q5. Repeat Q3 but with changing the preferences and production technology to:
- Let Crusoe have preferences represented by $u(x_1, x_2) = \min\{2x_1, x_2\}$.
- Let Crusoe's production technology be the ability to transform k units of good 1 into $\frac{5k^2}{8}$ units of good 2 .

a) Crusoe can spend k units of time working to produce $\frac{5k^2}{8}$ units of good 2 . Crusoe will optimise where he does not freely dispose (waste) either good and so letting $k \geq 0$ be the amount of time Crusoe devotes to labour, we solve

$$\max_{k \geq 0} u = \min\{2x_1, x_2\} \text{ s.t. } x_1 = 10 - k, x_2 = 2 + \frac{5k^2}{8}$$

This has solution at the kink of the indifference curve and so where

$$\begin{aligned} 2x_1 = x_2 &\iff 2(10-k) = 2 + \frac{5k^2}{8} \\ &\iff 0 = \frac{5k^2}{8} + 2k - 18 \\ &\iff 5k^2 + 16k - 144 = 0 \\ &\iff (k-4)(5k+36) = 0 \end{aligned}$$

So our solution is $k = 4$ and so we get our Pareto optimum:

$$(x_1, x_2) = (6, 12)$$

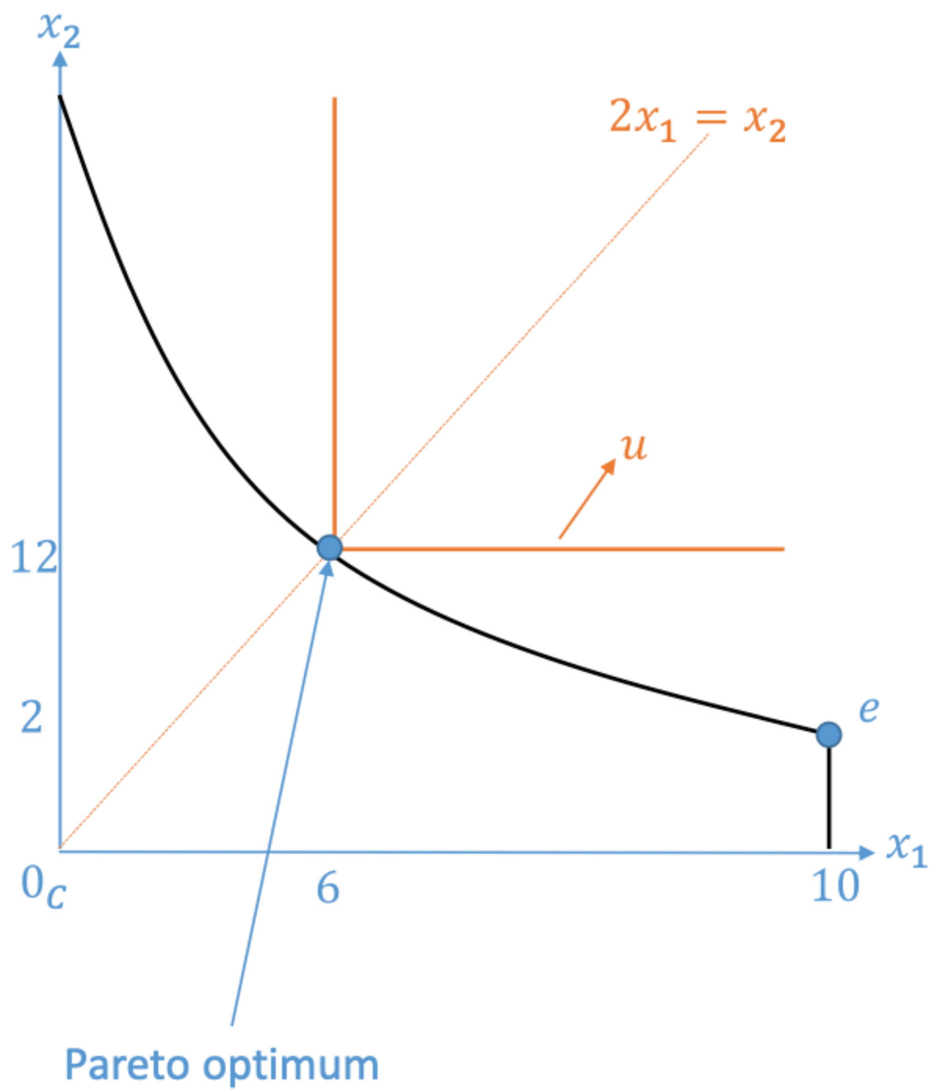
b) The firm has production set

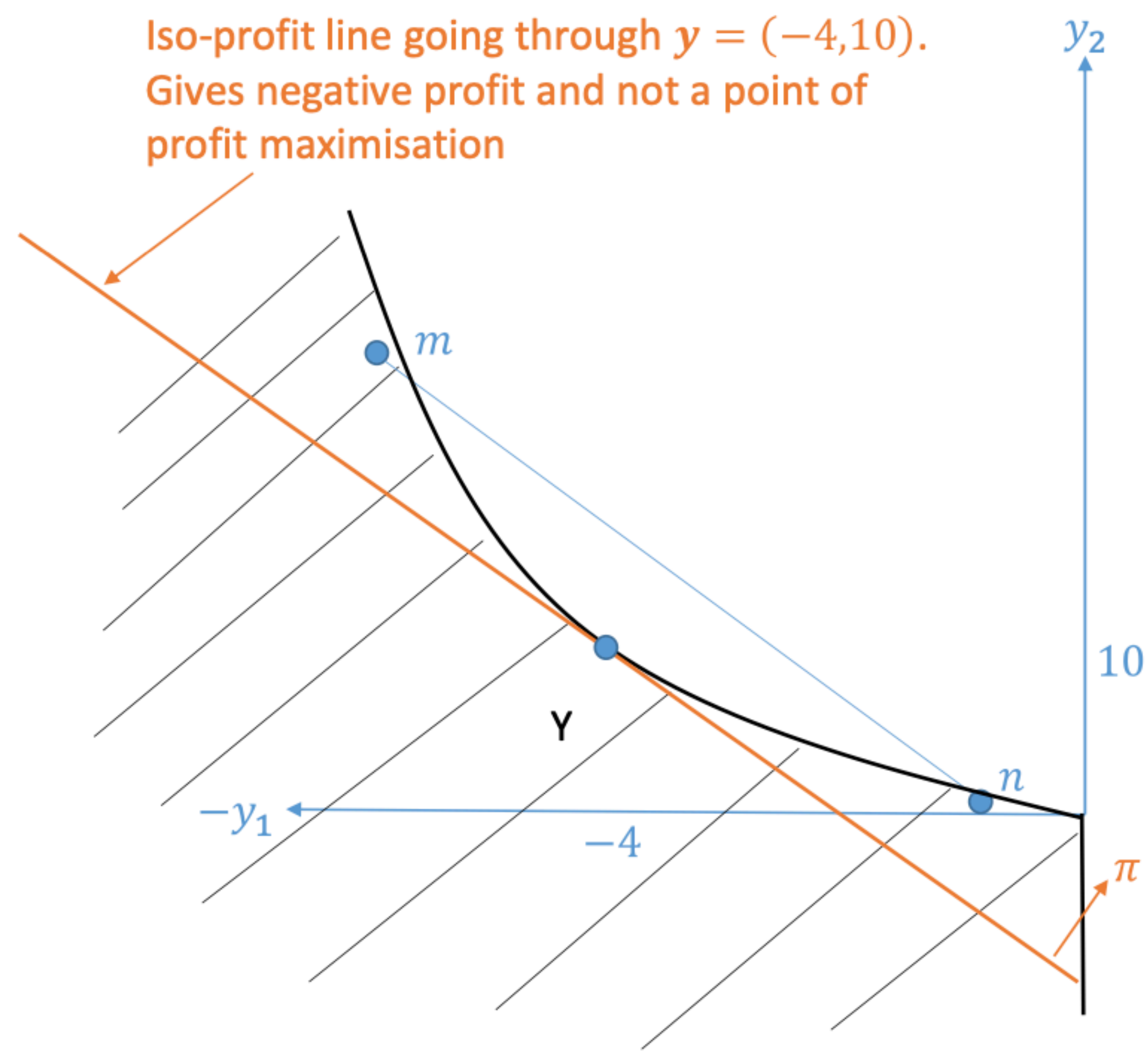
$$Y = \left\{ \mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0, y_2 \leq \frac{5(-y_1)^2}{8} \right\}$$

c) To apply the First Welfare Theorem, we need to show that preferences satisfy local non-satiation. We only have one utility function: $u(x_1, x_2) = \min\{2x_1, x_2\}$. Take any $\mathbf{x} \in \mathbb{R}^2_{\geq 0}$, and consider an $\varepsilon-$ ball around $\mathbf{x}$. For any $\varepsilon > 0$, the $\varepsilon-$ ball contains bundles with slightly more of both goods and any such bundle is strictly preferred to $\mathbf{x}$. Note that this argument actually shows the slightly stricter property of monotonicity. Alternatively, one could draw a diagram with indifference curves and shade in elements of the $\varepsilon-$ ball which are preferred to $\mathbf{x}$.
Since we have shown local non-satiation, we can conclude by the First Welfare Theorem, that all Walrasian Equilibria are Pareto efficient. As there is only one Pareto efficient allocation, this means that if a Walrasian Equilibrium exists, it must be at $(x_1, x_2) = (6, 12)$.

d) The Second Welfare Theorem cannot be applied because the production set violates convexity.

e) There is no Walrasian Equilibrium. As argued by the First Welfare Theorem, if a Walrasian Equilibrium exists, it must lie at $(x_1, x_2) = (6, 12)$, requiring the firm to produce $(y_1, y_2) = (-4, 10)$ which would not be a profit maximising output of the firm. Thus no Walrasian Equilibria exist. Alternatively one could argue that when $p_2 > 0$ the firm has no profit maximising output. While $p_2 = 0$ would violate market clearing as there would be excess demand of good 2.





Production set Y is not convex. For example m and n are both in Y but weighted averages of them lie outside Y.