EC202 Term 1 Problem set 2

October 21, 2022

Pre-class Questions

These are not covered in seminars unless time permits. They are here to give you extra practice. Solutions will be provided.

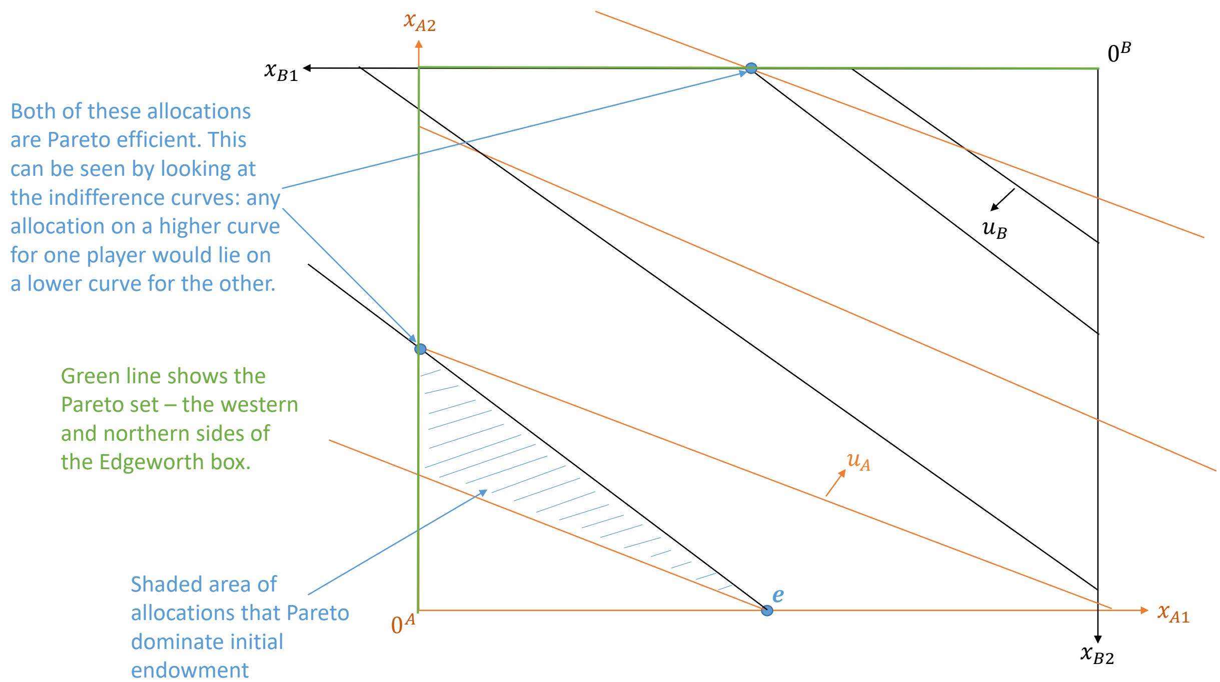
Q1. (Continuation of Exercise 4.2) Consider a 2×2 economy where initial endowments are $\mathbf{e}_A = (1, 0)$, $\mathbf{e}_B = (1, 2)$. Let preferences of each individual be be represented by $u_A, u_B : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ given by $u_A(x_{A1}, x_{A2}) = \alpha x_{A1} + x_{A2}$, $u_B(x_{B1}, x_{B2}) = \beta x_{B1} + x_{B2}$.

a) Suppose $0 < \alpha < \beta$. Draw the indifference curves through the initial endowment and in on an Edgeworth box. Is the initial endowment Pareto efficient? Find the Pareto set.

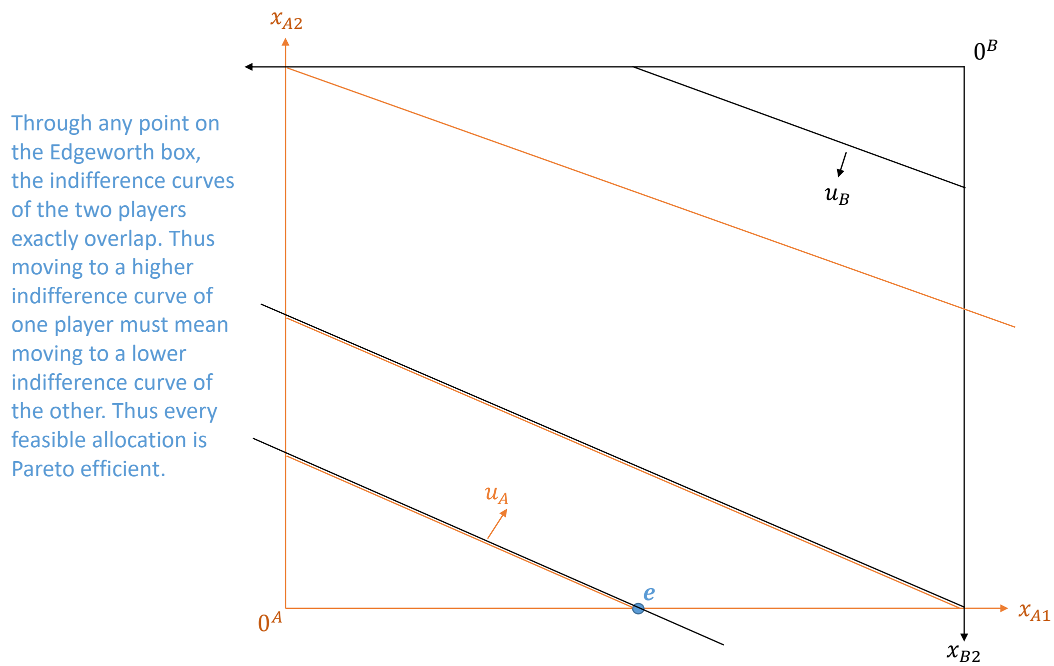
b) Suppose $\alpha = \beta$. Draw the indifference curves through the initial endowment on an Edgeworth box. Is the initial endowment Pareto efficient? Find the Pareto set.

Solution:

a) The diagram below shows that the initial endowment is not Pareto efficient since there is a lens of allocations that Pareto dominate it. It also shows that the Pareto Set is the northern and western sides of the Edgeworth box. This should make intuitive sense: Bob gets a higher utility than Andy from good 2, while they get the same utility from good 1. So a natural Pareto efficient allocation would be the top left of the Edgeworth box where Andy gets all of good 2 and Bob gets all of good 1. From this point, if we want to make Andy better off while remaining Pareto efficient then we give him more good 1, which traces out the allocations on the northern boundary of the Edgeworth box towards 0_B. Or from the top left point, if we want to make Bob better off while remaining Pareto efficient then we give him more good 2, which traces out the allocations on the western boundary of the Edgeworth box towards 0_A.



b) At every point in the Edgeworth box, the marginal rates of substitution for Andy and Bob are respectively α and β. Since $\alpha = \beta$ this means that the MRS of the two people are the same everywhere. The diagram shows that at every point in the Edgeworth box, the indifference curves of the two players exactly overlap. Therefore, every allocation in the Edgeworth box including the initial endowment is Pareto efficient. As well as diagramatically, this can also be seen mathematically: at every allocation, the sum of the utilities is $u_A + u_B = 2\alpha + 2$ and so any change that makes one player better off must make the other worse off. This is an unusual case: at no point in the Edgeworth box would there be any chance for players to mutuall gain from trade by moving to a Pareto dominating allocation.



Q2. Consider a 2×2 economy where initial endowments are $\mathbf{e}_A = (2, 1)$, $\mathbf{e}_B = (1, 2)$. Let preferences of each individual be be represented by $u_A, u_B : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$, where $u_A(x_{A1}, x_{A2}) = x_{A1}x_{A2}$. For each of the following utility functions for Bob: i) determine whether the initial endowwment is Pareto efficient and if not, shade the region of allocations which Pareto dominate it. ii) Find the Pareto Set and illustrate it on an Edgeworth box.

a) $u_B(x_{B1}, x_{B2}) = \min\{x_{B1}, x_{B2}\}$.
b) $u_B(x_{B1}, x_{B2}) = \min\{2x_{B1}, x_{B2}\}$.
c) $u_B(x_{B1}, x_{B2}) = x_{B1}x_{B2}^2$.
d) $u_B(x_{B1}, x_{B2}) = 2x_{B1} + x_{B2}$.

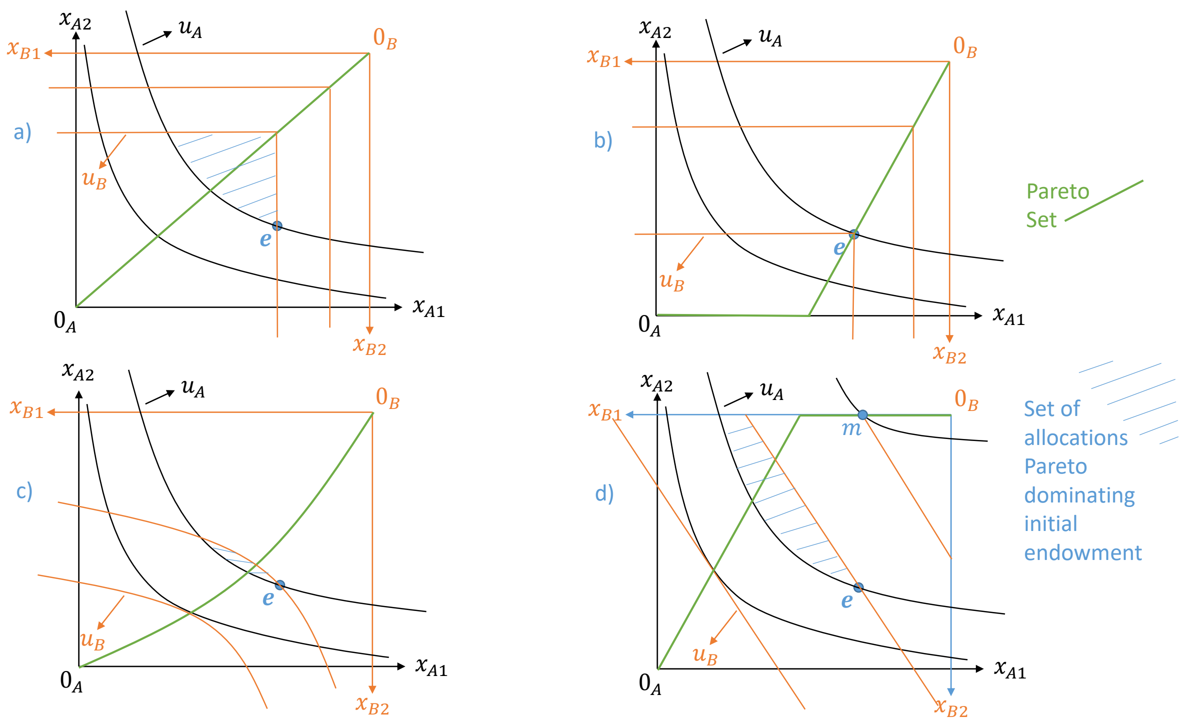
Solution: For all parts a) to d) note that for any allocation with $x_{A1} > 0$, the slope of Andy's indifference curve is

$$MRS^A_{1,2} = -\frac{MU_1}{MU_2} = -\frac{x_{A2}}{x_{A1}}$$

In parts a) and b) the utility functions of B are not differentiable at the kink of the indifference curves. While for parts c) and d), I calculate the slopes below, assuming $x_{B1} > 0$.

$$u_B(x_{B1}, x_{B2}) = x_{B1}^2 x_{B2} \Longrightarrow MRS^B_{1,2} = -\frac{x_{B2}^2}{2x_{B1}x_{B2}} = -\frac{x_{B2}}{2x_{B1}}$$

$$u_B\left(x_{B1}, x_{B2}\right) = 2x_{B1} + x_{B2} \Longrightarrow MRS^B_{1,2} = -2$$



a) The intial endowment is not on this line and hence is not Pareto efficient. An example of an allocation Pareto dominating it is $\mathbf{x}_A = (2,2)$, $\mathbf{x}_B = (1,1)$ where Bob is equally well off but Andy is strictly better off. The Pareto Set is

$$\left\{(\mathbf{x}_A, \mathbf{x}_B) \in \mathbb{R}^{2\times 2}_{\geq 0} \mid x_{A2} = x_{A1}, \mathbf{x}_A + \mathbf{x}_B = (3,3)\right\}$$

b) The initial enodwment is Pareto efficient as are other allocations on the $x_{B2} = 2x_{B1}$ line. Allocations along the southern edge with $2x_{B1} > x_{B2}$ are also Pareto efficient since at all these points $(u_A, u_B) = (0,3)$, so none Pareto dominate the other. At all these points Bob is getting highest possible utility, while moving to an allocation which would make Andy better off would require $x_{A2} > 0$ and so make Bob worse off. The Pareto Set is

$$\left\{(\mathbf{x}_A, \mathbf{x}_B) \in \mathbb{R}^{2\times 2}_{\geq 0} \mid x_{B2} = 2x_{B1}, \mathbf{x}_A + \mathbf{x}_B = (3,3)\right\} \cup$$
$$\left\{(\mathbf{x}_A, \mathbf{x}_B) \in \mathbb{R}^{2\times 2}_{\geq 0} \mid 3 = x_{B2} < 2x_{B1}, \mathbf{x}_A + \mathbf{x}_B = (3,3)\right\}$$

c) We see from our diagram that because preferences are convex, the interior Pareto optima lie at the points of tangency (also see Lecture 2, Lemma 4.2). So to find these Pareto optima we solve:

$$MRS_{1,2}^{A} = MRS_{1,2}^{B} \iff -\frac{x_{A2}}{x_{A1}} = -\frac{x_{B2}}{2x_{B1}}$$

We also require the feasibility condition $\mathbf{x}_A + \mathbf{x}_B = (3,3)$. Combining these two equations, we get

$$x_{B2} = \frac{6x_{B1}}{3 + x_{B1}}$$

This equation informs us of how to draw the Pareto Set line in the interior of the Edgeworth box. Note that the initial endowment is not Pareto efficient because it does not satisfy this equation. Alternatively, at the initial endowment we can calculate marginal rates of substitution as:

$$MRS_{1,2}^{A} = -\frac{1}{2} \qquad MRS_{1,2}^{B} = -1$$

This means that the Edgeworth box should look like that drawn where there are allocations that Pareto dominate the initial endowment where agents trade so that Andy gives Bob some good 1 in exchange for some good 2. The Pareto set can be described as the interior points of the Edgeworth box satisfying $x_{B2} = \frac{6x_{B1}}{3+x_{B1}}$ together with the two corners 0_A and 0_B. Formally, the Pareto Set is

$$\left\{(\mathbf{x}_A, \mathbf{x}_B) \in \mathbb{R}_{\geq 0}^{2\times 2} \mid x_{B2} = \frac{6x_{B1}}{3 + x_{B1}}, \mathbf{x}_A + \mathbf{x}_B = (3,3)\right\} \cup \\ \{((3,3),(0,0))\} \cup \{((0,0),(3,3))\}$$

d) Similar to c) in order to find Pareto optima we solve:

$$MRS_{1,2}^{A} = MRS_{1,2}^{B} \iff -\frac{x_{A2}}{x_{A1}} = -2$$

Therefore the interior Pareto optima satisfy $x_{A2} = 2x_{A1}$. We also need to look for Pareto optima on the boundary of the Edgeworth box. We see that both corners 0_A and 0_B are Pareto efficient and intuively one would expect the Pareto Set to also include points like m on the diagram lying between the $x_{A2} = 2x_{A1}$ line and 0_B so that the Pareto Set is a continuous line. By drawing indifference curves and noting that at all these points $MRS_{1,2}^{A} < MRS_{1,2}^{B}$, we see that these points are indeed Pareto efficient. Thus the Pareto set is

$$\left\{(\mathbf{x}_A, \mathbf{x}_B) \in \mathbb{R}_{\geq 0}^{2\times 2} \mid x_{A2} = 2x_{A1}, \mathbf{x}_A + \mathbf{x}_B = (3,3)\right\} \cup \\ \left\{(\mathbf{x}_A, \mathbf{x}_B) \in \mathbb{R}_{\geq 0}^{2\times 2} \mid 3 = x_{A2} < 2x_{A1}, \mathbf{x}_A + \mathbf{x}_B = (3,3)\right\}$$

Sidenote for exam:

There are many ways to express the Pareto Set in formal set notation. For the sake of an exam question, it just has to be clear from your diagram and description which points are in your Pareto Set and which are not. You don't need to be able to use formal set notation if you find it intimidating.

In-class Questions

These will be covered in seminar:

Q3. Consider a 2×2 pure exchange economy where the initial endowment is $\mathbf{e}_A = (3, 2)$,$\mathbf{e}_B = (2, 1)$ and preferences represented by $u_A, u_B : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ given by

$$u_A\left(x_{A1}, x_{A2}\right) = \frac{x_{A1}}{4} + \ln\left(1 + x_{A2}\right) \qquad u_B\left(x_{B1}, x_{B2}\right) = x_{B1}x_{B2}$$

a) Draw indifference curves on an Edgeworth box and verify that preferences are convex.

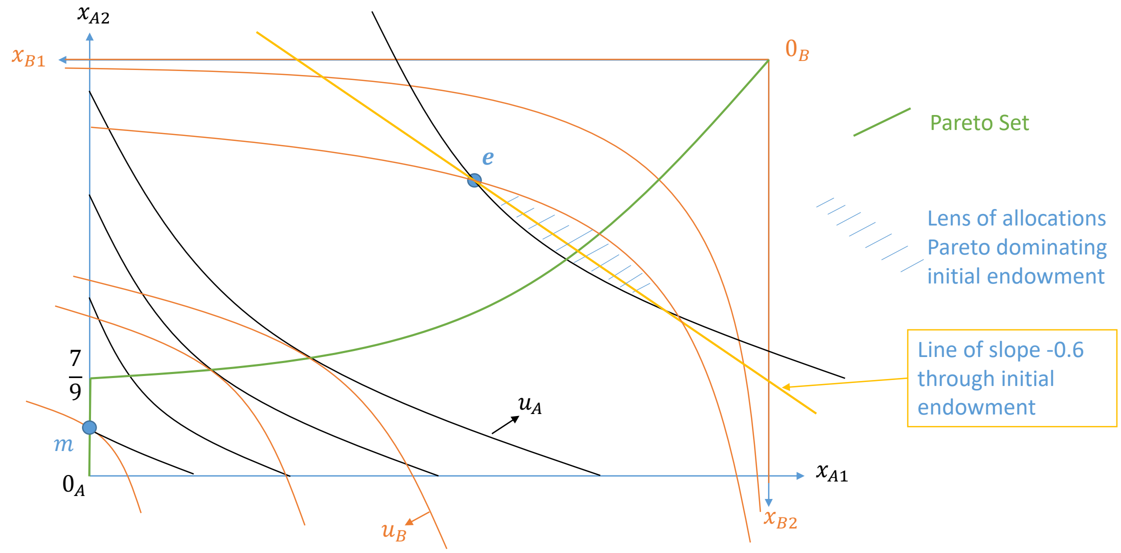
b) Calculate Marginal Rates of Substitution for both agents and evaluate them at the initial endowment.

c) Use b) to draw indifference curves through the initial endowment and on your Edgeworth box depict a lens of allocation that Pareto dominates the initial endowment.

d) Argue that we can find an allocation that Pareto dominates the initial endowment by having Andy give Bob 0.6ε good 2 in exchange for ε more good 1 for some small $\varepsilon > 0$. Test this by setting $\varepsilon = 0.1$ and calculating utilities at $\mathbf{x}_A = (3.1, 1.94)$,$\mathbf{x}_B = (1.9, 1.06)$. You should find this Pareto dominates the initial endowment and give some intuition for this.

e) Find the Pareto Set and depict it on your Edgeworth box.

Solution:



a) We can see from the diagram and the way indifference curves are curved that the upper level set taken from any indifference curve is a convex set. To verify the shape of the indifference curves, you can plot some or use the MRS.

b) Assuming $x_{B1} > 0$, the marginal rates of substitution are

$$MRS^A_{1,2} = -\frac{MU_1}{MU_2} = -\frac{\frac{1}{4}}{\frac{1}{1+x_{A2}}} = -\frac{1+x_{A2}}{4}$$

$$MRS^B_{1,2} = -\frac{MU_1}{MU_2} = -\frac{x_{B2}}{x_{B1}}$$

Evaluated at the initial endowment, we get $MRS^A_{1,2} = -\frac{3}{4}$ and $MRS^B_{1,2} = -\frac{1}{2}$.

c) see diagram

d) Observing the MRS of each agent, at the margin (ie for small $\varepsilon > 0$) Andy should be better off as long as he has to give up less than $\frac{3}{4}\varepsilon$ of good 2 for ε more good 1. While Bob is better off as long as he gets more than $\frac{1}{2}\varepsilon$ more good 2 for giving up ε of good 1. So exchanging in the ratio of 0.6 should make both better off for small $\varepsilon > 0$. This can be seen on the Edgeworth box, that there is section of allocations along the line with slope -0.6 going through the initial endowment that make both people better off. At initial endowment we calculate $(u_A, u_B) = (1.849, 2)$. At $\mathbf{x}_A = (3.1, 1.94)$,$\mathbf{x}_B = (1.9, 1.06)$ we calculate $(u_A, u_B) = (1.853, 2.014)$.

e) Lets start by finding interior Pareto optima. Since both preferences are convex any allocation where we have tangency of indifference curves will be

Pareto efficient. Incorporating this with $\mathbf{x}_A + \mathbf{x}_B = (5,3)$ gives

$$\begin{aligned} MRS^A_{1,2} = MRS^B_{1,2} &\iff -\frac{1+x_{A2}}{4} = -\frac{x_{B2}}{x_{B1}} \\ &\iff \frac{4-x_{B2}}{4} = \frac{x_{B2}}{x_{B1}} \\ &\iff x_{B2} = \frac{4x_{B1}}{4+x_{B1}} \end{aligned}$$

This equation gives us a line of points stretching from 0_B to $(x_{B1}, x_{B2}) = \left(5, \frac{20}{9}\right)$. We can also see that 0_A is Pareto efficient since this is the unique feasible allocation maximising Bob's utility. So intuitively we would expect allocation on the western edge of the form $(x_{B1}, x_{B2}) = (5, c)$ for $c \in \left(\frac{20}{9}, 3\right)$ to also be Pareto efficient. My Edgeworth box considers an example of such a point, labelled m where we can see that Andy's indifference curve is shallower than Bob's and so this is Pareto efficient. Formally we can describe the Pareto Set as

$$\left\{(\mathbf{x}_A, \mathbf{x}_B) \in \mathbb{R}^{2\times 2}_{\geq 0} \mid x_{B2} = \frac{4x_{B1}}{4+x_{B1}}, \mathbf{x}_A + \mathbf{x}_B = (5,3)\right\} \cup$$
$$\left\{(\mathbf{x}_A, \mathbf{x}_B) \in \mathbb{R}^{2\times 2}_{\geq 0} \mid \mathbf{x}_A = (0,c), c \in \left[0, \frac{7}{9}\right], \mathbf{x}_A + \mathbf{x}_B = (5,3)\right\}$$

(Although for purposes of an exam, if set notation is intimidating, I would accept a sketch of the Pareto set, with kink at $\mathbf{x}_A = \left(0, \frac{7}{9}\right)$ identified plus the formula $x_{B2} = \frac{4x_{B1}}{4+x_{B1}}$ given for the curve.)

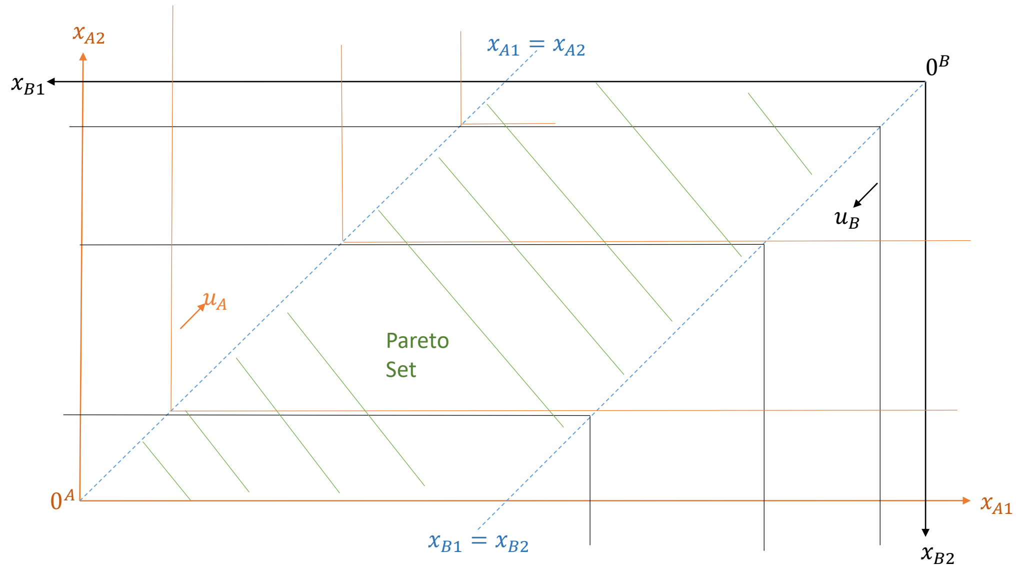
Q4. Consider a 2×2 pure exchange economy where there is 2 units of good 1 and 1 unit of good 2 in the economy. Let preferences be represented by $u_A, u_B : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ given by

$$u_A = \min\{x_{A1}, x_{A2}\} \quad u_B = \min\{x_{B1}, x_{B2}\}$$

a) Find the Pareto Set and illustrate it on an Edgeworth box.

b) Discuss why the Pareto Set takes the form it does.

Solution:
a)



As can be seen above, the Pareto Set is the plane of points

$$\left\{(\mathbf{x}_A, \mathbf{x}_B) \in \mathbb{R}^{2\times 2}_{\geq 0} \mid x_{A2} \leq x_{A1} \leq x_{A2}+1,\, x_{A2} \in [0,1]\,,\, \mathbf{x}_A + \mathbf{x}_B = (2,1)\right\}$$

b) Both people only value units of one good when combined with the equal amount of the other good. Since there is more good 1 than good in our economy, this means that at any feasible allocation there will be some units of good 1 which provide no extra benefit to the person holding them. To illustrate this, consider the allocation $\mathbf{x}_A = \mathbf{x}_B = \left(1, \frac{1}{2}\right)$. This is Pareto efficient because the only way to make one person better off would be to give them more good 2 which would make the other person worse off. At this allocation each player has half a unit surplus of good 1. So for example Andy could give Bob up to half a unit of good 1 without being worse off, but this doesn't help Bob and so doesn't Pareto dominate the previous allocation. Indeed all allocations of the form $\mathbf{x}_A = \left(c, \frac{1}{2}\right)$, $\mathbf{x}_B = \left(2-c, \frac{1}{2}\right)$ for any $c \in \left[\frac{1}{2}, \frac{3}{2}\right]$ give the same utilities as each other, namely $(u_A, u_B) = \left(\frac{1}{2}, \frac{1}{2}\right)$. Similar logic would hold at any other given share of good 2: For any split $x_{A2} = k$, $x_{B2} = 1-k$, there is a line of allocations of length 1 satisfying $x_{A1} \geq k$ and $x_{B1} \geq 1-k$ at which utilities are $(u_A, u_B) = (k, 1-k)$. All such points are Pareto efficient because to make one person better off would require them to be given more good 2 which would make the other person worse off.

Post-class question

Short essay question: Discuss the following statement: "All Pareto efficient allocations should be considered more socially desirable than any allocation that isn't Pareto efficient."

Solution: Pareto efficiency should be seen as a necessary condition for a socially desirable outcome but not a sufficient condition. Indeed many Pareto efficient allocations may be very socially undesirable. As an example, some Pareto efficient allocations will be very unequal where some people get great amounts of goods while others get hardly anything. Pareto efficiency imposes no equality or fairness criterior. As an example, consider a two person economy and two Pareto efficient allocations: allocation 1 where one person works very hard and gets 10% of the resources, while the other does no work, so gets lots of leisure time and 90% of the resources. Allocation 2: both work moderately hard and get 50% of the resources. But now consider modifying allocation 2 slightly to include some waste so that each person now only consumes 49% of the available resources - call this allocation 3.

While "socially desirable" is very subjective, I think most people would agree that allocation 2 is "better" than allocation 1, and would also regard allocation 3 as better than allocation 1 too. Although note that every non-Pareto efficient allocation (like allocation 3) is Pareto dominated by some Pareto efficient allocation (allocation 2).