EC202 Term 1 Problem set 3

October 18, 2022

Pre-class questions

These are not covered in seminars unless time permits. They are here to give you extra practice. Solutions will be provided.

Q1. Consider a 2×2 economy where initial endowments are $\mathbf{e}_A = (1, 0)$, $\mathbf{e}_B = (0, 1)$. Let preferences of each individual be represented by $u_A, u_B : \mathbb{R}^2_{\geq 0}$ given by $u_A(x_{A1}, x_{A2}) = \alpha x_{A1} + x_{A2}$, $u_B(x_{B1}, x_{B2}) = \beta x_{B1} + x_{B2}$.

a) Suppose $\alpha = \beta$. Draw the indifference curves and initial endowment on an Edgeworth box. Find the set of Walrasian Equilibria.

b) Suppose $\alpha < \beta < 1$. Draw the indifference curves and initial endowment on an Edgeworth box. Find the set of Walrasian Equilibria.

c) Suppose $\alpha < 1 < \beta$. Draw the indifference curves and initial endowment on an Edgeworth box. Find the set of Walrasian Equilibria.

(Hint: You can either do this algebraically or by using the Edgeworth box and trying lots of different price vectors and testing whether we have Walrasian Equilibrium. The second method will be easier!)

Solution: Algebraically, first for Andy:
He faces budget constraint

$$p_1 x_{A1} + p_2 x_{A2} \leq p_1$$

At the optimal bundle this will hold with equality and to see how he'll spend this budget we consider the "bang per buck" of each good.

$$\frac{MU_1}{p_1} = \frac{\alpha}{p_1} \text{ and } \frac{MU_2}{p_2} = \frac{1}{p_2}$$

Thus

$$\frac{MU_1}{p_1} \geq \frac{MU_2}{p_2} \iff \alpha \geq \frac{p_1}{p_2}$$

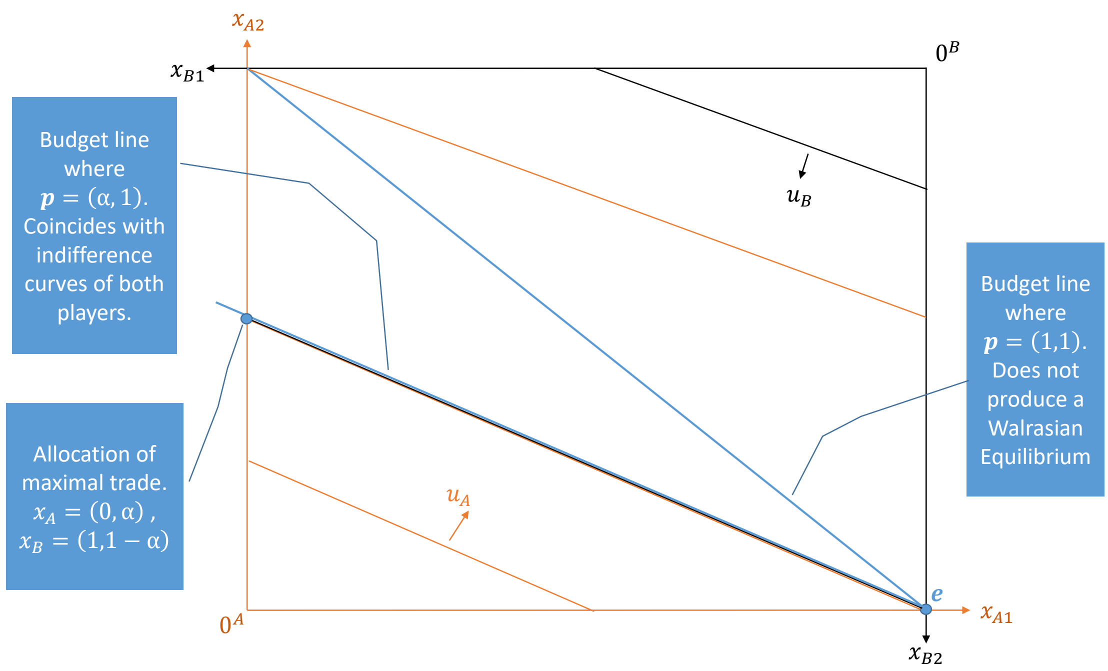
For his optimal bundle, Andy will choose to spend all his income on the good which gives the higher "bang per buck".

$$\mathbf{x}_A(\mathbf{p}) \begin{cases} (1,0) & \frac{p_1}{p_2} < \alpha \\ \left\{\mathbf{x}_A \in \mathbb{R}^2_{\geq 0} \mid p_1 x_{A1} + p_2 x_{A2} = p_1\right\} & \frac{p_1}{p_2} = \alpha \\ \left(0, \frac{p_1}{p_2}\right) & \frac{p_1}{p_2} > \alpha \end{cases}$$

Similar analysis finds Bob's optimal bundle to be

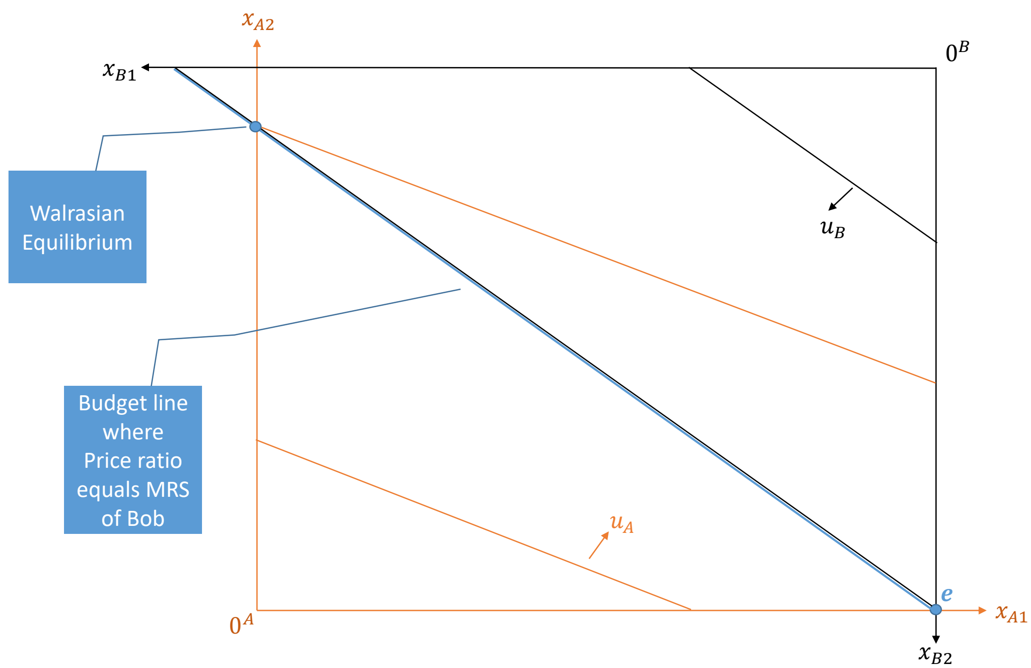
$$\mathbf{x}_B(\mathbf{p}) \begin{cases} \left(\frac{p_2}{p_1}, 0\right) & \frac{p_1}{p_2} < \beta \\ \left\{\mathbf{x}_B \in \mathbb{R}^2_{\geq 0} \mid p_1 x_{B1} + p_2 x_{B2} = p_2\right\} & \frac{p_1}{p_2} = \beta \\ (0,1) & \frac{p_1}{p_2} > \beta \end{cases}$$

a) As the diagram below shows (for case $\alpha = \beta < 1$ similar for other cases_ we have Walrasian Equilibria with prices $\mathbf{p} = (\alpha, 1)$. Here both agents can demand anywhere along the budget line and so any allocation on the Budget line inside the Edgeworth box is a Walrasian Equilibrium. Thus there infinitely many Walrasian Equilibrium allocations including the intial allocation and the allocation, the allocation of maximal trade marked below and any allocation on the line joining the two.

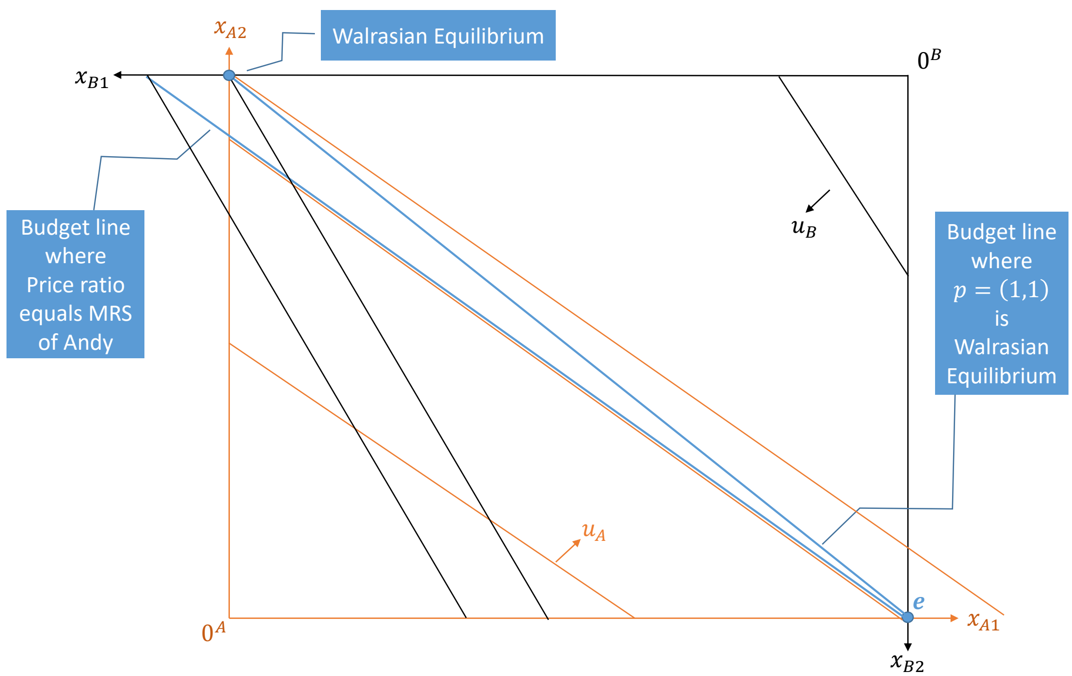


Note that $\mathbf{p} = (1,1)$ or any price vector steeper than this does not form a Walrasian Equilibrium as both agents would only demand good 2.

b) As seen below there is a unique Walrasian Equilibrium with $\mathbf{p} = (\beta, 1)$ and allocation $\mathbf{x}_A = (0, \beta)$, $\mathbf{x}_B = (1, 1-\beta)$. Here Andy has a unique optimal bundle while Bob can demand anywhere on his budget line including this point. No other prices would clear both markets.



c) As shown below, there is a Walrasian Equilibrium where $\mathbf{p} = (1, 1)$ and $\mathbf{x}_A = (0, 1)$, $\mathbf{x}_B = (1, 0)$. Notice that here both agents are consuming their unique optimal bundle. It relies upon the dual effects that: i) Each player is restricted from demanding negative amounts of their less preferred good and ii) neither player's budget set leaves the Edgeworth box.



There are no other Walrasian Equilibria. The diagram illustrates why setting price ratio equal to Andy's MRS is not a Walrasian Equilibrium: Bob would demand only good 1 at a point outside the Edgeworth box.

Q2. Consider a 2×2 economy where preferences are be represented by $u_A, u_B : \mathbb{R}^2_{\geq 0}$ where $u_A = x_{A1}^{\alpha} x_{A2}^{1-\alpha}$ and $u_B = \min\{2x_{B1}, x_{B2}\}$ and the initial endowment is $\mathbf{e}_A = (1, 1)$, $\mathbf{e}_B = (3, 3)$

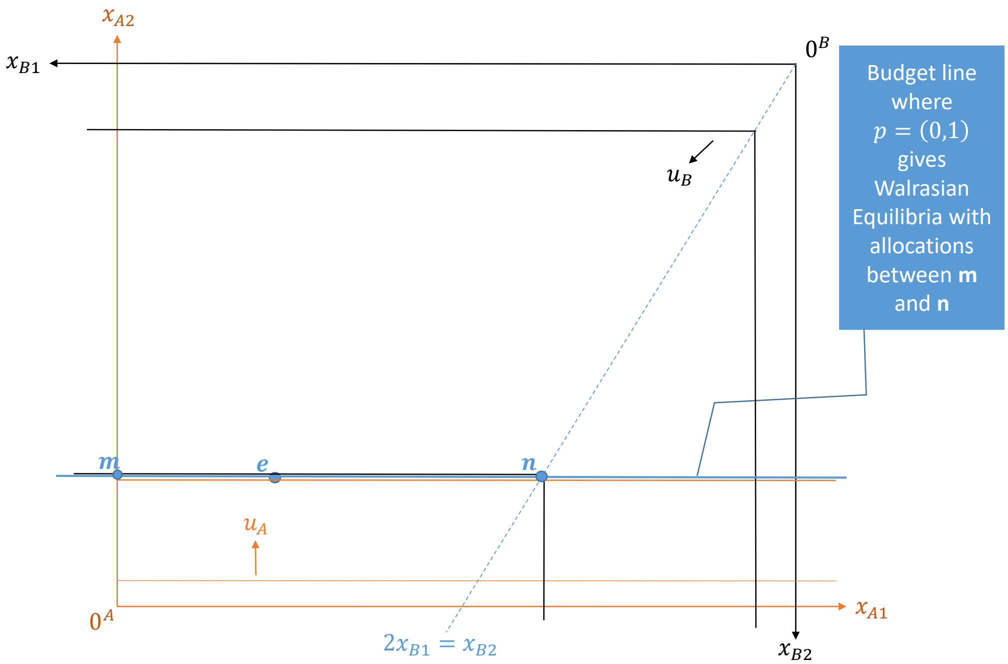
a) Let $\alpha = 0$. Draw the Edgeworth box and find the set of Walrasian Equilibria.

b) Let $\alpha = 1$. Draw the Edgeworth box and find the set of Walrasian Equilibria.

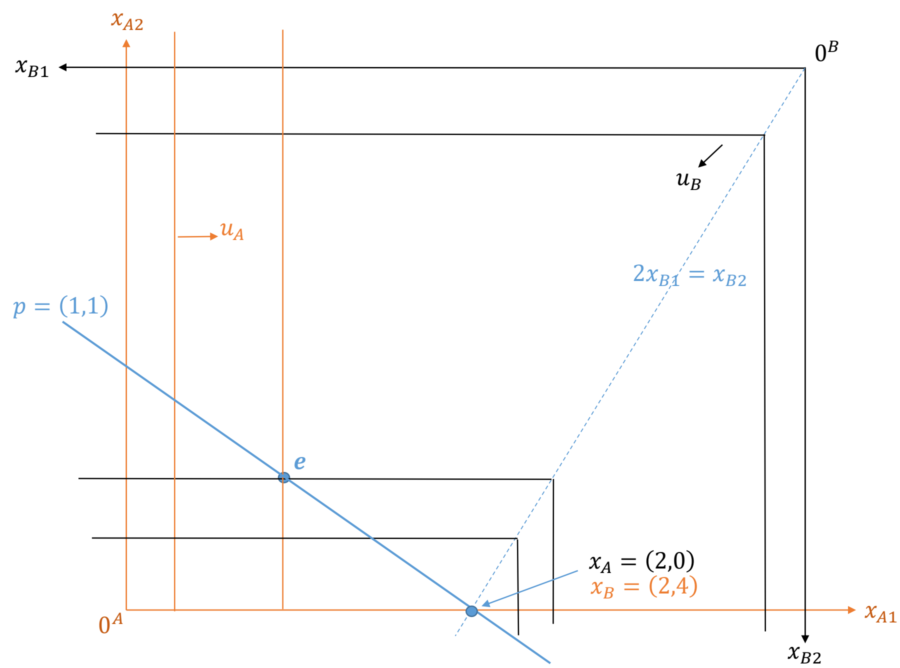
c) Let $\alpha = \frac{1}{2}$. Draw the Edgeworth box and find the set of Walrasian Equilibria.

Solution:

a) As seen below we have a set of Walrasian Equilibria where the Budget constraint is horizontal. The intuition for this is that at the initial endowment both agents have spare units of x_1: they don't lose utility from losing a bit of good 1 and don't gain utility from having more of good 1. For both agents, only extra good 2 will make them better off.



b) Here we have a unique Walrasian Equilibrium price vector $\mathbf{p} = (1, 1)$ and Walrasian Equilibrium allocation $\mathbf{x}_A = (2, 0)$, $\mathbf{x}_B = (2, 4)$ as pictured below.



This is the unique Equilibrium: for a shallower budget line there would be excess demand of good 1. For a steeper budget line Bob would demand more good 2 than exists in the economy.

c) Now we actually have to use some mathematics to calculate optimal demands. Note that we can't have a Walrasian Equilibrium where one of the prices is 0 since Andy wouldn't have a finite optimal demand. So we assume $\mathbf{p} \in \mathbb{R}^2_{>0}$ in what follows. By the Cobb-Douglas shortcut, Andy has income from the value of his endowment of $p_1 + p_2$ and spends half his income on each good. So we get

$$\mathbf{x}_A(\mathbf{p}) = \left(\frac{p_1+p_2}{2p_1}, \frac{p_1+p_2}{2p_2}\right)$$

Bob solves simultaneously two equations: being on the budget line and the kink of the indifference curve:

$$2x_{B1} = x_{B2} \qquad p_1 x_{B1} + p_2 x_{B2} = 3p_1 + 3p_2$$

$$\Longrightarrow p_1 x_{B1} + 2p_2 x_{B1} = 3p_1 + 3p_2$$

$$\Longrightarrow \mathbf{x}_B(\mathbf{p}) = \left(\frac{3p_1+3p_2}{p_1+2p_2}, \frac{6p_1+6p_2}{p_1+2p_2}\right)$$

Market clearing for good 1 gives:

$$\frac{p_1+p_2}{2p_1} + \frac{3p_1+3p_2}{p_1+2p_2} = 4$$

We can normalise one of the prices to be 1. Setting $p_1 = 1$ gives:

$$
\begin{aligned}
\frac{1+p_2}{2} + \frac{3+3p_2}{1+2p_2} = 4 \iff & \frac{(1+p_2)(1+2p_2) + 2(3+3p_2)}{2(1+2p_2)} = 4 \\
\iff & 1 + 3p_2 + 2p_2^2 + 6 + 6p_2 = 4[2(1+2p_2)] \\
\iff & 2p_2^2 + 9p_2 + 7 = 8 + 16p_2 \\
\iff & 2p_2^2 - 7p_2 - 1 = 0 \\
\iff & p_2^2 - \frac{7}{2}p_2 - \frac{1}{2} = 0 \\
\iff & \left(p_2 - \frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2 - \frac{1}{2} = 0 \\
\iff & \left(p_2 - \frac{7}{4}\right)^2 = \frac{57}{16} \\
\iff & p_2 = \sqrt{\frac{57}{16}} + \frac{7}{4} = \frac{\sqrt{57}+7}{4}
\end{aligned}
$$

So we get prices $\mathbf{p} = \left(1, \frac{\sqrt{57}+7}{4}\right) \simeq (1, 3.637)$ and subbing in (after much simplifying) gives

$$\mathbf{x}_A(\mathbf{p}) = \left(\frac{11+\sqrt{57}}{8}, \frac{2\sqrt{57}-10}{8}\right) \simeq (2.319, 0.637)$$

$$\mathbf{x}_A(\mathbf{p}) = \left(\frac{21-\sqrt{57}}{8}, \frac{42-2\sqrt{57}}{8}\right) \simeq (1.681, 3.363)$$

In an exam you shouldn't receive such complex algebra, and if you did, I would be ok with you giving answers to 3 decimal places.

In-class question

Q3. Consider a 2×2 economy where preferences are represented by $u_A, u_B : \mathbb{R}^2_{\geq 0}$ where $u_A = x_{A1}^{\alpha} x_{A2}^{1-\alpha}$ and $u_B = x_{B1}^{\beta} x_{B2}^{1-\beta}$ for some $\alpha, \beta \in (0,1)$. Assume both goods are in strctly positive supply.

a) Argue that, for any initial endowment, there are no Walrasian Equilibria where one good has price 0 and hence in an Walrasian Equilibrium we must have $(p_1, p_2) \in \mathbb{R}^2_{>0}$.

b) Let prices be $(p_1, p_2) \in \mathbb{R}^2_{>0}$ and incomes be M_A, M_B. Verify that optimal demands are

$$\mathbf{x}_A(\mathbf{p}, M_A) = \left(\frac{\alpha M_A}{p_1}, \frac{(1-\alpha) M_A}{p_2}\right)$$

$$\mathbf{x}_B(\mathbf{p}, M_B) = \left(\frac{\beta M_B}{p_1}, \frac{(1-\beta) M_B}{p_2}\right)$$

c) Consider the initial endowment $\mathbf{e}_A = (0, 1)$, $\mathbf{e}_B = (1, 0)$:

i) Find optimal demands $\mathbf{x}_A(\mathbf{p})$ and $\mathbf{x}_B(\mathbf{p})$. Show that these demands satisfy Walras' Law.

ii) Find the Walrasian Equilibrium and illustrate it on an Edgeworth box.

iii) Verify on your Edgeworth box and algebraically that both players prefer the Walrasian Equilibrium to their initial allocation.

d) Repeat b) for initial endowment $\mathbf{e}_A = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\mathbf{e}_B = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\alpha < \beta$.

Solution:

a) If the price of one good equals 0 then a consumer with positive endowment of the other good would not have a finite optimal demand, since they could get more and more utility by forever increasing their demand. Also, at least one of the two consumers must have a positive endowment of the other good.

b) By drawing indifference curves and Budget constraint, one can justify that the solution must be where slope of budget constraint (price ratio) equals slope of indifference curve (MRS). Also each agent must exhaust their budget. Hence for Andy we solve:

$$\text{MRS} = \frac{\alpha x_{A2}}{(1-\alpha)\, x_{A1}} = \frac{p_1}{p_2}$$

$$p_1 x_{A1} + p_2 x_{A2} = M_A$$

Solving these two equations simultaneously gives the optimal demand in the question:

$$\mathbf{x}_A(\mathbf{p}, M_A) = \left(\frac{\alpha M_A}{p_1}, \frac{(1-\alpha)\, M_A}{p_2}\right)$$

The analysis for Bob is identical.

c) i) Substituting in the value of the endowment as income we get:

$$\mathbf{x}_A(\mathbf{p}) = \left(\frac{\alpha p_2}{p_1}, \frac{(1-\alpha)\, p_2}{p_2}\right) = \left(\frac{\alpha p_2}{p_1}, (1-\alpha)\right)$$

$$\mathbf{x}_B(\mathbf{p}) = \left(\frac{\beta p_1}{p_1}, \frac{(1-\beta)\, p_1}{p_2}\right) = \left(\beta, \frac{(1-\beta)\, p_1}{p_2}\right)$$

Walras' Law states that the value of excess demand should equal 0. We will verify this holds:

$$
\begin{aligned}
p_1\left(x_{A1}+x_{B1}-1\right)+p_2\left(x_{A2}+x_{B2}-1\right)=&p_1\left(\frac{\alpha p_2}{p_1}+\beta-1\right)+p_2\left(\left(1-\alpha\right)+\frac{\left(1-\beta\right)p_1}{p_2}-1\right)\\
=&\alpha p_2+p_1\beta-p_1+p_2\left(1-\alpha\right)+\left(1-\beta\right)p_1-p_2\\
=&0
\end{aligned}
$$

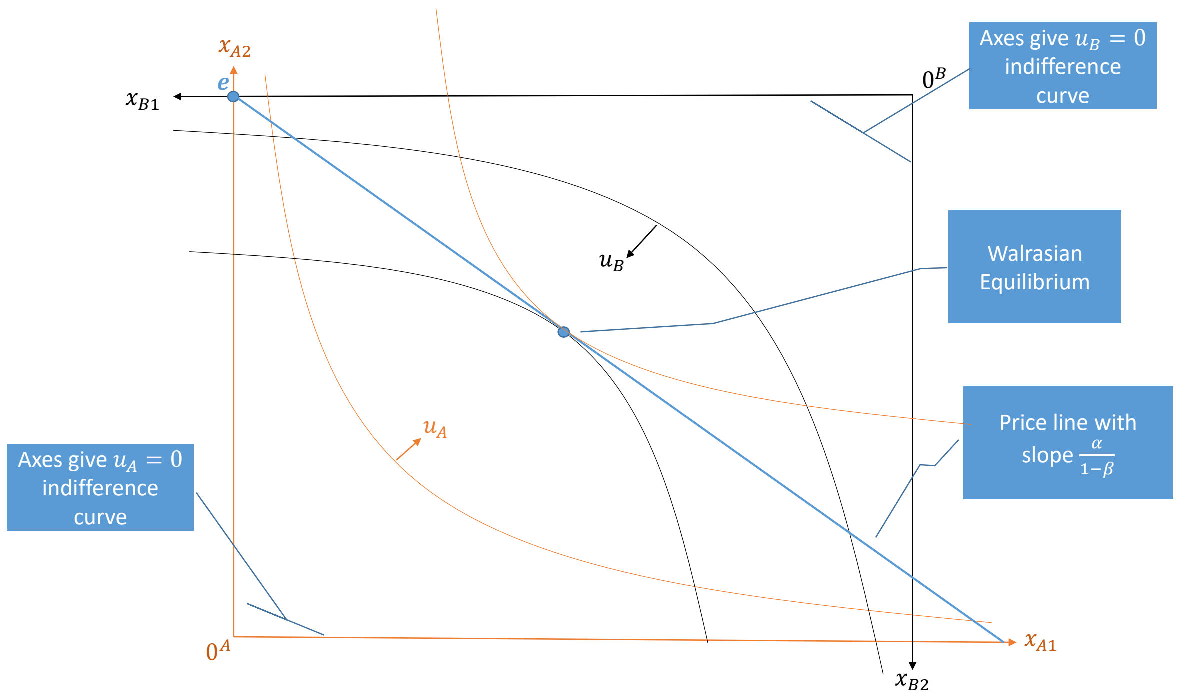
c) ii) In any Walrasian Equilibrium both prices must be positive. For example if $p_1=0$ then Andy would demand infinite amount of good 1 and we use the results above: We need markets to clear so:

$$\frac{\alpha p_2}{p_1}+\beta=1 \quad \text{(Market for good 1)}$$

$$\left(1-\alpha\right)+\frac{\left(1-\beta\right)p_1}{p_2}=1 \quad \text{(Market for good 2)}$$

As both prices must be positive, we can normalise $p_2=1$ and solve for p_1. When doing this for the market for good 1 we get $p_1=\frac{\alpha}{1-\beta}$ and as a check, you can verify that this clears the market for good 2 too. Thus our Walrasian Equilibrium is

$$\mathbf{p}=\left(\frac{\alpha}{1-\beta},1\right)\ \mathbf{x}_A=\left(\left(1-\beta\right),\left(1-\alpha\right)\right)\ \mathbf{x}_B=\left(\beta,\alpha\right)$$



This diagram gives a case for a particular value of α and β.

c) iii) From the diagram, we can see that the initial allocation lay on the lowest indifference curve of each agent, while the Walrasian Equilibrium lies on a much higher indifference curve. Algebraically Andy's utility has increased from 0 to $(1-\beta)^{\alpha}(1-\alpha)^{1-\alpha}$ while Bob's has increased from 0 to $\beta^{\beta}\alpha^{1-\beta}$.

d) Substituting in the value of the endowment as income we get:

$$\mathbf{x}_A(\mathbf{p}) = \left(\frac{\alpha\left(\frac{1}{2}p_1+\frac{1}{2}p_2\right)}{p_1}, \frac{(1-\alpha)\left(\frac{1}{2}p_1+\frac{1}{2}p_2\right)}{p_2}\right) = \left(\frac{\alpha}{2}+\frac{\alpha p_2}{2p_1}, \frac{(1-\alpha)p_1}{2p_2}+\frac{(1-\alpha)}{2}\right)$$

$$\mathbf{x}_B(\mathbf{p}) = \left(\frac{\beta\left(\frac{1}{2}p_1+\frac{1}{2}p_2\right)}{p_1}, \frac{(1-\beta)\left(\frac{1}{2}p_1+\frac{1}{2}p_2\right)}{p_2}\right) = \left(\frac{\beta}{2}+\frac{\beta p_2}{2p_1}, \frac{(1-\beta)p_1}{2p_2}+\frac{(1-\beta)}{2}\right)$$

Walras' Law states that the value of excess demand should equal 0. We will verify this holds:

$$\begin{aligned} p_1(x_{A1}+x_{B1}-1)+p_2(x_{A2}+x_{B2}-1) &= p_1\left(\frac{\alpha}{2}+\frac{\alpha p_2}{2p_1}+\frac{\beta}{2}+\frac{\beta p_2}{2p_1}-1\right) \\ &\quad + p_2\left(\frac{(1-\alpha)p_1}{2p_2}+\frac{(1-\alpha)}{2}+\frac{(1-\beta)p_1}{2p_2}+\frac{(1-\beta)}{2}-1\right) \\ &=0 \end{aligned}$$

In any Walrasian Equilibrium both prices must be positive for same reason. We need markets to clear so:

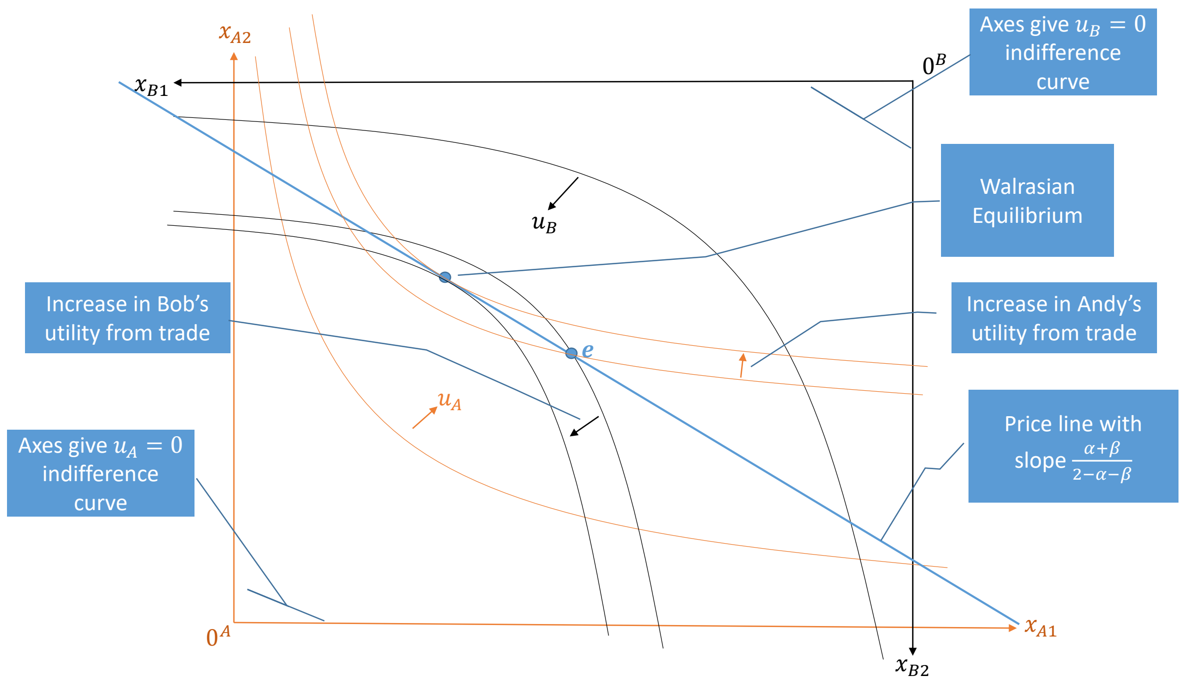
$$\frac{\alpha}{2} + \frac{\alpha p_2}{2p_1} + \frac{\beta}{2} + \frac{\beta p_2}{2p_1} = 1 \quad \text{(Market for good 1)}$$

$$\frac{(1-\alpha)\,p_1}{2p_2} + \frac{(1-\alpha)}{2} + \frac{(1-\beta)\,p_1}{2p_2} + \frac{(1-\beta)}{2} = 1 \quad \text{(Market for good 2)}$$

As both prices must be positive, we can normalise $p_2 = 1$ and solve for p_1. When doing this for the market for good 1 we get $p_1 = \frac{\alpha+\beta}{2-\alpha-\beta}$ (algebra skipped) and as a check, you can verify that this clears the market for good 2 too. Note that as well as $\mathbf{p} = \left(\frac{\alpha+\beta}{2-\alpha-\beta}, 1\right)$ we could write prices as $\mathbf{p} = (\alpha+\beta, 2-\alpha-\beta)$ Thus our Walrasian Equilibrium is

$$\begin{aligned}
\mathbf{p} &= \left(\frac{\alpha+\beta}{2-\alpha-\beta}, 1\right) \\
\mathbf{x}_A &= \left(\frac{\alpha}{2} + \frac{\alpha p_2}{2p_1}, \frac{(1-\alpha)\,p_1}{2p_2} + \frac{(1-\alpha)}{2}\right) \mid p_1 = \alpha+\beta \; p_2 = 2-\alpha-\beta \\
\mathbf{x}_B &= \left(\frac{\beta}{2} + \frac{\beta p_2}{2p_1}, \frac{(1-\beta)\,p_1}{2p_2} + \frac{(1-\beta)}{2}\right) \mid p_1 = \alpha+\beta \; p_2 = 2-\alpha-\beta
\end{aligned}$$

As a check, note that $\alpha = \beta = \frac{1}{2}$ gives $\mathbf{p} = (1,1)$, $\mathbf{x}_A = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\mathbf{x}_B = \left(\frac{1}{2}, \frac{1}{2}\right)$ as one would expect. Also note that as α and β increase, meaning agents put more weight on how much of good 1 they get, p_1 also increases. Indeed $p_1 \geq p_2 \iff \alpha + \beta \geq 1$ Also, note that as we are assuming $\alpha < \beta$ we see that Andy will get less of good 1 and more of good 2 than Bob. The diagram below pictures this for some $\alpha + \beta < 1$.



Note to seminar tutors when going through the question: encourage students to think about how our results in all parts depend on the exogenous parameters α, β. How do these parameters affect who gets more of each good? How do they affect prices?

Post-class question

Short essay question: In the real world, would you expect agents to trade more or less than predicted by the Walrasian Equilibrium model.

Sketch Solution: Recall the discussion about the assumptions that the Walrasian Equilibrium model makes in Section 1 of Lecture 2. I listed 9 assumptions. Of particular relevance to this question is our 7th assumption that there are no transaction costs. In reality there will be transaction costs to trade meaning less trade takes place. In addition we could discuss other factors not modeled here like incomplete information. If Andy doesn't know the quality of the goods that Bob is selling then that makes trade much harder: Andy might not know how much to pay. There could also be adverse selection effects if Bob is more likely to want to keep high quality goods for himself and sell low quality goods to Andy. This is a very open-ended question and one could describe many other effects that could conceivably have an impact which the Walrasian Eqilibrium model does not account for.