EC202 Term 1 Problem set 5

November 7, 2022

Pre-class Questions

Not covered in seminar unless time permits.

Q1. Consider a 2×2 pure exchange economy where there are 2 units of good 1 and 2 units of good 2. Let preferences be defined by the following utility functions that map $\mathbb{R}^2_{\geq 0}$ to $\mathbb{R} \cup \{-\infty\}$.[1]

$$u_A = \begin{cases} x_{A1} + \ln x_{A2} & x_{A2} > 0 \\ -\infty & x_{A2} = 0 \end{cases} \quad u_B = \begin{cases} \ln x_{B1} + \ln x_{B2} & x_{B1}x_{B2} > 0 \\ -\infty & x_{B1}x_{B2} = 0 \end{cases}$$

a) Find the Pareto Set and illustrate it on an Edgeworth box.

b) Starting from any initial allocation, will all Walrasian Equilibria be Pareto efficient?

c) Can every Pareto efficient allocation be sustained as a Walrasian Equilibrium?

Solution: Each agent has convex preferences so comparing marginal rates of substitution is the correct method, as one can confirm by sketching indifference curves on an Edgeworth box.

$$MRS^A_{12} = \frac{1}{x_{A2}^{-1}} = x_{A2} \quad x_{A2} \neq 0$$

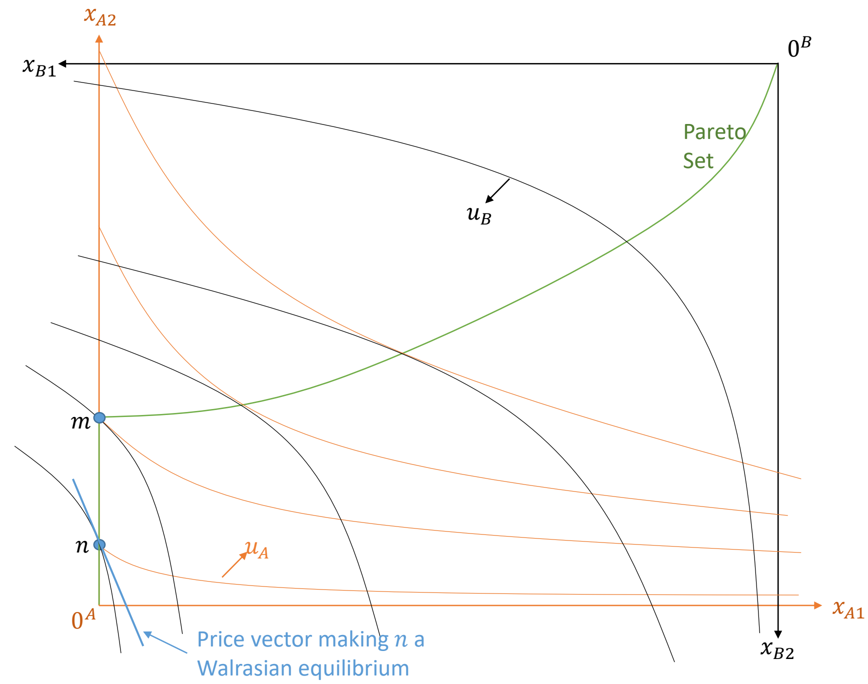
$$MRS^B_{12} = \frac{x_{B1}^{-1}}{x_{B2}^{-1}} = \frac{x_{B2}}{x_{B1}} \quad x_{B1}, x_{B2} \neq 0$$

Assuming $x_{A2}, x_{B1}, x_{B2} \neq 0$: Setting MRS for our agents equal to one another and substituting in $x_A + x_B = (2, 2)$ we can get x_{A1} in terms of x_{A2}

$$\begin{aligned} MRS^A_{12} = MRS^B_{12} &\iff x_{A2} = \frac{2 - x_{A2}}{2 - x_{A1}} \\ &\iff x_{A2}(2 - x_{A1}) = 2 - x_{A2} \\ &\iff x_{A1} = 2 - \frac{2 - x_{A2}}{x_{A2}} = 3 - \frac{2}{x_{A2}} \end{aligned}$$

[1] Ideally utility functions should map to $\mathbb{R}$ but I have extended this to $\mathbb{R} \cup \{-\infty\}$because I want allocations with $x_{A2} = 0$ or $x_{B1}x_{B2} = 0$ to be on the lowest possible indifference curves.

We can use this formula to plot a curve on our Edgeworth box. Note that with this formula $x_{A1} \geq 0 \iff x_{A2} \geq \frac{2}{3}$ and so we can't use this formula for $x_{A2} < \frac{2}{3}$ as it would direct us outside the Edgeworth box. So we have a line of allocations on the left boundary of the Edgeworth box, where at each of these allocations $MRS_{12}^{A} \leq MRS_{12}^{B}$



Ideally it should be drawn with indifference curves of same slope at all interior Pareto at optima and $m = (\mathbf{x}_A, \mathbf{x}_B) = \left(\left(0, \frac{2}{3}\right), \left(2, \frac{4}{3}\right)\right)$ and the slope of Andy's indifference curves is only dependent on x_{A2}, not x_{A1}. But at all other Pareto optima on the left boundary like n Bob's indifference curves are steeper than Andy's. The Pareto Set is

$$\left\{(\mathbf{x}_A, \mathbf{x}_B) \in \mathbb{R}_{\geq 0}^{2\times 2} \mid x_{A2} = \frac{x_{B2}}{x_{B1}}, \mathbf{x}_A + \mathbf{x}_B = (2,2)\right\} \cup \{((2,2), (0,0))\} \cup \{((0,0), (2,2))\}$$
$$\cup \left\{(\mathbf{x}_A, \mathbf{x}_B) = ((0,c), (2, 2-c)) \mid c \in \left[0, \frac{2}{3}\right]\right\}$$

b) Yes. Both agents preferences are locally non-satiated (since increasing consumption slightly of both goods must increase utility) and therefore we can apply the First Fundamental Theorem of Welfare Economics.

c) Yes. We can apply the Second Fundamental Theorem of Welfare Economics after arguing that both agents have convex and continuous preferences. Or we could show this directly. For m and all interior Pareto optima, we have a Walrasian Equilibrium at that point where the initial allocation is the Pareto optimal allocation we wish to support and the price ratio is equal to both agents' MRS. At boundary Pareto optima like n we again let out initial allocation be the Pareto optimum we want to support (in this case n). We set the price ratio equal to Bob's MRS so that Bob is choosing his optimal bundle here and note that Andy is also choosing his optimal bundle at this corner solution, since at point n, since indifference curve is shallower than budget constraint, we know

$$MRS_{1,2}^{A} = \frac{MU_1^A}{MU_2^A} < \frac{p_1}{p_2} \iff \frac{MU_1^A}{p_1} < \frac{MU_2^A}{p_2}$$

So Andy is correct not to consume any good 1.

Q2. Consider a 2×2 pure exchange economy where the initial endowment is $\mathbf{e}_A = (2,1)$,$\mathbf{e}_B = (3,2)$ and preferences are $u_A = \sqrt{x_{A1}x_{A2}}$, $u_B = \min\{x_{B1}, x_{B2}\}$

a) Find the Pareto Set and illustrate it on an Edgeworth box.

b) On your Edgeworth box find the lens of points of allocations that Pareto dominate the initial endowment and find so identify a set of allocations at which all Walrasian Equilibria must lie.

c) Find both agents' optimal bundles as a function of prices and find the Walrasian Equilibrium.

d) Use your Edgeworth box to illustrate the Walrasian Equilibria.

e) Show how we could have alternatively calculated the Walrasian Equilibrium by using the Edgeworth box and arguing:

i) The allocation must lie on the line $x_{B1} = x_{B2}$

ii) The equilibrium must satisfy slope of Andy's indifference curve equal to price ratio

iii) The budget line must intersect initial endowment and Walrasian Equilibrium

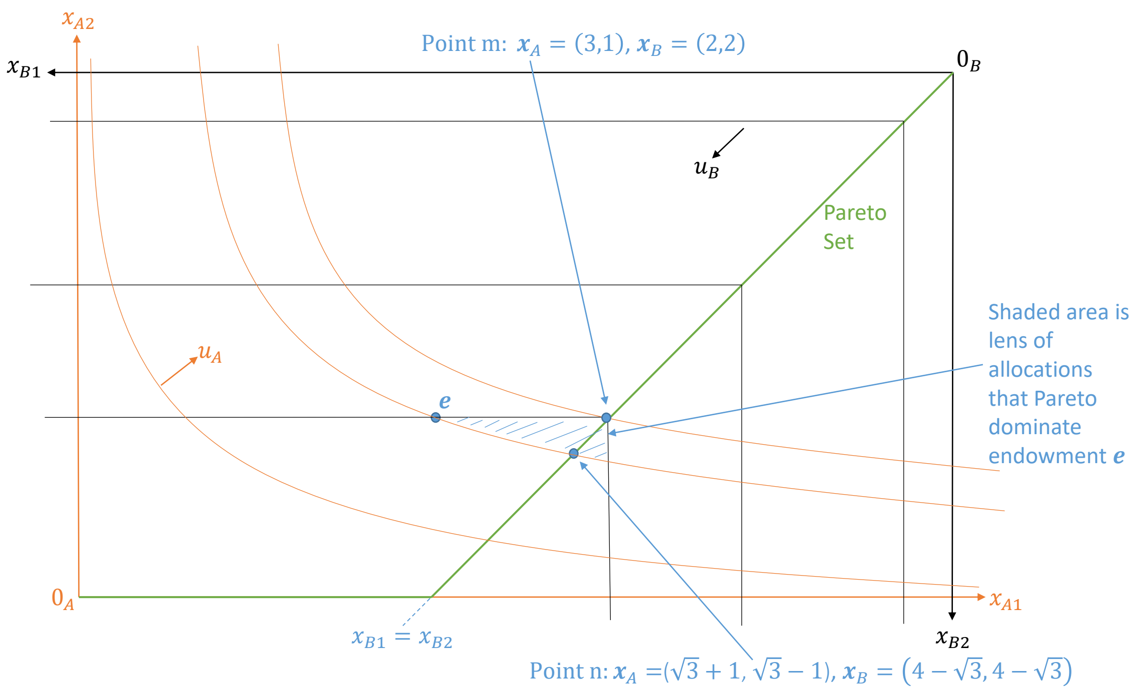
iv) Use i), ii) and iii) to find the Walrasian Equilibrium.

f) What does the Second Fundamental Theorem of Welfare Economics say? Does it hold in this case?

(You will need the formula for solving quadratic equations for mathematical parts of this question.)

Solution:
Mathematically the Pareto set is

$$\left\{(\mathbf{x}_A, \mathbf{x}_B) \in \mathbb{R}_{\geq 0}^{2\times 2} \mid x_{B1} = x_{B2} \in [0,3], \mathbf{x}_A + \mathbf{x}_B = (5,3)\right\} \cup \{(\mathbf{x}_A, \mathbf{x}_B) = ((c,0),(5-c,3)) \mid c \in [0,2]\}$$



Graphically this occurs on the line $x_{B1} = x_{B2}$, where Bob's indifference curves are kinked. The line of points along the bottom axis between the kink and 0_A are also Pareto efficient: along this line Bob is getting maximum utility and Andy is getting minimum utility. There is nothing that Pareto dominates these allocations, because if we made Andy better off, we must be giving him some good 2 and thus make bob worse off.

b) The diagram above gives the lens of points that Pareto dominate the initial endowment. The possible set of allocations for Walrasian Equilibria lie in this lens and along the Pareto set line. We can find the lowest of these points algebraically: it must give Andy some utility as initial endowment, which is $u_A = \sqrt{2}$ but also solves $x_{B1} = x_{B2}$. So $\mathbf{x}_A$ is given by

$$x_{A1}x_{A2} = 2 \text{ and } 5 - x_{A1} = 3 - x_{A2}$$

which gives $\mathbf{x}_A = \left(\sqrt{3}+1, \sqrt{3}-1\right)$, hence the labeling on the diagram - point n. The upper most point (point m) is on the same indifference curve of Bob but at $x_{B1} = x_{B2}$, thus at $\mathbf{x}_B = (2,2)$. So we can write the set of possible Walrasian Equilibrium allocations as the set of points between these two:

$$\left\{(\mathbf{x}_A, \mathbf{x}_B) = \alpha\left[\left(\sqrt{3}+1, \sqrt{3}-1\right), \left(4-\sqrt{3}, 4-\sqrt{3}\right)\right] + (1-\alpha)\left[(3,1), (2,2)\right] \mid \alpha \in [0,1]\right\}$$

c) Let prices be (p_1, p_2) then Andy solves

$$\max_{\mathbf{x}_A \in \mathbb{R}^2_{\geq 0}} \sqrt{x_{A1} x_{A2}} \text{ subject to } p_1 x_{A1} + p_2 x_{A2} \leq 2p_1 + p_2$$

From the diagram, we can see that, subject to any straight line budget constraint this must be solved when MRS equals price ratio (bang per buck of good 1 equals bang per buck of good 2) and where the budget constraint holds with equality. This gives us

$$p_1 x_{A1} = p_2 x_{A2} \text{ and } p_1 x_{A1} + p_2 x_{A2} = 2p_1 + p_2$$

Thus we get

$$\mathbf{x}_A (\mathbf{p}) = \left(1 + \frac{p_2}{2p_1}, \frac{p_1}{p_2} + \frac{1}{2}\right)$$

Bob solves

$$\max_{\mathbf{x}_B \in \mathbb{R}^2_{\geq 0}} \min \{x_{B1}, x_{B2}\} \text{ subject to } p_1 x_{B1} + p_2 x_{B2} \leq 3p_1 + 2p_2$$

This is solved where $x_{B1} = x_{B2}$ and $p_1 x_{B1} + p_2 x_{B2} = 3p_1 + 2p_2$, which gives

$$\mathbf{x}_B (\mathbf{p}) = \left(\frac{3p_1 + 2p_2}{p_1 + p_2}, \frac{3p_1 + 2p_2}{p_1 + p_2}\right)$$

At this stage, we can check our working thus far by verifying that Walras' law holds:

$$\begin{aligned}
p_1 z_1 + p_2 z_2 &= p_1 \left[1 + \frac{p_2}{2p_1} + \frac{3p_1 + 2p_2}{p_1 + p_2} - 5\right] + p_2 \left[\frac{p_1}{p_2} + \frac{1}{2} + \frac{3p_1 + 2p_2}{p_1 + p_2} - 3\right] \\
&= -4p_1 + \frac{p_2}{2} + p_1 - \frac{5p_2}{2} + (p_1 + p_2) \frac{3p_1 + 2p_2}{p_1 + p_2} = 0
\end{aligned}$$

Now market clearing requires:

$$1 + \frac{p_2}{2p_1} + \frac{3p_1 + 2p_2}{p_1 + p_2} = 5 \qquad \text{(Good 1)}$$

$$\frac{p_1}{p_2} + \frac{1}{2} + \frac{3p_1 + 2p_2}{p_1 + p_2} = 3 \qquad \text{(Good 2)}$$

Comparing these conditions, we see we need:

$$1 + \frac{p_2}{2p_1} = \frac{p_1}{p_2} + \frac{1}{2} + 2$$

Since it is clear both prices must be positive (or Andy demands infinite of the good that is free), we can normalise $p_2 = 1$ and solve to get $p_1 = \frac{\sqrt{17}-3}{4}$. Alternatively we could have normalised $p_1 = 1$. So we get

$$\text{Equilibrium prices: } \mathbf{p} = \left(\frac{\sqrt{17}-3}{4}, 1\right) \text{ or } \mathbf{p} = \left(1, \frac{\sqrt{17}+3}{2}\right)$$

Substituting this back into demands we get the Walrasian Equilibrium allocation is

$$\mathbf{x}_A = \left(\frac{\sqrt{17}+7}{4}, \frac{\sqrt{17}-1}{4}\right)$$

$$\mathbf{x}_B = \left(\frac{13-\sqrt{17}}{4}, \frac{13-\sqrt{17}}{4}\right)$$

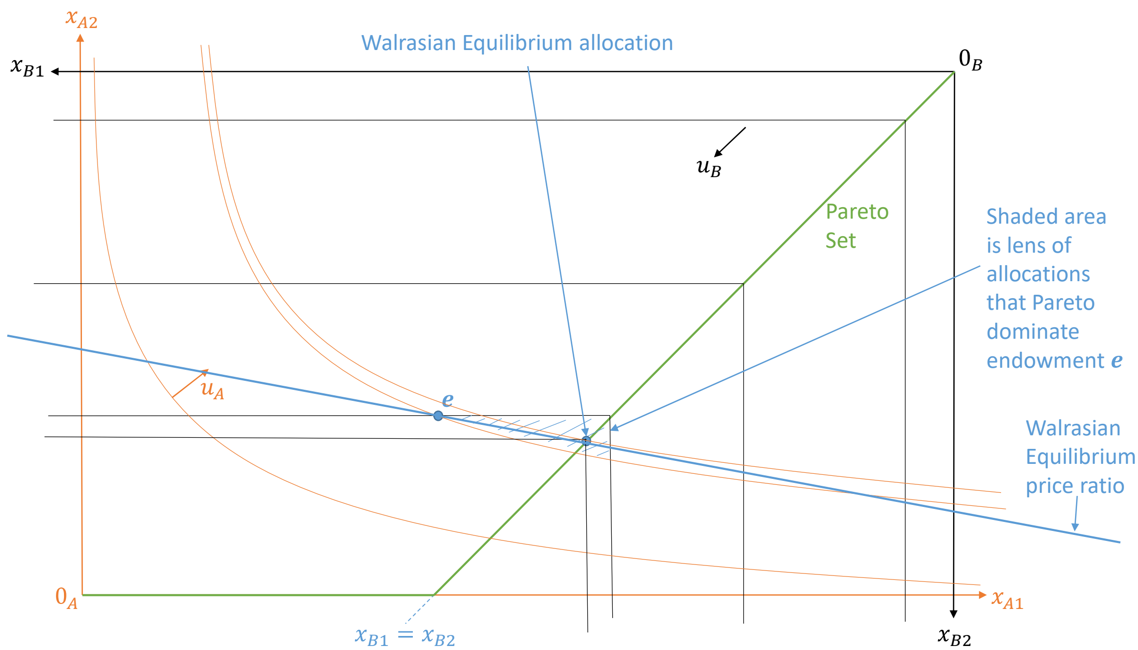
As a check on your algebra, you can verify that both Andy and Bob are better off here than under the initial endowment, where both got utility of 2. Their new utilities are

$$u_A = \frac{5+3\sqrt{17}}{8} > 2 \qquad u_B = \frac{13-\sqrt{17}}{4} > 2$$

Furthermore the Walrasian Equilibrium allocation is Pareto optimal as well as making both players better off and so lies on the set identified in b)

d) The Edgeworth box should display this information:

- Each agent maximises utility subject to budget constraint at same point in Edgeworth box.
  - For Andy budget line should be tangent to indifference curve.
  - For Bob, budget line should go through kink of indifference curve.
- Walrasian Equilibrium allocation lies in set identified in b). That is
  - lies on Pareto Set.
  - lies on higher indifference curve than initial allocation for both players.



e) i) Both Andy and Bob have locally non-satiated preferences so we can invoke the First Fundamental Theorem of Welfare Economics: if a Walrasian Equilibrium exists it must be Pareto efficient and make both players wakly better off than initial endowment. All elements of the contract curve satisfy $x_{B1} = x_{B2}$ and so a Walrasian Equilibrium must too if it exists.

ii) By definition of Walrasian Equilibrium Andy must be solving his UMP and given Andy's Cobb-Douglas preferences this must happen at an allocation where slope of indifference curve is equal to slope of budget constraint.

iii) Both Andy and Bob have locally non-satiated preferences so they each solve their UMP by spending all available money, thus at a point along the budget line. Therefore the budget line must go through each agents' optimal bundle, and if markets clear (as happens in Walrasian Equilibrium), then this defines the same point of the Edgeworth box. Meanwhile the initial endowment must be on the budget line by definition of budget line.

iv) From the above we know that if a Walrasian Equilibrium $(\mathbf{p}, \mathbf{x})$ exists, then by part i) (UMP of Bob and market clearing) it must satisfy:

$$5 - x_{A1} = 3 - x_{A2} \quad (1)$$

By ii) (UMP of Andy) it must satisfy

$$\frac{x_{A2}}{x_{A1}} = \frac{p_1}{p_2} \quad (2)$$

By iii) and calculating the slope of the budget constraint line it must satisfy

$$\frac{\text{change in good 2}}{\text{change in good 1}} = \frac{x_{A2} - 1}{x_{A1} - 2} = -\frac{p_1}{p_2}$$

This simplifies to

$$\frac{p_1}{p_2} = \frac{1 - x_{A2}}{x_{A1} - 2} \quad (3)$$

We can solve these 3 equations simultaneously to give $x_{A1} = \frac{\sqrt{17}+7}{4}$. Then we can substitute back into the above to calculate x_{A2}, $\frac{p_1}{p_2}$ and by market clearing $\mathbf{x}_B$. We get exactly the same as we calculated in c).

f) Preferences of both agents are convex, locally non-satiated and continuous. Therefore the Second Fundamental Theorem of Welfare Economics holds and says that every Pareto optimal allocation can be supported as a Walrasian Equilibrium. We can also show this directly: For allocations along the $x_{B1} = x_{B2}$ line, we let the initial endowment be the allocation we want to support and price ratio equal to Andy's MRS. For allocations along the bottom of the Edgeworth box where $x_{A2} = 0$, again we set initial allocation equal to the allocation we wish to support and we need a horizontal budget constraint, that is $p_1 = 0$, $p_2 > 0$.

In-class questions

For these questions we apply the Welfare Theorems to Robinson Crusoe economies. For the 2nd Welfare Theorem, note that with only one consumer there is no possible reallocation of initial resources. With three questions this might be too long for the seminar tutor to go through all in detail, In class, you will hopefully cover Q3 in detail and then the ideas behind Q4 and Q5.

Q3. Crusoe has 10 units of time (good 1) to allocate between work and leisure and 2 units of the consumption good (good 2). If he works for k hours he can produce $2\sqrt{k}$ units of the consumption good and can freely dispose of each good. Crusoe has utility function $u : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ where

$$u(x_1, x_2) = x_1^{\frac{1}{10}} x_2^{\frac{2}{10}}$$

a) Find the Pareto efficient bundle(s) and draw a diagram to illustrate them. (Hint: in this case algebra gets messy, so just show that the solution of Crusoe working 4 hours satisfies the first order condition.)

b) Write down the firm's production set.

c) What if anything can we learn about the Walrasian Equilibrium or Equilibria from the First Welfare Theorem?

d) What if anything can we learn about the Walrasian Equilibrium or Equilibria from the Second Welfare Theorem?

e) Find the Walrasian Equilibrium or Equilibria.

Solution: To start with, note that the utility function $u(x_1, x_2) = x_1^{\frac{1}{10}} x_2^{\frac{2}{10}}$ represents the same preferences as $v = x_1 x_2^2$ and so I will use the latter to make the algebra simpler.

a) Crusoe will optimise where he does not freely dispose (waste) either good and so letting $k \geq 0$ be the amount of time Crusoe devotes to labour, we solve

$$\max_{k \geq 0} v = x_1 x_2^2 \text{ s.t. } x_1 = 10 - k,\ x_2 = 2 + 2\sqrt{k}$$

Normally, I would solve by writing $v(k)$ and taking $\frac{dv}{dk} = 0$. While you can apply that method, the algebra gets a little messy and so it's easier to set slope of production function equal to slope of indifference curve:

$$\begin{aligned} |\text{MRS}| &= \frac{d}{dk}\left(2\sqrt{k}\right) \\ &\Longleftrightarrow \frac{x_2^2}{2x_1x_2} = k^{-\frac{1}{2}} \\ &\Longleftrightarrow \sqrt{k} = \frac{2x_1}{x_2} = \frac{2(10-k)}{2+2\sqrt{k}} = \frac{10-k}{1+\sqrt{k}} \\ \text{subbing in} k = 4 &\Longrightarrow 2 = \frac{10-4}{1+2} \end{aligned}$$

So we get the solution $k = 4$ meaning our Pareto optimum is

$$(x_1, x_2) = (6, 6)$$

b) The firm has production set

$$Y = \left\{\mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0,\ y_2 \leq 2\sqrt{-y_1}\right\}$$

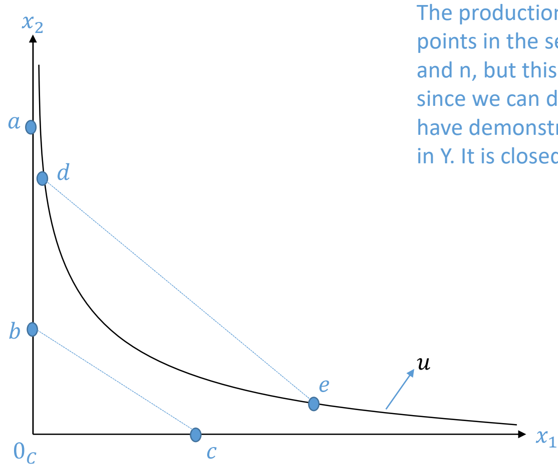
c) To apply the First Welfare Theorem, we need to show that preferences satisfy local non-satiation. We only have one utility function: $u(x_1, x_2) = x_1^{\frac{1}{10}} x_2^{\frac{2}{10}}$. Take any $\mathbf{x} \in \mathbb{R}^2_{\geq 0}$, and consider an $\varepsilon -$ ball around $\mathbf{x}$. For any $\varepsilon > 0$, the $\varepsilon -$ ball contains bundles with slightly more of both goods and any such bundle is strictly preferred to $\mathbf{x}$. Note that this argument actually shows the slightly stricter property of monotonicity. Alternatively, one could draw a diagram with indifference curves and shade in elements of the $\varepsilon -$ball which are preferred to $\mathbf{x}$, but to be rigorous enough should show this for points on both types of indifference curve: $u = 0$ and $u > 0$.

Since we have shown local non-satiation, we can conclude by the First Welfare Theorem, that all Walrasian Equilibria are Pareto efficient. As there is only one Pareto efficient allocation, this means that if a Walrasian Equilibrium exists, it must be at $(x_1, x_2) = (6, 6)$.

d) To apply the 2nd Welfare Theorem we need:

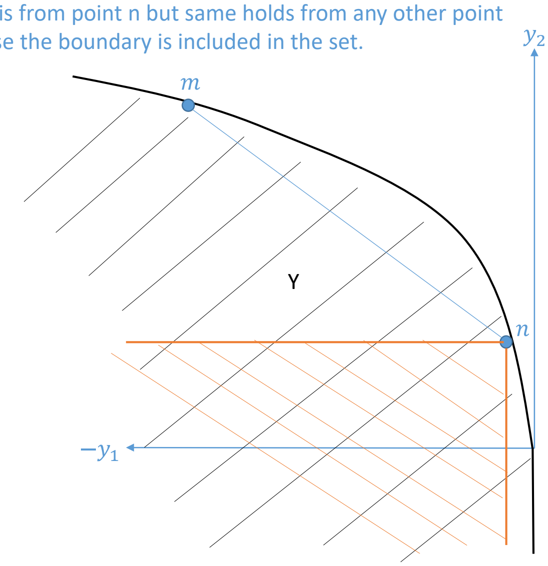
- Preferences are convex, continuous, locally non-satiated.
- Production sets are convex, closed and satisfy free disposal.

Local non-satiation has already been shown. Continuity is immediate as we have a continuous utility function. Covexity can be seen on the diagram below. We can see that the upper level sets are convex, or if we take any two bundles on the same indifference curve, the average of those bundles is weakly preferred to the original bundle. The diagram below also justifies that the three required properties of production sets hold too.



Bundles a,b,c are all along the $u_B = 0$ indifference curve. If we take a weighted average of b and c then all these bundles are strictly preferred to b and c. While averages of a and b are indifferent to a and b. For this reason, the $u_B = 0$ indifference curve is compatible with convexity but not strict convexity. Along $u_B > 0$ indifference curves we see averages are always strictly better than extremes as demonstrated at bundles d and e.

The production set is convex because the weighted average of any two points in the set remains in the set. I have demonstrated this with points m and n, but this would be true for any two points. It satisfies free disposal since we can dispose of units of one or both goods and remain in the set. I have demonstrated this from point n but same holds from any other point in Y. It is closed because the boundary is included in the set.



Since we have justified all the necessary assumptions, we can apply the 2nd Welfare Theorem: Every Pareto efficient allocation can be supported as a Walrasian Equilibrium for some reallocation of resources. Here there is only one Pareto efficient allocation - what makes Crusoe best off, which happens when $(x_1, x_2) = (6, 6)$. We also only have one possible allocation of resources - since there is only one consumer, Crusoe must own all the resources and the firm. Therefore starting from this allocation, we must have a Walrasian Equilibrium at the Pareto efficient allocation where $(x_1, x_2) = (6, 6)$.

e) For Walrasian Equilibrium, we need i) firm profit maximises, ii) Crusoe chooses optimal demand, iii) markets clear.

Firstly consider firm profit maximising: The firm's profit maximisation problem is $\max_{\mathbf{y} \in Y} \mathbf{p}.\mathbf{y}$. Letting the amount of input be k and substituting $y_1 = -k$

and $y_2 = 2\sqrt{k}$ we can solve this as follows:

$$\begin{aligned}
\pi = \mathbf{p}.\mathbf{y} &= -p_1 k + 2p_2\sqrt{k} \\
\frac{d\pi}{dk} = 0 &\iff -p_1 + p_2 k^{-\frac{1}{2}} = 0 \\
&\iff k = \left(\frac{p_2}{p_1}\right)^2 \\
&\iff \mathbf{y}\left(\mathbf{p}\right) = \left(-\left(\frac{p_2}{p_1}\right)^2, \frac{2p_2}{p_1}\right)
\end{aligned}$$

(We can argue that the first order condition is sufficient either by π being concave or by drawing a diagram and seeing that our maximum lies where the iso-profit line is tangential to the boundary of the production set.) The profit can be found by subbing $\mathbf{y}\left(\mathbf{p}\right)$ back into the profit function:

$$\pi = \mathbf{p}.\mathbf{y} = -p_1\left(\frac{p_2}{p_1}\right)^2 + p_2\left(\frac{2p_2}{p_1}\right) = \frac{p_2^2}{p_1}$$

Crusoe maiximises utility subject to budget constraint so solves:

$$\max_{\mathbf{x}\in\mathbb{R}^2_{\geq 0}} u = x_1^{\frac{1}{10}} x_2^{\frac{2}{10}} \text{ subject to } p_1x_1 + p_2x_2 \leq 10p_1 + 2p_2 + \frac{p_2^2}{p_1}$$

Crusoe maximises utility by spending $\frac{1}{3}$ of his income on good 1 and $\frac{2}{3}$ on good 2. This gives

$$\mathbf{x}\left(\mathbf{p}\right) = \left(\frac{10p_1 + 2p_2 + \frac{p_2^2}{p_1}}{3p_1}, \frac{2\left(10p_1 + 2p_2 + \frac{p_2^2}{p_1}\right)}{3p_2}\right)$$

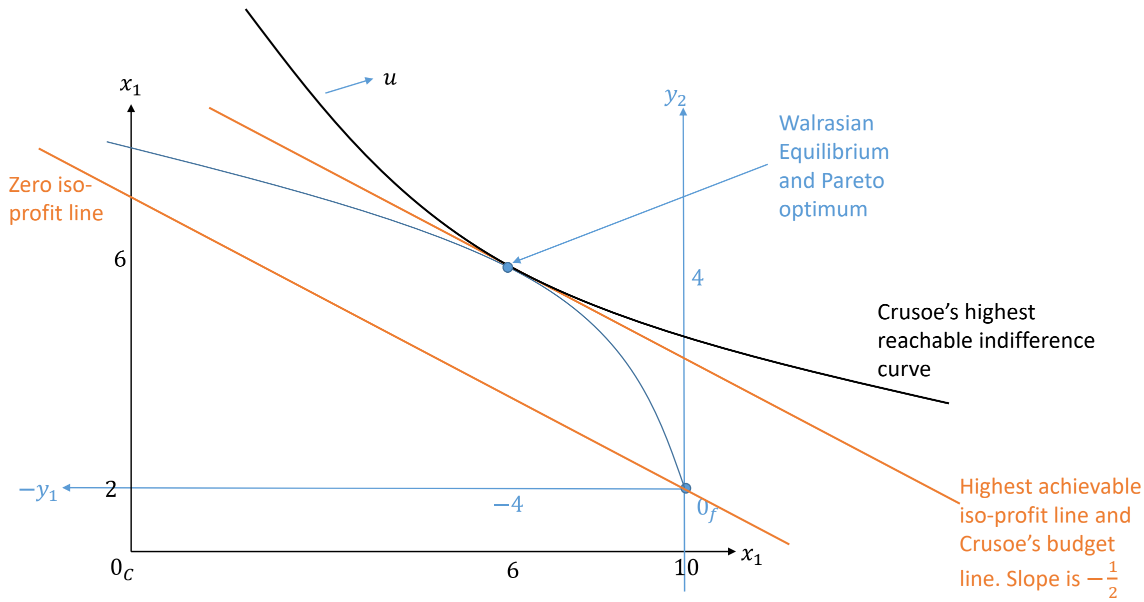
As we already know what the Walrasian Equilibrium allocation should be: we can just check that at this allocation, there are prices satisfying conditions i) to iii). We know we need $(x_1, x_2) = (6, 6)$ and so by market clearing we need $(y_1, y_2) = (-4, 4)$. This implies that $\mathbf{p} = (1, 2)$. The last thing to check is that subbing $\mathbf{p} = (1, 2)$ into $\mathbf{x}\left(\mathbf{p}\right)$ gives $(x_1, x_2) = (6, 6)$.

$$\begin{aligned}
\mathbf{x}\left(\mathbf{p}\right) &= \left(\frac{10p_1 + 2p_2 + \frac{p_2^2}{p_1}}{3p_1}, \frac{2\left(10p_1 + 2p_2 + \frac{p_2^2}{p_1}\right)}{3p_2}\right) \\
&\implies \mathbf{x}\left(1, 2\right) = \left(\frac{10 + 4 + 4}{3}, \frac{2\left(10 + 4 + 4\right)}{6}\right) = (6, 6)
\end{aligned}$$

Thus we have confirmed our Walrasian Equilibrium:

$$\mathbf{p} = (1, 2) \quad \mathbf{x} = (6, 6) \quad \mathbf{y} = (-4, 4)$$

By the First Welfare Theorem, this is the unique Walrasian Equilibrium allocation and therefore also the unique price ratio. Although students could check this by solving for Walrasian Equilibrium the same way as in Lecture 4 and Problem Set 4 - by writing market clearing conditions in terms of $\mathbf{x}(\mathbf{p})$ and $\mathbf{y}(\mathbf{p})$, normalising $p_1 = 1$ and calculating $p_2 = 2$.



Q4. Repeat Q3 but with changing the preferences and production technology to:

- Let Crusoe have preferences represented by $u(x_1, x_2) = 2x_1 + x_2$.
- Let Crusoe's production technology be the ability to transform k units of good 1 into $2k$ units of good 2.

Solution:

a) If Crusoe spends k units of time working to produce 2k units of good 2, relative to the initial endowment ($k = 0$), he loses $2k$ units of utility from good 1 and gains $2k$ units of utility from good 2 and so is indifferent between any such bundle. Thus anywhere along the boundary of the feasible set is Pareto efficient. Diagramatically the boundary of the feaisble set (production frontier) and the indifference curves of Crusoe are both straight lines of slope -2 and thus anywhere along the production frontier is a Pareto optimum. Thus the Pareto Set is

$$\left\{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \mid x_1 = 10 - k,\ x_2 = 2 + 2k,\ k \in [0, 1]\right\}$$

b) The firm has production set

$$Y = \left\{ \mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0,\, y_2 \leq -2y_1 \right\}$$

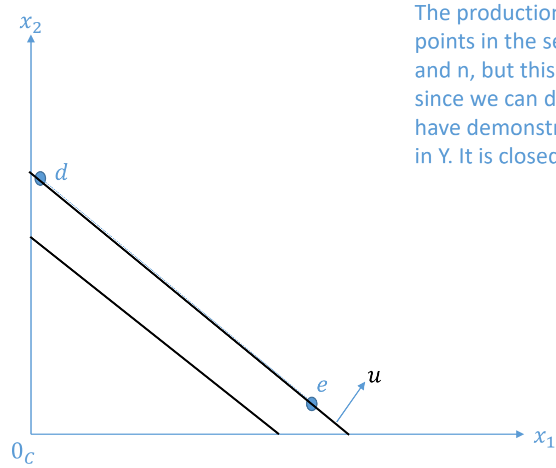
c) To apply the First Welfare Theorem, we need to show that preferences satisfy local non-satiation. We only have one utility function: $u(x_1, x_2) = 2x_1 + x_2$. Take any $\mathbf{x} \in \mathbb{R}^2_{\geq 0}$, and consider an $\varepsilon -$ ball around $\mathbf{x}$. For any $\varepsilon > 0$, the $\varepsilon -$ ball contains bundles with slightly more of both goods and any such bundle is strictly preferred to $\mathbf{x}$. Note that this argument actually shows the slightly stricter property of monotonicity. Alternatively, one could draw a diagram with indifference curves and shade in elements of the $\varepsilon -$ ball which are preferred to $\mathbf{x}$.

Since we have shown local non-satiation, we can conclude by the First Welfare Theorem, that all Walrasian Equilibria are Pareto efficient and so lies in the set identified in a).

d) To apply the 2nd Welfare Theorem we need:

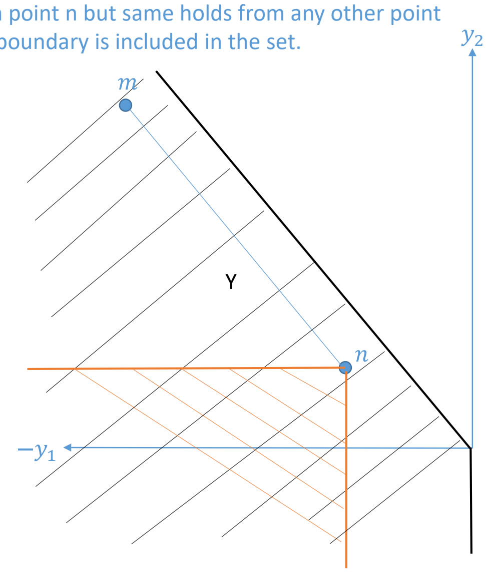
- Preferences are convex, continuous, locally non-satiated.
- Production sets are convex, closed and satisfy free disposal.

Local non-satiation has already been shown. Continuity is immediate as we have a continuous utility function. Covexity just holds since for any two bundles on the same indifference curve, the averages of those two bundles also lies on that indifference curve. We can also see that the upper level sets are convex. The diagram below also justifies that the three required properties of production sets hold too.



All the indifference curves are the same shape as each other. If we take two points on the same indifference curve like d and e then the weighted average of these two points is also on the same indifference curve.

The production set is convex because the weighted average of any two points in the set remains in the set. I have demonstrated this with points m and n, but this would be true for any two points. It satisfies free disposal since we can dispose of units of one or both goods and remain in the set. I have demonstrated this from point n but same holds from any other point in Y. It is closed because the boundary is included in the set.



Since we have justified all the necessary assumptions, we can apply the 2nd Welfare Theorem: Every Pareto efficient allocation can be supported as a Walrasian Equilibrium for some reallocation of resources. Here there are infinitely many Pareto efficient allocations as found in a). But still only one possible allocation of resources - since there is only one consumer, Crusoe must own all the resources and the firm. Therefore starting from this allocation, we must have the whole set of Pareto efficient allocations as Walrasian Equilibria.

e) In solving the profit maximisation problem, we have 3 cases, depending on the slope of the iso-profit lines compared to the slope of the production frontier: When the iso-profit lines are shallower than the production frontier ($\frac{p_1}{p_2} < 2$), there is no solution as profits keep increasing as we increase production. When the iso-profit lines are steeper than the production frontier ($\frac{p_1}{p_2} > 2$), the profit maximising output occurs at the firm's origin - ie doing nothing. When they are the same slope as each other, all points along the production frontier lie along the same iso-profit line (the zero iso-profit line) and the firm can choose any one of these. Thus we get:

$$\mathbf{y}\left(\mathbf{p}\right) = \begin{cases} \emptyset & \frac{p_1}{p_2} < 2 \\ \left\{\mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0,\, y_2 = -2y_1\right\} & \frac{p_1}{p_2} = 2 \\ \mathbf{0} & \frac{p_1}{p_2} > 2 \end{cases}$$

Crusoe maiximises utility subject to budget constraint so solves:

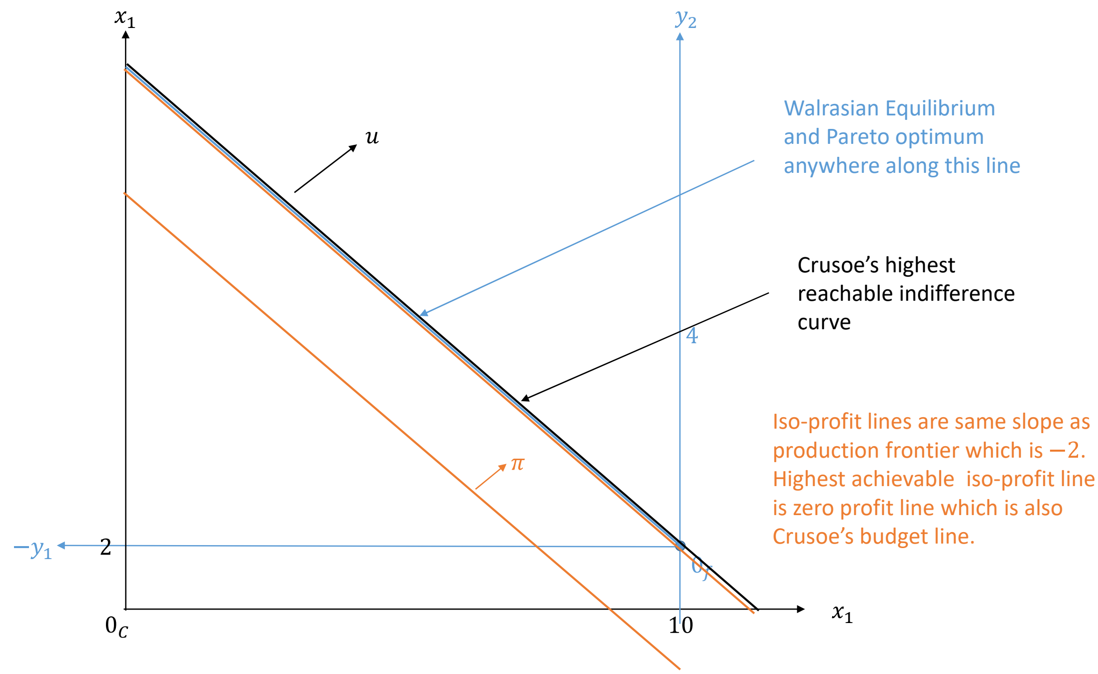
$$\max_{\mathbf{x}\in\mathbb{R}^2_{\geq 0}} u = 2x_1 + x_2 \text{ subject to } p_1x_1 + p_2x_2 \leq 10p_1 + 2p_2$$

Crusoe maximises utility by spending his entire income on whichever good gives the higher bang per buck: when $\frac{p_1}{p_2} < 2$ this is good 1; when $\frac{p_1}{p_2} > 2$ this is good 2; when $\frac{p_1}{p_2} = 2$ the bang per buck of each good is always the same and so any allocation along the budget line is a solution. So we get:

$$\mathbf{x}(\mathbf{p}) = \begin{cases} \left(\frac{10p_1+2p_2}{p_1}, 0\right) & \frac{p_1}{p_2} < 2 \\ \left\{\lambda\left(\frac{10p_1+2p_2}{p_1}, 0\right) + (1-\lambda)\left(0, \frac{10p_1+2p_2}{p_2}\right) \mid \lambda \in [0,1]\right\} & \frac{p_1}{p_2} = 2 \\ \left(0, \frac{10p_1+2p_2}{p_2}\right) & \frac{p_1}{p_2} > 2 \end{cases}$$

Looking for Walrasian Equilibrium in the three different cases: i) $\frac{p_1}{p_2} < 2$ is impossible due to the firm having no profit maximising output. iii) $\frac{p_1}{p_2} > 2$ is impossible because we then get excess demand of good 2 and excess supply of good 1. In case ii) Both the firm and Crusoe have infinitely many points solving their optimisation problems, but as long as they take compatible actions with each other we get a Walrasian Equilibrium. That is for each $k \in [0, 10]$ there is an Equilibrium of the following form where the firm demands k units of labour and Crusoe demands $10 - k$ units of leisure:

$$\mathbf{p} = (2, 1) \quad \mathbf{x} = (10-k, 2+2k) \quad \mathbf{y} = (-k, 2k)$$



Q5. Repeat Q3 but with changing the preferences and production technology to:

- Let Crusoe have preferences represented by $u(x_1, x_2) = \min\{2x_1, x_2\}$.
- Let Crusoe's production technology be the ability to transform k units of good 1 into $\frac{5k^2}{8}$ units of good 2.

Solution:

a) Crusoe can spend k units of time working to produce $\frac{5k^2}{8}$ units of good 2. Crusoe will optimise where he does not freely dispose (waste) either good and so letting $k \geq 0$ be the amount of time Crusoe devotes to labour, we solve

$$\max_{k\geq 0} u = \min\{2x_1, x_2\} \text{ s.t. } x_1 = 10 - k,\ x_2 = 2 + \frac{5k^2}{8}$$

This has solution at the kink of the indifference curve and so where

$$\begin{aligned}
&2x_1 = x_2 \iff 2(10-k) = 2 + \frac{5k^2}{8} \\
&\iff 0 = \frac{5k^2}{8} + 2k - 18 \\
&\iff 5k^2 + 16k - 144 = 0 \\
&\iff (k-4)(5k+36) = 0
\end{aligned}$$

So our solution is $k = 4$ and so we get our Pareto optimum:

$$(x_1, x_2) = (6, 12)$$

b) The firm has production set

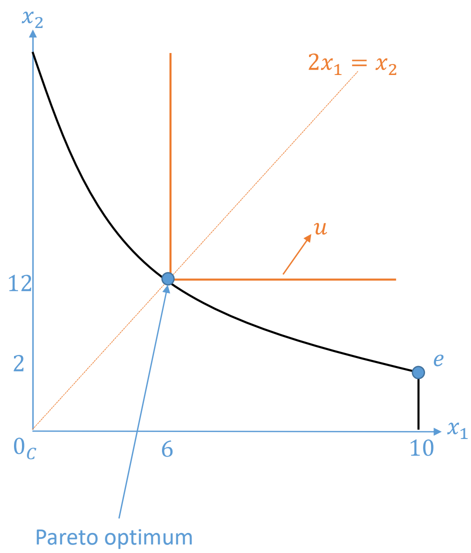
$$Y = \left\{ \mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0,\, y_2 \leq \frac{5\left(-y_1\right)^2}{8} \right\}$$

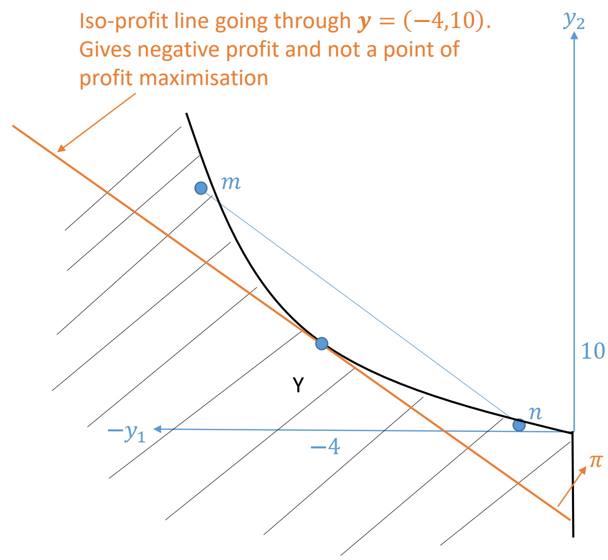
c) To apply the First Welfare Theorem, we need to show that preferences satisfy local non-satiation. We only have one utility function: $u\left(x_1, x_2\right) = \min\left\{2x_1, x_2\right\}$. Take any $\mathbf{x} \in \mathbb{R}^2_{\geq 0}$, and consider an $\varepsilon -$ ball around $\mathbf{x}$. For any $\varepsilon > 0$, the $\varepsilon -$ ball contains bundles with slightly more of both goods and any such bundle is strictly preferred to $\mathbf{x}$. Note that this argument actually shows the slightly stricter property of monotonicity. Alternatively, one could draw a diagram with indifference curves and shade in elements of the $\varepsilon -$ ball which are preferred to $\mathbf{x}$.

Since we have shown local non-satiation, we can conclude by the First Welfare Theorem, that all Walrasian Equilibria are Pareto efficient. As there is only one Pareto efficient allocation, this means that if a Walrasian Equilibrium exists, it must be at $(x_1, x_2) = (6, 12)$.

d) The Second Welfare Theorem cannot be applied because the production set violates convexity.

e) There is no Walrasian Equilibrium. As argued by the First Welfare Theorem, if a Walrasian Equilibrium exists, it must lie at $(x_1, x_2) = (6, 12)$, requiring the firm to produce $(y_1, y_2) = (-4, 10)$ which would not be a profit maximising output of the firm. Thus no Walrasian Equilibria exist. Alternatively one could argue that when $p_2 > 0$ the firm has no profit maximising output. While $p_2 = 0$ would violate market clearing as there would be excess demand of good 2.





Post-class question

Short essay question: Central to our analysis and the Welfare Theorems is the assumption that production sets are convex. To what extent do you think that is fair assumption to make?

Sketch Solution: For convexity of production sets we need 5 main things:

1. Nonincreasing returns to scale.

2. Free-disposal.

3. Infinitely divisible production space: inputs and outputs can be any real numbers, not just integers for example.

4. When we have more than 1 input, taking averages of inputs is better than taking extremes: for example, suppose we can produce a given level of output with either 10 units of input 1 or 10 units of input 2, then we could also produce that level of output with 5 units of each input.

5.When we have more than 1 output, if we can produce extremes of outputs then we could also produce averages: for example, suppose that for a given level of inputs, we can produce either 10 units of output 1 or 10 units of output 2, then with the same inputs we could also produce 5 units of each output.

You can then comment on how reasonable you judge each of these 5 assumptions to be.