EC202 Term 1 Problem set 7

November 22, 2021

Pre-class Questions

Not covered in seminar unless time permits.

Q1. Let $\mathcal{A}$ be the set of alternatives where $|\mathcal{A}| \geq 3$ and suppose we have $N \geq 2$ voters, each with a strict preference ordering over the set of alternatives and we are interested in defining an SWF giving an ordering over the set of alternatives (where indifference between alternatives is allowed). For each of the following say whether they are an SWF or not. If they are an SWF then which of Arrow's axioms (U), (IIA), (ND). Answer this assuming we have unrestricted domain (UD) meaning voters can have any preferences.

a) Given preferences, we select an agent at random, each with probability $\frac{1}{N}$. The SWF replicates the chosen agent's preferences.

b) Borda Count: (Lecture notes 8, Example 2.4)

c) Plurality: (Lecture notes 8, Example 2.3)

Justify your answers.

Solution:

a) This is not an SWF because it is not a deterministic function, except at the very few VPPs whre everyone has the same preferences.

b) This is an SWF because it assigns every alternative a score $\in \mathbb{R}$, uniquely defined for each VPP then ranks them by that score and "$\geq$" is transitive over $\mathbb{R}$. It satisfies (U) because if everybody ranks x higher than y then each voter assigns it a higher score, and when we sum these scores across voters to get its Borda score, that number must be higher too. The example below shows it violates (IIA):

	voter 1	voter 2	voter 3
1st preference	a	b	c
2nd preference	b	c	a
3rd preference	c	a	b

In this Example Borda Count gives $aIbIC$. If voter 3 changes their preferences over (a, b) then the SWF changes to $bPcPa$ changes its preferences over (b, c) to bPc even though no voter has changed its preferences over (b, c). It satisfies non-dictatorship, since in the preferences given in the table the SWF gives $aIbIC$ which isn't any voter's preference.

c) This is an SWF because it assigns every alternative a score $\in \mathbb{R}$, uniquely defined for each VPP then ranks them by that score and "$\geq$" is transitive over $\mathbb{R}$. It violates (U). As an example to see this suppose everybody ranks x as their second preference and y as their third preference. Then both x and y get a score of 0 and so xIy, however (U) requires xPy since everyone prefers x to y. It violates IIA. As an example of this, take the same VPP as in the table in b). Here each alternative gets 1 vote and so plurality gives $aIbIC$. However if we change voter 3's preferences over (c, a) then plurality gives $aPbPc$. So preferences over (a, b) have changed even though no voter changed their preferences over (a, b). It satisfies non-dictatorship, since in the preferences given in the table the SWF gives $aIbIC$ which isn't any voter's preference.

Q2. Let there be 2 voters and 3 alternatives $\{a, b, c\}$, each voter submits a strict prefererence ordering and then an alternative is chosen. Consider the following rules based on the idea of plurality voting system:

i) Select the alternative with the highest number of first preferences. If there is a tie then flip a coin to decide which of the two alternatives which tied is chosen.

ii) Select the alternative with the highest number of first preferences. If there is a tie then decide alphabetically - so in this case if a and b or a and c tie then a is chosen; if b and c tie then b is chosen.

iii) Select the alternative with the highest number of first preferences. If there is a tie then the alternative ranked higher in person 1's preferences is chosen.

For each of these rules, say whether or not they define an SCF and if they do, which of the SCF axioms (U), (SP), (ND) they satisfy. Justify your answers

Extra questions for students to consider on their own or in groups: How would your answers change if we had 4 voters instead of 2? How about, if instead of basing a SCF on plurality, we based it on the Borda Count or Alternative vote?

Solution:

i) This is not a SCF as there is a random element as to which alternative is chosen when the two voters rank a different alternative as top. SCF has to be a deterministic function.

ii) This is an SCF because, for each pair of preferences voters could report, the SCF does select a winner in a deterministic way. It satisfies (U) since if both players rank the same alternative as top then it is chosen. It violates (SP) as the example below shows:

	voter 1	voter 2
1st preference	c	b
2nd preference	a	c
3rd preference	b	a

Here if voter 1 submits his true ranking of $c \succ a \succ b$ then his least preferred alternative, b is chosen. Therefore he would be better off lying, reporting a as

his most preferred alternative in order to get his 2nd preference of a selected. It satisfies (ND) as in the table voter 1 doesn't get his preferred outcome and if player lies to put a top then voter doesn't get his preferred outcome. Intuitively, any SCF or SWF that treats all voters equally can't be a dictatorship.

iii) This is an SCF because, for each pair of preferences voters could report, the SCF does select a winner in a deterministic way. It satisfies (U) since if both players rank the same alternative top then that alternative is chosen. It violates (ND) since person 1 here is a dictator. Whichever alternative person 1 reports top is chosen, since even if person 2 disagrees, person 1 gets the deciding vote in case of a tie. Since it is dictatorial it will satisfy (SP): person 1 has a strong incentive to report truthfully; while what person 2 reports is irrelevant, which implies he has no strict incentive to lie. Reporting truthfully is a best response.

In-class question(s)

Your seminar tutor will cover Q3 and some portion of Q4 depending on time constraints.

Q3. Recall that for Arrow's Impossibility Theorem to hold, we need the following 6 things:

i) At least 3 alternatives,$|\mathcal{A}| \geq 3$.

ii) At least two voters, $N \geq 2$.

iii) Unrestricted Domain, (UD)

iv) Unanimity, (U)

v) Independence of Irrelevant Alternatives, (IIA)

vi) Non-dictatorship, (ND)

For each of these six, show why they are needed for Arrow's Impossibility Theorem to hold. In other words find an example of an SWF satisfying the other five.

Solution:

i) If only 2 alternatives, plurality, Borda Count, Alternative Vote, pairwise comparison all become the same SWF: we prefer whichever alternative gets the most votes. This SWF satisfies all axioms. As a sidepoint, note that there are other rules too that would work that treat the two alternatives asymmetrically. For example, in Florida to pass constitutional ammendments requires 60% of people to vote for the change over the status quo. Such a voting rule where the threshold is higher than 50% is called a "supermajority".

ii) With $N = 1$, technically Arrow's Theorem would still hold, since if we had just one voter, any SWF satisfying (U) must be a dictatorship where that one voter gets their way for every possible preference profile thy could have. However, having a dictatorship in this instance is no longer objectionable, while the problem of how to aggregate voters' preferences is now trivial and so ceases to be interesting.

iii) As we mentioned in the lecture notes, Single peaked preferences is a sufficient domain restriction for pairwise majority voting to now satisfy transitivity and hence be an SWF. Pairwise majority will also satisfy (U), (IIA), (ND). To see these each in turn: (U) is satisfied since if everyone prefers x to y, that implies a majority prefer x to y. (IIA) is satisfied by construction: in comparing x and y we only care whether a majority prefer x to y or vice-versa. Thus the only way to change the result of this SWF is by changing preferences over (x, y). Changing how voters feel over (x, z) or (y, z) has no effect. (ND) is satisfied since all voters are treated equally. To give an example, if we consider the VPP where all voters apart from i have the same preferences as each other, but i has different preferences then voter i does not get their way. We can find such a VPP for every $i \in N$ assuming (UD).

iv) Consider 3 alternatives $\{a, b, c\}$ and the SWF defined as: for every VPP we have the ranking $aPbPc$. This means that regardless of what our voters think, we always get $aPbPc$. This example was for 3 alternatives, but similar would exist for any number of alternatives. This defines an SWF over (UD), since for every VPP it gives a well-defined, deterministic ordering, namely $aPbPc$. It violates (U) since there are VPPs where everybody has different preferences. For example take a VPP where everyone prefers b to a. It satisfies (IIA) trivially: to contradict (IIA) we need to construct an example where the SWF changes its ranking over (x, y) despite no voters changing their ranking over (x, y). That cannot here since the SWF never changes its ranking. It satisfies (ND) for similar reasons to (U). Take a VPP where everyone prefers b to a then nobody is getting their way and so nobody is a dictator.

v) There are numerous examples of this. One is Borda Count, as we showed in Q1b).

vi) Let voter 1 be a dictator. This is a well-defined SWF over (UD) since it always gives a well-defined deterministic ranking, namely voter 1's ranking. It satisifies (U) since if everyone prefers x to y, it means voter 1 prefers x to y and so xPy. It satisfies (IIA) since we cannot construct a counterexample to (IIA): in order to change the SWF's ranking over (x, y), we need to change voter 1's preferences over (x, y). It violates (ND) by definition.

Q4. This question is about the merits of the plurality voting system compared to Alternative Vote. We show some pros and cons of each.

Let there be 3 alternatives and a large number of voters. Each voter has a strict preference ordering so fits into one of the following 6 categories:

	I	II	III	IV	V	VI
1st Preference	a	a	b	b	c	c
2nd Preference	b	c	a	c	a	b
3rd Preference	c	b	c	a	b	a

By varying the percentages of voters in each category, we demonstrate how we can get examples of some of the disadvantages of each system:[1]

[1] In your examples, don't set percentages so that there are tied votes so that we can avoid the complications of tie-breaking rules.

a) Consider the voting profiles below.

Profile 1	I	II	III	IV	V	VI
	20%	15%	20%	12%	13%	20%
1st Preference	a	a	b	b	c	c
2nd Preference	b	c	a	c	a	b
3rd Preference	c	b	c	a	b	a

Profile 2	I	II	III	IV	V	VI
	20%	17%	20%	12%	11%	20%
1st Preference	a	a	b	b	c	c
2nd Preference	b	c	a	c	a	b
3rd Preference	c	b	c	a	b	a

a1) Show that under Profile 1 Alternative vote elects a. However, if 2% of the population change their preferences over (a, c) in favour of a, to give Profile 2 then Alternative Vote will elect b instead of a.

a2) To summarise what happened in i), a was winning under Alternative Vote. We changed how voters felt about (a, c) in favour of a without changing how voters feel about (a, b) or (b, c) and as a result a is no longer elected. Explain why plurality could never fall victim to this peculiarity.[2]

b1) Show an example where assuming people vote truthfully, plurality elects a, however if some voters change their preferences over how they rank b compared to c, then plurality would elect b instead.

b2) Comment on your findings in a1) and how this links to the (IIA) axiom for SWFs and (SP) axiom for SCFs.

c) How does Alternative Vote fare with these two axioms?

d) Show that when a Condorcet winner exists, plurality might fail to elect it. And furthermore it could elect a Condorcet loser.

e) Show that when a Condorcet winner exists, Alternative Vote might fail to elect it. However it cannot elect a Condorcet loser.

Solution:

a1) Under AV using the Profile 1: 1st Round a gets 35%, b gets 32%, c gets 33% so b gets eliminated. In the final round a beats c with 55% to 45% so a is elected.

Under AV using the Profile 2: 1st Round a gets 37%, b gets 32%, c gets 31% so c gets eliminated. In the final round b beats a with 52% to 48% and so b is elected.

In this example a was winning, gained more support and as a result lost, because it gained more support at the expense of the candidate it beats in pairwise comparison and so in the final under AV faced the candidate it loses to in pairwise comparison.

a2) Plurality can never fail this criterion, since if the only change in preferences is to make a more popular then this increases the number of voters it

[2]Social choice theorists have defined a "monotonicity" axiom (beyond the scope of the course). The "peculiarity" here is an example of Alternative Vote violating this monotonicity axiom.

gets, while no other candidate gets more votes.

b1) Examples of this are easy to come by. I give a particularly extreme example to highlight the flaws of plurality. Supppose we have 34% in category I, and 33% in category IV and 33% in category VI. That gives what I have called Profile 3.

Profile 3	I	II	III	IV	V	VI
	34%	0%	0%	33%	0%	33%
1st Preference	a	a	b	b	c	c
2nd Preference	b	c	a	c	a	b
3rd Preference	c	b	c	a	b	a

With these preferences plurality elects a with 34% beating its two rivals who each get 33%. Consider these changed preferences over (b, c) moving some voters from category VI to category IV so that we get

Profile 4	I	II	III	IV	V	VI
	34%	0%	0%	35%	0%	31%
1st Preference	a	a	b	b	c	c
2nd Preference	b	c	a	c	a	b
3rd Preference	c	b	c	a	b	a

In making this change, only the ranking over (b, c) has changed, not over (a, b) or (a, c). But now in profile 4, we get b elected with 35% of the vote.

b2) This can be seen as an example of a failure of (IIA) since in moving from Profile 3 to Profile 4, voter's preferences over (a, b) haven't changed but the social ranking over (a, b) has changed. It also demonstrates why the plurality system fails (SP), in other words, gives voters an incentive to misreport their preferences. Suppose true preferences were as in the top table of b1) then voters in category VI have an incentive to lie and so give us reported preferences like those in the bottom table. Although in this particular example, voters might face coordination problems in practice. If this is an election where the opinion polls are showing such a close race then voters in categories IV and VI might not know whether they should vote tactically to get b elected or c elected. If the number of category IV people who tactically vote for c eqauls the number of category VI people who vote tactically for b then a still gets elected.

c) As we saw in a), Alternative Vote violates both. To show this, consider moving from Profile 1 to Profile 2. In making this change only preferences over (a, c) have changed. Preferences over (a, b) have remained the same and yet the Social ranking now has b top instead of a. This shows that any SWF based on AV must violate (IIA). To see strategy proofness (SP) violated, assume true preferences are as in Profile 2, then a's supporters have an incentive lie and put c top of their ranking so that we get reported preferences like Profile 1 where a gets elected. However, in practice, a's supporters face a co-ordination problem: if too many of them switch to c, then a would not get enough votes to reach the final round.

d) In Profile 3 the Condorcet winner is b but plurality elects a. In fact by electing a plurality elects the Condorcet loser, since 66% of people think this is

the worst alternative and so it loses pairwise comparisons to both b and c.

e) In Profile 1 the Condorcet winner is b but AV elects a. However AV cannot elect a Condorcet loser, because the winner under AV has to win in the final round where it faces another alternative in pairwise comparison.

Post-class question

Short essay question: In 2011 the UK had a referendum about whether to move from the current system of electing MPs by a plurality system in each constituency to electing MPs by implementing the Alternative Vote (AV) system in each constituency. Discuss the pros and cons of each system. If you were campaigning in that referendum, which side would you have been on and what arguments would you have made to try to persuade the public.

Sketch Solution: Some of the advantages and disadvantages of the two systems can be seen in Q4.

From a theorist's perspective I would favour AV over plurality. As we saw in Profile 3 of Q4, an alternative can be elected under plurality even when the vast majority of people despise it. As the number of alternatives increase, this problem grows: an alternative can win with a lower and lower share of the vote and so we could end up with something that most people do not like. This theory also suggests that the plurality system gives parties an incentive to put forth policies that don't have majority support, but will energise a portion of the electorate to vote for them. An example of this can be seen in Scottish politics where in 2015 the SNP won 56 out of 59 seats in Scotland while being pro-indendence despite the Independence referendum 2 years earlier showing that a majority in Scotland support remaining part of the UK[3]. They benefit from being the only pro-independence party, getting all the pro-independence vote, while the other parties split the pro-union vote. AV on the other hand can never elect a Condorcet Loser and so rewards parties who put forward more moderate policies in order to attract voters' 2nd preferences.

In my opinion, the best reason to support AV over plurality comes from looking at the incentives to vote tactically. As we have demonstrated in Q4, both AV and plurality violate strategy proofness, however I think this is more likely to matter in the case of plurality. To construct examples like Profile 1 required quite specific preferences and even if voters know other voters' preferences, they may be wary of voting tactically for the reasons discussed in Q4c. However there are plenty of actual real life examples of constituencies where voters have a clear, unambiguous incentive to vote tactically and much anecdotal evidence from voters suggesting they do. Consider a constituency that historically has swung been close between Conservatives and Labour - lets say each getting around 40% of the vote. If you are a voter in this constituency, who supports a 3rd party like Lib Dems or Greens, you know there is no point in voting that way because that party would never be elected - in general conversation, this is often

[3] Although in the post-Brexit world of 2020 this may be changing. Also they may attract voters who vote for them for other reasons

called “a wasted vote”. Instead you have an incentive to vote for whichever one of Labour or Conservative you like more (or dislike less). As a result, the way people vote may not be a fair reflection of their actual preferences and even if 3rd parties start becoming more popular with the electorate, they will never get many votes if a large portion of their supporters vote tactically for the 2 larger parties. While under AV, this problem would be solved since a voter could vote for their actual preferred candidate as first preference and use their 2nd or lower preference to register who they prefer out of the 2 potential winners. Thus if a 3rd party does start to become more popular in a constituency, then that will be fairly represented in the election results and it will be easier for that party to mount a challenge in future elections.

On the other hand, there are certain practical advantages of plurality over AV. The first is simplicity: plurality is a very simple and easy system for people to understand, whereas AV is much more complicated. In a world where many people are already disengaged from politics, it is feared that having a more complex voting system would encourage more people to switch off. Other advantages of plurality linked to its simplicity, is it makes it quicker and easier to see who has won, and voters only need to form opinions on which candidate they like the most instead of giving a complete list from top to bottom. From a theorist’s perspective, the main advantage of plurality over AV can be seen in Q4a.

Unfortunately the academic arguments weren’t really made in the election campaign. Party politics took over instead. You can read wikipedia’s summary here: https://en.wikipedia.org/wiki/2011_United_Kingdom_Alternative_Vote_referendum