EC202 Term 1 Problem set 4

October 29, 2022

Pre-class Questions

Not covered in seminar unless time permits.

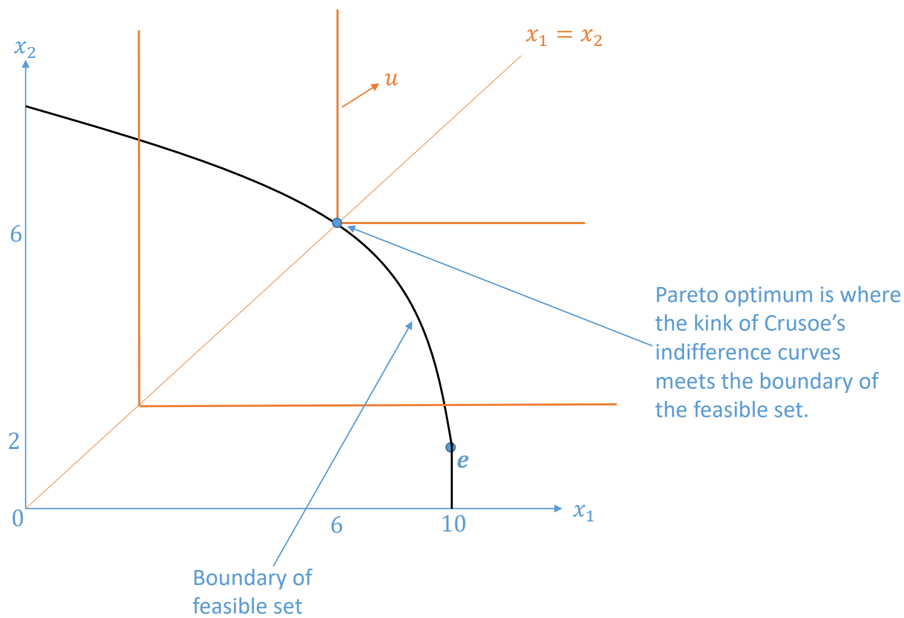
Q1. Consider a Robinson Crusoe economy where Crusoe has initial endowment of $(e_1, e_2) = (10, 2)$ and is able to turn $k \geq 0$ units of good 1 into $2\sqrt{k}$units of good 2. Let Crusoe have preferences represented by $u : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ where $u(x_1, x_2) = \min\{x_1, x_2\}$.

a) Draw the feasible set, Crusoe's indifference curves and find the Pareto optimum.

b) Consider the Walrasian Equilibrium model, where the firm has Crusoe's production technology and satisfies free-disposal. Write down the firm's production set Y, assuming free disposal, and say what type of returns to scale are exhibited.

c) For reasons that you will see in Lecture 5, if a Walrasian Equilibrium exists it will be Pareto efficient. Assuming the Walrasian Equilibrium allocation is the Pareto efficient allocation, find the Walrasian Equilibrium if one exists.

Solution:
a)



Mathematically we find the Pareto optimum by solving

$$\begin{aligned}
&x_1 = x_2 \text{ and } x_2 = 2 + 2\sqrt{10 - x_1} \Longrightarrow (x_2 - 2)^2 = 4\,(10 - x_2) \\
\Longleftrightarrow & x_2^2 - 4x_2 + 4 - 40 + 4x_2 = 0 \\
\Longleftrightarrow & x_2^2 = 36
\end{aligned}$$

Thus we get solution $\mathbf{x} = (6, 6)$.

b) Production set is

$$Y = \left\{\mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0,\, y_2 \leq 2\sqrt{-y_1}\right\}$$

This exhibits non-increasing returns to scale.

c) The firm's profit maximisation problem is $\max_{\mathbf{y}\in Y} \mathbf{p}.\mathbf{y}$. Letting the amount of input be k and substituting $y_1 = -k$ and $y_2 = 2\sqrt{k}$ we can solve this as follows:

$$\begin{aligned}
\pi &= \mathbf{p}.\mathbf{y} = -p_1 k + 2p_2\sqrt{k} \\
\frac{d\pi}{dk} &= 0 \Longleftrightarrow -p_1 + p_2 k^{-\frac{1}{2}} = 0 \\
\Longleftrightarrow & k = \left(\frac{p_2}{p_1}\right)^2 \\
\Longleftrightarrow & \mathbf{y}\,(\mathbf{p}) = \left(-\left(\frac{p_2}{p_1}\right)^2, \frac{2p_2}{p_1}\right)
\end{aligned}$$

(We can argue that the first order condition is sufficient either by π being concave or by drawing a diagram and seeing that our maximum lies where the iso-profit line is tangential to the boundary of the production set.) The profit can be found by subbing $\mathbf{y}(\mathbf{p})$ back into the profit function:

$$\pi = \mathbf{p}.\mathbf{y} = -p_1 \left(\frac{p_2}{p_1}\right)^2 + p_2 \left(\frac{2p_2}{p_1}\right) = \frac{p_2^2}{p_1}$$

Crusoe maiximises utility subject to budget constraint so solves:

$$\max u = \min \{x_1, x_2\} \text{ subject to } p_1 x_1 + p_2 x_2 \leq 10p_1 + 2p_2 + \frac{p_2^2}{p_1}$$

This has solution where

$$x_1 = x_2 \text{ and } p_1 x_1 + p_2 x_2 = 10p_1 + 2p_2 + \frac{p_2^2}{p_1}$$
$$\Longrightarrow \mathbf{x}(\mathbf{p}) = \left(\frac{10p_1 + 2p_2 + \frac{p_2^2}{p_1}}{p_1 + p_2}, \frac{10p_1 + 2p_2 + \frac{p_2^2}{p_1}}{p_1 + p_2} \right)$$

To have a Walrasian Equilibrium where the firm produces the efficient quantity of good 2, we need $\mathbf{x} = (6, 6)$ and so by market clearing $\mathbf{y} = (-4, 4)$. This allows us to solve for prices:

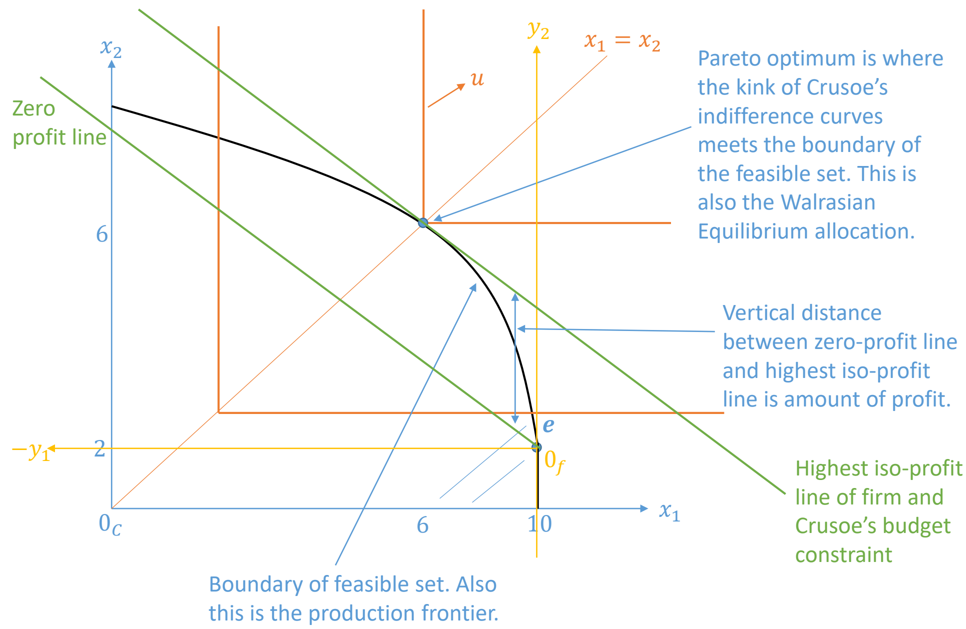
$$\mathbf{y}(\mathbf{p}) = \left(-\left(\frac{p_2}{p_1}\right)^2, \frac{2p_2}{p_1} \right) = (-4, 4)$$
$$\Longleftrightarrow \frac{2p_2}{p_1} = 4$$
$$\Longleftrightarrow \frac{p_2}{p_1} = 2 \Longleftrightarrow \frac{p_1}{p_2} = \frac{1}{2}$$

So we get prices $\mathbf{p} = (1, 2)$. The last piece of the jigsaw is to check that subbing in $\mathbf{p} = (1, 2)$ into $\mathbf{x}(\mathbf{p})$ does give $\mathbf{x} = (6, 6)$:

$$\mathbf{x}(\mathbf{p}) = \left(\frac{10p_1 + 2p_2 + \frac{p_2^2}{p_1}}{p_1 + p_2}, \frac{10p_1 + 2p_2 + \frac{p_2^2}{p_1}}{p_1 + p_2} \right)$$
$$= \left(\frac{10 + 4 + 4}{3}, \frac{10 + 4 + 4}{3} \right)$$
$$= (6, 6)$$

Thus we have a Walrasian Equilibrium

$$\mathbf{p} = (1, 2) \quad \mathbf{x} = (6, 6) \quad \mathbf{y} = (-4, 4)$$



Q2. Repeat Q1 but with a different technology for turning units of good 1 into good 2:

i) Now, instead suppose he can turn $k \geq 0$ units of good 1 into k units of good 2.

ii) Now, instead suppose he can turn $k \geq 0$ units of good 1 into $\frac{k^2}{4}$ units of good 2.

Solution:

Part i)

a) Mathematically we find the Pareto optimum by solving

$$x_1 = x_2 \text{ and } x_2 = 2 + (10 - x_1) \Longrightarrow x_1 = x_2 = 6$$

Thus we get solution $\mathbf{x} = (6, 6)$.

b) Production set is

$$Y = \left\{ \mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0, \, y_2 \leq -y_1 \right\}$$

This exhibits constant returns to scale.

c) Firstly profit maximisation for the firm: we can draw a diagram and compare slope of production frontier which is -1 to slope of iso-profit line which is $-\frac{p_1}{p_2}$. If $\frac{p_1}{p_2} < 1$ then there is no finite solution to the profit maximisation problem as we can keep increasing profit by moving higher and higher up the $y_2 = -y_1$ line. Thus we get

$$\mathbf{y}(\mathbf{p}) = \begin{cases} \emptyset & \frac{p_1}{p_2} < 1 \\ \left\{\mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0,\, y_2 = -y_1\right\} & \frac{p_1}{p_2} = 1 \\ \mathbf{0} & \frac{p_1}{p_2} > 1 \end{cases}$$

In the bottom 2 cases, where we do find a finite solution, the firm makes no profit, as can be seen by noting that in the $\frac{p_1}{p_2} = 1$ case, revenue equals cost, or diagrammatically all the solutions lie along the iso-profit line going through $\mathbf{y} = (0, 0)$ which is the zero iso-profit line. As there is no profit, Crusoe's only income is from the value of the endowment and so Crusoe solves:

$$\max u = \min\{x_1, x_2\} \text{ subject to } p_1x_1 + p_2x_2 \leq 10p_1 + 2p_2$$

This has solution where

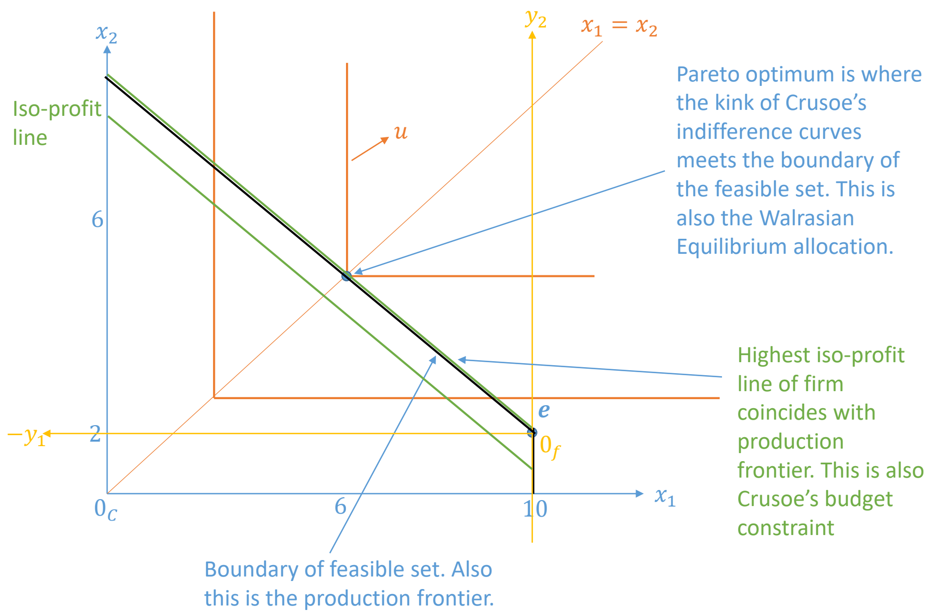
$$x_1 = x_2 \text{ and } p_1x_1 + p_2x_2 = 10p_1 + 2p_2$$
$$\Longrightarrow \mathbf{x}(\mathbf{p}) = \left(\frac{10p_1 + 2p_2}{p_1 + p_2}, \frac{10p_1 + 2p_2}{p_1 + p_2}\right)$$

Now, testing different price ratios for Walrasian Equilibrium, we see:

1. No Equilibrium where $\frac{p_1}{p_2} < 1$ as no profit maximising output.
2. No Equilibrium where $\frac{p_1}{p_2} > 1$ as excess demand of good 2, excess supply of good 1. (You can calculate this formally observe that Crusoe wants equal amounts of the 2 goods but there is more good 1 supplied than good 2).

So the only chance of Walrasian Equilibrium occurs where $\frac{p_1}{p_2} = 1$. At this price ratio there are many optimal actions for the firm but only one optimal action for Crusoe so our plan of action will be to find Crusoe's demand, apply market clearing to find what we need the firm to do and then show that the firm is profit maximising by doing so. Since $\frac{p_1}{p_2} = 1$ we can let prices be $\mathbf{p} = (1, 1)$ and subbing this into $\mathbf{x}(\mathbf{p})$ we get $\mathbf{x} = (6, 6)$. Since the initial endowment is $\mathbf{e} = (10, 2)$, it means that market clearing requires $\mathbf{y} = (-4, 4)$. This is in the set $\left\{\mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0,\, y_2 = -y_1\right\}$ and so is a profit maximising output. So in conclusion we have found a Walrasian Equilibrium:

$$\mathbf{p} = (1, 1) \quad \mathbf{x} = (6, 6) \quad \mathbf{y} = (-4, 4)$$

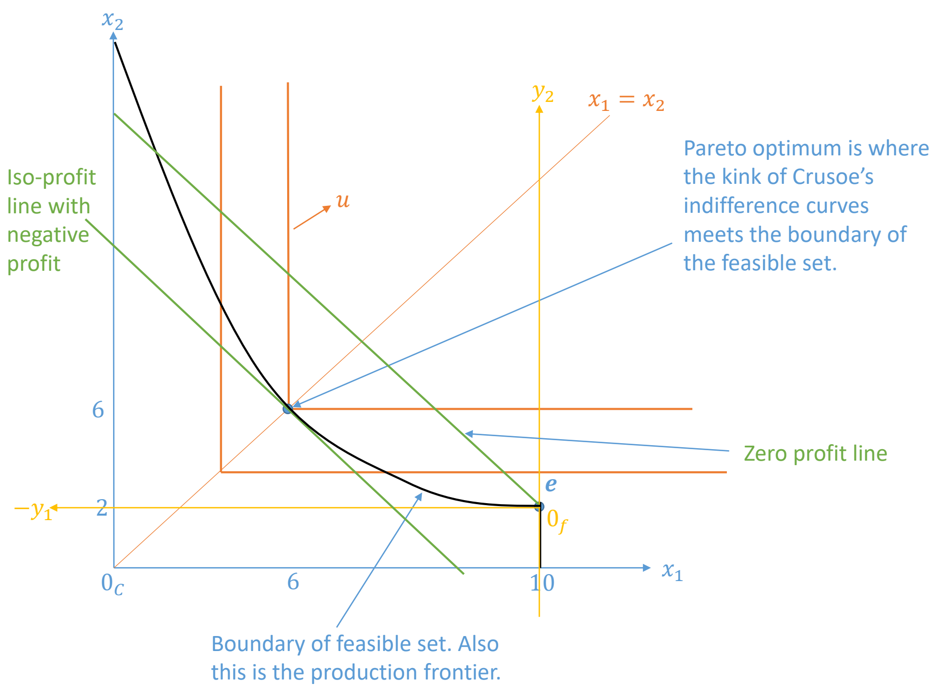


Part ii) With this production function we can still find the Pareto optimum by solving

$$\begin{aligned} & x_1 = x_2 \text{ and } x_2 = 2 + \frac{(10 - x_1)^2}{4} \\ \Longrightarrow & 4(x_1 - 2) = (10 - x_1)^2 \\ \Longleftrightarrow & 0 = 108 - 24x_1 + x_1^2 \\ \Longleftrightarrow & 0 = (x_1 - 6)(x_1 - 18) \end{aligned}$$

Since we have 10 units of good 1 in the endowment we must have $x_1 \in [0, 10]$ thus our Pareto optimum is $\mathbf{x} = (6, 6)$.

We can't have a Walrasian Equilibrium since we have no finite solution to the firm's optimisation problem when $p_2 > 0$. This is because no matter how steep the $-\frac{p_1}{p_2}$ iso-profit line is, it will always be possible to find a point on the production frontier with positive profits. Then profits keep increasing as we keep moving up the production frontier. Meanwhile we cannot have a Walrasian Equilibrium with $p_2 = 0$ because then the firm wouldn't produce anything and we would have excess demand of good 2. The diagram below illustrates the Pareto optimum and we can see that we cannot have the firm producing at this point as this is not a point of profit maximisation. If one tried to solve the profit maximissation problem by taking the first order condition, we would indeed find the Pareto efficient level of production, but as the diagram shows, the firm makes a negative profit there.



In-class question

Q3. Crusoe has 12 units of time (good 1) to allocate between work and leisure. If he works for k hours he can produce $2k$ units of the consumption good (good 2) and can freely dispose of each good. Crusoe has utility function $u : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ where

$$u(x_1, x_2) = \frac{e^{\sqrt{x_1 x_2}}}{17} + 2\pi (x_1 x_2)^5 - 6$$

a) Find the Pareto efficient bundle and draw a diagram to illustrate it.

b) We model this situation using Walrasian Equilibrium:

i) Assuming free-disposal, write down the Production set and find the firm's profit maximising output vector as a function of prices $\mathbf{y}(\mathbf{p})$ and the profits the firm makes. Draw a diagram to illustrate this.

ii) Find Crusoe's optimal bundle $\mathbf{x}(\mathbf{p})$.

iii) Write down the market clearing conditions and verify that Walras' law holds.

iv) Find the Walrasian equilibrium prices and allocation. Illustrate it on a diagram.

v)[1] Consider the case where Crusoe has some good 2 in the initial endowment, that is $\mathbf{e} = (12, c)$ for $c > 0$. Describe what would happen to equilibrium prices as c increases and give some intuition for this.

Solution: To start with observe that we could also represent the preferences u by $v(x_1, x_2) = x_1 x_2$ since u is an increasing transformation of v. Or for any 2 bundles $\mathbf{x}, \hat{\mathbf{x}}$, we have

$$u(\mathbf{x}) \geq u(\hat{\mathbf{x}}) \iff v(\mathbf{x}) \geq v(\hat{\mathbf{x}})$$

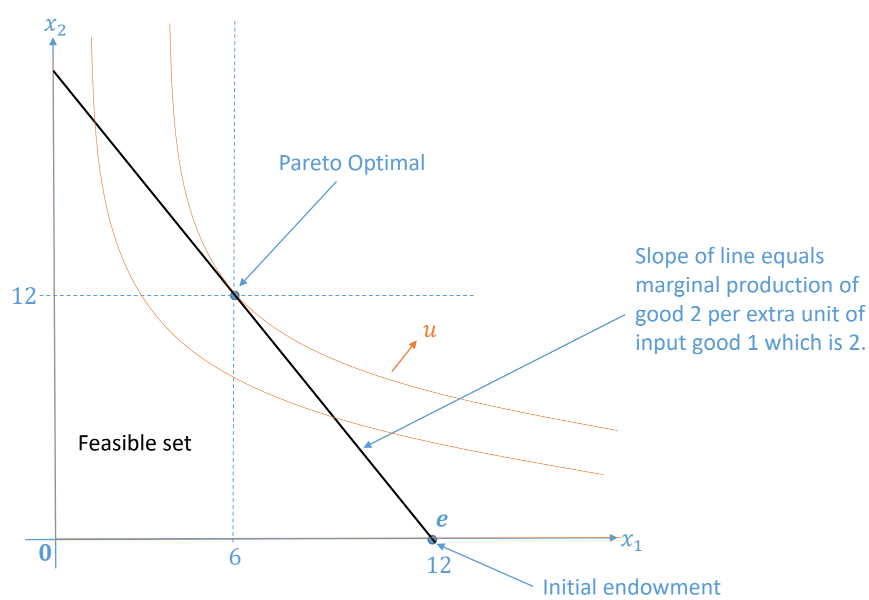
This is because both u and v only care about the product of the two goods, $x_1 x_2$. For reasons of simplicity, we choose to work with v instead of u from now on.

a) Crusoe will optimise where he does not freely dispose (waste) either good and so solves

$$\max_{k \geq 0} v = x_1 x_2 \text{ s.t. } x_1 = 12 - k,\ x_2 = 2k$$

which gives solution $k = 6$ and thus $\mathbf{x} = (6, 12)$. Note that at this point, the slope of the indifference curve equals the slope of the production function which is Crusoe's Marginal Product of Labour (MPL).

$$MRS_{1,2} = \frac{x_2}{x_1} = 2 = \frac{d}{dk}(2k) = MPL$$



b) We use the model as in Lecture 4: we have a firm with production function determined by Crusoe's marginal product of labour which buys labour off Crusoe to produce good 2. Crusoe gets money from selling his labour plus any profits the firm makes, which he spends on buying good 2 from the firm.

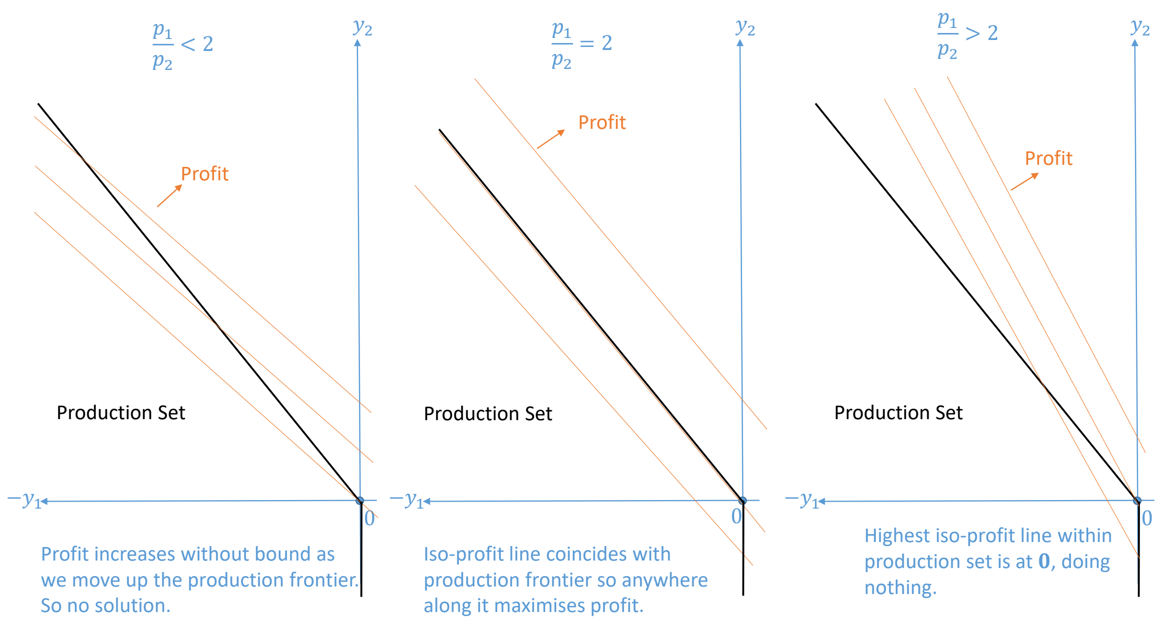
i) Since we assumed free disposal, the firm has technology

$$Y = \left\{ \mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0,\ y_2 \leq -2y_1 \right\}$$

In solving the profit maximisation problem, we have 3 cases, depending on the slope of the iso-profit lines compared to the slope of the production frontier:

As seen from the diagrams below we get

$$\mathbf{y}(\mathbf{p}) = \begin{cases} \emptyset & \frac{p_1}{p_2} < 2 \\ \left\{ \mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0,\ y_2 = -2y_1 \right\} & \frac{p_1}{p_2} = 2 \\ \mathbf{0} & \frac{p_1}{p_2} > 2 \end{cases}$$



ii) From the $\mathbf{y}(\mathbf{p})$ found above, we can see that for $\frac{p_1}{p_2} < 2$ profits are unbounded. Since these profits should be added into the income side of Crusoe's budget constraint, it means his UMP is not well-defined for $\frac{p_1}{p_2} < 2$. So in what follows, I assume $\frac{p_1}{p_2} \geq 2$. In all these cases the firm makes no profit and so Crusoe solves

$$\max_{\mathbf{x} \in \mathbb{R}^2_{\geq 0}} v = x_1 x_2 \text{ s.t. } p_1 x_1 + p_2 x_2 \leq 12 p_1$$

Using the knowledge that with this utility function we spend half our income on each good, we get the solution

$$\mathbf{x} = \left(6, \frac{6p_1}{p_2}\right)$$

iii) The market clearing conditions are:

$$z_1(\mathbf{p}) = 6 - y_1 - 12 = 0 \quad \text{(Good 1)}$$

$$z_2(\mathbf{p}) = \frac{6p_1}{p_2} - y_2 = 0 \quad \text{(Good 2)}$$

Walras' Law says $\mathbf{p}.\mathbf{z}(\mathbf{p}) = 0$, which we can confirm:

$$\mathbf{p}.\mathbf{z}(\mathbf{p}) = p_1(6 - y_1 - 12) + p_2\left(\frac{6p_1}{p_2} - y_2\right)$$

For $\frac{p_1}{p_2} = 2$, we have $y_2 = -2y_1$ thus

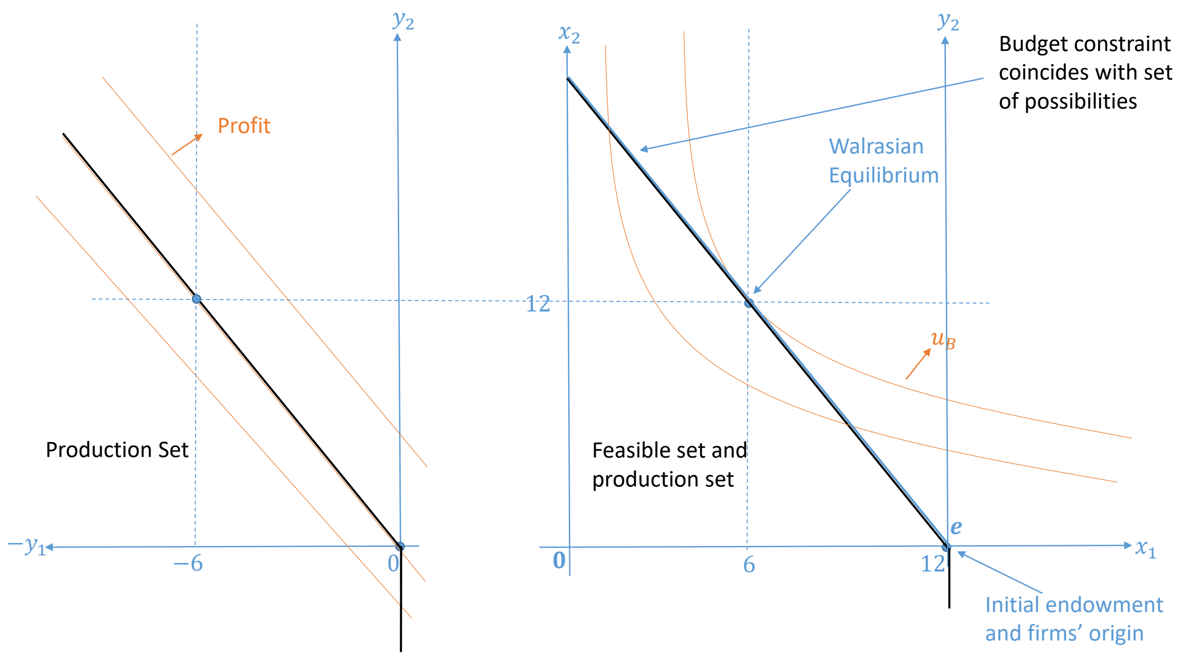
$$\mathbf{p}.\mathbf{z}(\mathbf{p}) = 2p_2(6 - y_1 - 12) + p_2\left(\frac{6(2p_2)}{p_2} + 2y_1\right) = 0$$

For $\frac{p_1}{p_2} > 2$, we have $y_1 = y_2 = 0$ thus

$$\mathbf{p}.\mathbf{z}(\mathbf{p}) = p_1(6 - 12) + p_2\left(\frac{6p_1}{p_2}\right) = 0$$

iv) By market clearing of good 1, $y_1 = -6$. Thus $\frac{p_1}{p_2} = 2$ and $y_2 = 12$. Note this also clears the market of good 2, confirming our answer. Thus the Walrasian Equilibrium is:

$$\mathbf{p} = (2, 1) \quad \mathbf{x} = (6, 12) \quad \mathbf{y} = (-6, 12)$$



v) In general, one expects that as c increases, meaning good 2 becomes less scarce, its price will fall and hence $\frac{p_1}{p_2}$ increases. However in this particular instance, due to the nature of the production technology, equilibrium prices can only be of the form $\frac{p_1}{p_2} = 2$ or $\frac{p_1}{p_2} > 2$. Intuitively, for small $c > 0$, the Walrasian Equilibrium will be similar to the above where $\frac{p_1}{p_2} = 2$ and the firm transforms some units of good 1 into good 2. As c increases, the amount of production the firm needs to do so that markets clear decreases. While if c was really high, then there would be no role for the firm at all. Between the two, there is a critical value of c which is the upper bound on the values of c for which we have $\frac{p_1}{p_2} = 2$ in equilibrium. At this level of c, the firm will produce nothing. So we need that Crusoe is maximising utility at the initial endowment and $\frac{p_1}{p_2} = 2$, so we let $\mathbf{p} = (2, 1)$. For Crusoe to be solving his UMP we need the bang per buck of each good to be the same:

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} \iff \frac{x_2}{2} = \frac{x_1}{1}$$

We've fixed the endoment of good 1 at 12 and so the critical value of c is $c = 24$. So for $c \in [0, 24]$ the equilibrium price ratio remains at $\frac{p_1}{p_2} = 2$. For $c > 24$ we need prices to be such that Crusoe's initial endowment solves his UMP and this will require $\frac{p_1}{p_2} > 2$ and increasing in c. In this latter case we have a rather trivial economy with no trade becasue the only entity Crusoe could trade with (the firm) doesn't have a production technology that can help Crusoe.

Revision tip: Give yourself more examples like Q1, 2 and Q3 where we have different preferences and production functions and perhaps an endowment where Crusoe already has some of the consumption good. Ask yourself the same things as in Q3. If you work in groups on such exercises, you can check each others' answers.

Post-class question

Short essay question: Discuss the strengths and limitations of the Walrasian Equilibrium model.

Sketch Solution: The strengths are that it is a relatively simple compelling model to explain how prices will fluctuate with the underlying factors like agents' preferences, firm's technology and how scarce or plentiful different commodities are in the economy. It gives a powerful narrative to explain several effects:

i) At a given price schedule, if there is excess demand of a good then this is because its price is too low and we would expect it to increase (relative to the prices of other goods).

ii) At a given price schedule, if there is excess supply of a good then this is because its price is too high and we would expect it to decrease (relative to the prices of other goods).

iii) If a good becomes scarcer, we would expect it's price to increase; while if a good becomes more plentiful, we would expect its price to decrease. We can also apply this logic to explain why the price of some goods (like platinum) is so much higher than the price of other goods (like copper).

For limitations, any model is only as good as its assumptions and so the most natural attack would be to look at the assumptions (I gave you a large swathe of text in lecture notes 3 on assumptions in pure exchange economies and some additional assumptions in Lecture notes 4 when we add production. One could talk about Walrasian Equilibria not necessarily existing or being unique. A good answer would take a balanced view of the strengths and weaknesses and maybe conclude by saying something like "it provides a good base model on top of which further research can build, if one wants to try to allow for some of the assumptions mentioned when discussing the limitations.