Three Types of Real Option Problems

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Here I summarize the general approach to solving agency theory problems, with an concrete example from 2022 assignment.

1 First things first: identify who is who

An agent theory game usually includes two players: a (rich and dumb) principal, who pays the agent to work for him/her, and a (poor and clever) agent, who wants to maximize utility and minimize effort.

Only when you are absolutely clear about who is the principal and who is agent will you characterize the problem properly. No matter how the setting changes, find who is paying money and who is exerting effort. Let's see an example.

2022 Assignment

A private investor (PIR) is interested in taking over the Warwick Parkway to London Marylebone rail line. The revenues of operating the plant are determined by the passenger demand - a random parameter ω distributed uniformly on $[0, 1]$ - and by the electricity (e) used; the electricity is provided by a Cornish supplier (EDC). The revenue function is ω e, the cost of electricity supply is $\frac{e^2-1}{2}$ and the supplier's outside option is $\overline{U} > 0$. PIR needs to specify a contract paying EDC an amount π (if the railway produces revenue R, PIR gets $R - \pi$ and the electricity supplier EDC gets π. Suppose, moreover, that EDC must accept or reject the contract before observing ω, but chooses the amount of electricity after observing it.

2022 Assignment: streamlined version

Principal: PIR

Agent: EDC

Principal's payment to Agent (which is also her choice): π

Agent's effort level (which is also her choice): e

Principal's utility: $R - \pi$

Agent's utility: $\pi - \frac{e^2-1}{2}$, or the ouside option $\bar{U}$, whichever is larger

Revenue: $R = \omega e$

Randomness: $R = \omega e$, where ω is uniformaly distributed

Question

Assume that PIR can observe the revenue R but cannot separately observe $\boldsymbol{e}$ or $\boldsymbol{\omega}$. You may assume that the contract takes the form $\boldsymbol{\pi} = \boldsymbol{\alpha R}$ where $\boldsymbol{\alpha}$ is a (not necessarily positive) parameter. Find EDC's optimal quantity as a function of $\alpha, \bar{U}$ and R, and use this to compute the value of α that maximises the PIR's expected profit. Also, find the optimised expected value of the railway line as a function of $\bar{U}$.

2 How does the game proceed?

The key assumption about the uncertainty, which varies among problems and is something you should pay extra attention to, is that the Agent chooses the amount of effort after observing the uncertainty, but has to accept or reject the Principal's offer before observing it.

What this changes is that, once the Agent observes the uncertainty, nothing is uncertain anymore – the Agent will maximize her own utility without any uncertainty. In this question, the agent will

$$\max \pi - \frac{e^2 - 1}{2}$$

inserting the assumption in a. that $\pi = \alpha R; R = \omega e$, the question becomes a simple quadratic function:

$$-\frac{e^2}{2} + \alpha\omega e + \frac{1}{2}$$

which is maximized at $e = \alpha\omega$.

This implies that, the Agent will always choose this level of effort, because when she is exerting effort she already knows the true value of ω.

Very importantly, knowing that the Agent will act after she learns the uncertainty, the Principal will have to choose her best response based on this.

Therefore, the Principal will maximize $R - \pi$, where $\pi = \alpha R; R = \omega e; e = \alpha\omega$

$$\max \omega\alpha\omega - \alpha\omega\alpha\omega = -\omega^2\alpha^2 + \omega^2\alpha$$

Remember, the choice of the Principal is simply the value of α – intuitively, what portion of the total revenue she is willing to give to the Agent.

As the offer is made before the uncertainty is observed, the Principal has no choice but to calculate his expected payoff based on the distribution of the uncertainty – we need to calculate integral (remember, doing intergration on the uncertainty, not α!).

$$(1 - \alpha)\alpha \int_0^1 \omega^2 d\omega = \frac{\alpha(1 - \alpha)}{3}$$

and then the Principal wants to maximize her *expected utility* by choosing the proper α. With simply calculation, she will choose $\alpha = \frac{1}{2}$

However, this question does not stop here. You will also check the actions satisfy *participation compatibility*. What we've done so far is to make sure that both players don't have the incentive to deviate, and thus satisfying the incentive compatibility. By testing participation compatibility, you need to make sure that 1) the Principal is not getting negative utility; 2) the Agent is not getting anything less than the outside option, $\bar{U}$. Due to the fact that we don't know the exact value of $\bar{U}$, we will need to further discuss potential values of it. The results will be

$$\alpha^* = \begin{cases} \frac{1}{2} \text{ if } \bar{U} \in \left(0, \frac{13}{24}\right) \\ \sqrt{6\bar{U} - 3} \text{ if } \bar{U} \in \left(\frac{13}{24}, \frac{2}{3}\right) \\ 0 \text{ (no contract) otherwise} \end{cases}$$

$$V = \begin{cases} \frac{1}{12} \text{ if } \bar{U} \in \left(0, \frac{13}{24}\right) \\ \frac{(\sqrt{6\bar{U}-3})(1-\sqrt{6\bar{U}-3})}{3} \text{ if } \bar{U} \in \left(\frac{13}{24}, \frac{2}{3}\right) \\ 0 \text{ otherwise} \end{cases}$$

For other questions, all you need to do is carefully define who knows the uncertainty at which point, and as a result how would they change their best responses.