EC202 Week 6
Covering Materials from Week 4

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Production

Definition 1.1. A firm has *technology* or *production set* $Y \subseteq \mathbb{R}^J$ describing the feasible net outputs of the firm. Note that with this notation, for $\mathbf{y} = (y_1, \ldots, y_J) \in Y$, if $y_j < 0$, it means that the firm is using good j as as input; while if $y_j > 0$ then the firm is producing good j as an output.

The above convention that negative amounts of goods mean inputs is made so that we can define the profit that a firm makes when producing $\mathbf{y}$ at price vector $\mathbf{p}$ as:

$$\text{Firm's profit} = \mathbf{p}.\mathbf{y} = p_1 y_1 + p_2 y_2 + \ldots p_J y_J$$

To understand this better, we could decompose this into revenue and cost[1]:

$$\text{Revenue} = \sum_{i:y_i>0} p_i y_i \qquad \text{Cost} = -\sum_{i:y_i<0} p_i y_i$$

$$\Longrightarrow \text{Revenue} - \text{Cost} = \sum_{i:y_i>0} p_i y_i + \sum_{i:y_i<0} p_i y_i = \mathbf{p}.\mathbf{y}$$

Production

- Walrasian Equilibrium with production
  - Recall Walrasian Equilibrium
    - Each player reaches his/her maximum utility
    - Market clear
  - Formal Setup

Definition 2.2. Given a production economy with fundamentals described in Definition 2.1, a Walrasian Equilibrium is a triple $\left(\mathbf{p}, (\mathbf{x}_i)_{i \in I}, (\mathbf{y}_m)_{m \in M}\right)$ such that:

1. For each consumer $i \in I$, the choice of bundle $\mathbf{x}_i$ solves the UMP:

$$\max_{\mathbf{x}_i \in \mathbb{R}^J_{\geq 0}} u_i(\mathbf{x}_i) \text{ subject to } \mathbf{p}.\mathbf{x}_i \leq \mathbf{p}.\mathbf{e}_i + \sum_{m \in M} s_{i,m} \mathbf{p}.\mathbf{y}_m$$

2. For each firm $m \in M$, bundle $\mathbf{y}_m$ solves the profit maximisation problem:

$$\max_{\mathbf{y}_m \in Y_m} \mathbf{p}.\mathbf{y}$$

3. All markets clear:

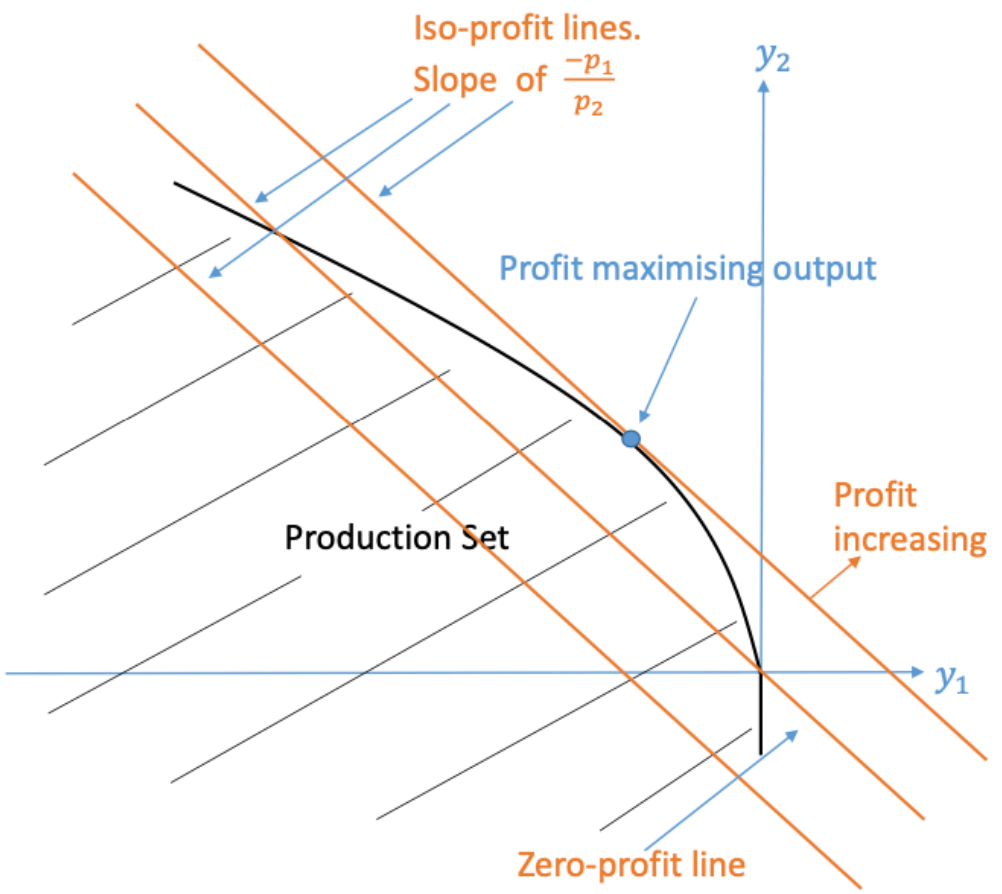
$$\sum_{i \in I} \mathbf{x}_i(\mathbf{p}) = \sum_{i \in I} \mathbf{e}_i + \sum_{m \in M} \mathbf{y}_m(\mathbf{p})$$

Profit Maximization

Given the above defintion of Walrasian Equilibrium, we would like to know what it means for a firm to be profit maximising. The firm is profit maximising with respect to production set Y if it solves

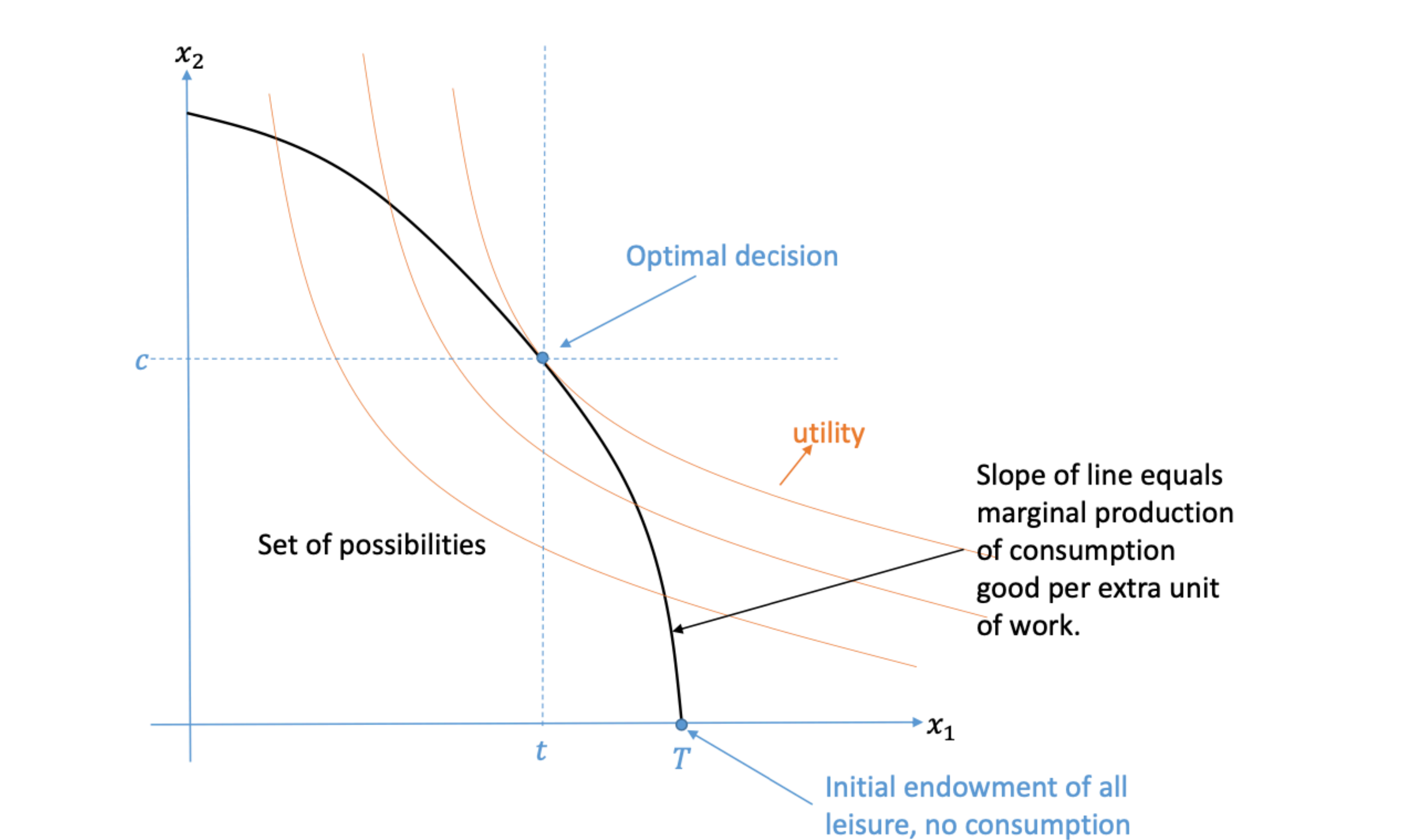
$$\max_{\mathbf{y} \in Y} \mathbf{p}.\mathbf{y}$$

We show graphically what this means for a nonincreasing returns to scale, convex production set where the firm uses good 1 as an input to produce good 2 as an output:



Robinson Crusoe Economies

- The Model
  - Two goods
    - Leisure and Consumption
  - Constraint
    - Fixed amount of time to spend between working and leisure
  - Marginal output
    - Decreasing marginal output
  - Pareto efficiency



In-class Question

- Q3. Crusoe has 12 units of time (good 1) to allocate between work and leisure. If he works for k hours he can produce 2k units of the consumption good (good 2) and can freely dispose of each good. Crusoe has utility function:

$u : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ where

$$u(x_1, x_2) = \frac{e^{\sqrt{x_1 x_2}}}{17} + 2\pi (x_1 x_2)^5 - 6$$

a) Find the Pareto efficient bundle and draw a diagram to illustrate it.
b) We model this situation using Walrasian Equilibrium:
i) Assuming free-disposal, write down the Production set and find the firm's profit maximising output vector as a function of prices $\mathbf{y}(\mathbf{p})$ and the profits the firm makes. Draw a diagram to illustrate this.
ii) Find Crusoe's optimal bundle $\mathbf{x}(\mathbf{p})$.
iii) Write down the market clearing conditions and verify that Walras' law holds.
iv) Find the Walrasian equilibrium prices and allocation. Illustrate it on a diagram.
v) 1 Consider the case where Crusoe has some good 2 in the initial endowment, that is $\mathbf{e} = (12, c)$ for $c > 0$. Describe what would happen to equilibrium prices as c increases and give some intuition for this.

Solution: To start with observe that we could also represent the preferences u by $v(x_1, x_2) = x_1 x_2$ since u is an increasing transformation of v. Or for any 2 bundles $\mathbf{x}, \hat{\mathbf{x}}$, we have

$$u(\mathbf{x}) \geq u(\hat{\mathbf{x}}) \iff v(\mathbf{x}) \geq v(\hat{\mathbf{x}})$$

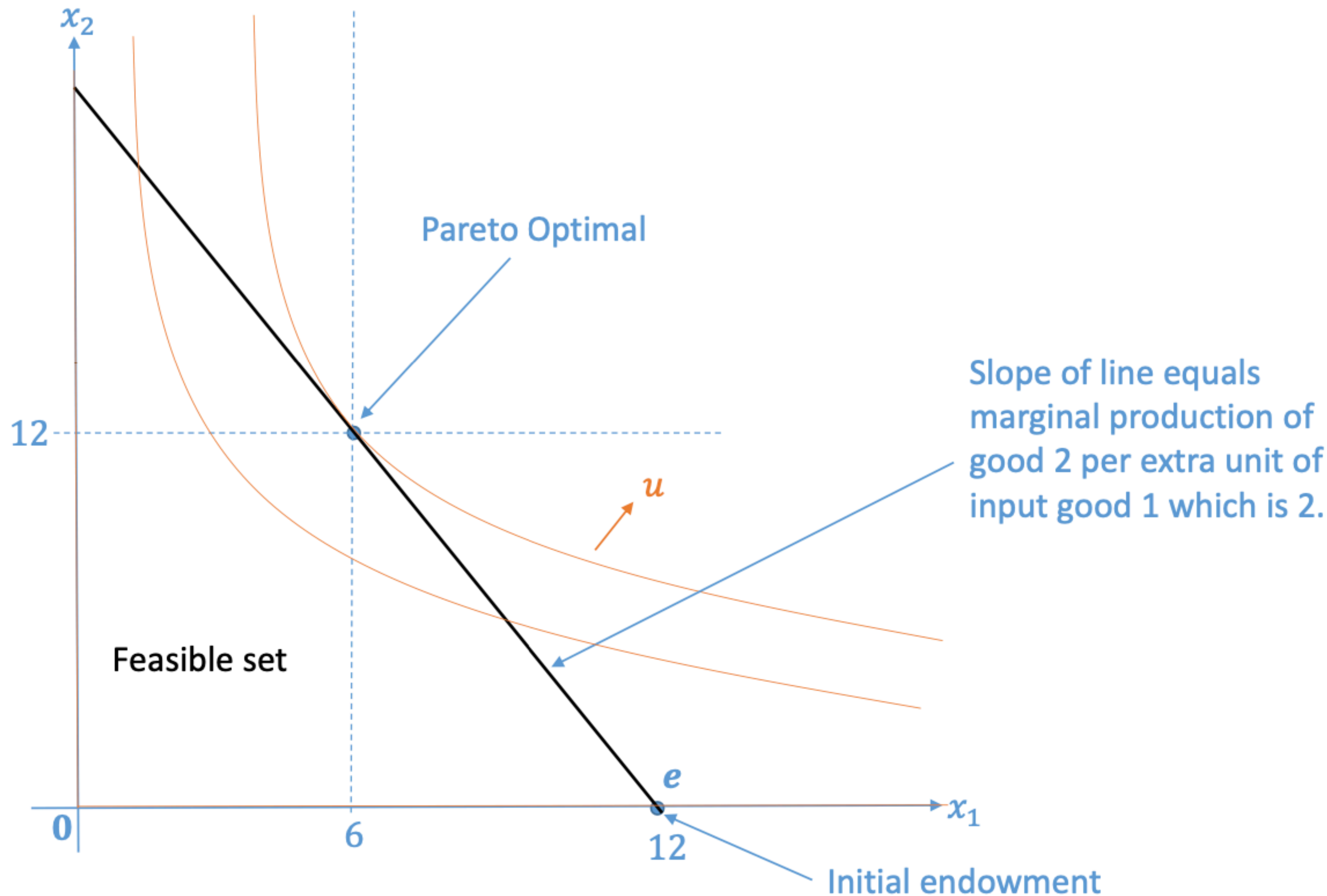
This is because both u and v only care about the product of the two goods, $x_1 x_2$. For reasons of simplicity, we choose to work with v instead of u from now on.

a) Crusoe will optimise where he does not freely dispose (waste) either good and so solves

$$\max_{k \geq 0} v = x_1 x_2 \text{ s.t. } x_1 = 12 - k, x_2 = 2k$$

which gives solution $k = 6$ and thus $\mathbf{x} = (6, 12)$. Note that at this point, the slope of the indifference curve equals the slope of the production function which is Crusoe's Marginal Product of Labour (MPL).

$$MRS_{1,2} = \frac{x_2}{x_1} = 2 = \frac{d}{dk}(2k) = MPL$$



x_2
Pareto Optimal
Slope of line equals marginal production of good 2 per extra unit of input good 1 which is 2.
12
u
Feasible set
e
0
6
12
x_1
Initial endowment

b) We use the model as in Lecture 4: we have a firm with production function determined by Crusoe's marginal product of labour which buys labour off Crusoe to produce good 2. Crusoe gets money from selling his labour plus any profits the firm makes, which he spends on buying good 2 from the firm.

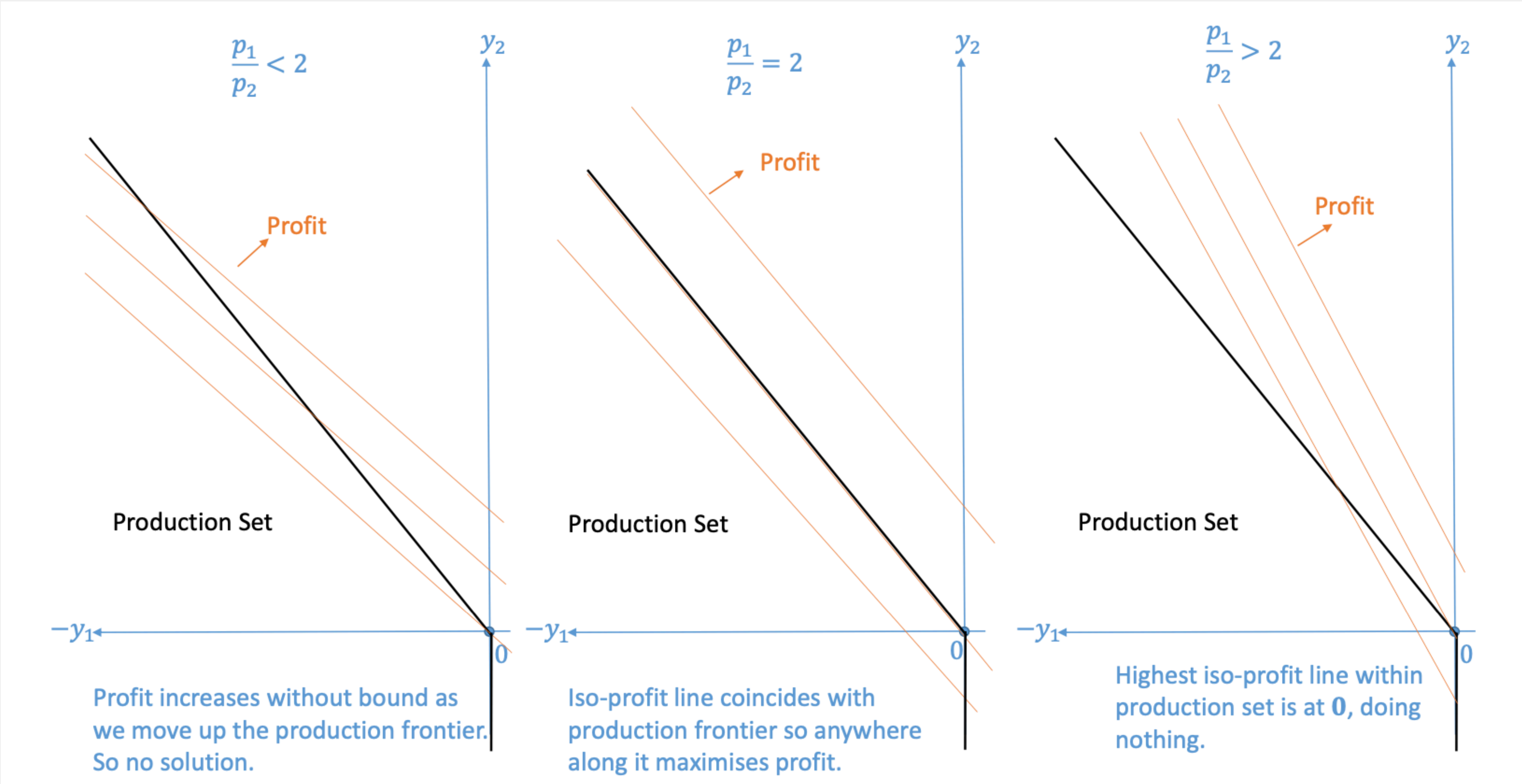
i) Since we assumed free disposal, the firm has technology

$$Y = \{\mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0, y_2 \leq -2y_1\}$$

In solving the profit maximisation problem, we have 3 cases, depending on the slope of the iso-profit lines compared to the slope of the production frontier:

As seen from the diagrams below we get

$$\mathbf{y}(\mathbf{p}) = \begin{cases} \emptyset & \frac{p_1}{p_2} < 2 \\ \{\mathbf{y} \in \mathbb{R}^2 \mid y_1 \leq 0, y_2 = -2y_1\} & \frac{p_1}{p_2} = 2 \\ \mathbf{0} & \frac{p_1}{p_2} > 2 \end{cases}$$


$\frac{p_1}{p_2} < 2$
y_2
Profit
Production Set
$-y_1$
0
Profit increases without bound as we move up the production frontier. So no solution.
$\frac{p_1}{p_2} = 2$
y_2
Profit
Production Set
$-y_1$
0
Iso-profit line coincides with production frontier so anywhere along it maximises profit.
$\frac{p_1}{p_2} > 2$
y_2
Profit
Production Set
$-y_1$
0
Highest iso-profit line within production set is at $\mathbf{0}$, doing nothing.

ii) From the $\mathbf{y}(\mathbf{p})$ found above, we can see that for $\frac{p_1}{p_2} < 2$ profits are unbounded. Since these profits should be added into the income side of Crusoe's budget constraint, it means his UMP is not well-defined for $\frac{p_1}{p_2} < 2$. So in what follows, I assume $\frac{p_1}{p_2} \geq 2$. In all these cases the firm makes no profit and so Crusoe solves

$$\max_{\mathbf{x} \in \mathbb{R}^2_{\geq 0}} v = x_1 x_2 \text{ s.t. } p_1 x_1 + p_2 x_2 \leq 12 p_1$$

Using the knowledge that with this utility function we spend half our income on each good, we get the solution

$$\mathbf{x} = \left(6, \frac{6p_1}{p_2}\right)$$

iii) The market clearing conditions are:

$$z_1(\mathbf{p}) = 6 - y_1 - 12 = 0 \quad (\text{ Good } 1)$$

$$z_2(\mathbf{p}) = \frac{6p_1}{p_2} - y_2 = 0 \quad (\text{Good } 2)$$

Walras' Law says $\mathbf{p.\,z(p)} = 0$, which we can confirm:

$$\mathbf{p.\,z(p)} = p_1(6 - y_1 - 12) + p_2\left(\frac{6p_1}{p_2} - y_2\right)$$

For $\frac{p_1}{p_2} = 2$, we have $y_2 = -2y_1$ thus

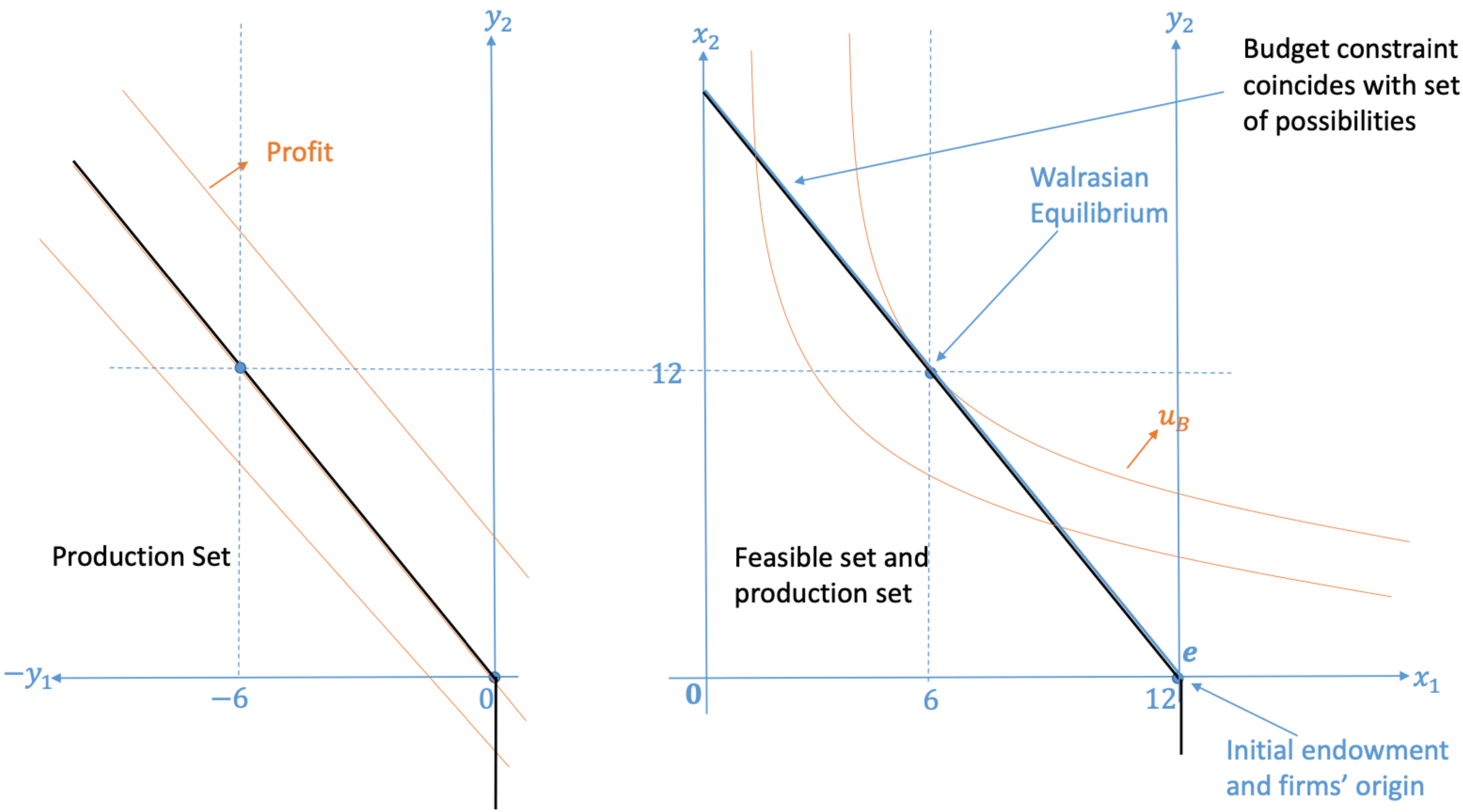
$$\mathbf{p.\,z(p)} = 2p_2(6 - y_1 - 12) + p_2\left(\frac{6(2p_2)}{p_2} + 2y_1\right) = 0$$

For $\frac{p_1}{p_2} > 2$, we have $y_1 = y_2 = 0$ thus

$$\mathbf{p.\,z(p)} = p_1(6 - 12) + p_2\left(\frac{6p_1}{p_2}\right) = 0$$

iv) By market clearing of good 1, $y_1 = -6$. Thus $\frac{p_1}{p_2} = 2$ and $y_2 = 12$. Note this also clears the market of good 2, confirming our answer. Thus the Walrasian Equilibrium is:

$$\mathbf{p} = (2, 1) \quad \mathbf{x} = (6, 12) \quad \mathbf{y} = (-6, 12)$$



v) In general, one expects that as c increases, meaning good 2 becomes less scarce, its price will fall and hence $\frac{p_1}{p_2}$ increases. However in this particular instance, due to the nature of the production technology, equilibrium prices can only be of the form $\frac{p_1}{p_2} = 2$ or $\frac{p_1}{p_2} > 2$. Intuitively, for small $c > 0$, the Walrasian Equilibrium will be similar to the above where $\frac{p_1}{p_2} = 2$ and the firm transforms some units of good 1 into good 2 . As c increases, the amount of production the firm needs to do so that markets clear decreases. While if c was really high, then there would be no role for the firm at all. Between the two, there is a critical value of c which is the upper bound on the values of c for which we have $\frac{p_1}{p_2} = 2$ in equilibrium. At this level of c, the firm will produce nothing. So we need that Crusoe is maximising utility at the initial endowment and $\frac{p_1}{p_2} = 2$, so we let $\mathbf{p} = (2, 1)$. For Crusoe to be solving his UMP we need the bang per buck of each good to be the same:

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} \iff \frac{x_2}{2} = \frac{x_1}{1}$$

We've fixed the endoment of good 1 at 12 and so the critical value of c is $c = 24$. So for $c \in [0, 24]$ the equilibrium price ratio remains at $\frac{p_1}{p_2} = 2$. For $c > 24$ we need prices to be such that Crusoe's initial endowment solves his UMP and this will require $\frac{p_1}{p_2} > 2$ and increasing in c. In this latter case we have a rather trivial economy with no trade becasue the only entity Crusoe could trade with (the firm) doesn't have a production technology that can help Crusoe.