EC202 Term 1 Problem set 1 Solutions

October 16, 2022

Pre-class questions

These are not covered in seminars unless time permits. They are here to give you extra practice. Solutions will be provided.

Q1. For each of the following preferences, say whether it satisfies: i) convexity, ii) strict convexity, iii) strong monotonicity, iv) monotonicity, v) local non-satiation.

a) Perfect substitutes defined over domain $\mathbb{R}^2_{\geq 0}$: Preferences are represented by $u : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ given by $u(x_1, x_2) = x_1 + x_2$.

b) Perfect complements defined over domain $\mathbb{R}^2_{\geq 0}$: Preferences are represented by $u : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ given by $u(x_1, x_2) = \min\{x_1, x_2\}$.

c) Cobb-Douglas defined over domain $\mathbb{R}^2_{>0}$: Preferences are represented by $u : \mathbb{R}^2_{>0} \to \mathbb{R}$ given by $u(x_1, x_2) = x_1^a x_2^{1-a}$ for some $a \in (0, 1)$.

d) Cobb-Douglas defined over domain $\mathbb{R}^2_{\geq 0}$: Preferences are represented by $u : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ given by $u(x_1, x_2) = x_1^a x_2^{1-a}$ for some $a \in (0, 1)$.

e) Only care about good 1 over domain $\mathbb{R}^2_{\geq 0}$: Preferences are represented by $u : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ given by $u(x_1, x_2) = x_1$. (Note this fits the Cobb-Douglas equation with $a = 1$.)

f) Indifferent between all bundles in $\mathbb{R}^2_{\geq 0}$: Preferences are represented by $u : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ given by $u(x_1, x_2) = c$, for some constant $c \in \mathbb{R}$.

g) Lexicographic preferences over domain $\mathbb{R}^2_{\geq 0}$. That is $(x_1, x_2) \succeq (y_1, y_2)$ iff $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$. That is we primarily look at how much of good 1 the consumer has and if equal then look at the amount of good 2.

Solution: For parts a) to e) you can draw indifference curves and justify the following:

	Property					
		i)	ii)	iii)	iv)	v)
Preferences	a)	yes	no	yes	yes	yes
	b)	yes	no	no	yes	yes
	c)	yes	yes	yes	yes	yes
	d)	yes	no	no	yes	yes
	e)	yes	no	no	yes	yes

For part f) the whole space $\mathbb{R}^2_{\geq 0}$ is on the same indifference curve, while in g) every bundle is in a separate indifference curve. For both of these parts, you need to pay attention to the precise mathematical definitions (Lecture 1 Definitions 3.8 to 3.12) instead of just the intuitive idea of the properties.

f) This satisfies convexity quite trivially since we have weak preference between any two bundles we have weak preference. The other four properties are violated as we never have strict preference.

g) This satisfies all 5 properties.

Q2. Show that preferences being strongly monotone implies monotonicity which in turn implies local non-satiation. Give counter-examples to show the converse implications do not hold.

Solution:

Strong monotonicity implies monotonicity since it imposes a restriction on more pairs of bundles. To see this, note

$$x_j < \hat{x}_j \ \forall j \implies x_j \leq \hat{x}_j \ \forall j \text{ and } \exists j \text{ s.t. } x_j < \hat{x}_j$$

For examples of monotone but not strongly monotone, consider the preferences in Q5a) when $\alpha \in \{0, 1\}$ or the preferences in Q5c).

Monotonicity implies local non-satiation since it imposes a restriction on more pairs of bundles. To see this, note that for any bundle x and any $\varepsilon > 0$, the ε-neighbourhood of x will always contain bundles which have slightly more of every commodity. For an example of locally non-satiated but not monotone preferences, consider one of the commodities being a “bad” instead of a “good”. For example $u(x_1, x_2) = x_1 - x_2$.

Q3. Consider the equation $x + 2 = \sqrt{4 - x}$. Solve this equation using your algebraic skills, writing $\implies$ or $\iff$ between each line. You should be able to justify why $x = 0$ is the unique solution and in particular why $x = -5$ is not a solution.

Solution:

$$\begin{aligned} x+2=\sqrt{4-x} &\Longrightarrow (x+2)^2=4-x \\ &\Longleftrightarrow x^2+4x+4=4-x \\ &\Longleftrightarrow x^2+5x=0 \\ &\Longleftrightarrow x\,(x+5)=0 \\ &\Longleftrightarrow x\in\{0,-5\} \end{aligned}$$

This tells us

$$x+2=\sqrt{4-x}\Longrightarrow x\in\{0,-5\}$$

Although, note the implication only goes one way. Taking the contrapositive of this we get

$$x\notin\{0,-5\}\Longrightarrow x+2\neq\sqrt{4-x}$$

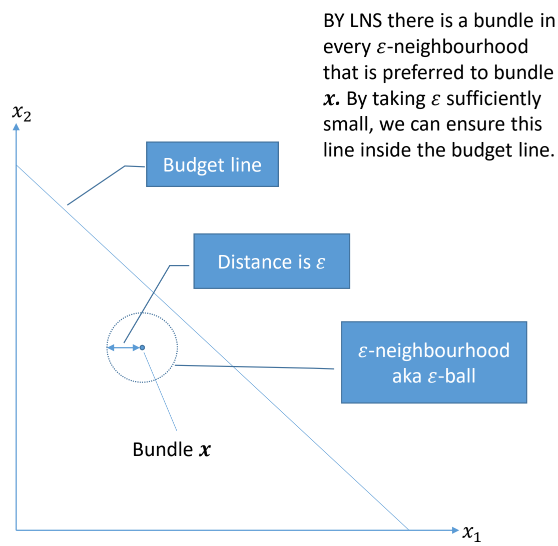
Thus only $x=0$ or $x=-5$ could be solutions to $x+2=\sqrt{4-x}$. To check whether $x=0$ or $x=-5$ are actually solutions, we plug these numbers into the equation and find that only $x=0$ is a solution.

$$x+2\mid_{x=0}=\sqrt{4-x}\mid_{x=0}\Longleftrightarrow 2=2$$

$$x+2\mid_{x=-5}=\sqrt{4-x}\mid_{x=-5}\Longleftrightarrow -3=3$$

Q4. Lemma 3.1 of Lecture 1 says "If preferences satisfy local non-satiation then the consumer must expend all her budget to maximise utility." Explain why we need local non-satiation for this result to hold.

Solution: I show this for the 2 good case. The general J good case is conceptually the same.



The diagram above shows that a bundle $\mathbf{x}$which lies inside the budget line, ie where consumer doesn't expend all her budget, cannot maximixse utility, hence showing the statement in the question. To see why LNS is needed, consider the following two examples which both have preferences violating LNS and have optimal bundles where not all income is expended:

1. $u(\mathbf{x}) = 0$ or any other constant. This is the preference relation saying that all bundles are indifferent to all other bundles. Here every bundle solves the UMP regardless of whether they spend all income or not.
2. $u(x_1, x_2) = -(x_1 - 2)^2 - (x_2 - 2)^2$ and budget set given by $(p_1, p_2) = (1, 1)$ and $M = 10$. Here the unique optimal bundle is $(x_1, x_2) = (2, 2)$.

In-class questions

Q5. Consider a student deciding on their housing choices. We model this as having a budget of M to split between two goods: the first being accommodation, where the more luxurious a place the student rents, the more they have to pay, and the second being the composite good (ie money to spend on all other goods). We let h denote units of housing quality. A basic model could measure this in square metres or a more sophisticated measurement would include things like condition of the house, location, amenities etc. Let g be units of the composite good, that is money to spend on all other things. Let the price of housing be p_h per unit and the price of the composite good be 1.

We let our consumption set be $X = \mathbb{R}^2_{\geq 0}$. For each of the following utility functions $u : X \to \mathbb{R}$, draw indifference curves and budget constraint, write down the utility maximisation problem and solve it.

a) $u(g,h) = g^{\alpha}h^{1-\alpha}$ for some exogeneous $\alpha \in [0,1]$. (Hint you might want to first consider the cases $\alpha = 0$ and $\alpha = 1$ and note $0^0 = 1$.)

b) $u(g,h) = \begin{cases} \alpha \ln g + (1-\alpha)\ln h & g,h > 0 \\ -\infty & g = 0 \text{ or } h = 0 \end{cases}$ for some exogeneous $\alpha \in (0,1)$

c) $u(g,h) = \min\{\alpha g, h\}$ for some exogeneous $\alpha > 0$.

d) $u(g,h) = \alpha g + h$ for some exogeneous $\alpha > 0$.

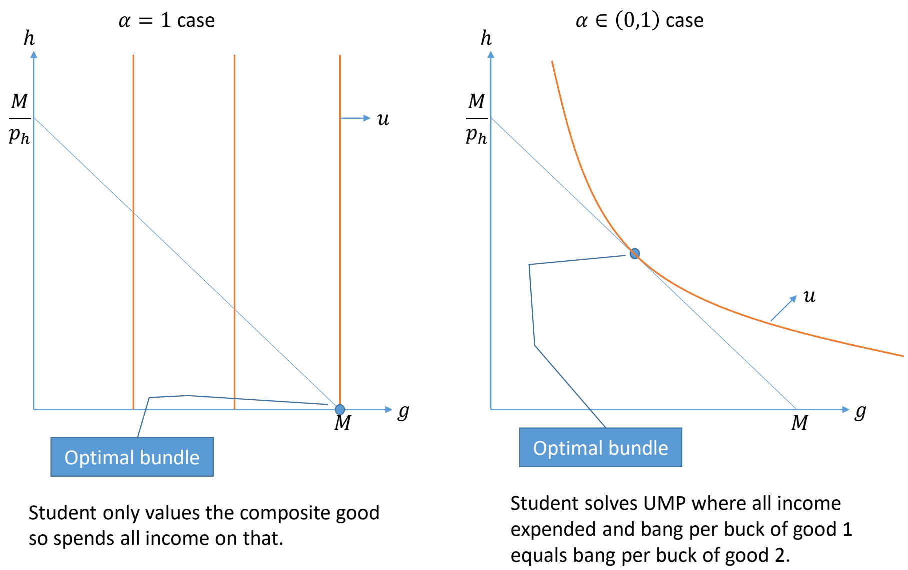
e) $u(g,h) = g + h^{\alpha}$ where $\alpha \in \left\{\frac{1}{2}, 2\right\}$.

Solution:

In all parts, the UMP is

$$\max_{(g,h)\in\mathbb{R}^2_{\geq 0}} u(g,h) \text{ s.t. } p_h h + g \leq M$$

a) We have 3 different cases depending on α. The below diagram displays 2 of them:



The $\alpha = 1$ case can be seen from the diagram. The optimal bundle is $(g,h)^* = (M, 0)$.

The $\alpha = 0$ case is conceptually similar: now the studegnt only cares about housing and so indifference curves are horizontal and so the optimal bundle is $(g,h)^* = \left(0, \frac{M}{p_h}\right)$.

For $\alpha \in (0,1)$ we set

$$\begin{aligned}\frac{MU_h}{p_h} = \frac{MU_g}{p_g} &\iff \frac{(1-\alpha)\, g^{\alpha} h^{-\alpha}}{p_h} = \frac{\alpha g^{\alpha-1} h^{1-\alpha}}{1} \\ &\iff (1-\alpha)\, g = \alpha h p_h\end{aligned}$$

We can substitute this back into $p_h h + g = M$ to get the optimal bundle is

$$(g,h)^* = \left(M\alpha, \frac{M\,(1-\alpha)}{p_h}\right)$$

Note that we could have also used MRS equals price ratio or the Langrangian to get the same result. These are called Cobb-Douglas preferences and have the property that the student spends proportion α of their income on good g and proportion $1-\alpha$ on good h.

b) These preferences are simply an increasing transformation of those in part a) since $f(x) = \ln x$ is an increasing function and $\ln\left(g^{\alpha} h^{1-\alpha}\right) = \alpha \ln g + (1-\alpha)\ln h$ when $g, h > 0$, while the second line of the utility function ensures that if either g or h is 0 then this is the lowest possible utility, just as in the preferences in a). We could check this by calculating we get the same answer from equating bang per buck of each good:

$$\begin{aligned}\frac{MU_h}{p_h} = \frac{MU_g}{p_g} &\iff \frac{(1-\alpha)}{h p_h} = \frac{\alpha}{g} \\ &\iff (1-\alpha)\, g = \alpha h p_h\end{aligned}$$

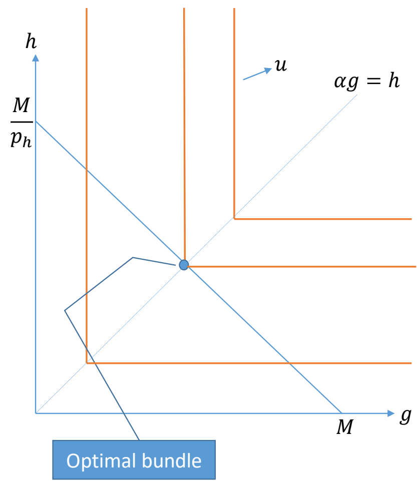
c) As seen in the diagram below the optimal bundle satisfies

$$\alpha g = h \text{ and } p_h h + g = M$$

Solving these two equations simultaneously gives

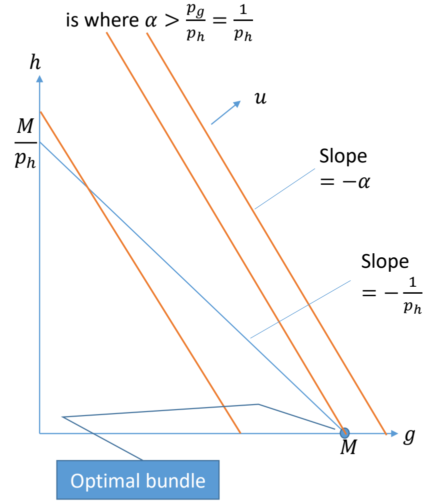
$$(g,h)^* = \left(\frac{M}{1+\alpha p_h}, \frac{\alpha M}{1+\alpha p_h}\right)$$

c) Perfect complements



Optimal bundle lies at kink of indifference curve, on budget line.

d) Perfect substitutes. Case drawn is where $\alpha > \frac{p_g}{p_h} = \frac{1}{p_h}$



Maximise utility by spending all income on the good with higher bang per buck.

d) Here the indifferences curve are linear and so of the same slope everywhere along them. We get 3 different cases depending on whether the slope of the indifference curve is steeper, shallower or equal to the slope of the budget constraint. The diagram above shows the case when $\alpha > \frac{1}{p_h}$ and we can see from the diagram that the optimal bundle is $(g, h)^* = (M, 0)$.

The case when the indifference curve is shallower than the budget line is conceptually similar and would lead us to the optimal bundle at $(g, h)^* = \left(0, \frac{M}{p_h}\right)$.

Finally, the indifference curve could be the same slope as the budget line. Here the budget line and the highest chievable indifference curve exactly coincide and so utility is maximised anywhere along the indifference curve.

All of these results can be seen clearly upon drawing the relevant diagram, although another way to approach this is to compare the bang per buck the consumer gets from spending on each good.

$$\frac{MU_g}{p_g} \geq \frac{MU_h}{p_h} \iff \alpha \geq \frac{1}{p_h}$$

So when $\alpha > \frac{1}{p_h}$, the bang per buck on the composite good is higher and so this is why the consumer spends all income on that good. For $\alpha < \frac{1}{p_h}$ the bang per buck on housing is greater and so the consumer spends all income on that. When $\alpha = \frac{1}{p_h}$ the bang per buck on each good is the same and so the consumer can spend all their money on either one or a mixture of the two.

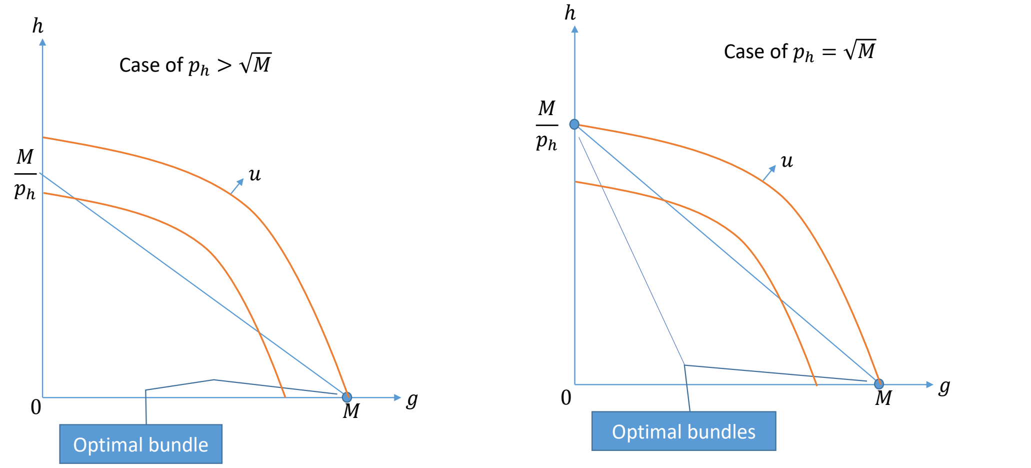
e) First the $\alpha = 2$ case. Here the indifference curves are curved in the opposite direction to normal and so equating slope of indifference curve to slope of budget line or equivalent method would be solving a utility minimisation instead of maximisation problem. But instead we can see the solution graphically. Observe that only the corners of the budget set, $(g, h) = (M, 0)$ or $\left(0, \frac{M}{p_h}\right)$ can be optimal bundles. So we simply compare their utilities: $u(M, 0) = M$, while $u\left(0, \frac{M}{p_h}\right) = \left(\frac{M}{p_h}\right)^2$. This brings us to the following results:

If $p_h > \sqrt{M}$ then $(g, h)^* = (M, 0)$.

If $p_h < \sqrt{M}$ then $(g, h)^* = \left(0, \frac{M}{p_h}\right)$.

If $p_h = \sqrt{M}$ then both $\left(0, \frac{M}{p_h}\right)$ and $(M, 0)$ are optimal bundles. Two of these three cases are depicted below. Students should be able to use these diagrams to draw the diagram for the $p_h < \sqrt{M}$ case.

e) $\alpha = 2$



Indifference curve is curved the opposite way from normal. The two corners of budget set are the candidates for optimal bundles. You should check utility at each and compare.

It is also possible for both corners to give the same utility as each other and so for both to be optimal bundles. Although this is unlikely and only happens at one very specific price.

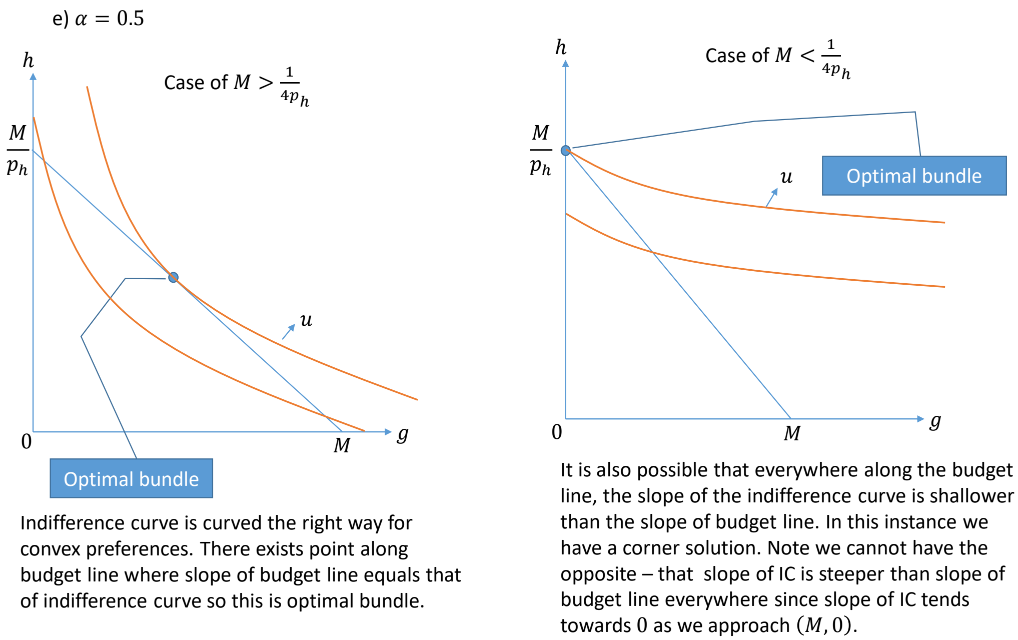
Next, the $\alpha = \frac{1}{2}$ case. Here preferences are convex and so we can obtain our correct result by equating slope of budget line with slope of budget constraint. However, as we will see below, it is possible that the slope of the indifference curve could always be shallower than the slope of the budget constraint, leading to a corner solution. In my view, the most intutitive way to look at this is by considering the bang per buck of each good:

$$\frac{MU_g}{p_g} \geq \frac{MU_h}{p_h} \iff 1 \geq \frac{1}{2p_h\sqrt{h}}$$
$$\iff h \geq \frac{1}{4p_h^2}$$

So we also have

$$\frac{MU_g}{p_g} < \frac{MU_h}{p_h} \iff h < \frac{1}{4p_h^2}$$

This tells us the consumer consumes only good h up until the point that $h = \frac{1}{4p_h^2}$, which costs $\frac{1}{4p_h}$, and thereafter consumes only good g with any remaining income. Thus if $M < \frac{1}{4p_h}$ then the consumer doesn't get any good g and so we have a corner solution. While if $M > \frac{1}{4p_h}$, the consumer will get both goods and we have an interior solution where slope of indifference curve equals slope of budget constraint. So we have 2 cases as depicted below:



From the diagram we can see that if $M < \frac{1}{4p_h}$ then $(g, h)^* = \left(0, \frac{M}{p_h}\right)$.

If $M > \frac{1}{4p_h}$ then the solution is where the consumer spends all their income and the bang per buck of the two goods is the same. So we solve the following two simultaneous equations:

$$h = \frac{1}{4p_h^2} \text{ and } p_h h + g = M$$

This gives the solution $(g, h)^* = \left(M - \frac{1}{4p_h}, \frac{1}{4p_h^2}\right)$.

Finally, to mention the $M = \frac{1}{4p_h}$ case. Here the consumer spends all income on h but we also have the that slope of indifference curve equals slope of budget line at this point. Therefore we could describe it by either equation. Note that

$$M = \frac{1}{4p_h} \implies \left(M - \frac{1}{4p_h}, \frac{1}{4p_h^2}\right) = \left(0, \frac{M}{p_h}\right)$$

Q6. Let $X = \mathbb{R}^3_{\geq 0}$ Consider perfect substitutes in the 3 good case: $u : X \to \mathbb{R}$ is defined by $u(x_1, x_2, x_3) = \alpha x_1 + \beta x_2 + x_3$ for some exogeneous $\alpha > 0$, $\beta > 0$. Find the optimal bundle as a function of income and prices.

Solution: Preferences are convex (just about as linear) so we get correct answer by considering bang per buck of each good:

$$\frac{MU_1}{p_1} = \frac{\alpha}{p_1} \quad \frac{MU_2}{p_2} = \frac{\beta}{p_2} \quad \frac{MU_3}{p_3} = \frac{1}{p_3}$$

Our consumer spends all their income on whichever good has the highest bang per buck. If two or more goods are tied for highest then the consumer can do any mixture amongst those goods.

Post-class question

Short essay question: discuss which of the utility functions in Q5 would best model the student's housing dilemma.

Solution: Subjective but I'll give my view. It is reasonable to assume the student would prefer more of either the composite good or housing keeping the quantity of the other constant. In other words, that the student's preferences satisfy strong monotonicity. This rules out the eontieff preferemces in c) while also ruling out the preferences in a) when $\alpha \in \{0, 1\}$. The Cobb-Douglas preferences in a) with $\alpha \in (0, 1)$ are strongly monotone at all interior points but not when g or h equals 0 and so over the interior may be reasonable. I would also rule out the preferences of d) and e) when $\alpha = 2$ on the grounds that they give corner solutions, while in reality the student wouldn't want to either be homeless or have no money for other things. This leaves us with a choice of $u = g^\alpha h^{1-\alpha}$, $\alpha \in (0, 1)$ or $u(g, h) = g + \sqrt{h}$. A key difference between the two is what happens when the student gets extra income. Under the first, they spend proportion $(1 - \alpha)$ of that extra income on housing whereas under the second, once the amount spent on housing reaches its threshold, all extra income goes to the composite good. Depending on which you think is more realistic, you have your answer.