EC202 Seminar Week 9

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A Bit of Logistics

- There will be no seminars next week, nor office hours.
- Per James' request, another seminar and corresponding office hours will be held in term 3. Related information will be updated via email (and also Tabula, I suppose).

Some Key Concepts

- Different voting systems: plurality, alternative, plurality runoff
- Voting preference profile and Social Welfare Function
- Dictatorship and Non-Dictatorship
- Unanimity
- Independence of Irrelevant Alternatives
- Arrow's Impossibility Theorem

Theorem 2.1. *Arrow's Impossibility Theorem*

Let the number of voters be $N \geq 2$ *and the number of alternatives be* $|\mathcal{A}| \geq 3$ *then assuming (UD), there is no SWF satisfying (U), (IIA) and (ND)*

- Condorcet winner and loser; Condorcet Cycle
- Single peaked preference

Some Key Concepts

- Black's median voter Theorem

Theorem 2.2. *Black's median voter Theorem*

If preferences are single peaked then the median voter's ideal point is the Condorcet winner.

- Violation of Arrow's Impossibility Theorem
- Strategy Proofness
- Gibbard-Satterthwaite Theorem

Theorem 3.1. *The Gibbard-Satterthwaite Theorem:*

Let the number of voters be $N \geq 2$ *and the number of alternatives be* $|\mathcal{A}| \geq 3$ *then assuming (UD), there is no surjective SCF satisfying (SP) and (ND).*

In-class Question

Q3. Recall that for Arrow's Impossibility Theorem to hold, we need the following 6 things:

i) At least 3 alternatives, $|\mathcal{A}| \geq 3$.

ii) At least two voters, $N \geq 2$.

iii) Unrestricted Domain, (UD)

iv) Unanimity, (U)

v) Independence of Irrelevant Alternatives, (IIA)

vi) Non-dictatorship, (ND)

For each of these six, show why they are needed for Arrow's Impossibility Theorem to hold. In other words find an example of an SWF satisfying the other five.

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i) If only 2 alternatives, plurality, Borda Count, Alternative Vote, pairwise comparison all become the same SWF: we prefer whichever alternative gets the most votes. This SWF satisfies all axioms. As a sidepoint, note that there are other rules too that would work that treat the two alternatives asymmetrically. For example, in Florida to pass constitutional ammendments requires 60% of people to vote for the change over the status quo. Such a voting rule where the threshold is higher than 50% is called a "supermajority".

ii) With $N = 1$, technically Arrow's Theorem would still hold, since if we had just one voter, any SWF satisfying (U) must be a dictatorship where that one voter gets their way for every possible preference profile thy could have. However, having a dictatorship in this instance is no longer objectionable, while the problem of how to aggregate voters' preferences is now trivial and so ceases to be interesting.

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iii) As we mentioned in the lecture notes, Single peaked preferences is a sufficient domain restriction for pairwise majority voting to now satisfy transitivity and hence be an SWF. Pairwise majority will also satisfy (U), (IIA), (ND). To see these each in turn: (U) is satisfied since if everyone prefers x to y, that implies a majority prefer x to y. (IIA) is satisfied by construction: in comparing x and y we only care whether a majority prefer x to y or vice-versa. Thus the only way to change the result of this SWF is by changing preferences over (x, y). Changing how voters feel over (x, z) or (y, z) has no effect. (ND) is satisfied since all voters are treated equally. To give an example, if we consider the VPP where all voters apart from i have the same preferences as each other, but i has different preferences then voter i does not get their way. We can find such a VPP for every $i \in N$ assuming (UD).

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For each of these six, show why they are needed for Arrow's Impossibility Theorem to hold. In other words find an example of an SWF satisfying the other five.

iv) Consider 3 alternatives $\{a, b, c\}$ and the SWF defined as: for every VPP we have the ranking $aPbPc$. This means that regardless of what our voters think, we always get $aPbPc$. This example was for 3 alternatives, but similar would exist for any number of alternatives. This defines an SWF over (UD), since for every VPP it gives a well-defined, deterministic ordering, namely $aPbPc$. It violates (U) since there are VPPs where everybody has different preferences. For example take a VPP where everyone prefers b to a. It satisfies (IIA) trivially: to contradict (IIA) we need to construct an example where the SWF changes its ranking over (x, y) despite no voters changing their ranking over (x, y). That cannot here since the SWF never changes its ranking. It satisfies (ND) for similar reasons to (U). Take a VPP where everyone prefers b to a then nobody is getting their way and so nobody is a dictator.

In-class Question

Q3. Recall that for Arrow's Impossibility Theorem to hold, we need the following 6 things:

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v) Independence of Irrelevant Alternatives, (IIA)

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For each of these six, show why they are needed for Arrow's Impossibility Theorem to hold. In other words find an example of an SWF satisfying the other five.

v) There are numerous examples of this. One is Borda Count, as we showed in Q1b).

vi) Let voter 1 be a dictator. This is a well-defined SWF over (UD) since it always gives a well-defined deterministic ranking, namely voter 1's ranking. It satisifies (U) since if everyone prefers x to y, it means voter 1 prefers x to y and so xPy. It satisfies (IIA) since we cannot construct a counterexample to (IIA): in order to change the SWF's ranking over (x, y), we need to change voter 1's preferences over (x, y). It violates (ND) by definition.

Q4. This question is about the merits of the plurality voting system compared to Alternative Vote. We show some pros and cons of each.

Let there be 3 alternatives and a large number of voters. Each voter has a strict preference ordering so fits into one of the following 6 categories:

	I	II	III	IV	V	VI
1st Preference	a	a	b	b	c	c
2nd Preference	b	c	a	c	a	b
3rd Preference	c	b	c	a	b	a

By varying the percentages of voters in each category, we demonstrate how we can get examples of some of the disadvantages of each system:[1]

a) Consider the voting profiles below.

Profile 1	I	II	III	IV	V	VI
	20%	15%	20%	12%	13%	20%
1st Preference	a	a	b	b	c	c
2nd Preference	b	c	a	c	a	b
3rd Preference	c	b	c	a	b	a

Profile 2	I	II	III	IV	V	VI
	20%	17%	20%	12%	11%	20%
1st Preference	a	a	b	b	c	c
2nd Preference	b	c	a	c	a	b
3rd Preference	c	b	c	a	b	a

a1) Show that under Profile 1 Alternative vote elects a. However, if 2% of the population change their preferences over (a, c) in favour of a, to give Profile 2 then Alternative Vote will elect b instead of a.

a2) To summarise what happened in i), a was winning under Alternative Vote. We changed how voters felt about (a, c) in favour of a without changing how voters feel about (a, b) or (b, c) and as a result a is no longer elected. Explain why plurality could never fall victim to this peculiarity.[2]

b1) Show an example where assuming people vote truthfully, plurality elects a, however if some voters change their preferences over how they rank b compared to c, then plurality would elect b instead.

b2) Comment on your findings in a1) and how this links to the (IIA) axiom for SWFs and (SP) axiom for SCFs.

c) How does Alternative Vote fare with these two axioms?

d) Show that when a Condorcet winner exists, plurality might fail to elect it. And furthermore it could elect a Condorcet loser.

e) Show that when a Condorcet winner exists, Alternative Vote might fail to elect it. However it cannot elect a Condorcet loser.

a1) Under AV using the Profile 1: 1st Round a gets 35%, b gets 32%, c gets 33% so b gets eliminated. In the final round a beats c with 55% to 45% so a is elected.

Under AV using the Profile 2: 1st Round a gets 37%, b gets 32%, c gets 31% so c gets eliminated. In the final round b beats a with 52% to 48% and so b is elected.

In this example a was winning, gained more support and as a result lost, because it gained more support at the expense of the candidate it beats in pairwise comparison and so in the final under AV faced the candidate it loses to in pairwise comparison.

a2) Plurality can never fail this criterion, since if the only change in preferences is to make a more popular then this increases the number of voters it gets, while no other candidate gets more votes.

b1) Examples of this are easy to come by. I give a particularly extreme example to highlight the flaws of plurality. Supppose we have 34% in category I, and 33% in category IV and 33% in category VI. That gives what I have called Profile 3.

Profile 3	I	II	III	IV	V	VI
	34%	0%	0%	33%	0%	33%
1st Preference	a	a	b	b	c	c
2nd Preference	b	c	a	c	a	b
3rd Preference	c	b	c	a	b	a

With these preferences plurality elects a with 34% beating its two rivals who each get 33%. Consider these changed preferences over (b, c) moving some voters from category VI to category IV so that we get

Profile 4	I	II	III	IV	V	VI
	34%	0%	0%	35%	0%	31%
1st Preference	a	a	b	b	c	c
2nd Preference	b	c	a	c	a	b
3rd Preference	c	b	c	a	b	a

In making this change, only the ranking over (b, c) has changed, not over (a, b) or (a, c). But now in profile 4, we get b elected with 35% of the vote.

b2) This can be seen as an example of a failure of (IIA) since in moving from Profile 3 to Profile 4, voter's preferences over (a, b) haven't changed but the social ranking over (a, b) has changed. It also demonstrates why the plurality system fails (SP), in other words, gives voters an incentive to misreport their preferences. Suppose true preferences were as in the top table of b1) then voters in category VI have an incentive to lie and so give us reported preferences like those in the bottom table. Although in this particular example, voters might face coordination problems in practice. If this is an election where the opinion polls are showing such a close race then voters in categories IV and VI might not know whether they should vote tactically to get b elected or c elected. If the number of category IV people who tactically vote for c eqauls the number of category VI people who vote tactically for b then a still gets elected.

c) As we saw in a), Alternative Vote violates both. To show this, consider moving from Profile 1 to Profile 2. In making this change only preferences over (a, c) have changed. Preferences over (a, b) have remained the same and yet the Social ranking now has b top instead of a. This shows that any SWF based on AV must violate (IIA). To see strategy proofness (SP) violated, assume true preferences are as in Profile 2, then a's supporters have an incentive lie and put c top of their ranking so that we get reported preferences like Profile 1 where a gets elected. However, in practice, a's supporters face a co-ordination problem: if too many of them switch to c, then a would not get enough votes to reach the final round.

d) In Profile 3 the Condorcet winner is b but plurality elects a. In fact by electing a plurality elects the Condorcet loser, since 66% of people think this is the worst alternative and so it loses pairwise comparisons to both b and c.

e) In Profile 1 the Condorcet winner is b but AV elects a. However AV cannot elect a Condorcet loser, because the winner under AV has to win in the final round where it faces another alternative in pairwise comparison.